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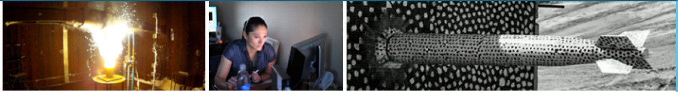
Sandia National Laboratories

Optimization-based multigrid-enabled finite element solvers for the Monge-Ampère equation



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Introduction to the Monge-Ampère Equation

Optimal transport problem:

$$\min_{\phi} \int_{\Omega} \mu_0(\mathbf{x})|\phi(\mathbf{x}) - \mathbf{x}|^2 d\mathbf{x},$$
$$\text{subject to } \det(\nabla\phi(\mathbf{x}))\mu_1(\phi(\mathbf{x})) - \mu_0(\mathbf{x}) = 0,$$

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$$\det(\nabla\phi) = \det(\nabla(\nabla u)) = \det(H(u)) \quad = \det\begin{pmatrix} \frac{\partial}{\partial x} u_x & \frac{\partial}{\partial x} u_y \\ \frac{\partial}{\partial y} u_x & \frac{\partial}{\partial y} u_y \end{pmatrix} \text{ in 2D}$$

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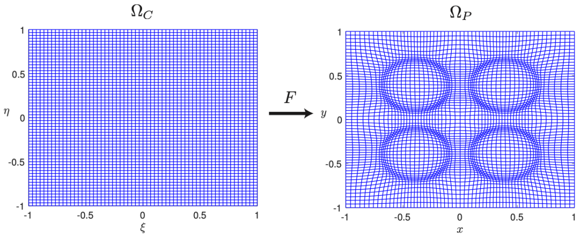
Obtain the Monge-Ampère equation (MAE) with Dirichlet boundary conditions

$$\det(H(\mathbf{u})) = \frac{\mu_0}{\mu_1} = \mathbf{f}, \quad \mathbf{u} \in \Omega,$$
$$\mathbf{u} = g, \quad \mathbf{u} \in \partial\Omega.$$

Applications of the Monge-Ampère Equation

Several applications for the Monge-Ampère equation:

- Inverse problems in seismic imaging.
- Differential geometry.
- Wasserstein Neural Networks.
- Mesh adaptation
  - If u is a convex solution to the Monge-Ampère equation, then $\vec{\mathbf{x}} = \nabla \mathbf{u}$ gives an adaptive mesh in the physical space (assumption of a transport boundary condition).
  - Mesh point locations are determined through a density function $f = \frac{\mu_0}{\mu_1}$.



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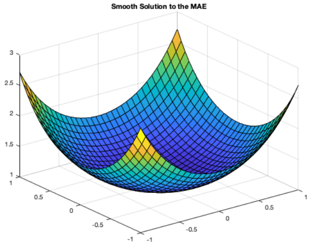

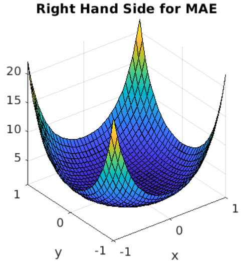

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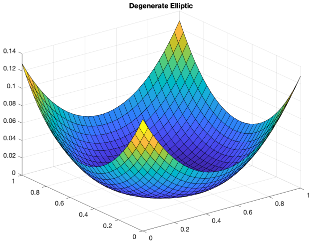
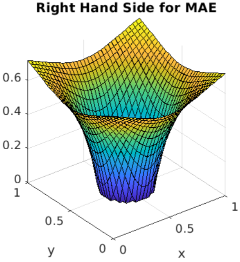
    - Weak solution type.
    - Requires continuity of $f \in C(\Omega)$.
    - Based on theory of sub and super solutions and elliptic operators.





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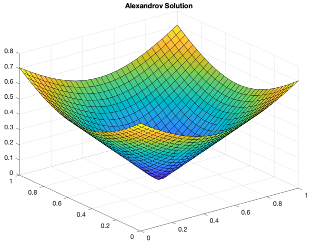
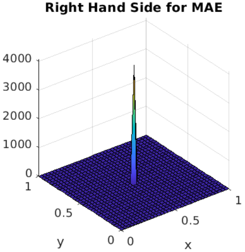
  - Based on theory of sub and super solutions and elliptic operators.
- Alexandrov solutions: $u \in C(\Omega)$.
  - Weakest solution type.
  - **Stable with respect to weak convergence**.
  - Does not require continuity in the source term f.





Conditional Ellipticity for the MAE

Define the Monge-Ampère operator as

$$F(\mathbf{x}, r, H(\mathbf{u})) = \begin{cases} \det H(\mathbf{u}) - f(\mathbf{x}), & \mathbf{x} \in \Omega \\ g(\mathbf{x}) - r, \ \mathbf{x} \in \partial\Omega. \end{cases}$$

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We will “sidestep” convexity through a robust nonlinear solver.

Highlighted Numerical Method

- Focus on the mixed finite element method (or *nonvariational* finite element method) by Lakkis and Pryer †.
- Mixed in that we have auxiliary variables that define the Hessian matrix – thus resulting in a system of equations.
- Remaining talk summary:
  1. Mixed finite element method for linear elliptic problems that use the Hessian matrix.
  2. Application to general nonlinear elliptic systems.
  3. Focusing specifically on the Monge-Ampère equation, propose an **optimization-based multigrid-enabled nonlinear solver**.
  4. Present a variety of 2D and 3D examples.

† **Lakkis, Pryer (2011)**, *A finite element method for second order nonvariational elliptic problems*, SISC.

Mixed Finite Element Method for Elliptic Problems

This method directly applies conforming finite elements to elliptic problems dependent on the Hessian matrix †. Uses the notation D^2u to denote the Hessian of u.

General formulation entails finding **u** such that

$$\mathbf{A} : D^2\mathbf{u} = f \text{ in } \Omega \quad \text{and } \mathbf{u} = g \text{ on } \partial\Omega,$$

for $\mathbf{A} : \Omega \to \mathbb{R}^{d\times d}$.

$\mathbf{A} : D^2\mathbf{u} = \text{trace}(\mathbf{A}^T D^2\mathbf{u})$ is the Frobenius inner product.

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Requires a finite element approximation for the Hessian $D^2\mathbf{u}$.

Allows us to solve an *augmented system* for the elliptic PDE and the Hessian.

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Mixed Finite Element Method for Elliptic Problems

Define a piecewise polynomial finite element space $\mathbb{P}^k$ for elements K in triangulation $\mathcal{T}$.

$$\mathbb{V} := \{\Phi \in H^1(\Omega) : \Phi|_K \in \mathbb{P}^k \, \forall K \in \mathcal{T}\},$$

$$\mathring{\mathbb{V}} := \mathbb{V} \cap H_0^1(\Omega) = \{\Phi \in \mathbb{V} : \Phi|_{\partial\Omega} 0\}.$$

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For test function $\phi \in H_0^1(\Omega)$, an FE approximation for the problem is given by

$$\langle \mathbf{A} : D^2\mathbf{u}, \phi \rangle = \langle f, \phi \rangle.$$

Still need an approximation for the Hessian $D^2\mathbf{u}$.

Finite Element Hessian

Consider the smooth function $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and apply integration by parts to show the Hessian $D^2 v$ of v satisfies

$$\langle D^2 v, \varphi \rangle = -\langle \nabla v D\varphi \rangle + \langle \nabla v \mathbf{n}^T \varphi \rangle_{\partial\Omega}, \text{ for each } \varphi \in H^1(\Omega).$$

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Define the generalized Hessian of v, where $D^2 v$ maps to $\mathbb{R}^{d\times d}$.

Generalize for functions $v \in H^1(\Omega)$ with $\nabla v n^T|_{\partial\Omega}$ *in* $\left(H^{1/2}(\partial\Omega)'\right)^{d\times d}$.

$$\langle D^v | \varphi \rangle := -\langle \nabla v D\varphi \rangle + \langle \nabla v n^T | \varphi \rangle_{(H^{1/2}(\partial\Omega))\times H^{1/2}(\partial\Omega)} \text{ for each } \varphi \in H^1(\Omega)$$

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. Define the finite element Hessian for $v \in \mathbb{V}$,

$$\langle \mathbf{H} v, \Phi \rangle_{\Omega_\mathcal{T}} = \langle D^2 v | \Phi \rangle = \langle \nabla V \otimes \nabla \Phi \rangle + \langle \nabla V \otimes \mathbf{n} \Phi \rangle_{\partial\Omega} \quad \forall \Phi \in \mathbb{V},$$

where $a \otimes b = ab^T$ for vectors a, b.

Finite Element Convexity for Low-Order Elements

Aguilera and Morin [†] define the potential limitation for low-order finite elements to accurately approximate convex functions .

Define the discrete finite element Hessian:

$$\langle Hu, \varphi \rangle_{ij} = -\int_{\Omega} \partial_i u(\mathbf{x}) \partial_j \phi(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} \partial_i u(\mathbf{x}) \varphi(\mathbf{x}) \mathbf{n}_j dS.$$

Finite element convexity: A function $u \in V_h$ is finite element convex with respect to test and trial functions $\{\phi_r^h\}, \{\varphi_s^h\}$ if $H_s^h \succeq 0$ for all $s \in I_{test}^h$.

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They note in their experiments that FE convexity is not **always** guaranteed when using linear elements so they cannot prove convergence.

However this does not mean that it is impossible for linear elements to be finite element convex.

[†] **Aguilera, Morin (2009)**, *On convex functions and the finite element method*, SINUM.

Mixed FEM System

Discretized augmented system for a Dirichlet boundary condition, $\mathbf{E}\mathbf{v} = \mathbf{b} - \mathbf{E}_{\mathbf{dc}}\mathbf{b}_{\mathbf{dc}}$.

$$\mathbf{E} = \begin{bmatrix} \mathbf{M} & \mathbf{0} & \dots & \mathbf{0} & -\mathbf{C}_{1,1} \\ \mathbf{0} & \mathbf{M} & \dots & \mathbf{0} & -\mathbf{C}_{1,2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M} & -\mathbf{C}_{d,d} \\ \mathbf{B}^{1,1} & \mathbf{B}^{1,2} & \dots & \mathbf{B}^{d,d} & \mathbf{0}. \end{bmatrix}, \quad \mathbf{E}_{\mathbf{dc}} = \begin{bmatrix} -\mathbf{C}^{\mathbf{dc}}_{1,1} \\ -\mathbf{C}^{\mathbf{dc}}_{1,2} \\ \vdots \\ -\mathbf{C}^{\mathbf{dc}}_{d,d} \\ \mathbf{0}. \end{bmatrix}.$$

$$\mathbf{v} = (\mathbf{h}_{1,1}, \mathbf{h}_{1,2}, \dots, \mathbf{h}_{d,d}, \mathbf{u})^T, \mathbf{b} = (\mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \mathbf{f}), \mathbf{b}_{\mathbf{dc}} = [\mathbf{g}]^T.$$

$$\mathbf{B}^{\alpha,\beta} := \langle \mathring{\Phi}, \mathbf{A}^{\alpha,\beta}\Phi^T \rangle \in \mathbb{R}^{\mathring{N}\times N},$$

$$\mathbf{M} := \langle \Phi, \Phi^T \rangle \in \mathbb{R}^{N\times N},$$

$$\mathbf{C}_{\alpha,\beta} := -\langle \partial_\beta \Phi, \partial_\alpha \mathring{\Phi}^T \rangle + \langle \Phi n_\beta, \partial_\alpha \mathring{\Phi}^T \rangle_{\partial\Omega} \in \mathbb{R}^{\mathring{N}\times N}$$

$$\mathbf{f} := \langle f, \mathring{\Phi} \rangle \in \mathbb{R}^{\mathring{N}}$$

Schur Complement System

If low order elements are used, then $\mathbf{M}$ can be diagonalized using mass lumping and a Schur complement can be used to solve for $\mathbf{u}$ and the auxiliary Hessian variables are recovered separately.

$$\mathbf{D}\mathbf{u} := \sum_{\alpha=1}^{d}\sum_{\beta=1}^{d} \mathbf{B}^{\alpha,\beta}\mathbf{M}^{-1}\mathbf{C}_{\alpha,\beta}\mathbf{u} = \mathbf{f}.$$

$$\begin{aligned}
\mathbf{B}^{\alpha,\beta} &:= \langle \mathring{\Phi}, \mathbf{A}^{\alpha,\beta}\Phi^{T}\rangle \in \mathbb{R}^{\mathring{N}\times N}, \\
\mathbf{M} &:= \langle \Phi, \Phi^{T}\rangle \in \mathbb{R}^{N\times N}, \\
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\mathbf{f} &:= \langle f, \mathring{\Phi}\rangle \in \mathbb{R}^{\mathring{N}}
\end{aligned}$$

Note: We can reduce the size of the system by using the symmetry of the Hessian, i.e. $h_{a,b} = h_{b,a}$.

Mixed FEM Equivalence to Standard FEM

Equivalence to standard FEM

†For the second order elliptic problem $\mathbf{A} : D^2 u = f$, if the problem coefficients in $\mathbf{A}$ are piecewise constant then the mixed finite element method coincides with the standard FEM $\mathbf{A} : D^2 u = \text{div}(\mathbf{A}\nabla u)$. This implies that $\mathbf{u}$ solves both

$$\mathbf{Du} = \mathbf{f} \quad \text{and} \quad \mathbf{Su} = \mathbf{f}$$

$$\text{where} \quad \mathbf{S} = \sum_{\alpha,\beta=1}^{d} \langle \partial_\beta \mathbf{\Phi}, a^{\alpha,\beta} \partial_\alpha \mathbf{\Phi}^T \rangle.$$

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$$\text{where} \quad \mathbf{S} = \sum_{\alpha,\beta=1}^{d} \langle \partial_\beta \mathbf{\Phi}, a^{\alpha,\beta} \partial_\alpha \mathbf{\Phi}^T \rangle.$$

Opens up the possibility to use **algebraic multigrid** to solve the Schur complement system, due to its structure, which resembles the structure of the **discrete Laplacian**.

[†]**Lakkis, Pryer (2011)**, *A finite element method for second order nonvariational elliptic problems*, SISC.

Mixed FEM for Nonlinear Elliptic Problems

General nonlinear elliptic problem††:

$$\mathcal{N}[\mathbf{u}] = F(D^2\mathbf{u}) - \mathbf{f} = 0 \quad \text{in } \Omega.$$

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Previous applications use a Newton's method to obtain the linear system of equations

$$\begin{aligned} \mathbf{N}(D^2\mathbf{u}^n) : D^2\mathbf{u}^{n+1} = g(D^2\mathbf{u}^n), \\ \mathbf{N}(\mathbf{X}) := F'(\mathbf{X}), \\ g(\mathbf{X}) := f - F(\mathbf{X}) + F'(\mathbf{X}) : \mathbf{X}. \end{aligned}$$

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Solve using the mixed finite element method, given an initial guess $U^0 := \Pi_0 u^0$ for each $n \in \mathbb{N}_0$ find $(U^{n+1}, \mathbf{H}[U^{n+1}]) \in \mathbb{V} \times \mathbb{V}^{d\times d}$ such that,

$$\begin{aligned} < \mathbf{H}[U^{n+1}], \Phi > + \int_\Omega \nabla U^{n+1} \otimes \nabla \Phi - \int_{\partial\Omega} \nabla U^{n+1} \otimes \mathbf{n}\Phi = 0 \quad \forall \Phi \in \mathbb{V}, \\ < \mathbf{N}(\mathbf{H}[U^n]) : \mathbf{H}[U^{n+1}], \Psi >=< g(\mathbf{H}[U^n]), \Psi > \quad \forall \Psi \in \mathbb{V}, \end{aligned}$$

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Mixed FEM Applied to the MAE

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$$< \mathbf{H}[U^{n+1}], \Phi > + \int_{\Omega} \nabla U^{n+1} \otimes \Phi - \int_{\partial\Omega} \nabla U^{n+1} \otimes \mathbf{n}\Phi = \mathbf{0} \quad \forall \Phi \in \mathbb{V}$$

$$< \mathbf{Cof}(D^2 U^n) : \mathbf{H}[U^{n+1}], \Psi >= < f + \mathbf{det} D^2 U^n, \Psi > \ \forall \Psi \in \mathring{\mathbb{V}}.$$

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Where $\mathbf{Cof}(D^2 U^n)$ is the cofactor matrix given by

$$2D: \quad \mathbf{Cof}(D^2 U^n) = \begin{pmatrix} U^n_{yy} & -U^n_{xy} \\ -U^n_{yx} & U^n_{xx} \end{pmatrix},$$

$$3D: \quad \mathbf{Cof}(D^2 U^n) = \begin{pmatrix} U_{yy}U_{zz} - U^2_{yz} & U_{yz}U_{xz} - U_{xy}U_{zz} & U_{xy}U_{yz} - U_{yy}U_{xz} \\ U_{yz}U_{xz} - U_{xy}U_{zz} & U_{xx}U_{zz} - U^2_{xz} & U_{xy}U_{xz} - U_{xx}U_{yz} \\ U_{xy}U_{yz} - U_{yy}U_{xz} & U_{xy}U_{xz} - U_{xx}U_{yz} & U_{xx}U_{yy} - U^2_{xy} \end{pmatrix}$$

The Mixed FEM for $\mathbb{P}^1$ Elements

Previous applications of the method are limited to $\mathbb{P}^k$ elements with $k \geq 2$. [†]

Further asserted in a follow-up paper [††] that $\mathbb{P}^1$ elements can only be used with a gradient recovery operator.

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Motivates using a robust optimization-based solver for the low-order system.

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An Optimization-based Nonlinear Solver

- Combine auxiliary and primal variables into $x = (\mathbf{h}_{1,1}, \mathbf{h}_{1,2}, \ldots, \mathbf{h}_{d,d-1}, \mathbf{h}_{d,d}, \mathbf{u})$. We solve the nonlinear equation:

$$c(x) = 0,$$

where c is the nonlinear residual function, $c : \mathcal{X} \to \mathcal{C}$.

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- Verified that Newton's method and damped Newton's method do not converge for $\mathbb{P}^1$ mixed finite element discretizations.
- **Observation:** Solving the problem

$$\begin{aligned} \min \;\; & 0 \\ \text{subject to} \;\; & c(x) = 0, \end{aligned}$$

  with a *composite-step sequential quadratic programming (SQP)* solver **converges without exception** for a variety of MAE examples, using $\mathbb{P}^1$ elements.
- Simplify the nonlinear solution method by using the normal step, and eliminating the tangential step, of the SQP method†.

† **Heinkenschloss, Ridzal (2014).** *A matrix-free trust-region SQP method for equality constrained optimization*, SIOPT.

SQP Summary

- The SQP solver is a **composite-step method** that coordinates two steps: tangential step, which improves optimality, and a normal step, which improves feasibility.
- At every nonlinear iteration k, the feasibility step s solves the trust-region subproblem:

$$\begin{aligned} \min \quad & \|c'(x_k)s + c(x_k)\|_{\mathcal{C}}^2 \\ \text{subject to} \quad & \|s\|_{\mathcal{X}} \le \Delta_k, \end{aligned}$$

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- Significantly more robust than damped Newton.
- We compute the feasibility step using **Powell's dogleg method**.

Dogleg Method

Cauchy point

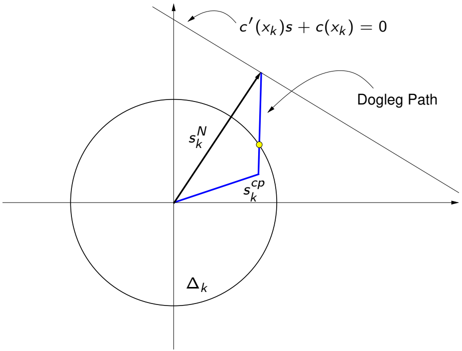
$$s_k^{cp} = \min_{\alpha \geq 0, \|s\|_{\mathcal{X}} \leq \Delta_k} \|c'(x_k)s + c(x_k)\|_{\mathcal{C}}^2$$

$$\text{subject to} \quad s = -\alpha c'(x_k)^* c(x_k)$$

Newton point

$s_k^N =$ minimum norm solution of

$$\min_s \quad \|c'(x_k)s + c(x_k)\|_{\mathcal{C}}^2$$



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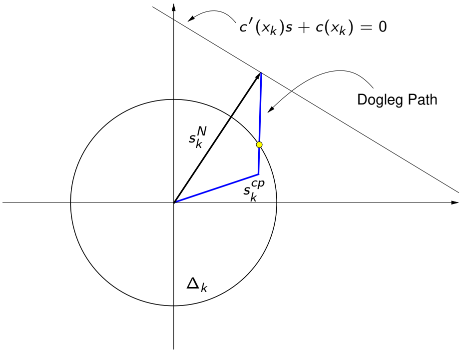

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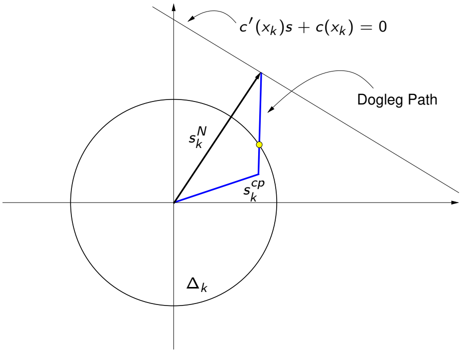

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Step: $s_k = s_k^{cp} + s_k^N$.

If $\|s_k\|_{\mathcal{X}} > \Delta_k$, compute $s_k = s_k^{cp} + \tau s_k^N$, with $\left\| s_k^{cp} + \tau (s_k^N - s_k^{cp}) \right\|_{\mathcal{X}}^2 = \Delta_k^2$.

Step Acceptance

Calculate ratio of actual and predicted reduction,

$$r = \frac{\|c(x_k)\|_{\mathcal{C}}^2 - \|c(x_k + s_k)\|_{\mathcal{C}}^2}{\|c(x_k)\|_{\mathcal{C}}^2 - \|c'(x_k)s_k + c(x_k)\|_{\mathcal{C}}^2} .$$

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If $r > \eta$, for some specified threshold $\eta > 0$, accept the step, $x_{k+1} = x_k + s_k$.

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Algorithm terminates if the norm of the residual is less than a specific tolerance or the maximum number of iterations is reached.

Cauchy Point Computation

Recall $c'(x_k)$ is the Jacobian of the discretized Monge-Ampère equation evaluated at the step x_k, $c'(x_k)^*$ is the adjoint of the Jacobian, $c(x_k)$ is the residual of the Monge-Ampère equation at x_k, and Δ_k is the trust region radius at step k.

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First compute

$$s_k^{cp} = -\frac{\|c'(x_k)^* c(x_k)\|_{\mathcal{X}}^2}{\|c'(x_k)c'(x_k)^* c(x_k)\|_{\mathcal{C}}^2} c'(x_k)^* c(x_k).$$

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If $\|s_k\|_{\mathcal{X}}^{cp} > \Delta_k$, then set $s_k = \frac{\Delta_k}{\|s_k^{cp}\|} s_k^{cp}$.

Cauchy point computation is inexpensive, as it does not involve application of $(c'(x_k))^{-1}$.

Global Convergence

Algorithm converges from any initial guess under the following assumptions:

The step s_k must satisfy the boundedness condition

$$\|s_k\|_{\mathcal{X}} \leq \kappa_1 \|c(x_k)\|_{\mathcal{C}},$$

and the sufficient decrease condition

$$\|c(x_k)\|_{\mathcal{C}}^2 - \|c'(x_k)s_k + c(x_k)\|_{\mathcal{C}}^2 \geq \kappa_2 \|c(x_k)\|_{\mathcal{C}} \min\left\{\kappa_3 \|c(x_k)\|_{\mathcal{C}}, \Delta_k\right\},$$

where $\kappa_1, \kappa_2, \kappa_3 > 0$ are independent of k.

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Satisfied by the Cauchy point!!!

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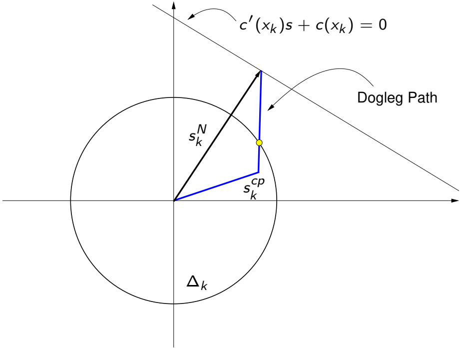
Must ensure that the Newton point does not destroy the progress made by the Cauchy point.

Newton Point Computation

Newton point: $s_k^N = -(c'(x_k))^{-1} c(x_k)$



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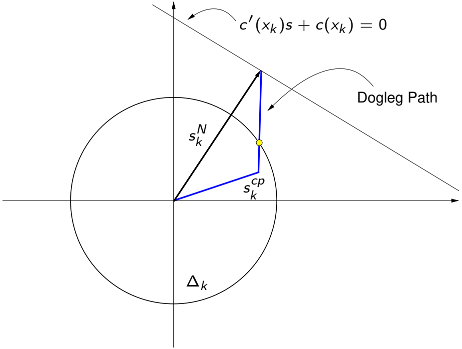
Instead of solving for s_k^N, we solve for

$$\delta s_k = s_k^N - s_k^{cp}$$



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Instead of solving for s_k^N, we solve for

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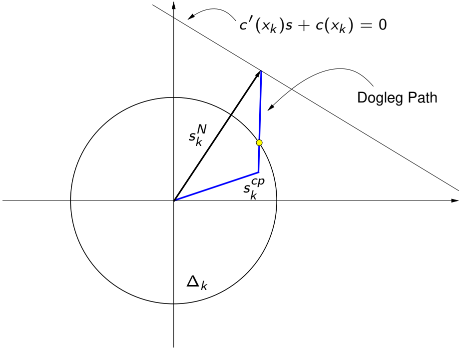
In other words, solve

$$c'(x_k)\delta s_k = -(c'(x_k)s_k^{cp} + c(x_k))$$



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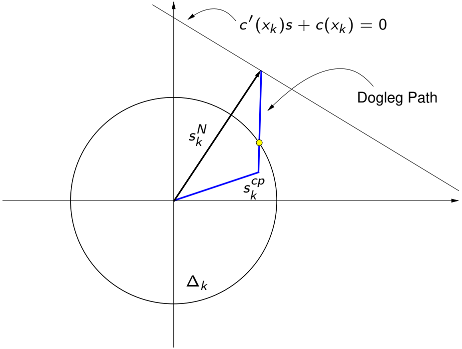

If the linear system is solved iteratively with nonzero error vector e, i.e.,

$$c'(x_k)\delta s_k = -(c'(x_k)s_k^{cp} + c(x_k)) + e,$$

then for global convergence it is sufficient to require $\|e\|_{\mathcal{C}} \,/\, \|c'(x_k)s_k^{cp} + c(x_k)\|_{\mathcal{C}} \leq 1$.
$\longrightarrow$ Any reasonable relative residual will do!!!

Efficient Solution of the Linear System

- Solve the linear system using the Schur complement,
  $\mathbf{Du} := \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \mathbf{B}^{\alpha,\beta} \mathbf{M}^{-1} \mathbf{C}_{\alpha,\beta} \mathbf{u} = \mathbf{f}$, for $\alpha, \beta = 1, \ldots, d$,

$$\mathbf{B}^{\alpha,\beta} := \langle \mathring{\mathbf{\Phi}}, \mathbf{Cof}(\mathbf{H}[U])^{\alpha,\beta}) \mathbf{\Phi}^T \rangle \in \mathbb{R}^{\mathring{N} \times N},$$

$$\mathbf{C}_{\alpha,\beta} := -\langle \partial_\beta \Phi, \partial_\alpha \mathring{\Phi}^T \rangle + \langle \Phi n_\beta, \partial_\alpha \mathring{\Phi}^T \rangle_{\partial\Omega} \in \mathbb{R}^{\mathring{N} \times N}$$

$$\mathbf{M} := \langle \Phi, \Phi^T \rangle \in \mathbb{R}^{N \times N}, \quad \mathbf{f} := \langle f, \mathring{\Phi} \rangle \in \mathbb{R}^{\mathring{N}},\ f = -(c'(x_k) s_k^{cp} + c(x_k))$$

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- Schur complement of the linear system $\mathbf{Du} = \mathbf{f}$ is solved using an **algebraic multigrid method**.

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$$\mathbf{B}^{\alpha,\beta} := \langle \mathring{\boldsymbol{\Phi}}, \mathbf{Cof}(\mathbf{H}[U])^{\alpha,\beta})\boldsymbol{\Phi}^T\rangle \in \mathbb{R}^{\mathring{N}\times N},$$

$$\mathbf{C}_{\alpha,\beta} := -\langle \partial_\beta \Phi, \partial_\alpha \mathring{\Phi}^T \rangle + \langle \Phi n_\beta, \partial_\alpha \mathring{\Phi}^T \rangle_{\partial\Omega} \in \mathbb{R}^{\mathring{N}\times N}$$

$$\mathbf{M} := \langle \Phi, \Phi^T \rangle \in \mathbb{R}^{N\times N}, \quad \mathbf{f} := \langle f, \mathring{\Phi} \rangle \in \mathbb{R}^{\mathring{N}}, \; f = -(c'(x_k)s_k^{cp} + c(x_k))$$

- Schur complement of the linear system $\mathbf{D}\mathbf{u} = \mathbf{f}$ is solved using an **algebraic multigrid method**.
- Use the Trilinos package Muelu to set up and solve the resulting system.
  - Symmetric Gauss-Siedel smoother.
  - GMRES used for solving the resulting system.
  - Future plans to explore Petrov-Galerkin methods for nonsymmetric linear systems.[†]

[†] **Sala, Tuminaro (2008)**, *A new Petrov-Galerkin smoothed aggregation preconditioner for nonsymmetric linear systems*, SISC.

Efficient Solution of the Linear System

- Solve the linear system using the Schur complement,
  $\mathbf{Du} := \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \mathbf{B}^{\alpha,\beta}\mathbf{M}^{-1}\mathbf{C}_{\alpha,\beta}\mathbf{u} = \mathbf{f}$, for $\alpha, \beta = 1, \ldots, d$,

$$\mathbf{B}^{\alpha,\beta} := \langle \mathring{\boldsymbol{\Phi}}, \mathbf{Cof}(\mathbf{H}[U])^{\alpha,\beta})\boldsymbol{\Phi}^T \rangle \in \mathbb{R}^{\mathring{N}\times N},$$

$$\mathbf{C}_{\alpha,\beta} := -\langle \partial_\beta \Phi, \partial_\alpha \mathring{\Phi}^T \rangle + \langle \Phi n_\beta, \partial_\alpha \mathring{\Phi}^T \rangle_{\partial\Omega} \in \mathbb{R}^{\mathring{N}\times N}$$

$$\mathbf{M} := \langle \Phi, \Phi^T \rangle \in \mathbb{R}^{N\times N}, \quad \mathbf{f} := \langle f, \mathring{\Phi} \rangle \in \mathbb{R}^{\mathring{N}}, \; f = -(c'(x_k)s_k^{cp} + c(x_k))$$

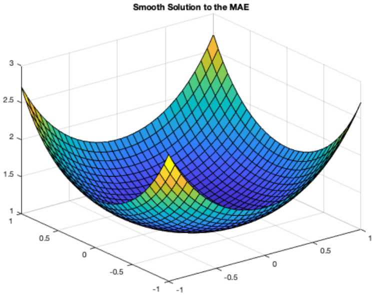
- Schur complement of the linear system $\mathbf{Du} = \mathbf{f}$ is solved using an **algebraic multigrid method**.
- Use the Trilinos package Muelu to set up and solve the resulting system.
  - Symmetric Gauss-Siedel smoother.
  - GMRES used for solving the resulting system.
  - Future plans to explore Petrov-Galerkin methods for nonsymmetric linear systems.[†]
- Hessian terms $\mathbf{h}_{\alpha,\beta}$ are recovered separately using the definition of the finite element Hessian.

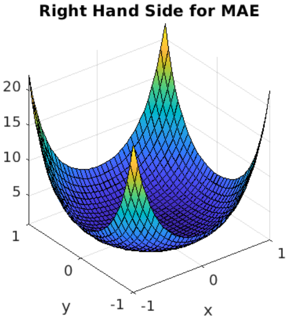
[†]**Sala,Tuminaro (2008)**, *A new Petrov-Galerkin smoothed aggregation preconditioner for nonsymmetric linear systems*, SISC.

Results - 2D Smooth Solution

Exact Solution: $u = \exp\left(\frac{|\mathbf{x}|^2}{2}\right), \ u \in C^\infty.$

Classical solution of the Monge-Ampère equation.





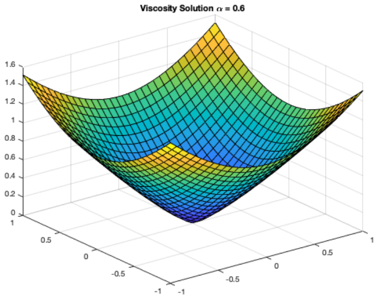
Results - 2D Smooth Solution

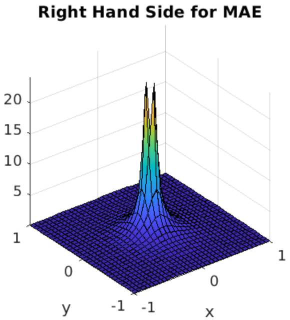
Grid Size	$\|u-u_e\|_2^2$	Rate	$\|u-u_e\|_\infty$	Rate
$2^5 \times 2^5$	3.40e-03	-	4.40e-03	-
$2^6 \times 2^6$	8.60e-04	1.98	1.10e-03	2
$2^7 \times 2^7$	2.16e-04	2.0	2.88e-04	1.93
$2^8 \times 2^8$	5.40e-05	2.0	7.27e-05	1.99

Standard smooth problem used in every numerical paper for the MAE, we attain **optimal** L_2 error estimates.

Results - 2D Viscosity Solution

Exact solution: $u(\mathbf{x}) = |\mathbf{x}|^{2\alpha}$, $\alpha \in (1/2, 3/4)$, singular derivatives at $(0,0)$, $\alpha = 0.6$.





Results - 2D Viscosity Solution

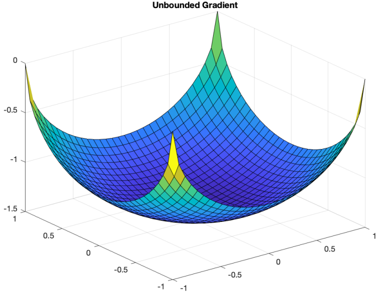
Grid Size	$\|u - u_e\|_2^2$	L_2 Rate	LP L_2 Rate[†]	$\|u - u_e\|_\infty$	∞ Error Rate
$2^5 \times 2^5$	0.1249	-	-	0.1841	-
$2^6 \times 2^6$	0.0937	0.41	0.42	0.1381	0.41
$2^7 \times 2^7$	0.0705	0.41	0.41	0.1036	0.41
$2^8 \times 2^8$	0.0531	0.41	0.41	0.0777	0.42

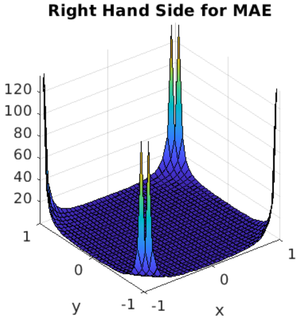
Compared to the paper by Lakkis and Pryer[†] which uses $\mathbb{P}^2$ elements to solve for the Monge-Ampère equation using a Newton's method. We attain similar errors with $\mathbb{P}^1$ element method.

Lakkis, Pryer (2011), *A finite element method for second order nonvariational elliptic problems*, SJSC.

Results - 2D Unbounded Gradient

Exact solution: $u(x, y) = -\sqrt{2 - x^2 - y^2}$, $u \in C^\infty(\Omega) \cap C^0(\Omega)$.





Results - 2D Unbounded Gradient

Grid Size	$\|u-u_e\|_2^2$	L_2 Rate	LP Rate†	$\|u-u_e\|_\infty$	FO Max Error††
$2^5 \times 2^5$	6.4e-03	-	-	8.62e-02	1.74e-03
$2^6 \times 2^6$	2.3e-03	1.48	1.62	6.10e-02	5.9e-04
$2^7 \times 2^7$	8.2460e-04	1.48	1.60	4.32e-02	2.0e-04
$2^8 \times 2^8$	2.9253e-04	1.50	1.55	3.05e-02	1.5e-04

Compared to the $\mathbb{P}^2$ mixed finite element method by Lakkis and Pryer † and the wide stencil, monotone finite difference schemes by Froese and Oberman ††.

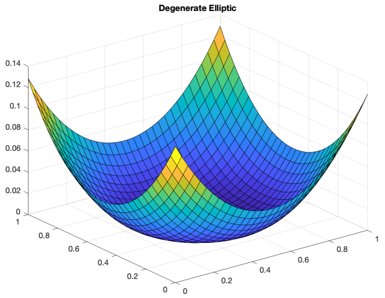
Lakkis, Pryer (2011), *A finite element method for second order nonvariational elliptic problems*, SJSC.

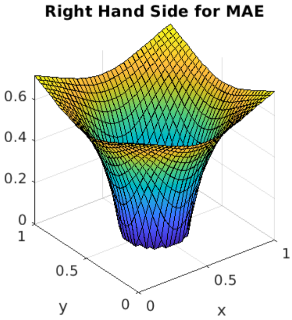
Froese, Oberman (2013), *Convergent filtered schemes for the Monge-Ampère partial differential equation*, SJNA.

Results - 2D Degenerate Elliptic

Exact solution: $u(\mathbf{x}) = \frac{1}{2}(\max(|\mathbf{x} - \mathbf{x}_0| - 0.2, 0))^2$, $\mathbf{x}_0 = (0.5, 0.5)$.

Forcing term: $f(\mathbf{x}) = \max\left(1 - \frac{0.2}{|\mathbf{x}-\mathbf{x}_0|}, 0\right)$.





Results - 2D Degenerate Elliptic

Grid Size	$\|u - u_e\|_2^2$	Rate	$\|u - u_e\|_\infty$	NNZ Max Error†	FO Max Error††
$2^5 \times 2^5$	2.18e-04	-	4.22e-04	5.4e-04	3.73e-04
$2^6 \times 2^6$	6.04e-05	1.8516	1.39e-04	2.8e-04	1.51e-04
$2^7 \times 2^7$	2.00e-05	1.5915	5.51e-05	1.5e-4	9.2e-05
$2^8 \times 2^8$	6.48e-06	1.6282	2.22e-05	7.8e-05	3.8e-05

Compared to two-scale (unstructured wide scale) finite element method by Nochetto et al. † and the wide stencil, montone finite differences schemes by Frose and Oberman ††.

Nochetto, Ntogkas, Zhang (2019), *Two-Scale method for the Monge-Ampère equation: convergence to the viscosity solution*. Mathematics of Computation.

Froese, Oberman (2013), *Convergent filtered schemes for the Monge-Ampère partial differential equation*, SJNA.

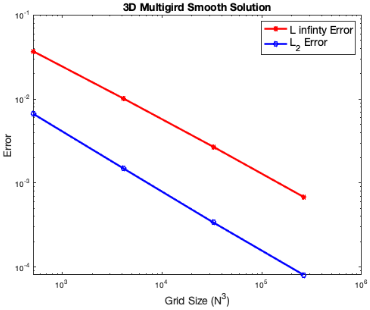
Results - 3D Smooth Solution

$$f = (1 + x^2 + y^2 + z^2)\exp\left(\frac{3(x^2 + y^2 + z^2)}{2}\right)$$

$$u = \exp\left(\frac{x^2 + y^2 + z^2}{2}\right)$$

$$\Omega = (0, 1)^3$$

Taken from: **Liu et al. (2016)**, *A multigrid scheme for 3D Monge-Ampère equations*, IJCM.



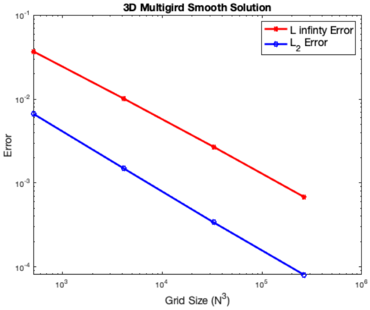
Convergence results for the 3D smooth problem.

Grid Size N	$\|u - u_e\|_2^2$	Rate	$\|u - u_e\|_\infty$	Rate	Max GMRES
8	6.70e-03	-	3.72e-02	-	3
16	1.50e-03	2.16	1.02e-02	1.87	3
32	3.39e-04	2.15	2.70e-03	1.89	3
64	8.06e-05	2.07	6.84e-04	1.92	5

Results - 3D Smooth Solution

$$f = (\sin(x)+1)(\sin(y)+1)(\sin(z)+1)$$
$$u = -\sin(x) - \sin(y) - \sin(z) + (x^2+y^2+z^2)/2$$
$$\Omega = (0,\pi)^3$$

Taken from: **Liu et al. (2016)**, *A multigrid scheme for 3D Monge-Ampère equations*, IJCM.



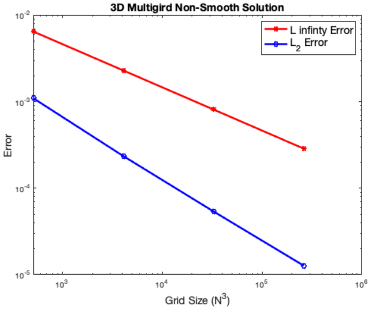
Convergence results for the 3D smooth problem.

Grid Size N	$\|u-u_e\|_2^2$	Rate	$\|u-u_e\|_\infty$	Rate	Max GMRES
8	2.09e-01	-	7.12e-02	-	4
16	4.84e-02	2.11	1.82e-02	1.97	4
32	1.16e-02	2.06	4.60e-03	1.98	3
64					

Results - 3D Non-smooth Solution

$$f = (1/16)(x^2 + y^2 + z^2)^{-3/4}$$
$$u = (1/3)(x^2 + y^2 + z^2)^{3/4}$$
$$\Omega = (0,1)^3$$

Taken from: **Liu et al. (2016)**, *A multigrid scheme for 3D Monge-Ampère equations*, IJCM.



Convergence results for the 3D non-smooth problem.

Grid Size N	$\|u - u_e\|_2^2$	Rate	$\|u - u_e\|_\infty$	Rate	Max GMRES
8	1.10e-03	-	6.50e-03	-	3
16	2.35e-04	1.50	2.30e-03	2.23	4
32	5.38e-05	1.50	8.13e-04	2.13	5
64	1.27e-05	1.50	1.27e-05	2.08	6

Conclusion

- Presented an optimization-based solver for any standard low-order finite element discretization of nonlinear elliptic problems.
- Our optimization-based algorithm combined with a $\mathbb{P}^1$ discretization demonstrates good results with solid numerical error and convergence rates.
- While the method does not directly impose convexity like other discretizations for the Monge-Ampère equation, we have reason to believe that it is provably convergent to a solution of the MAE.
- Future work will further leverage the built in machinery for inexact methods that can offer additional speed ups for the methods.
- Potential to build in convexity constraints or other boundary conditions (such as the optimal transport condition) directly into the objective function of the optimization solver.

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