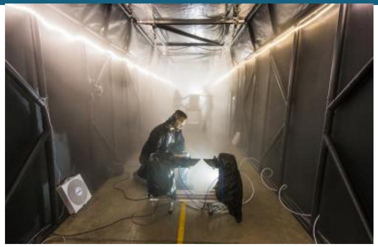
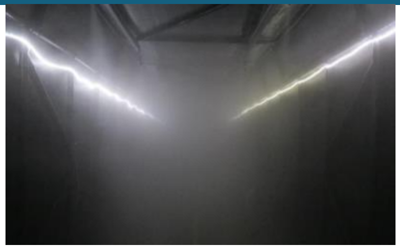
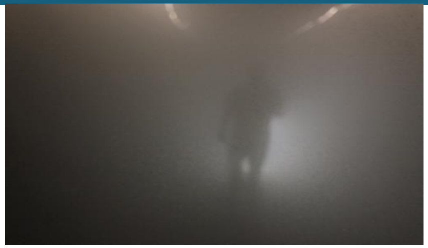
SAND2022-9190C

Optical Characterization of the Sandia Fog Facility for Computational Sensing (Invited Talk)







Optical Sensors and Sensing Congress, LF1C.3, July 15, 2022

Machine Learning and Computational Sensing (COSI and LACSEA)

PRESENTED BY

Brian Z. Bentz

Christian A. Pattyn, Brian J. Redman, John Zenker, Elihu Deneke, Andres L. Sanchez, Karl Westlake, John van der Laan, and Jeremy B. Wright

Sandia National Laboratories

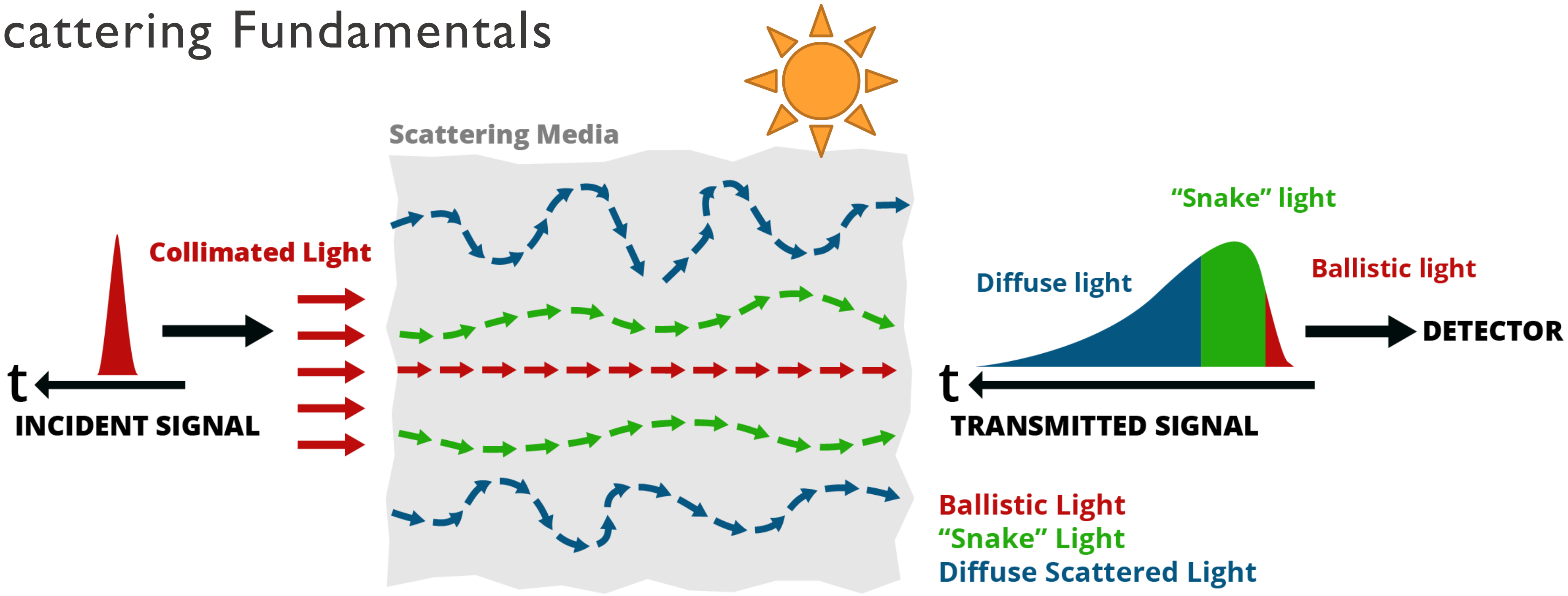
Light Scattering Limits Situational Awareness

- Aerosols like fog create **degraded visual environments** and cause unacceptable down-time for critical systems or operations
- Information is lost due to the random scatter of photons from tiny particles
- Impacts physical security, surveillance, navigation, tactical scenarios, remote sensing, and more

Simulated degraded visual environment at the Sandia Fog Chamber Facility

[1] A. Mosk, Y. Silberberg, K. J. Webb, and C. Yang, *Defense Technical Information Center* ADA627354, 2015
[2] C. Dunsby and P. M. W. French, *J. Phys. D: Appl. Phys.* **36**, R207-R227, 2003

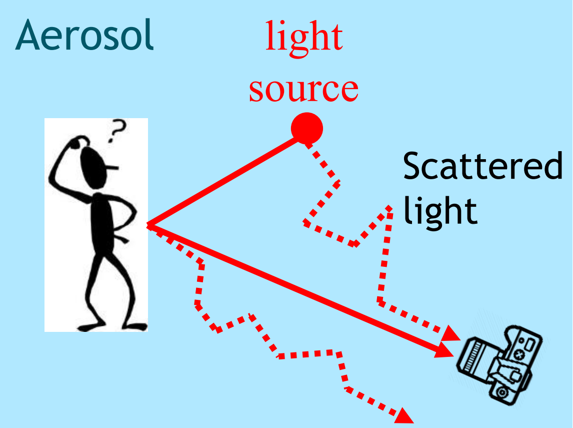
Light Scattering Fundamentals

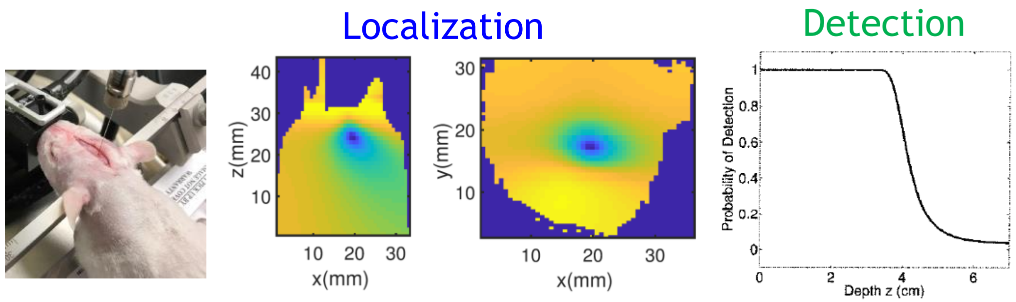


- Ballistic light is exponentially attenuated with distance: $I = I_o \exp(-L/\mathrm{MFP})$ [Beer-Lambert law]
- Sandia Fog Chamber Mean Free Path (MFP)~ 1 m
- Time and coherence gating reject scattered light limiting imaging to $L \sim 10$ MFP
- Using all photons could allow optical imaging to $L \sim 30$ MFP or 2-4 times further
- Increased background from sunlight scattered into the detector

Detection and Localization with Scattered Light

- **Key question**
  - **What can be done with scattered light in aerosols like fog?**
- Interpretation of measured data with light transport model provides new information
  - Detection: is an object there?
  - Localization: where is the object?
  - Imaging: what is the object?
    - Potentially provided in real time for rapid decision making using existing infrastructure

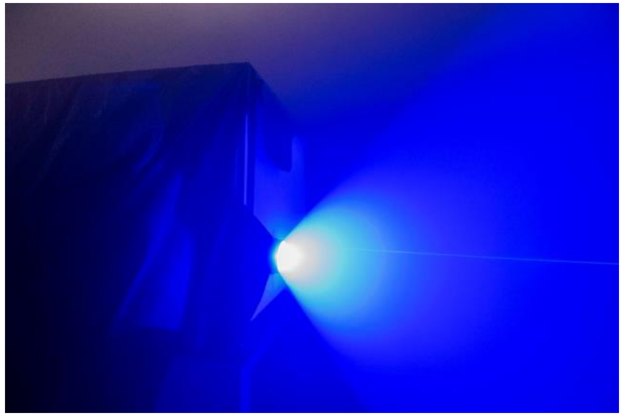


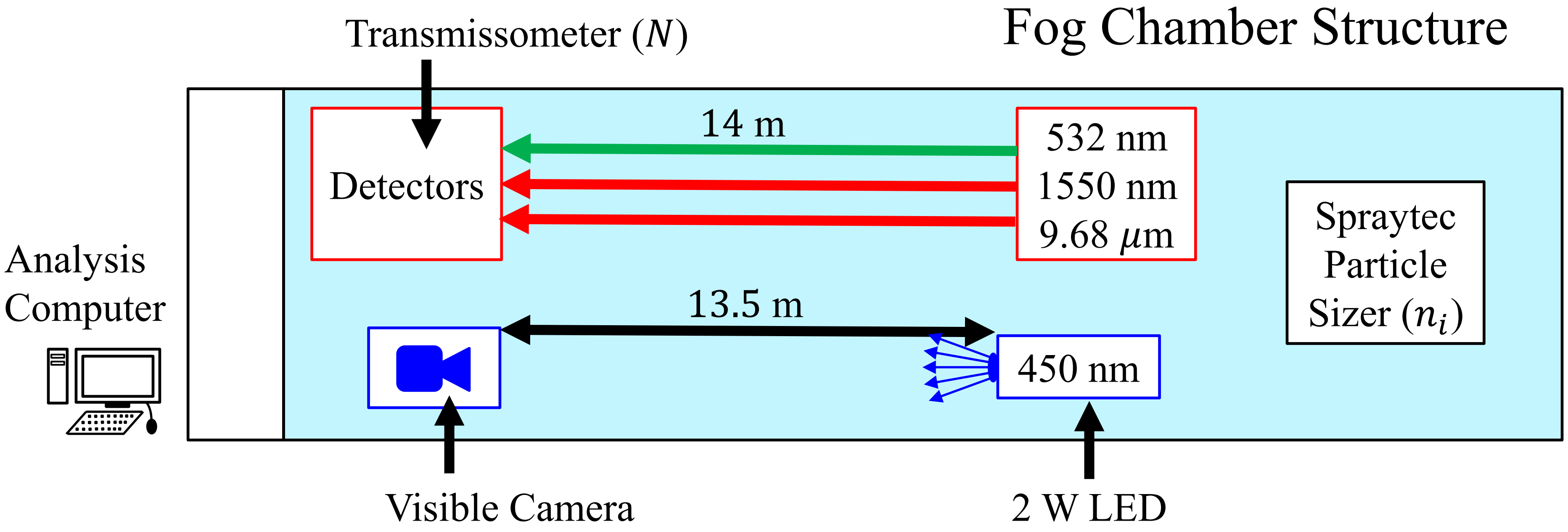


Examples from biomedical imaging [1,2]

[1] A. B. Milstein, M. D. Kennedy, P. S. Low, C. A. Bouman, and K. J. Webb, *Applied Optics* **44**(12), 2300-2310 (2005).
[2] B. Z. Bentz, et al., *IEEE Trans. Med. Imaging* **39**(7), 2472-2481 (2020).

Sandia National Laboratory Fog Chamber Facility (SNLFC)





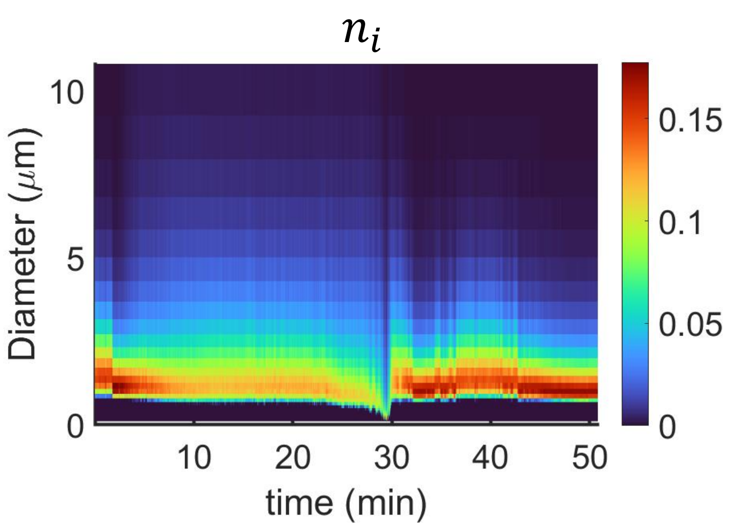
Measuring the Fog Scattering Parameters

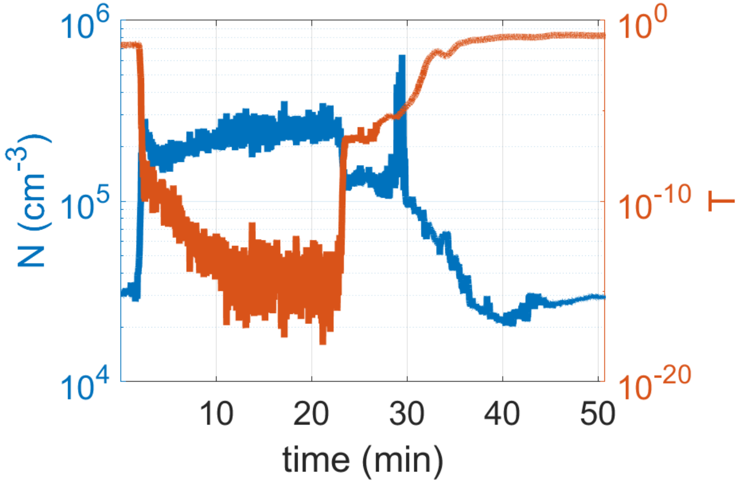
$$\mu_s = N \sum_i \sigma_{s_i} n_i$$

$$\mu_a = N \sum_i \sigma_{a_i} n_i$$

Where

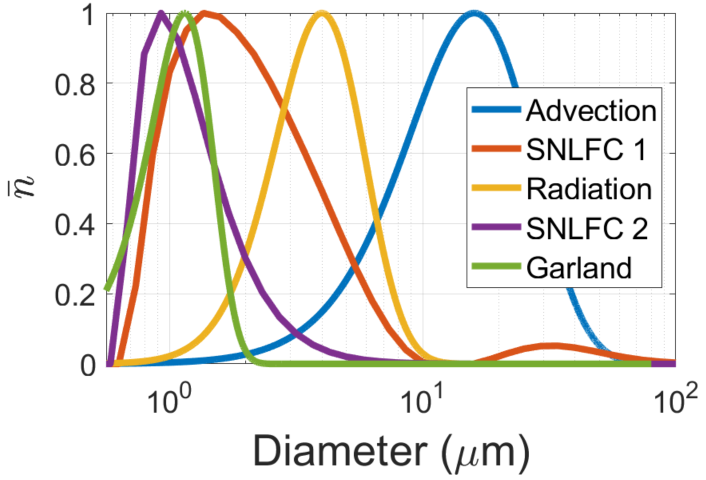
- μ_s is the scattering coefficient (m^{-1})
- μ_a is the absorption coefficient (m^{-1})
- N is the particle density (cm^{-3})
- n_i is the fraction of N contributed by particles of diameter d_i
- σ_{s_i} and σ_{a_i} are the scattering and absorption cross sections computed with Mie theory



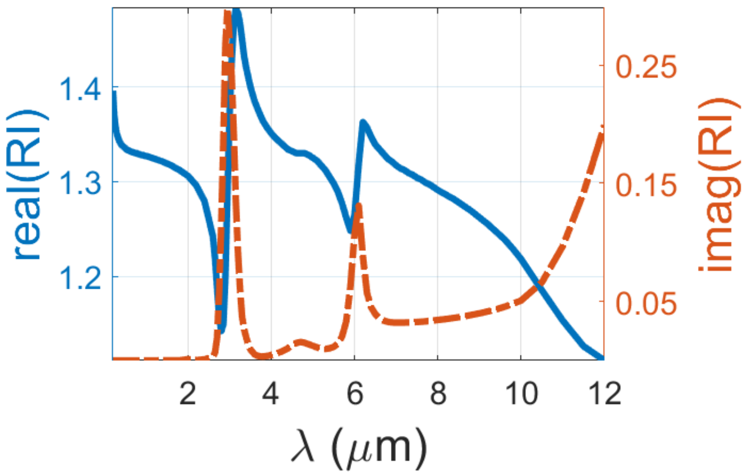


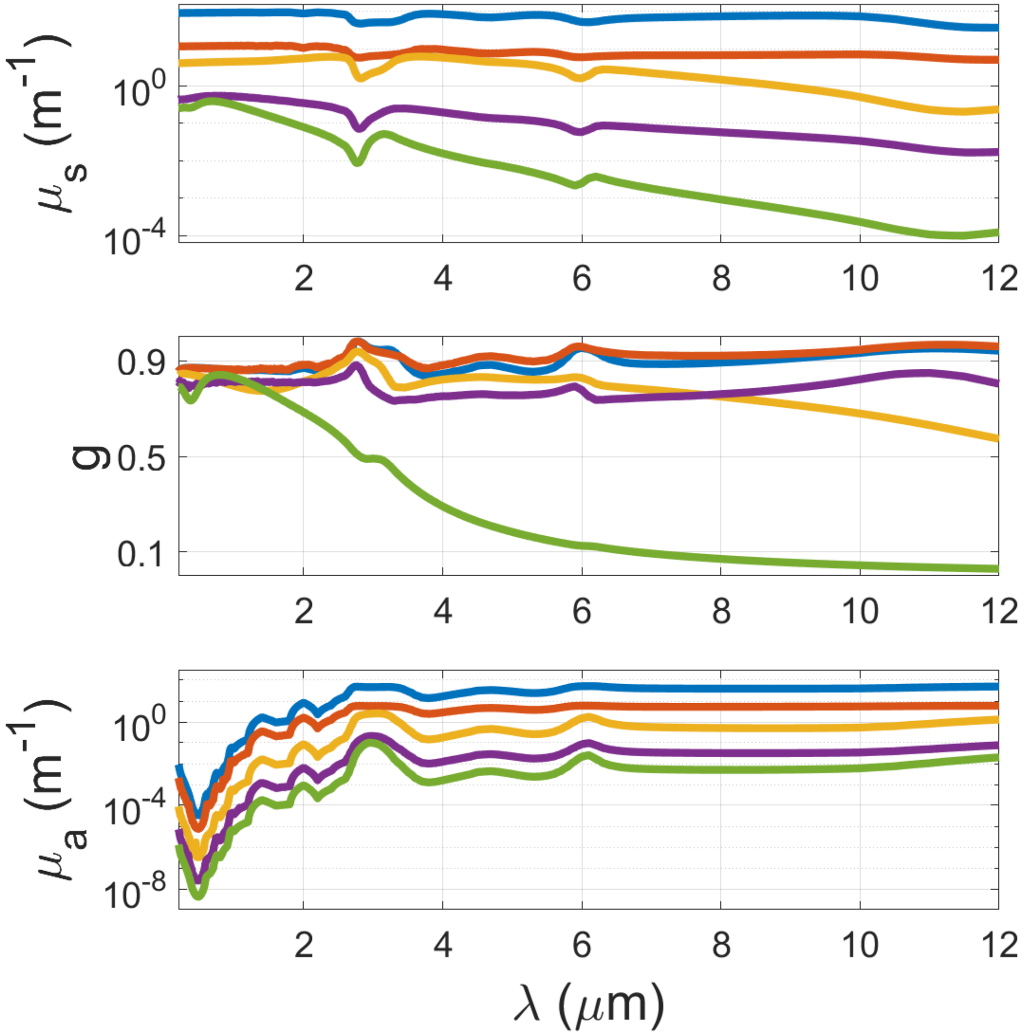
Fog Optical Properties are Highly Dependent on the Size Distribution

Particle Size Distributions



Water Refractive Index





Radiative Transfer Equation (RTE)

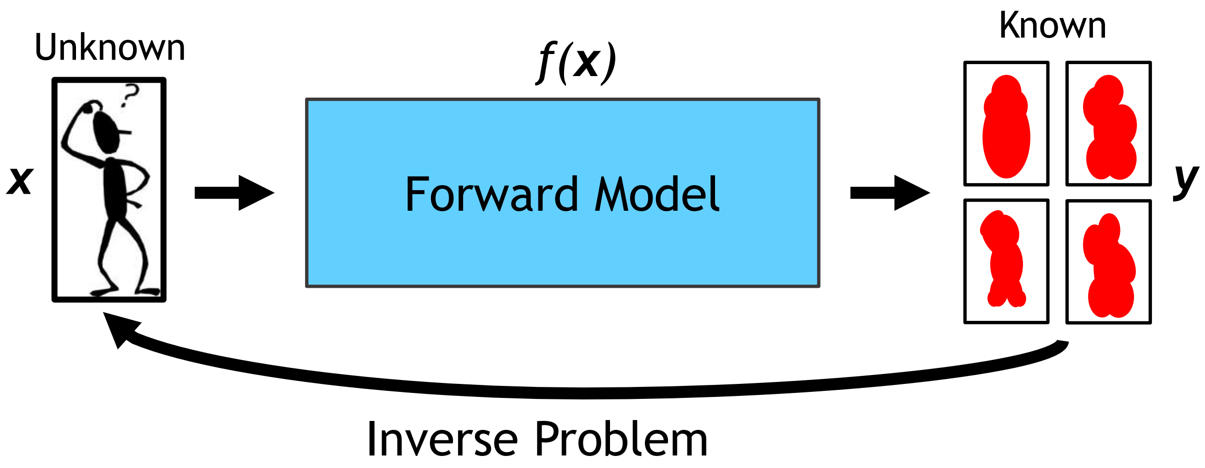
$$\widehat{\boldsymbol{\Omega}} \cdot \nabla I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) + (\mu_a + \mu_s) I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = \mu_s \int_{4\pi} d\widehat{\boldsymbol{\Omega}}' f(\widehat{\boldsymbol{\Omega}}' \to \widehat{\boldsymbol{\Omega}}) I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}') + Q(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$$

Where

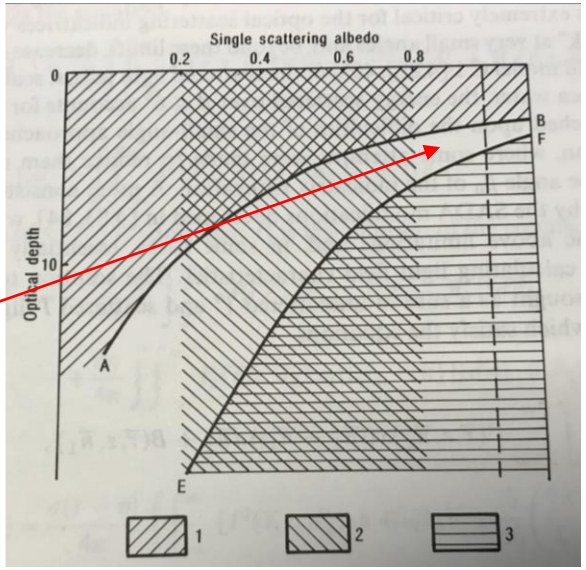
- $I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$ is the radiance (J/m^2/sr) at position $\boldsymbol{r}$ in direction $\widehat{\boldsymbol{\Omega}}$
- μ_a is the absorption coefficient (m^{-1})
- μ_s is the scattering coefficient (m^{-1})
- $f(\widehat{\boldsymbol{\Omega}}' \to \widehat{\boldsymbol{\Omega}})$ is the scattering phase function for incident direction $\widehat{\boldsymbol{\Omega}}'$ and scattering direction $\widehat{\boldsymbol{\Omega}}$
- $Q(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$ is the radiance source function (J/m^3/sr)

[1] G. I. Bell and S. Glasstone, *Nuclear Reactor Theory*, U.S. Atomic Energy Commission, Washington DC (1970).

New Light Transport Model Enables Computational Sensing in Fog



Applicability domains of approximate methods [1]:
(1) small-angle approximation
(2) small-angle diffusion approximation
(3) diffusion approximation



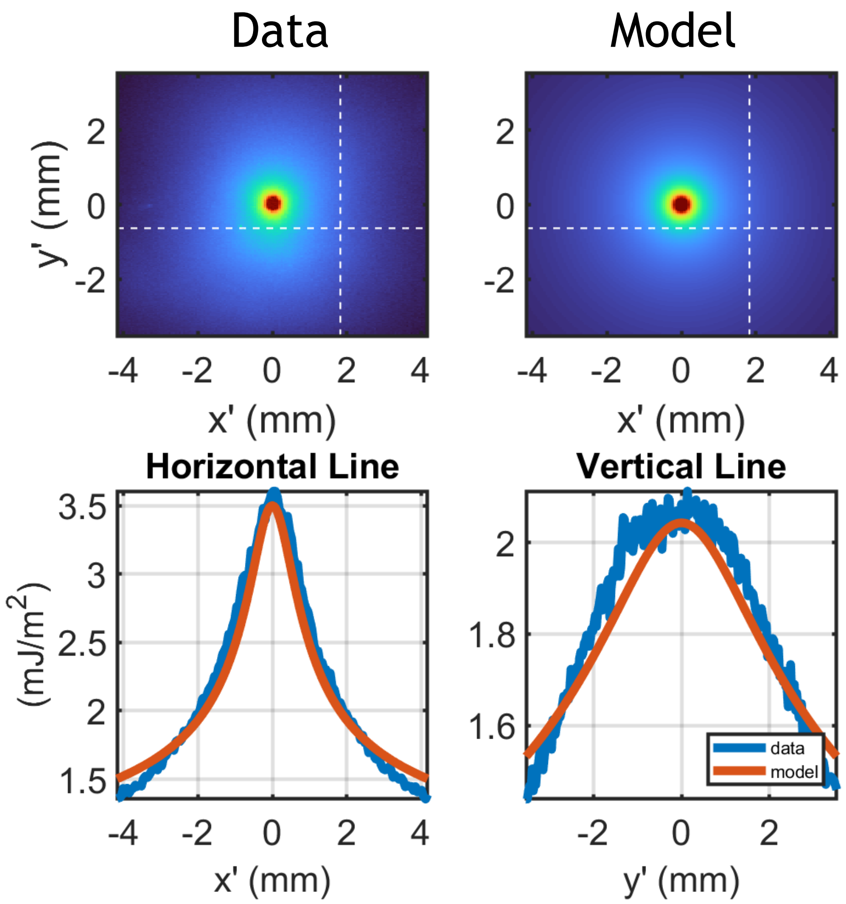
Fog

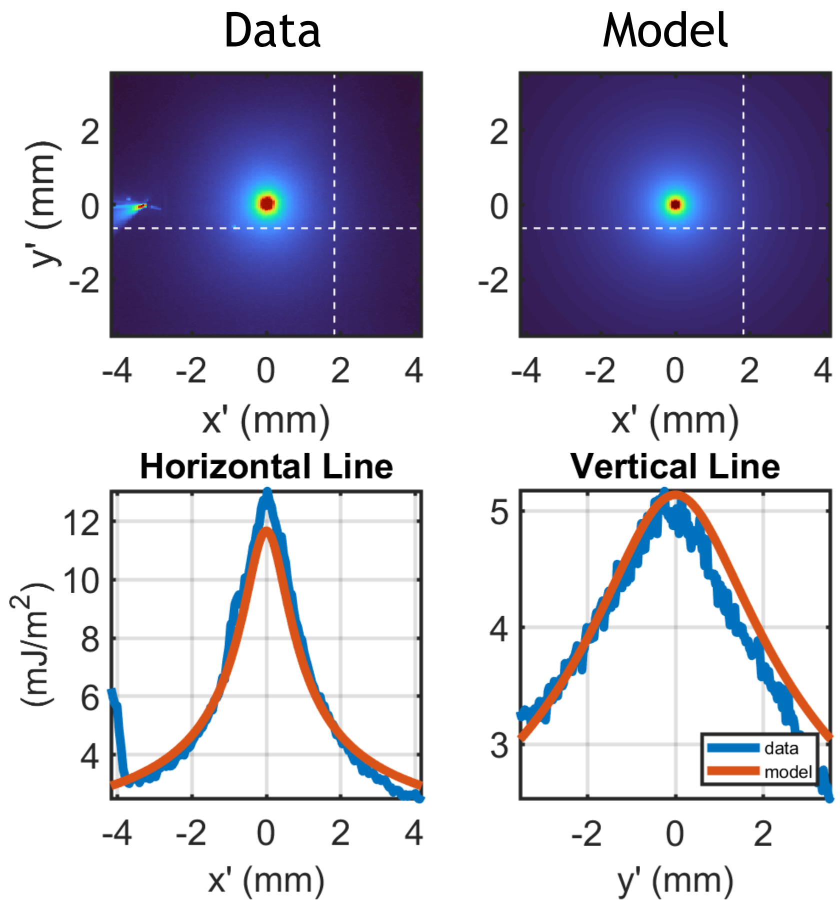
$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi}\int_{\mathbf{0}}^{\infty} dR\,\exp[-(\mu_s+\mu_a)R]\left[\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}\right]$$

- ϕ is the fluence rate (J/m^2)
- $\boldsymbol{J}$ is the flux density (J/m^2)

[1] E. P. Zege, et. al, Vol. 349. New York: Springer-Verlag, 1991.
[2] B. Z. Bentz, et. al, *Optics Express* **29**(9), 13231-13245, 2021.

Experimental Verification





<u>Fog Optical Parameters:</u>

$\mu_s = 0.323$ m^{-1}

$\mu_a = 1.84 \times 10^{-8}$ m^{-1}

$g = 0.797$

<u>Fog Optical Parameters:</u>

$\mu_s = 0.186$ m^{-1}

$\mu_a = 1.12 \times 10^{-8}$ m^{-1}

$g = 0.795$

Modeling Objects

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r}_o, d) + \Gamma \boldsymbol{R}(\boldsymbol{r}_o, d)$$

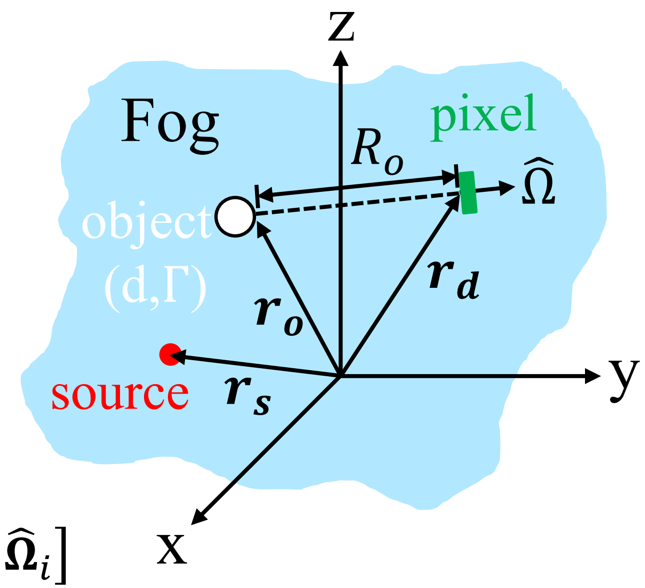
Measurement without object

Blocked in-scattered light

Additional in-scattered reflected light

Where

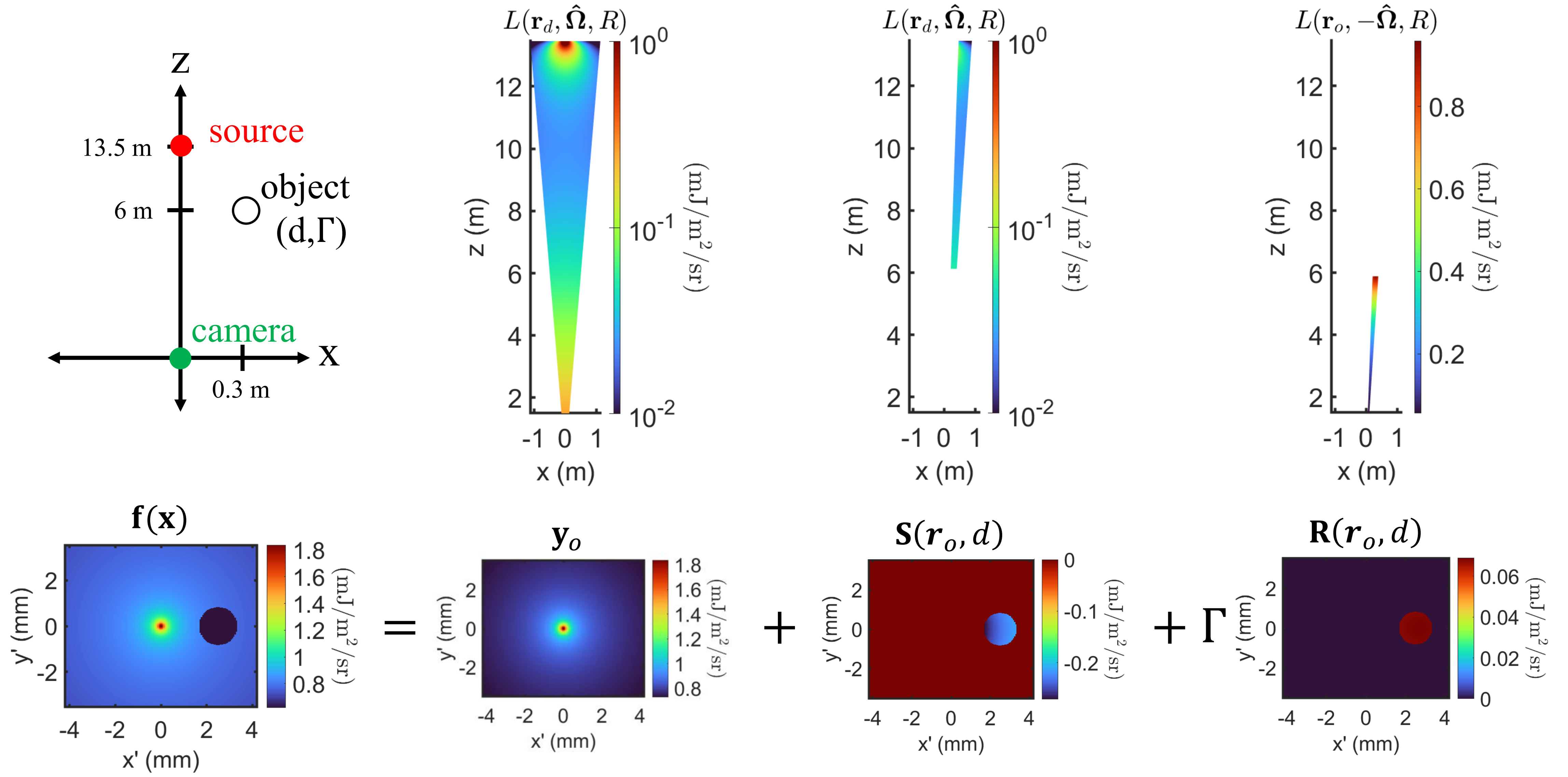
- $\boldsymbol{x} = [\boldsymbol{r}_o, d, \Gamma]$, d is the object size and Γ is the reflection coefficient
- $y_{o_i} = \frac{\mu_s}{4\pi} \int_0^\infty dR_i \exp[-(\mu_s + \mu_a)R_i] \left[\phi\left(\boldsymbol{r}_d - R_i \widehat{\boldsymbol{\Omega}}_i\right) + 3g\boldsymbol{J}\left(\boldsymbol{r}_d - R_i \widehat{\boldsymbol{\Omega}}_i\right) \cdot \widehat{\boldsymbol{\Omega}}_i\right]$
- $S_i(\boldsymbol{r}_o, d) = -\frac{\mu_s}{4\pi} \int_{R_o}^\infty dR_i \exp[-(\mu_s + \mu_a)R_i] \left[\phi\left(\boldsymbol{r}_d - R_i \widehat{\boldsymbol{\Omega}}_i\right) + 3g\boldsymbol{J}\left(\boldsymbol{r}_d - R_i \widehat{\boldsymbol{\Omega}}_i\right) \cdot \widehat{\boldsymbol{\Omega}}_i\right]$
- $R_i(\boldsymbol{r}_o, d) = I\left(\boldsymbol{r}_o, -\widehat{\boldsymbol{\Omega}}_i\right) \exp[-(\mu_s + \mu_a)R_o]$



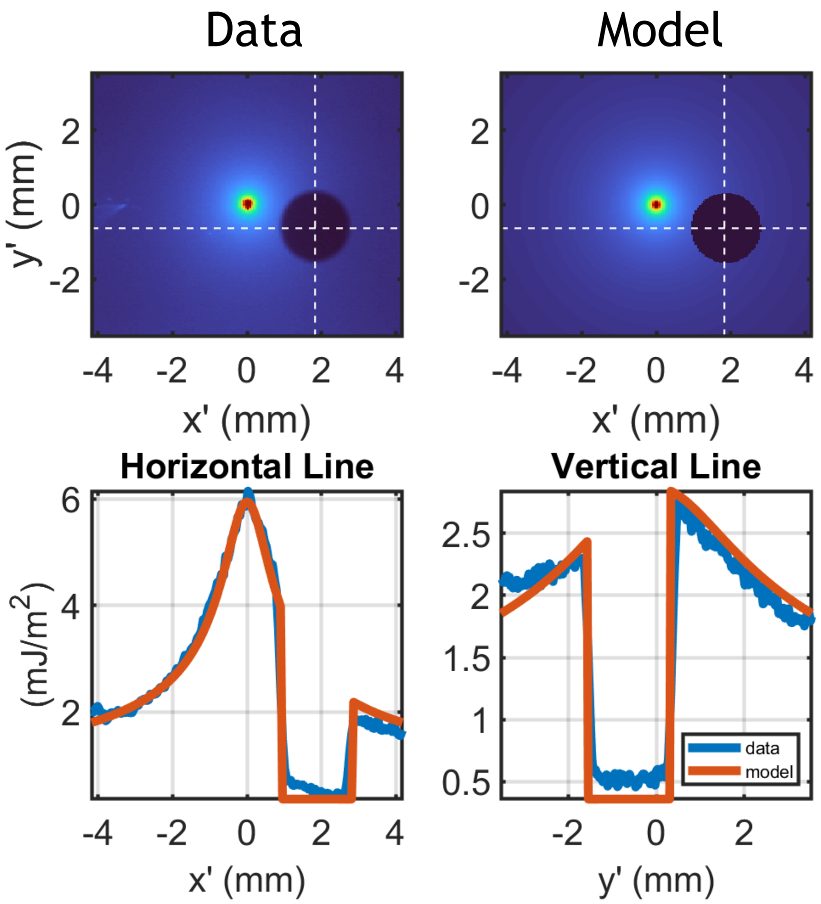
[1] B. Z. Bentz, et. al, *Optics Letters* ***47***(8), 2000-2003, 2022.

Fog Transmission Simulation ($\eta = 0.9, d = 0.2$ m)

Computation: 4 sec



Experimental Verification



Fog Optical Parameters:

$\mu_s = 0.236$ m^{-1}

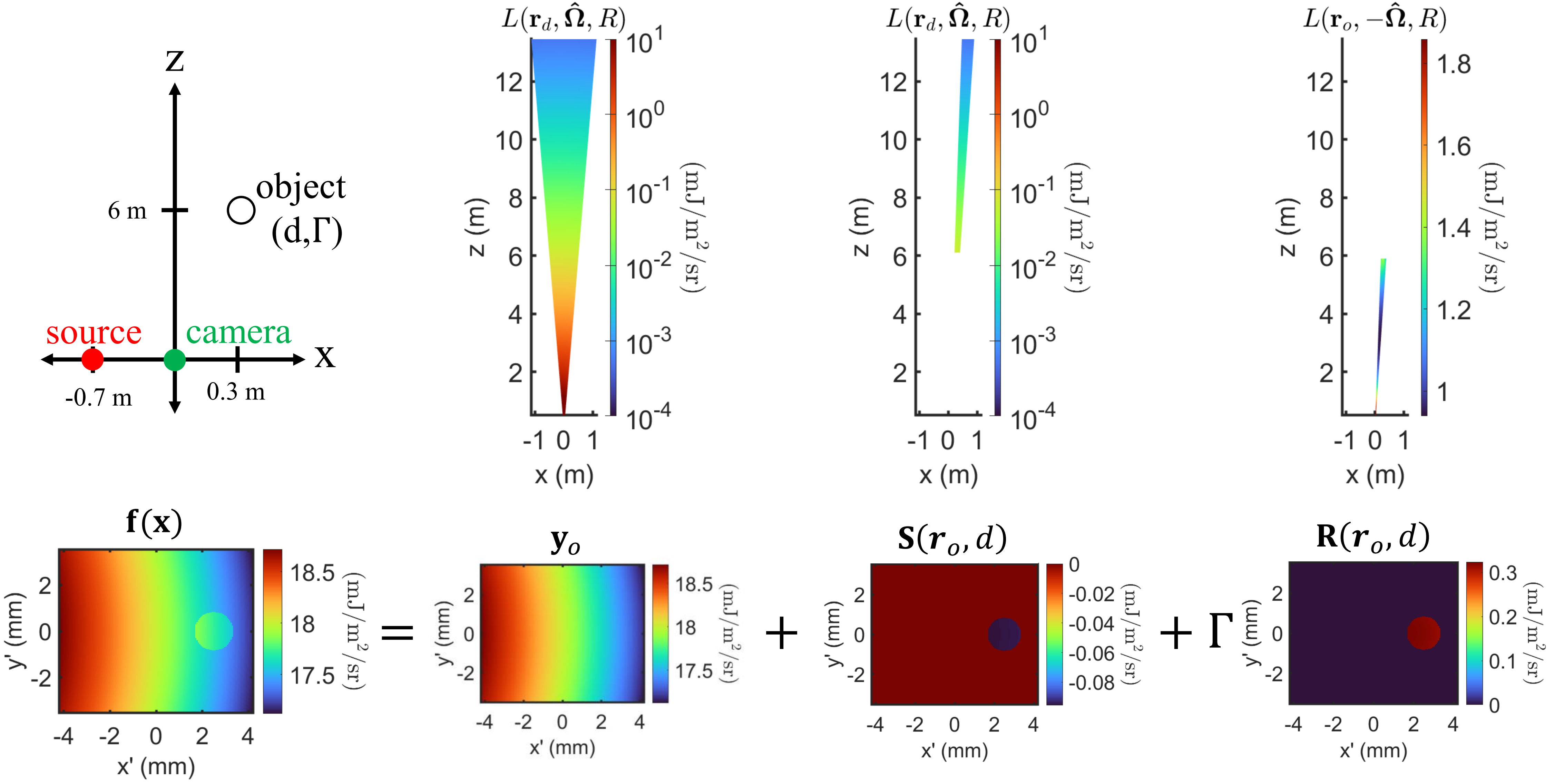
$\mu_a = 1.44 \times 10^{-8}$ m^{-1}

$g = 0.802$

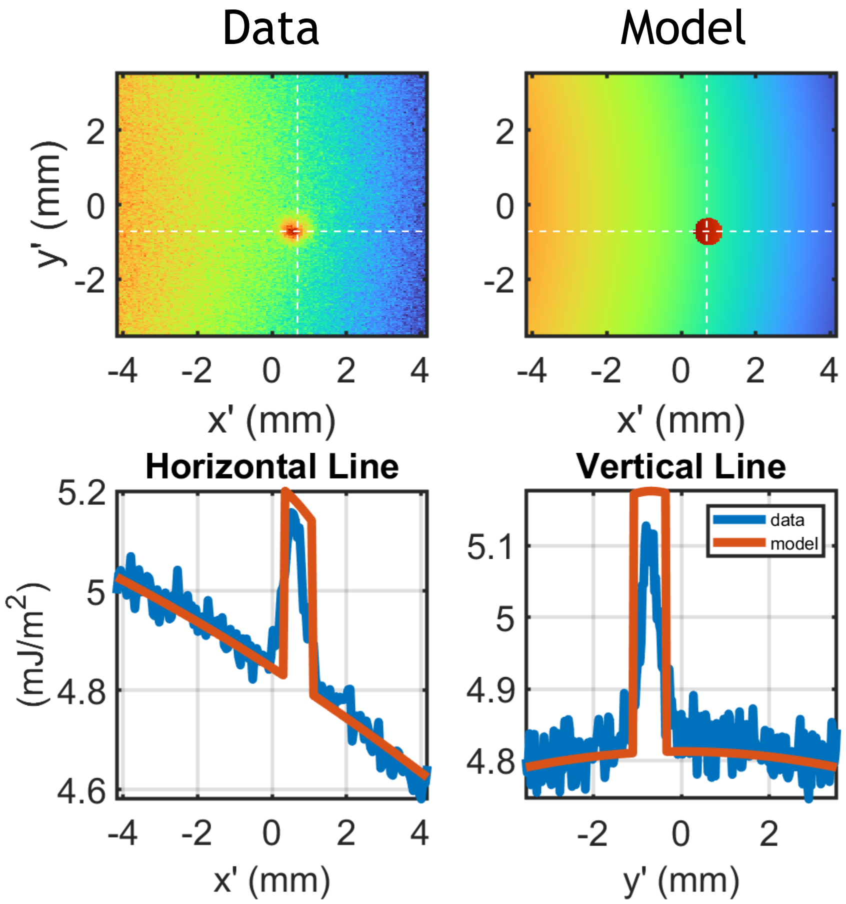
[1] B. Z. Bentz, et. al, *Optics Letters* ***47***(8), 2000-2003, 2022.

Fog Reflection Simulation ($\eta = 0.9$, $d = 0.2$ m)

Computation: 4 sec



Experimental Verification



Fog Optical Parameters:

$\mu_s = 1.03$ m^{-1}

$\mu_a = 1.48 \times 10^{-7}$ m^{-1}

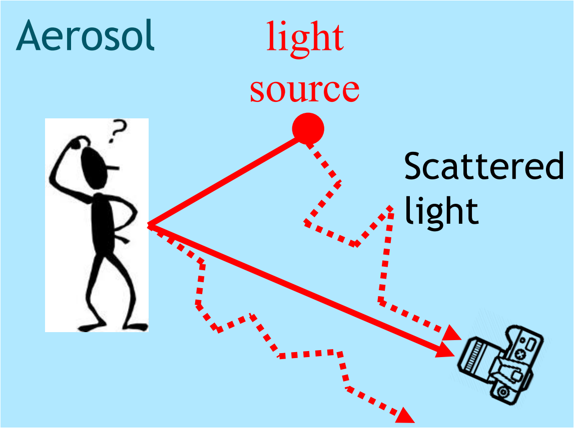
$g = 0.831$

[1] B. Z. Bentz, et. al, *Optics Letters* ***47***(8), 2000-2003, 2022.

All models are wrong, but some are useful!

- George Edward Pelham Box

- Detection: is an object there?
- Localization: where is the object?
- Imaging: what is the object?



Measurement Model (binary hypothesis)

$$p_0(\boldsymbol{y}) = \frac{1}{\sqrt{(2\pi)^P|\boldsymbol{\Upsilon}|}} \exp\left(-\frac{1}{2}\|\boldsymbol{y} - w\boldsymbol{y}_o\|^2_{\boldsymbol{\Upsilon}^{-1}}\right)$$

$$p_{1,\boldsymbol{x}}(\boldsymbol{y}) = \frac{1}{\sqrt{(2\pi)^P|\boldsymbol{\Upsilon}|}} \exp\left\{-\frac{1}{2}\|\boldsymbol{y} - w\boldsymbol{f}(\boldsymbol{x})\|^2_{\boldsymbol{\Upsilon}^{-1}}\right\}$$

Where

- $\boldsymbol{y}$ is the measurement vector of length P
- $\boldsymbol{y}_o$ is the expected measurement in the absence of an object
- $\boldsymbol{f}(\boldsymbol{x})$ is the expected measurement in the presence of an object
- $\boldsymbol{x} = [\boldsymbol{r}_o, d, \Gamma]$ is the opaque object parameters
- $\boldsymbol{\Upsilon}$ is the covariance matrix with $[\Upsilon]_{ii} = \alpha|y_i|$
- α is a scaler parameter of the measurement system
- w is a proportionality constant
- $\|\boldsymbol{u}\|^2_{\boldsymbol{\Upsilon}^{-1}} = \boldsymbol{u}^T\boldsymbol{\Upsilon}^{-1}\mathbf{u}$ and T is the transpose

[1] A. B. Milstein, M. D. Kennedy, P. S. Low, C. A. Bouman, and K. J. Webb, *Applied Optics* **44**(12), 2300-2310 (2005).
[2] G. Cao, V. Gaind, K. J. Webb, and C. A. Bouman, *Optics Letters* **32**(20), 3026-3028 (2007).

Object Detection

- Let $\theta = [\boldsymbol{x}, w]$ be the parameters of interest
- Neyman-Pearson lemma:
  - Likelihood ratio test (LRT) produces the highest probability of detection for specified **false alarm rate $\boldsymbol{P_F}$**

$$L(\boldsymbol{y}, \theta\ ) = \ln \frac{p_{1,\boldsymbol{x}}(\boldsymbol{y})}{p_0(\boldsymbol{y})} > \tilde{k}_{P_F}(\theta)$$

$$= [w\boldsymbol{f}(\boldsymbol{x}) - w\boldsymbol{y}_o]^T \boldsymbol{\Upsilon}^{-1}[\boldsymbol{y} - w\boldsymbol{y}_o] - \frac{1}{2}\|w\boldsymbol{f}(\boldsymbol{x}) - w\boldsymbol{y}_o\|^2_{\Upsilon^{-1}}$$

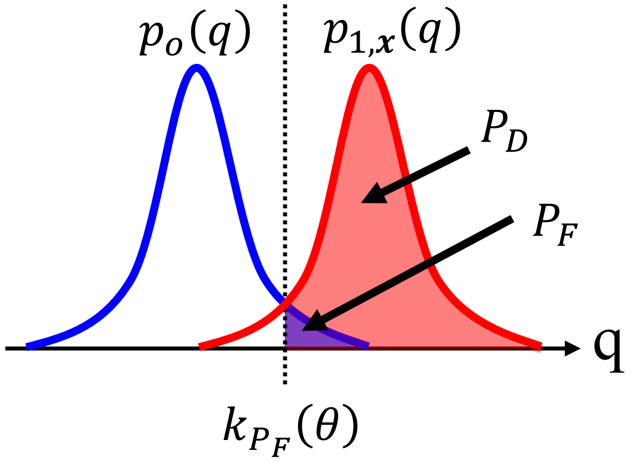
$$= \boldsymbol{h}^T(\theta)[\boldsymbol{y} - w\boldsymbol{y}_o] - c(\theta)$$

- Then, since $c(\theta)$ is a constant, the detection threshold becomes

$$\boldsymbol{h}^T(\theta)[\boldsymbol{y} - w\boldsymbol{y}_o] > k_{P_F}(\theta)$$

Detection:

$$\boldsymbol{h}^T(\theta)[\boldsymbol{y} - w\boldsymbol{y}_o] > k_{P_F}(\theta)$$



Object Detection

- We define $q = \boldsymbol{h}^T(\theta)[\boldsymbol{y} - w\boldsymbol{y}_o]$ as the decision statistic to compare to $k_{P_F}(\theta)$ threshold
- q has a normal distribution under the two hypothesis

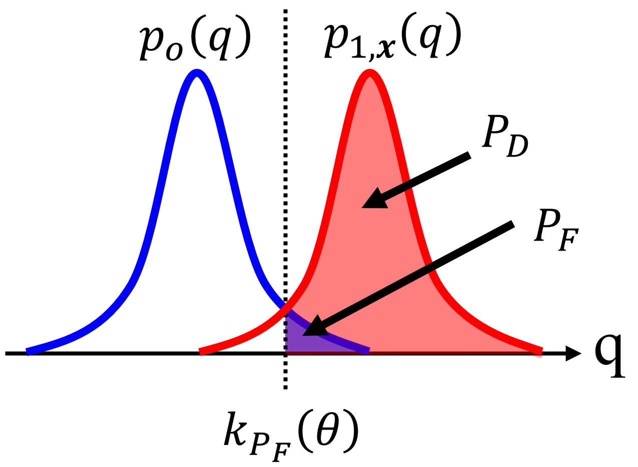
$$p_0(q) = \frac{1}{\sqrt{2\pi}\sigma_q} \exp\left(-\frac{q^2}{2\sigma_q^2}\right)$$

$$p_{1,\boldsymbol{x}}(q) = \frac{1}{\sqrt{2\pi}\sigma_q} \exp\left[-\frac{(q-\bar{q})^2}{2\sigma_q^2}\right]$$

- Where $\bar{q} = \sigma_q^2 = \|w\boldsymbol{f}(\boldsymbol{x}) - w\boldsymbol{y}_o\|_{\Upsilon^{-1}}^2$

Detection:

$q > k_{P_F}(\theta)$



Object Detection

$$P_F = \int_{k_{P_F}(\theta)}^{\infty} p_o(q)\,dq$$

$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{k_{P_F}(\theta)}{\sqrt{2}\sigma_q}\right)$$

$$P_D = \int_{k_{P_F}(\theta)}^{\infty} p_{1,\boldsymbol{x}}(q)\,dq$$

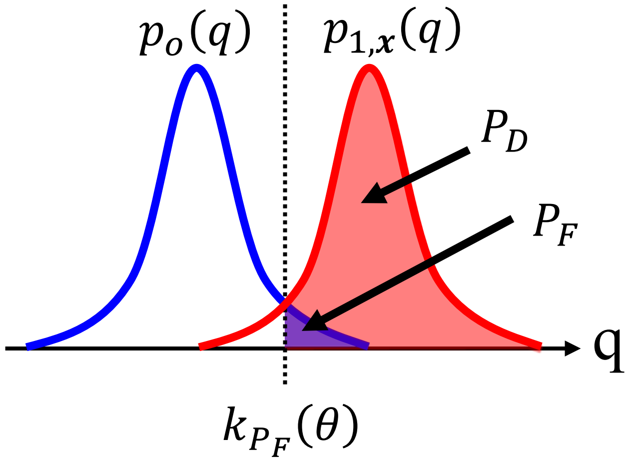
$$= \frac{1}{2}\,\mathrm{erfc}\left(\frac{k_{P_F}(\theta) - \bar{q}}{\sqrt{2}\sigma_q}\right)$$

$$k_{P_F}(\theta) = \sqrt{2}\,\sigma_q \mathrm{erfinv}(2\mathrm{P_F})$$

Error Functions

Detection:

$q > k_{P_F}(\theta)$

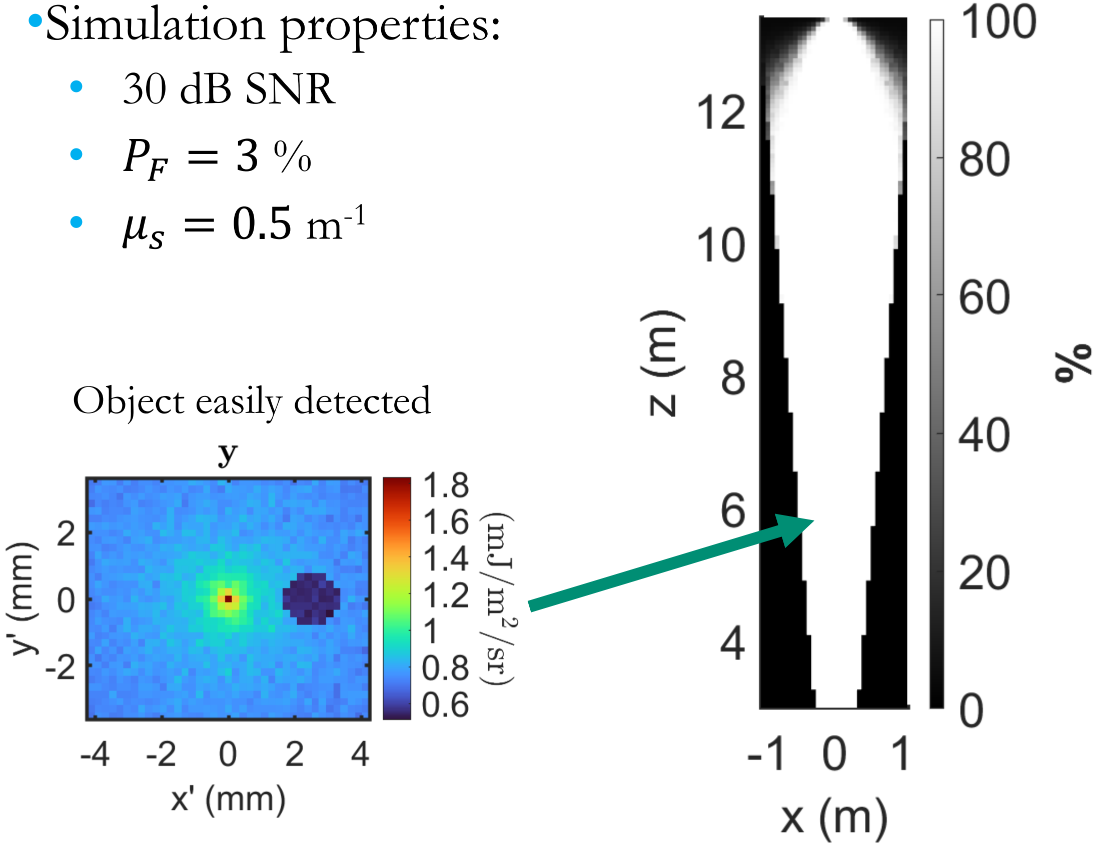


Simulated Probability of Detection (P_D)

Transmission

- Simulation properties:
  - 30 dB SNR
  - $P_F = 3\ \%$
  - $\mu_s = 0.5\ \mathrm{m}^{-1}$

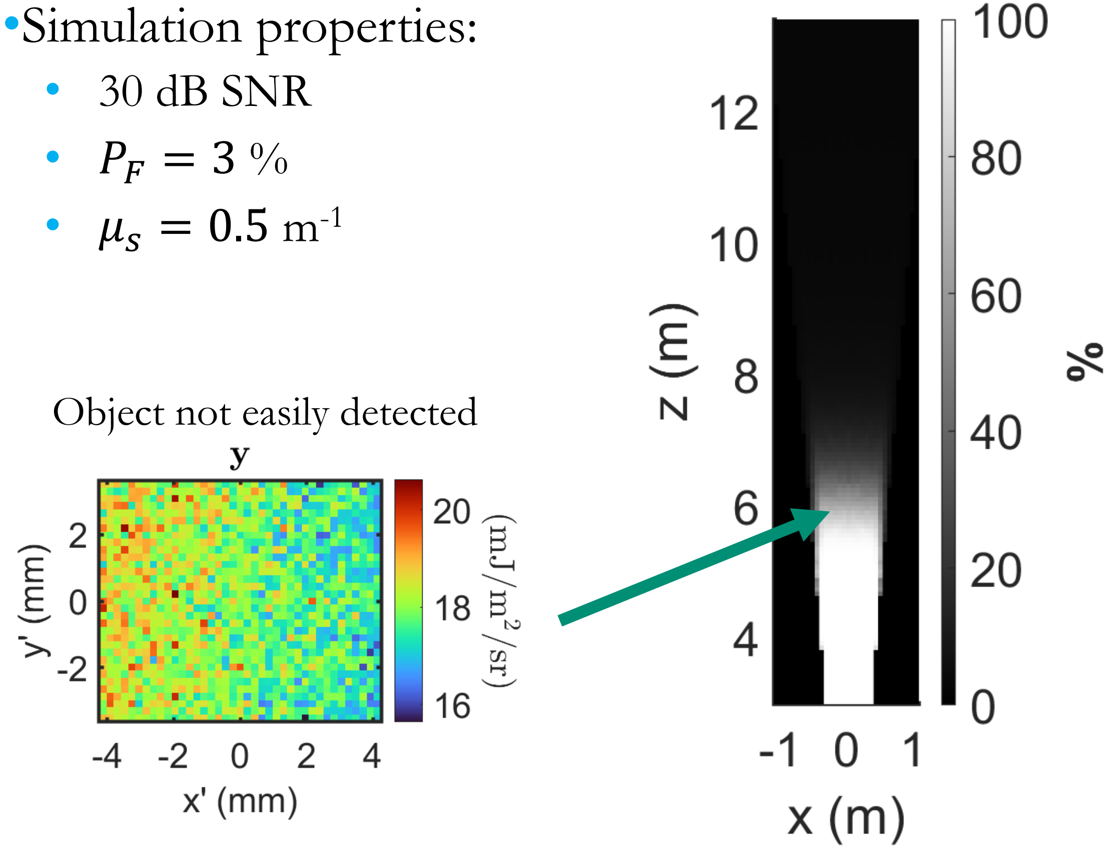
Object easily detected



Reflection

- Simulation properties:
  - 30 dB SNR
  - $P_F = 3\ \%$
  - $\mu_s = 0.5\ \mathrm{m}^{-1}$

Object not easily detected

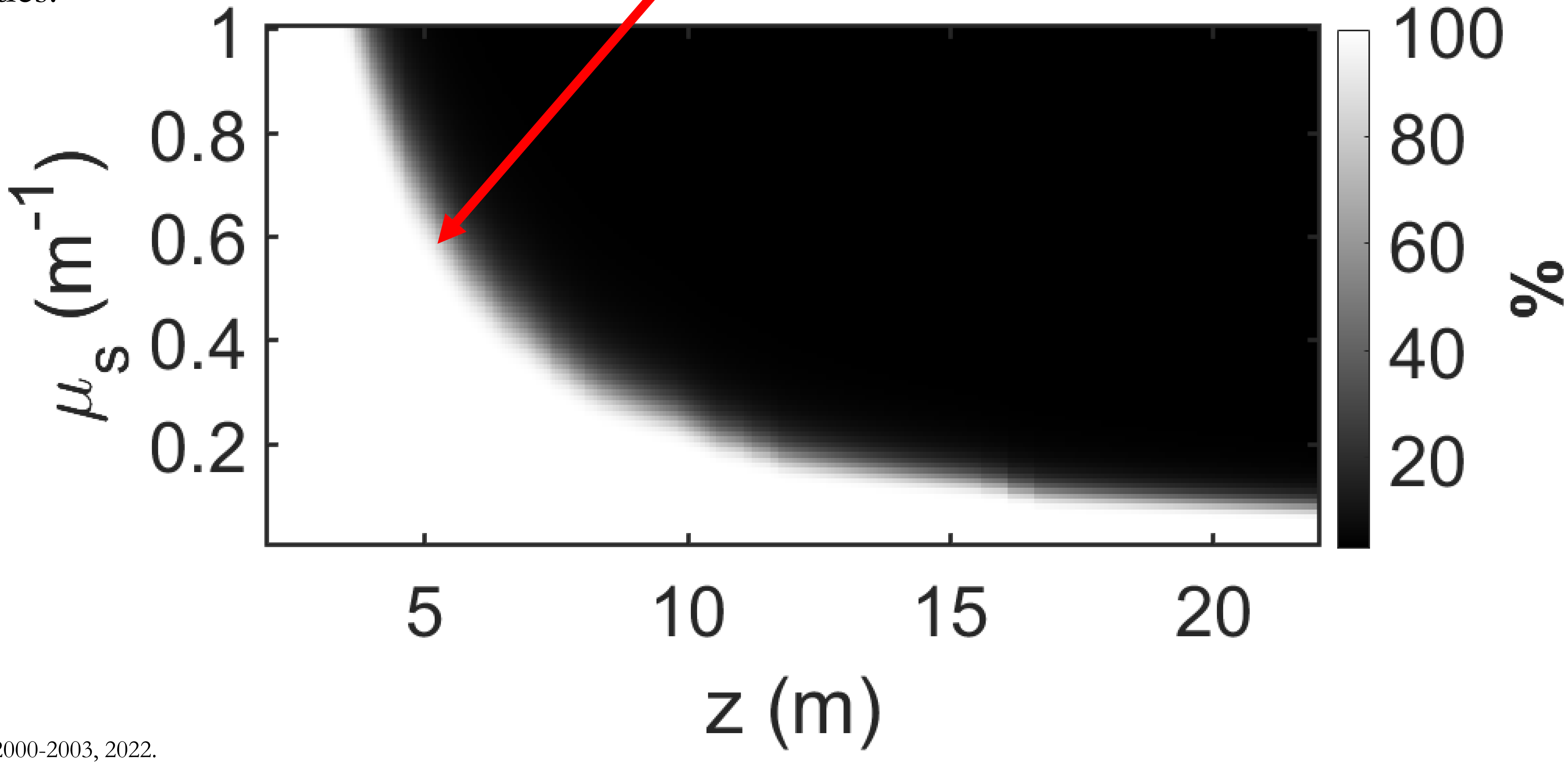


Fog Chamber P_D as a Function of Depth and μ_s

Reflection Geometry

- Simulation properties:
  - 30 dB SNR
  - $P_F = 3\ \%$
  - $g = 0.8$
  - $\Gamma = 0.9$
  - $d = 0.2\ \text{m}$

Detection Limit: $z = 22\exp(-12\mu_s) + 9\exp(-\mu_s)$



[1] B. Z. Bentz, et. al, *Optics Letters* ***47***(8), 2000-2003, 2022.

Localization using the Maximum Likelihood Estimate (MLE)

- $\boldsymbol{r_o}$ and d can be estimated by minimizing

$$c(\boldsymbol{r}_o, d) = \min_{\Gamma, w}\left[\|\boldsymbol{y} - wy_o - w\boldsymbol{S}(\boldsymbol{r_o}, d) - w\Gamma\boldsymbol{R}(\boldsymbol{r_o}, d)\|_{\Upsilon^{-1}}^2\right]$$

- Take derivatives and solve for closed form estimates of w and Γ
- Call them $\widetilde{w}$ and $\tilde{\Gamma}$

$$\widetilde{w} = \frac{\left[\boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r_o}, d) + \tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r_o}, d)\right]^T \boldsymbol{\Upsilon}^{-1}\boldsymbol{y}}{\left\|\boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r_o}, d) + \tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r_o}, d)\right\|_{\Upsilon^{-1}}^2}$$

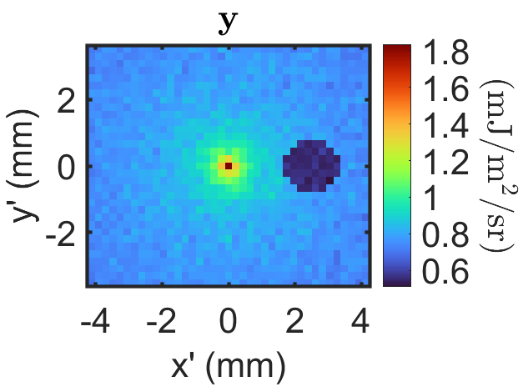
$$\tilde{\Gamma} = \frac{[\boldsymbol{y} - \widetilde{w}\boldsymbol{y}_o - \widetilde{w}\boldsymbol{S}(\boldsymbol{r_o}, d)]^T \boldsymbol{\Upsilon}^{-1}\boldsymbol{R}(\boldsymbol{r_o}, d)}{\widetilde{w}\|\boldsymbol{R}(\boldsymbol{r_o}, d)\|_{\Upsilon^{-1}}^2}$$

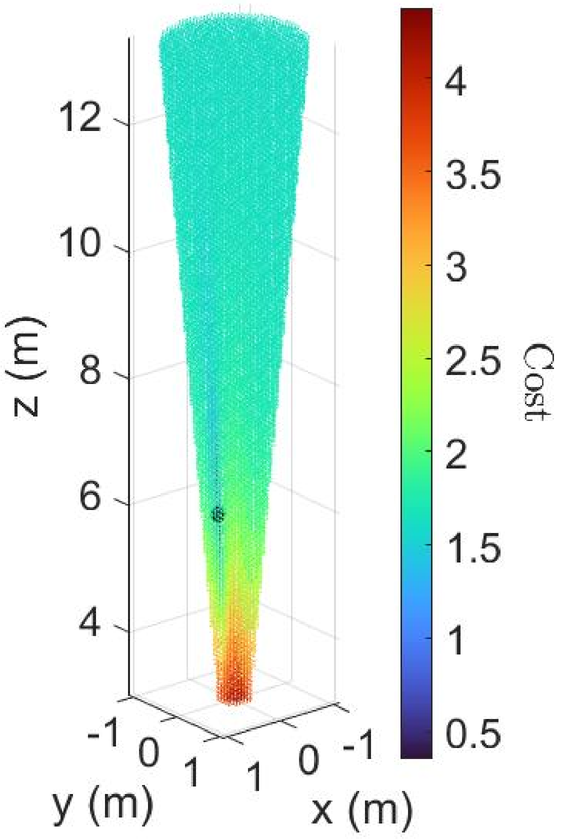
$$c(\boldsymbol{r}_o, d) = \left\|\boldsymbol{y} - \widetilde{w}y_o - \widetilde{w}\boldsymbol{S}(\boldsymbol{r_o}, d) - \widetilde{w}\tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r_o}, d)\right\|_{\Upsilon^{-1}}^2$$

Object Position, Diameter, and Reflection Coefficient Estimated Correctly

Transmission

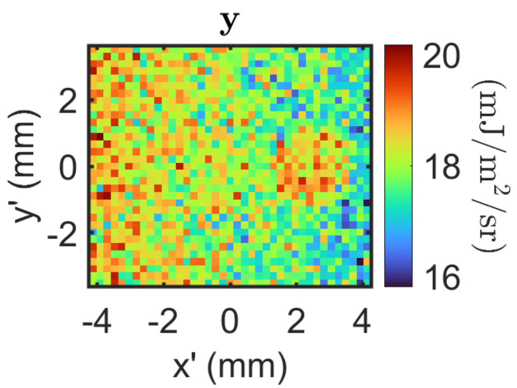
- True values:
  - $\boldsymbol{r_o} = (0.3,0,6)$ m
  - $d = 0.2$ m
  - $\mu_s = 0.5$ m^{-1}
- 30 dB SNR





Reflection

- True values:
  - $\boldsymbol{r_o} = (0.3,0,4)$ m
  - $d = 0.2$ m
  - $\Gamma = 0.9$
  - $\mu_s = 0.5$ m^{-1}
- 30 dB SNR



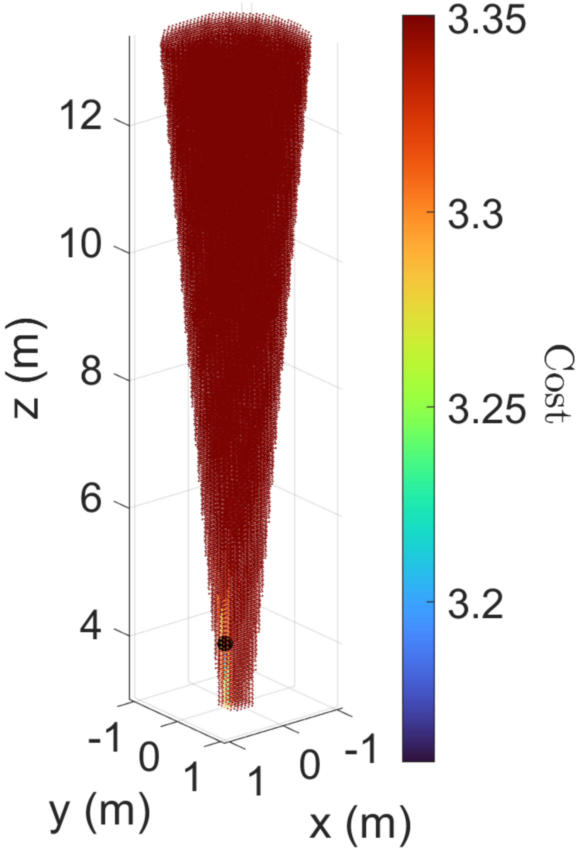


Image "Denoising"

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r}_o, d) + \Gamma \boldsymbol{R}(\boldsymbol{r}_o, d)$$

Measurement without object (background)

Blocked in-scattered light

Additional in-scattered reflected light

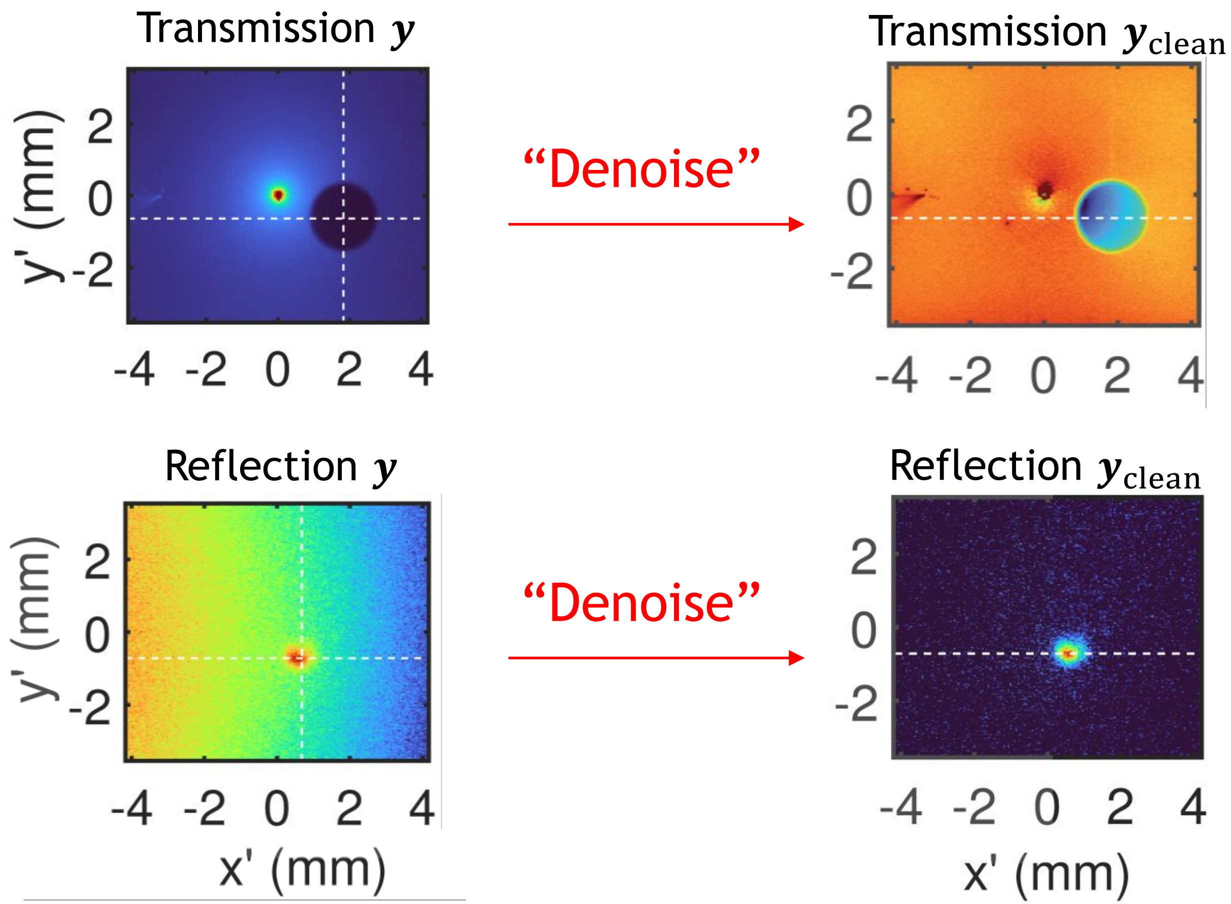
- Measurement without object is "noise"
- Model can predict this contribution
- Influence can be removed from the image

Image "Denoising"

$$\hat{\mu}_s = \arg\min_{\mu_s} \|\boldsymbol{y} - w y_o(\mu_s)\|^2_{\Upsilon^{-1}}$$

$$w = \frac{\boldsymbol{y}_o^T \boldsymbol{\Upsilon}^{-1} \boldsymbol{y}}{\boldsymbol{y}_o^T \boldsymbol{\Upsilon}^{-1} \boldsymbol{y}_o} \qquad \widehat{w} = \widetilde{w}(\hat{\mu}_s)$$

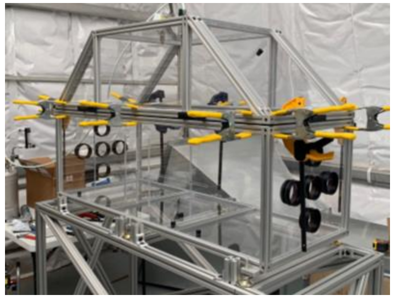
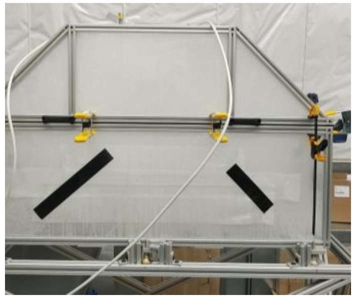
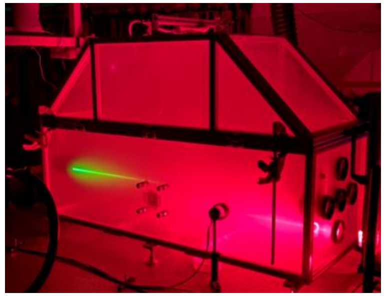
$$\boldsymbol{y}_{\text{clean}} = \boldsymbol{y} - \widehat{w} y_o(\hat{\mu}_s)$$



- Maximum likelihood estimate of μ_s minimizes difference between measured and modeled y_o
- Reduced error improves image contrast for interpretation

Conclusions

- The weak angular dependence approximation to the RTE can be sufficient for modeling photon transport in fog and with objects present
  - **This model can be leveraged for detection and localization**
- The model can be leveraged within a computational sensing framework
  - Detection: is an object there?
  - Localization: where is the object?
  - Imaging: what is the object?
- **Future work:** study convergence, leverage tabletop fog chamber, pursue machine learning inversion


Thank you!

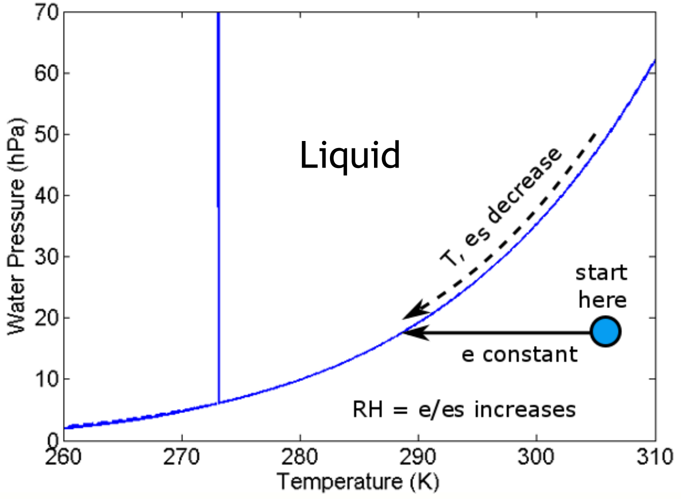
Supported by:
Sandia Laboratory Directed Research and Development (LDRD)
and the
Sandia Academic Alliance (SAA)

Additional Slides

What is fog?

- Micron sized water droplets suspended in air
  - < 1 km visibility
- Radiation fog – humid air cooled by emitted radiation reaches the dew point temperature
- Advection fog –humid air cooled while passing over colder wet surface

Water Phase Diagram



[1] J. B. Wright, J. D. Van der Laan, A. Sanchez, S. A. Kemme, and D. A. Scrymgeour, *Proc. SPIE* **10197**, 2017

Enhancing Fog Droplet Generation Techniques

Mechanical Technique

Droplets are generated via pressurized spray nozzle with salt water.

Instrumentation:
- Malvern Spraytec
- Three-wavelength Transmissometer

Measured Parameters:
- Droplet size distribution
- Transmission coefficients

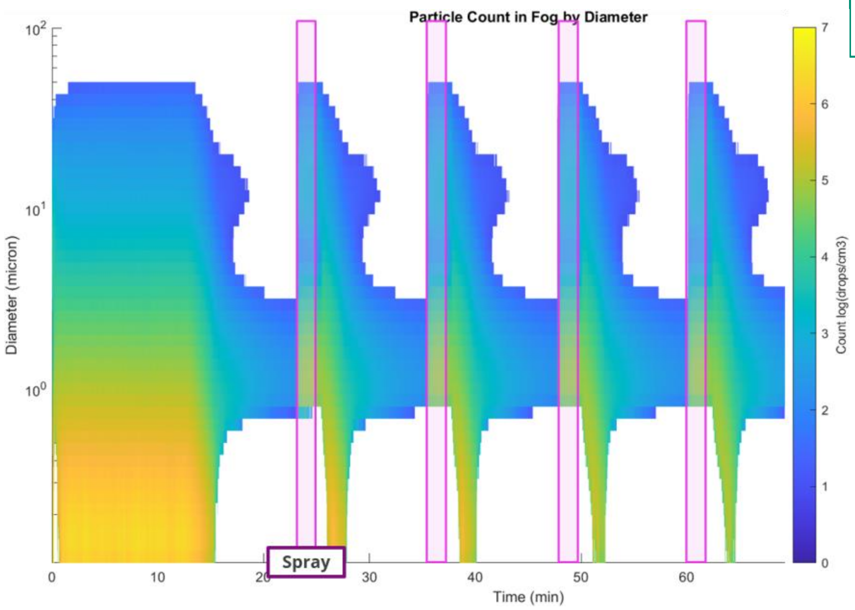
Injection Technique

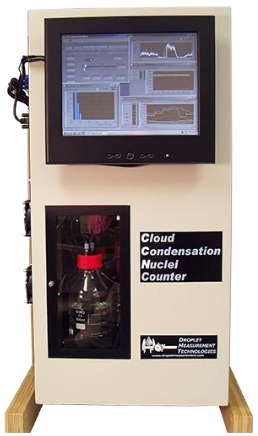
Dry aerosols are injected into a humid/supersaturated environment.

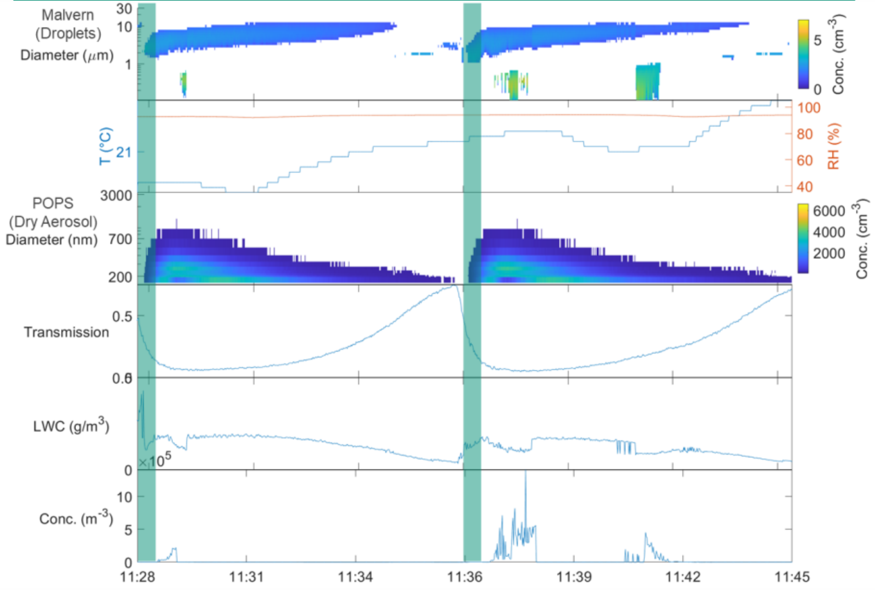
Design improvements:
- Insulation
- Cooling coil
- Heated humidifiers

New Instrumentation:
- SMPS (10 to 480 nm)
- APS/OPCs (~0.5 to >20 μm)
- CCNC

Benefits of Injection Technique
- Fog is more relevant to ambient formation mechanism
- Can observe droplet nucleation and diffusional growth
- Able to control T, RH, and dry aerosol concentration
- Stable RH conditions allow for the generation of unimodal fog (RH drops quickly after spray using mechanical technique)

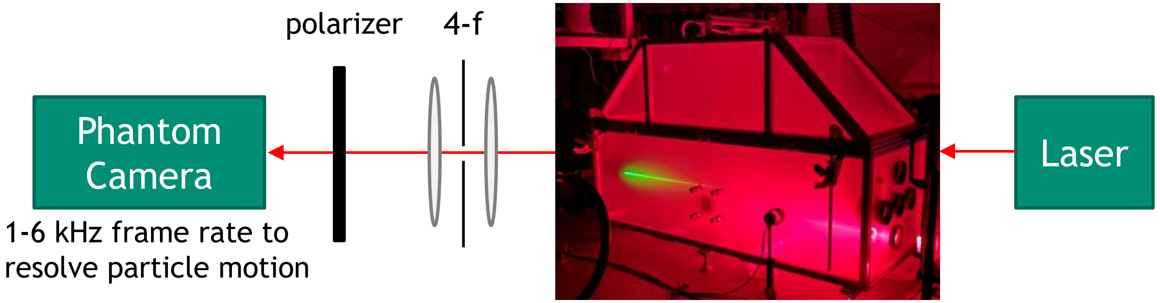


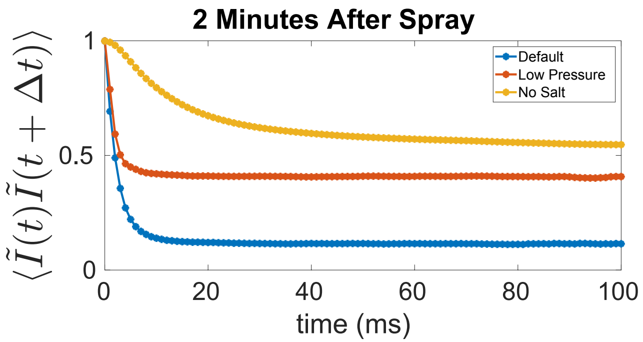




*NaCl injection periods are shaded in green.

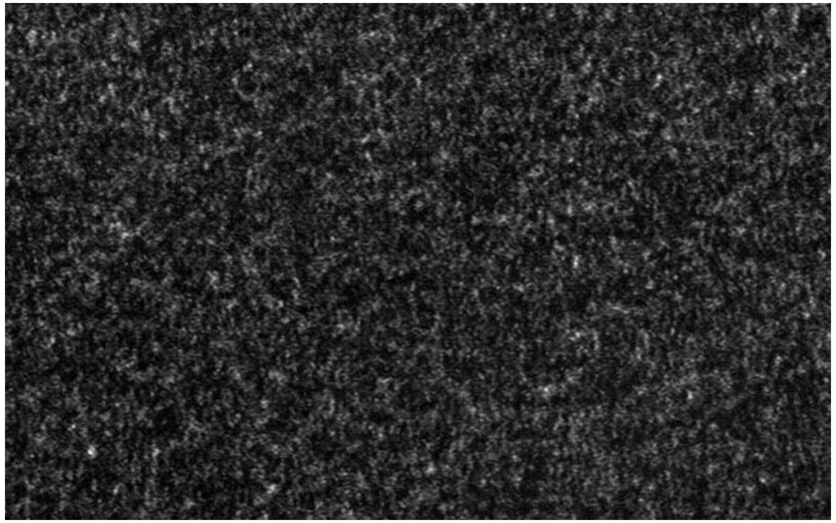
With Purdue University[1]: Speckle Correlation Imaging through Fog



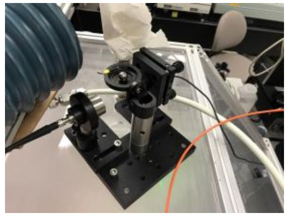


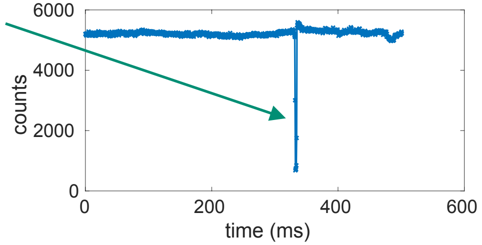
- Speckle decorrelation is due to fog particle motion (velocity)
- Plateau depends on fog particle density

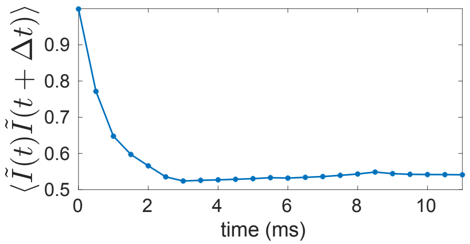
Coherent speckle pattern

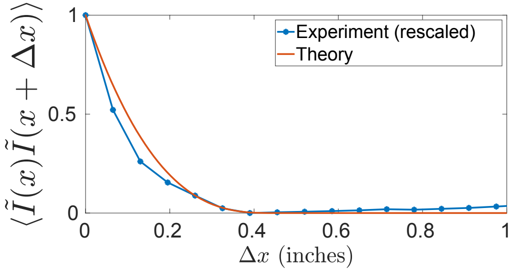


Triggered ball drop

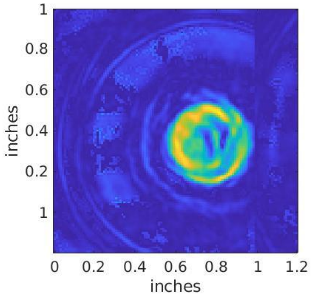








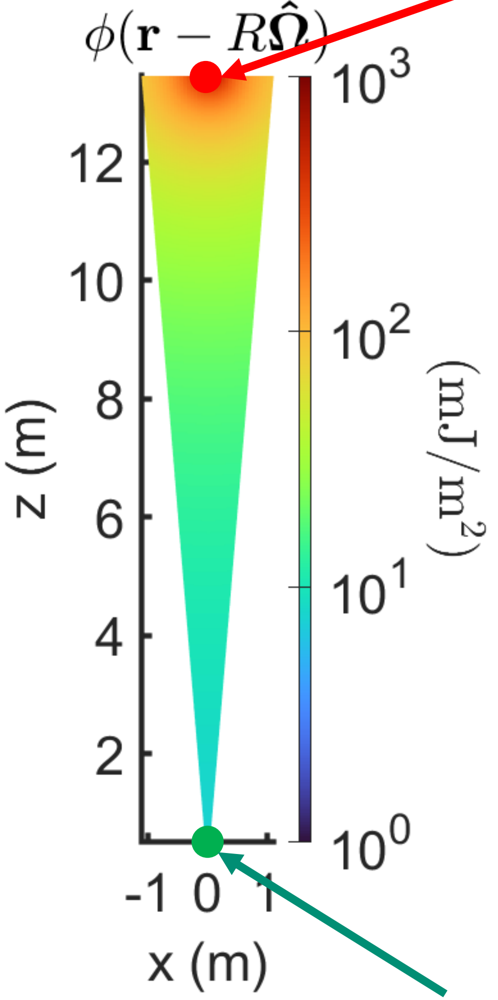
Retrieval



[1] Q. Luo and K. J. Webb, *Phys. Rev. Res.* **2**, 033148 (2020).

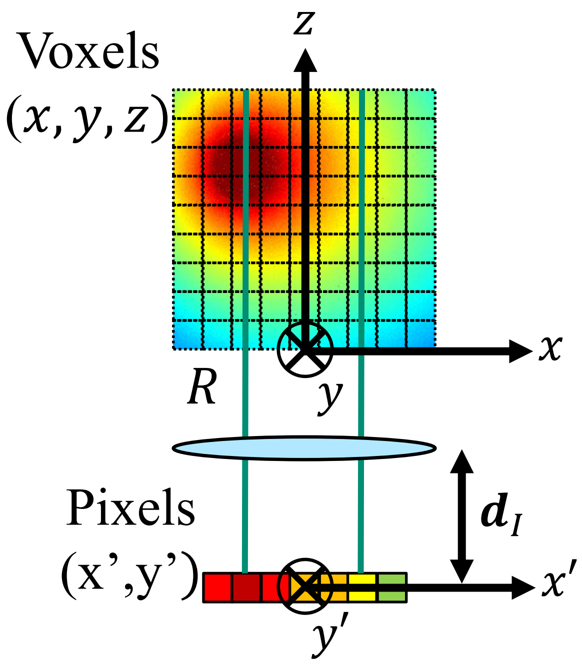
Transmission Simulation in Homogeneous Fog

Light source here



Camera here

Pinhole camera model



$$x = -\frac{z}{d_I}x'$$

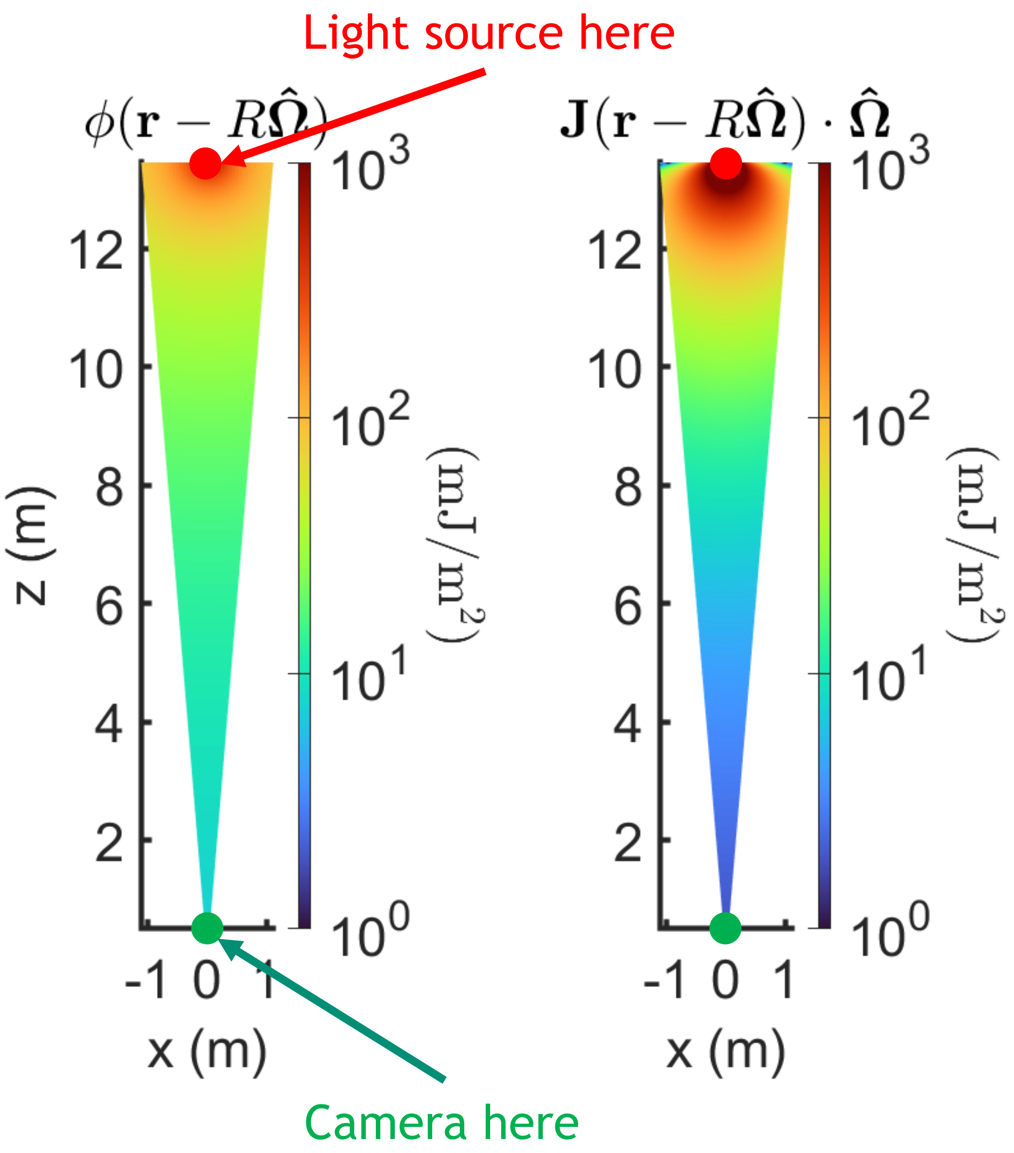
$$y = -\frac{z}{d_I}y'$$

Parameters (450 nm):

- MOR = 6 m
- Fog Density = 10^5 cm^{-3}
- Average Particle Diameter = 1.34 μm
- μ_s = 0.5 m^{-1}
- $\mu_a = 10^{-8}$ m^{-1}
- $g = 0.8$
- MFP = $1/\mu_s$ = 2 m
- $l_t = 1/[\mu_s(1-g)]$ = 10 m

- $S_o = 2\ (W) \times 2\ (\mathrm{sec}) = 4$ J

Transmission Simulation in Homogeneous Fog

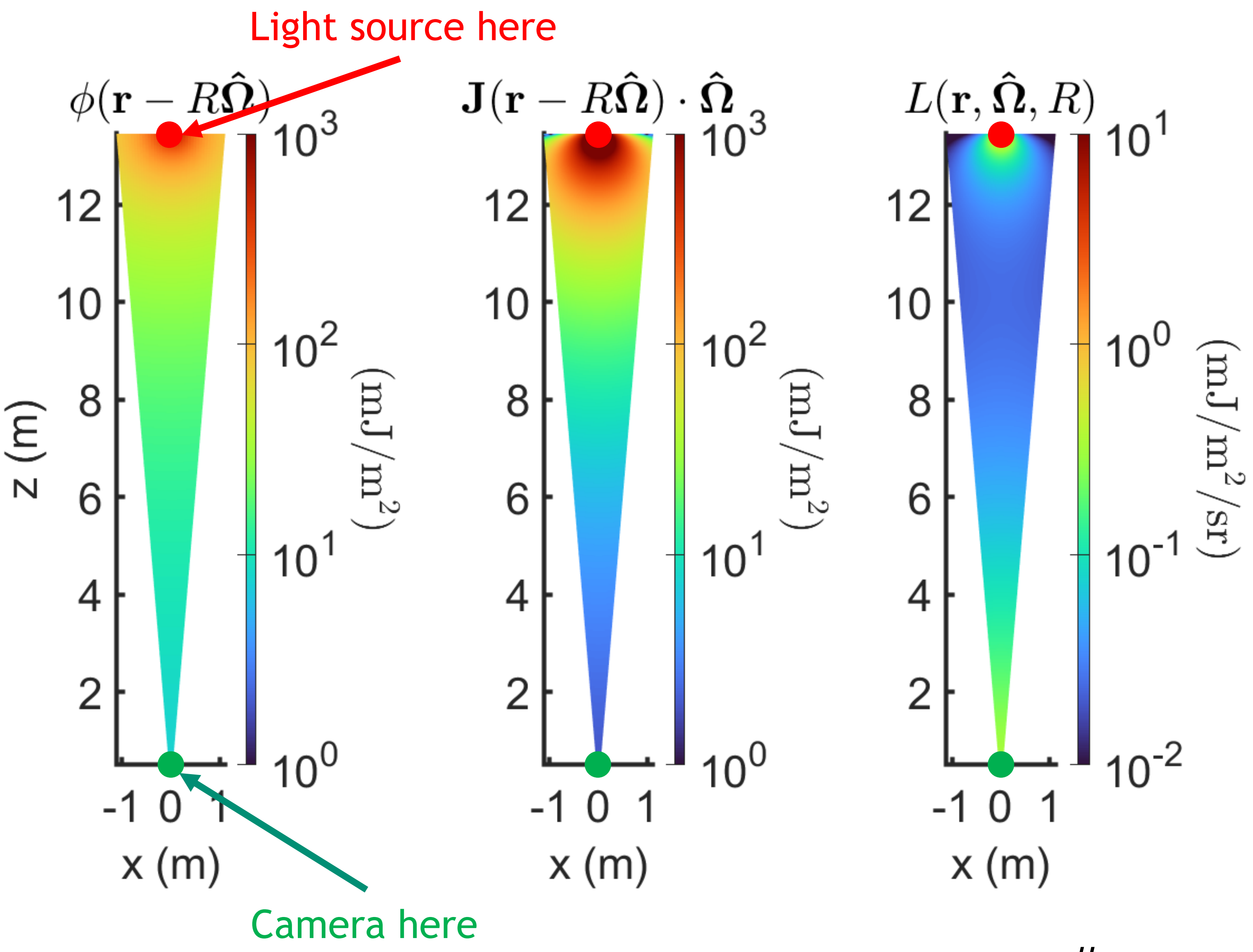


$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi}\int_{\boldsymbol{0}}^{\infty} dR\,\exp[-(\mu_s+\mu_a)R]\left[\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}\right]$$

$$\phi(\boldsymbol{r}) = \left(\frac{S_o}{4\pi D}\right)\frac{\exp\left(-\sqrt{\frac{\mu_a}{D}}\,|\boldsymbol{r}-\boldsymbol{r}_s|\right)}{|\boldsymbol{r}-\boldsymbol{r}_s|}$$

$$\boldsymbol{J}(\boldsymbol{r}) = \left[\frac{S_o(\boldsymbol{r}-\boldsymbol{r}_s)}{4\pi}\right]\left(\frac{\sqrt{\mu_a/D}}{|\boldsymbol{r}-\boldsymbol{r}_s|^2} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|^3}\right)\exp\left(-\sqrt{\frac{\mu_a}{D}}\,|\boldsymbol{r}-\boldsymbol{r}_s|\right)$$

Transmission Simulation in Homogeneous Fog

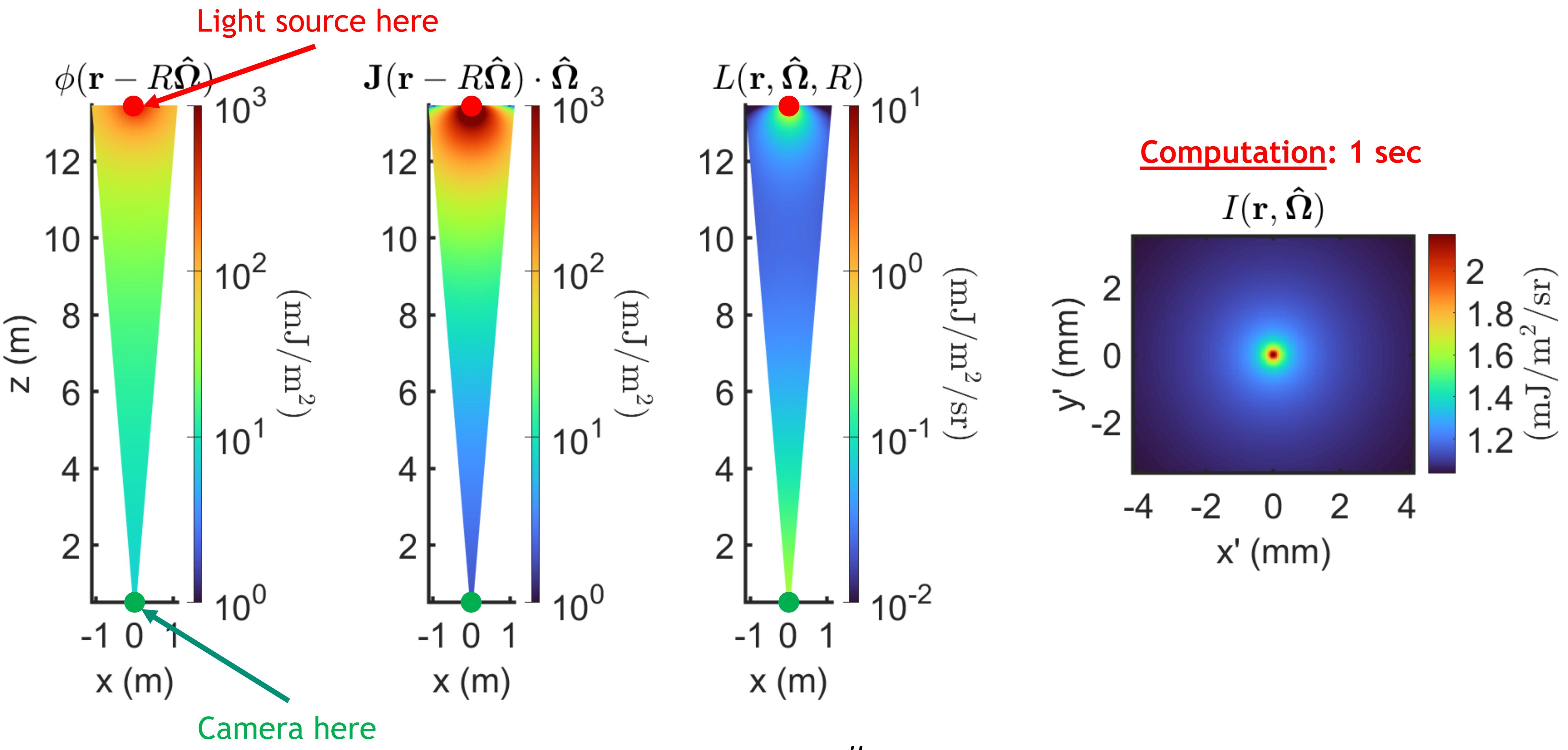


$$I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi} \int_0^\infty dR \exp[-(\mu_s + \mu_a)R] \times [\phi(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) \cdot \widehat{\boldsymbol{\Omega}}]$$

$$I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = \int_0^\infty dR\, L(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}, R)$$

$$L(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}, R) = \frac{\mu_s}{4\pi} \exp[-(\mu_s + \mu_a)R]\, [\phi(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) \cdot \widehat{\boldsymbol{\Omega}}]$$

Transmission Simulation in Homogeneous Fog



$$L(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}, R) = \frac{\mu_s}{4\pi} \exp[-(\mu_s + \mu_a)R] \left[\phi(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) \cdot \widehat{\boldsymbol{\Omega}}\right]$$

Radiative Transfer Equation (RTE)

$$\hat{\mathbf{\Omega}} \cdot \nabla I(\boldsymbol{r}, \hat{\mathbf{\Omega}}) + (\mu_a + \mu_s) I(\boldsymbol{r}, \hat{\mathbf{\Omega}}) = \mu_s \int_{4\pi} d\hat{\mathbf{\Omega}}' f(\hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}}) I(\boldsymbol{r}, \hat{\mathbf{\Omega}}') + Q(\boldsymbol{r}, \hat{\mathbf{\Omega}})$$

Where

- $I(\boldsymbol{r}, \hat{\mathbf{\Omega}})$ is the radiance (J/m^2/sr) at position $\boldsymbol{r}$ in direction $\hat{\mathbf{\Omega}}$
- μ_a is the absorption coefficient (m^{-1})
- μ_s is the scattering coefficient (m^{-1})
- $f(\hat{\mathbf{\Omega}}' \to \hat{\mathbf{\Omega}})$ is the scattering phase function for incident direction $\hat{\mathbf{\Omega}}'$ and scattering direction $\hat{\mathbf{\Omega}}$
- $Q(\boldsymbol{r}, \hat{\mathbf{\Omega}})$ is the radiance source function (J/m^3/sr)

[1] G. I. Bell and S. Glasstone, *Nuclear Reactor Theory*, U.S. Atomic Energy Commission, Washington DC (1970).

Continuity and Diffusion Equations

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) + \mu_a(\boldsymbol{r})\Phi(\boldsymbol{r}) = S(\boldsymbol{r})$$

Where

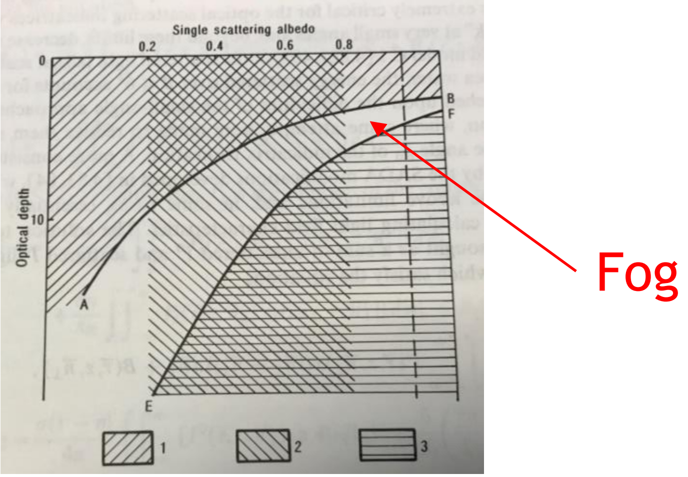
- $\phi(\boldsymbol{r}) = \int_{4\pi} d\widehat{\boldsymbol{\Omega}} I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$ is the fluence rate (J/m^2)
- $\boldsymbol{J}(\boldsymbol{r}) = \int_{4\pi} d\widehat{\boldsymbol{\Omega}} \widehat{\boldsymbol{\Omega}} I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$ is the flux density (J/m^2)
- $S(\boldsymbol{r}) = \int_{4\pi} d\widehat{\boldsymbol{\Omega}} Q(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$ is the source (J/m^3)

If: $\boldsymbol{J}(\boldsymbol{r}) = -D(\mathbf{r})\nabla\phi(\boldsymbol{r})$ (Fick's first law of diffusion)

- $D = 1/3(\mu_s' + \mu_a)$ is the diffusion coefficient (m)
- $\mu_s' = \mu_s(1-g)$ is the reduced scattering coefficient for anisotropy g

$$-\nabla \cdot D(\boldsymbol{r})\nabla\phi(\boldsymbol{r}) + \mu_a(\boldsymbol{r})\phi(\boldsymbol{r}) = S(\boldsymbol{r})$$

Applicability domains of approximate methods [1]:
(1) small-angle approximation
(2) small-angle diffusion approximation
(3) diffusion approximation



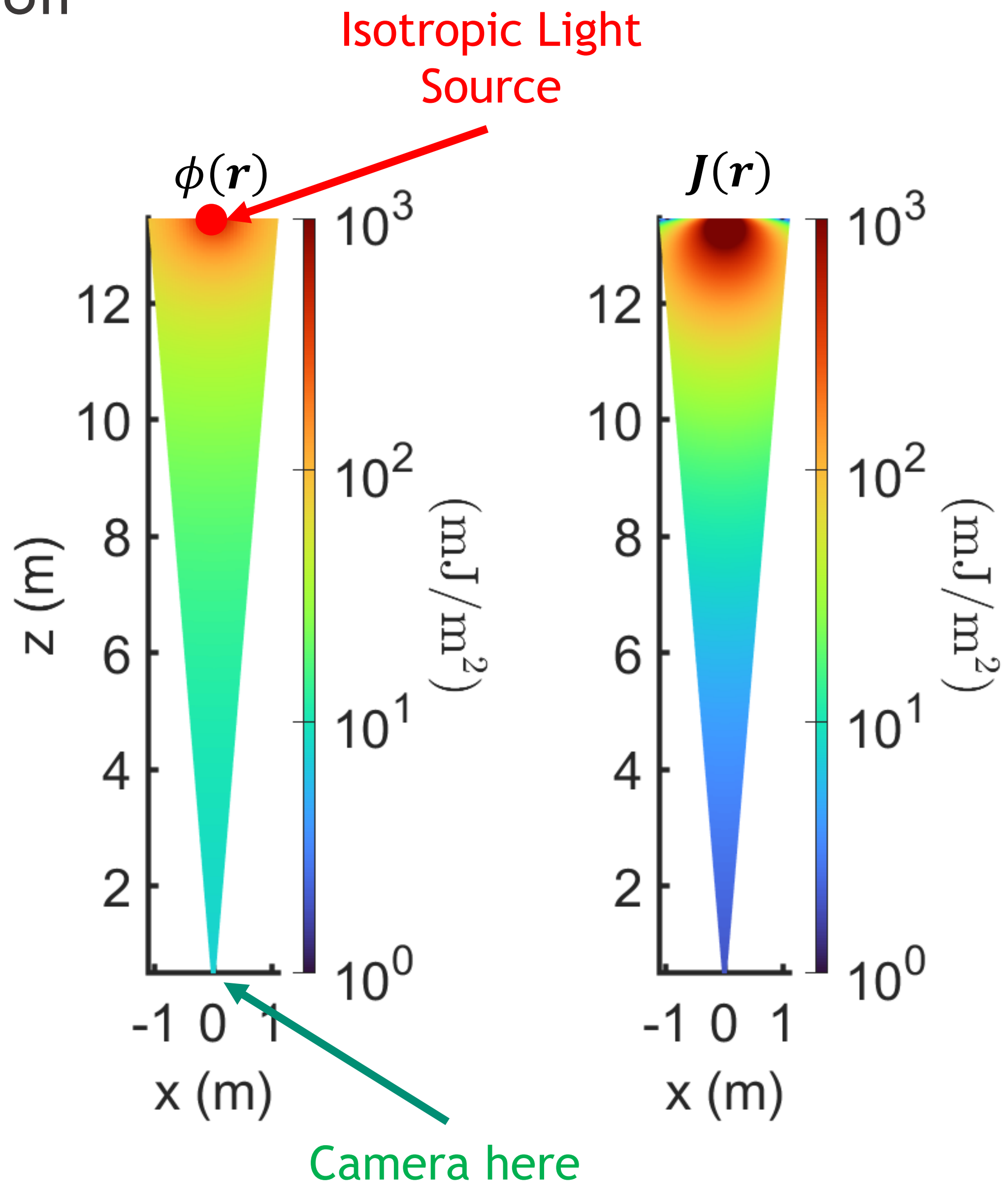
[1] E. P. Zege, A. P. Ivanov, and I. L. Katsev, New York: Springer-Verlag, (1991).

Analytic Solutions to the Diffusion Equation

- The diffusion equation has well known solutions
- In an infinite homogeneous media with a source located at $\boldsymbol{r}_s$ with power S_o (J)

$$\phi(\boldsymbol{r}) = \left(\frac{S_o}{4\pi D}\right)\frac{\exp\left(-\sqrt{\frac{\mu_a}{D}}|\boldsymbol{r}-\boldsymbol{r}_s|\right)}{|\boldsymbol{r}-\boldsymbol{r}_s|}$$

$$\boldsymbol{J}(\boldsymbol{r}) = \left[\frac{S_o(\boldsymbol{r}-\boldsymbol{r}_s)}{4\pi}\right]\left(\frac{\sqrt{\mu_a/D}}{|\boldsymbol{r}-\boldsymbol{r}_s|^2}+\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|^3}\right)\exp\left(-\sqrt{\frac{\mu_a}{D}}|\boldsymbol{r}-\boldsymbol{r}_s|\right)$$



Radiative Transfer Equation (RTE)

$$\widehat{\boldsymbol{\Omega}} \cdot \nabla I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) + (\mu_a + \mu_s) I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = \mu_s \int_{4\pi} d\widehat{\boldsymbol{\Omega}}' f(\widehat{\boldsymbol{\Omega}}' \to \widehat{\boldsymbol{\Omega}}) I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}')$$

$$\widehat{\boldsymbol{\Omega}} \cdot \nabla I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) + (\mu_a + \mu_s) I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = Q_s(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}})$$

$$\widehat{\boldsymbol{\Omega}} \cdot \nabla I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) \exp[(\mu_a + \mu_s) \widehat{\boldsymbol{\Omega}} \cdot \boldsymbol{r}] + (\mu_a + \mu_s) I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) \exp[(\mu_a + \mu_s) \widehat{\boldsymbol{\Omega}} \cdot \boldsymbol{r}] = Q_s(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) \exp[(\mu_a + \mu_s) \widehat{\boldsymbol{\Omega}} \cdot \boldsymbol{r}]$$

$$\widehat{\boldsymbol{\Omega}} \cdot \nabla \{ I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) \exp[(\mu_a + \mu_s) \widehat{\boldsymbol{\Omega}} \cdot \boldsymbol{r}] \} = Q_s(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) \exp[(\mu_a + \mu_s) \widehat{\boldsymbol{\Omega}} \cdot \boldsymbol{r}]$$

Derivative in $\widehat{\boldsymbol{\Omega}}$ direction

Define Line-of-Sight

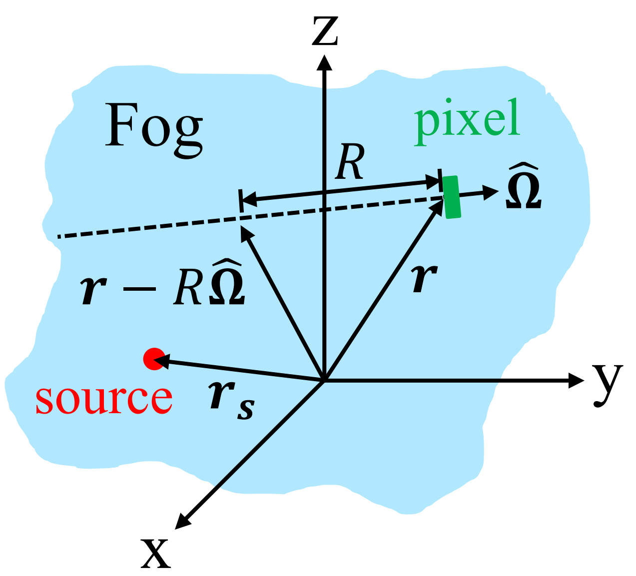
$$\hat{\mathbf{\Omega}} \cdot \nabla \{ I(\boldsymbol{r}, \hat{\mathbf{\Omega}}) \exp[(\mu_a + \mu_s)\hat{\mathbf{\Omega}} \cdot \boldsymbol{r}] \} = Q_s(\boldsymbol{r}, \hat{\mathbf{\Omega}}) \exp[(\mu_a + \mu_s)\hat{\mathbf{\Omega}} \cdot \boldsymbol{r}]$$

$$\hat{\mathbf{\Omega}} \cdot \nabla \rightarrow \hat{\mathbf{\Omega}} \cdot \frac{\partial}{\partial R}\hat{\mathbf{\Omega}} \rightarrow \frac{\partial}{\partial R}$$

$$I(\boldsymbol{r}, \hat{\mathbf{\Omega}}) \exp[(\mu_a + \mu_s)\hat{\mathbf{\Omega}} \cdot \boldsymbol{r}] = \int_0^\infty dR\, Q_s(\boldsymbol{r} - R\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}) \exp[(\mu_a + \mu_s)\hat{\mathbf{\Omega}} \cdot (\boldsymbol{r} - R\hat{\mathbf{\Omega}})]$$

$$I(\boldsymbol{r}, \hat{\mathbf{\Omega}}) = \int_0^\infty dR\, Q_s(\boldsymbol{r} - R\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}) \exp[-(\mu_a + \mu_s) R]$$

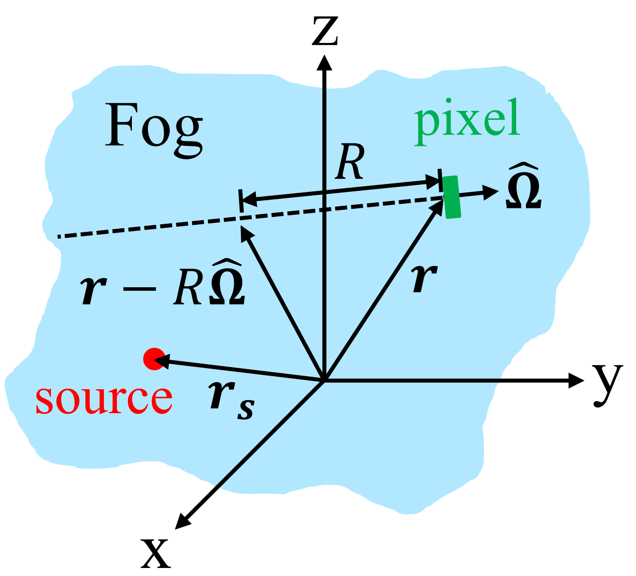
$$I(\boldsymbol{r}, \hat{\mathbf{\Omega}}) = \mu_s \int_0^\infty dR \exp[-(\mu_a + \mu_s) R] \int_{4\pi} d\hat{\mathbf{\Omega}}' f(\hat{\mathbf{\Omega}}' \rightarrow \hat{\mathbf{\Omega}}) I(\boldsymbol{r} - R\hat{\mathbf{\Omega}}, \hat{\mathbf{\Omega}}')$$



Solving the RTE with Weak Angular Dependence

- Radiance at a detector at position $\boldsymbol{r}$ in homogeneous fog

$$I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = \mu_s \int_{\boldsymbol{0}}^{\infty} dR \exp[-(\mu_s + \mu_a)R] \int_{4\pi} d\widehat{\boldsymbol{\Omega}}' \, f(\widehat{\boldsymbol{\Omega}}' \rightarrow \widehat{\boldsymbol{\Omega}}) I(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}, \widehat{\boldsymbol{\Omega}}')$$



- Assuming isotropic scatter

$$I(\boldsymbol{r}, \widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi} \int_{\boldsymbol{0}}^{\infty} dR \exp[-(\mu_s + \mu_a)R] \, \phi(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}})$$

- Assume weak angular dependence

$$I(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}, \widehat{\boldsymbol{\Omega}}) \approx \frac{1}{4\pi} \phi(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) + \frac{3}{4\pi} \boldsymbol{J}(\boldsymbol{r} - R\widehat{\boldsymbol{\Omega}}) \cdot \widehat{\boldsymbol{\Omega}}$$

[1] B. Z. Bentz, et. al, *Optics Express* **29**(9), 13231-13245, 2021.

Solving the RTE with Weak Angular Dependence

$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \mu_s \int_{\boldsymbol{0}}^{\infty} dR \exp[-(\mu_s+\mu_a)R] \int_{4\pi} d\widehat{\boldsymbol{\Omega}}' \, f(\widehat{\boldsymbol{\Omega}}' \to \widehat{\boldsymbol{\Omega}}) \left[\frac{1}{4\pi}\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}}) + \frac{3}{4\pi}\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}\right]$$

$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \mu_s \int_{\boldsymbol{0}}^{\infty} dR \exp[-(\mu_s+\mu_a)R] \left[\frac{\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})}{4\pi} \underbrace{\int_{4\pi} d\widehat{\boldsymbol{\Omega}}' \, f(\widehat{\boldsymbol{\Omega}}' \to \widehat{\boldsymbol{\Omega}})}_{=1} + \frac{3\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})}{4\pi} \cdot \underbrace{\int_{4\pi} d\widehat{\boldsymbol{\Omega}}'\widehat{\boldsymbol{\Omega}}' f(\widehat{\boldsymbol{\Omega}}' \to \widehat{\boldsymbol{\Omega}})}_{=g\widehat{\boldsymbol{\Omega}}}\right]$$

$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \mu_s \int_{\boldsymbol{0}}^{\infty} dR \exp[-(\mu_s+\mu_a)R] \left[\frac{1}{4\pi}\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}}) + \frac{3g}{4\pi}\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}\right]$$

$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi} \int_{\boldsymbol{0}}^{\infty} dR \exp[-(\mu_s+\mu_a)R] \left[\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}\right]$$

Additional anisotropic term

[1] B. Z. Bentz, et. al, *Optics Express* **29**(9), 13231-13245, 2021.

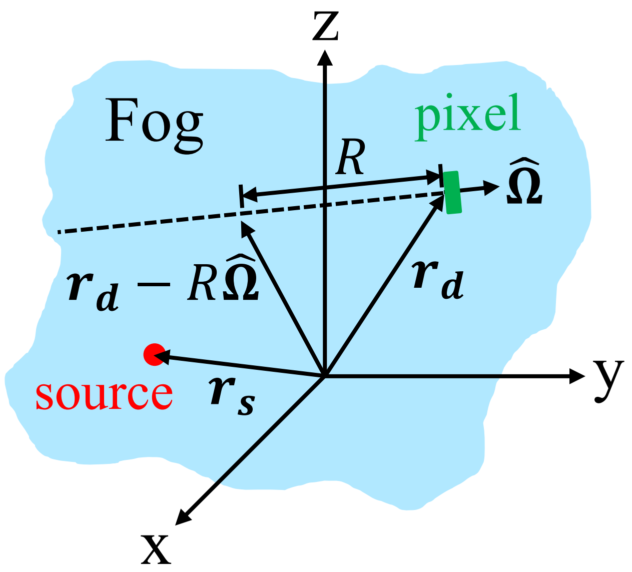
Solving the RTE with Weak Angular Dependence

$$I(\boldsymbol{r}_d, \widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi}\int_{\mathbf{0}}^{\infty} dR\, \exp[-(\mu_s + \mu_a)R]\left[\phi\left(\boldsymbol{r_d} - R\widehat{\boldsymbol{\Omega}}\right) + 3g\boldsymbol{J}\left(\boldsymbol{r_d} - R\widehat{\boldsymbol{\Omega}}\right)\cdot\widehat{\boldsymbol{\Omega}}\right]$$

- Diffusion equation analytic solutions

$$\phi(\boldsymbol{r}) = \left(\frac{S_o}{4\pi D}\right)\frac{\exp\left(-\sqrt{\frac{\mu_a}{D}}|\boldsymbol{r} - \boldsymbol{r}_s|\right)}{|\boldsymbol{r} - \boldsymbol{r}_s|}$$

$$\boldsymbol{J}(\boldsymbol{r}) = \left[\frac{S_o(\boldsymbol{r} - \boldsymbol{r}_s)}{4\pi}\right]\left(\frac{\sqrt{\mu_a/D}}{|\boldsymbol{r} - \boldsymbol{r}_s|^2} + \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_s|^3}\right)\exp\left(-\sqrt{\frac{\mu_a}{D}}|\boldsymbol{r} - \boldsymbol{r}_s|\right)$$



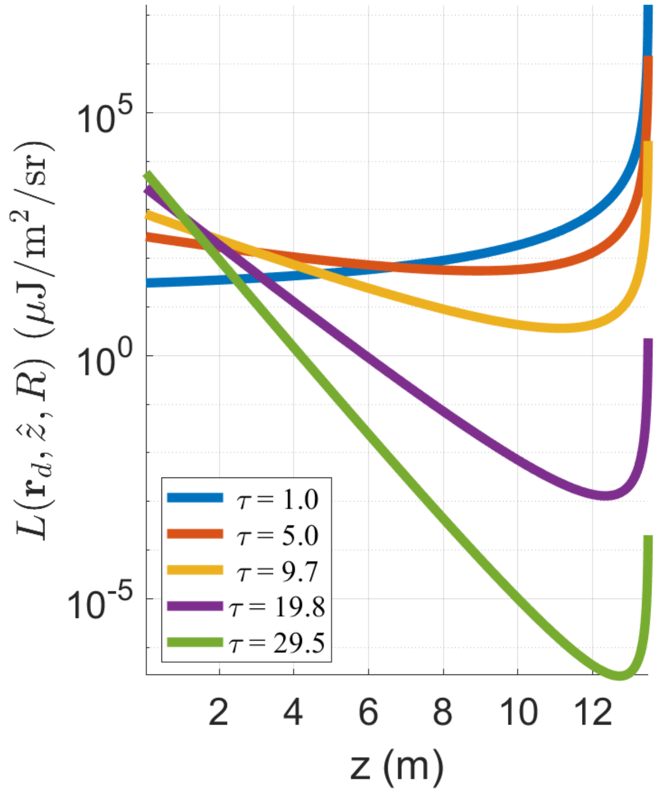
[1] B. Z. Bentz, et. al, *Optics Express* **29**(9), 13231-13245, 2021.

Spatial Distribution of Light Integrated at Detector

$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi}\int_{\mathbf{0}}^{\infty} dR\, \exp[-(\mu_s+\mu_a)R]\left[\phi(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}}) + 3g\boldsymbol{J}(\boldsymbol{r}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}\right]$$

$$I(\boldsymbol{r},\widehat{\boldsymbol{\Omega}}) = \int_{\mathbf{0}}^{\infty} dR\, L(\boldsymbol{r},\widehat{\boldsymbol{\Omega}},R)$$

- Contribution of in-scattered light as a function of depth
- $\tau = 13.5/\mathrm{MFP}$ is the number of mean free path lengths between the source at 13.5 m and the detector at the origin
- Transition from moderately scattering regime ($\tau < 10$) to highly scattering regime ($\tau > 10$)
  - $\tau < 10$: small angle approximation, memory effect, shower curtain effect
  - $\tau > 10$: diffusion approximation



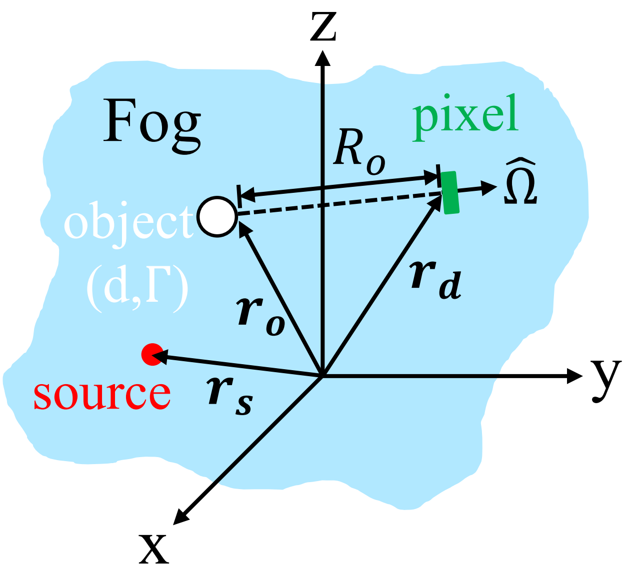
[1] B. Z. Bentz, et. al, *Optics Express* **29**(9), 13231-13245, 2021.

Modeling Opaque Objects

$$I(\boldsymbol{r_d},\widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi}\int_0^{R_o} dR\,\exp[-(\mu_s+\mu_a)R]\,[\phi(\boldsymbol{r_d}-R\widehat{\boldsymbol{\Omega}}) + 3gJ(\boldsymbol{r_d}-R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}] + \Gamma I(\boldsymbol{r}_o,-\widehat{\boldsymbol{\Omega}})\exp[-(\mu_s+\mu_a)R_o]$$

Pixel line of sight only extends from $\boldsymbol{r}$ to $\boldsymbol{r_o}$

Object surface reflection



Where

- Γ is the reflection coefficient at object position $\boldsymbol{r_o}$ and

$$I(\boldsymbol{r}_o,-\widehat{\boldsymbol{\Omega}}) = \frac{\mu_s}{4\pi}\int_{\boldsymbol{0}}^{R_o} dR\,\exp[-(\mu_s+\mu_a)R]\,[\phi(\boldsymbol{r}_o+R\widehat{\boldsymbol{\Omega}}) - 3gJ(\boldsymbol{r}_o+R\widehat{\boldsymbol{\Omega}})\cdot\widehat{\boldsymbol{\Omega}}]$$

Integration along pixel line of sight from $\boldsymbol{r}_o$ to $\boldsymbol{r}$

For simplicity, we assume ϕ and $\boldsymbol{J}$ unaffected by object

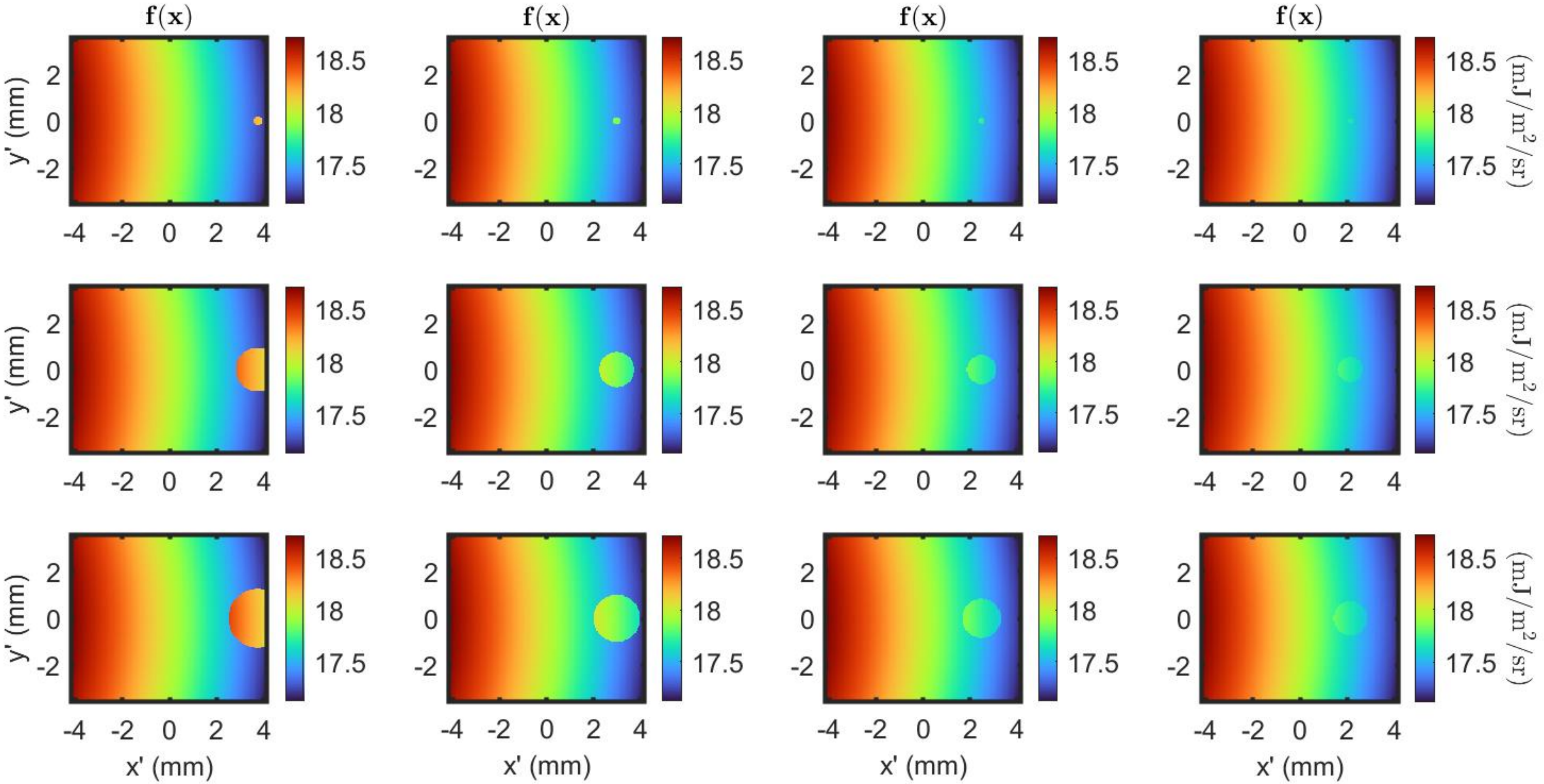
Reflection: Varying Object Distance along $\hat{z}$ and Diameter

$\boldsymbol{r_o} = (-0.2,0,4)$ m | $\boldsymbol{r_o} = (-0.2,0,5)$ m | $\boldsymbol{r_o} = (-0.2,0,6)$ m | $\boldsymbol{r_o} = (-0.2,0,7)$ m

$d = 0.03$ m

$d = 0.15$ m

$d = 0.2$ m



Localization using the Maximum Likelihood Estimate (MLE)

- Let $\theta = [\boldsymbol{x}, w]$ be the parameters of interest

$$\hat{\theta} = \arg\max_{\theta}\left[p_{1,\boldsymbol{r}_o}(\boldsymbol{y})\right]$$

$$= \arg\max_{\theta}\left[\ln p_{1,\boldsymbol{r}_o}(\boldsymbol{y})\right]$$

$$= \arg\max_{\theta}\left[-\frac{1}{2}\ln[(2\pi)^P|\Upsilon|] - \frac{1}{2}\|\boldsymbol{y} - w\boldsymbol{f}(\boldsymbol{x})\|^2_{\Upsilon^{-1}}\right]$$

$$= \arg\min_{\theta}\left[\|\boldsymbol{y} - w\boldsymbol{f}(\boldsymbol{x})\|^2_{\Upsilon^{-1}}\right]$$

Localization using the Maximum Likelihood Estimate (MLE)

- $\boldsymbol{r_o}$ and d can be estimated by minimizing

$$c(\boldsymbol{r}_o, d) = \min_{\Gamma, w}\left[\|\boldsymbol{y} - wy_o - w\boldsymbol{S}(\boldsymbol{r_o}, d) - w\Gamma\boldsymbol{R}(\boldsymbol{r_o}, d)\|_{\Upsilon^{-1}}^2\right]$$

- Take derivatives and solve for closed form estimates of w and Γ
- Call them $\widetilde{w}$ and $\tilde{\Gamma}$

$$\widetilde{w} = \frac{\left[\boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r_o}, d) + \tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r_o}, d)\right]^T \boldsymbol{\Upsilon}^{-1}\boldsymbol{y}}{\left\|\boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r_o}, d) + \tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r_o}, d)\right\|_{\Upsilon^{-1}}^2}$$

$$\tilde{\Gamma} = \frac{[\boldsymbol{y} - \widetilde{w}\boldsymbol{y}_o - \widetilde{w}\boldsymbol{S}(\boldsymbol{r_o}, d)]^T \boldsymbol{\Upsilon}^{-1}\boldsymbol{R}(\boldsymbol{r_o}, d)}{\widetilde{w}\|\boldsymbol{R}(\boldsymbol{r_o}, d)\|_{\Upsilon^{-1}}^2}$$

$$c(\boldsymbol{r}_o, d) = \left\|\boldsymbol{y} - \widetilde{w}y_o - \widetilde{w}\boldsymbol{S}(\boldsymbol{r_o}, d) - \widetilde{w}\tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r_o}, d)\right\|_{\Upsilon^{-1}}^2$$

Localization using the Maximum Likelihood Estimate (MLE)

- Summary

$$c(\boldsymbol{r}_o, d) = \left\| \boldsymbol{y} - \tilde{w} y_o - \tilde{w}\boldsymbol{S}(\boldsymbol{r}_o, d) - \tilde{w}\tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r}_o, d) \right\|^2_{\Upsilon^{-1}}$$

$$\tilde{w} = \frac{\left[\boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r}_o, d) + \tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r}_o, d)\right]^T \boldsymbol{\Upsilon}^{-1}\boldsymbol{y}}{\left\| \boldsymbol{y}_o + \boldsymbol{S}(\boldsymbol{r}_o, d) + \tilde{\Gamma}\boldsymbol{R}(\boldsymbol{r}_o, d) \right\|^2_{\Upsilon^{-1}}}$$

$$\tilde{\Gamma} = \frac{[\boldsymbol{y} - \tilde{w}\boldsymbol{y}_o - \tilde{w}\boldsymbol{S}(\boldsymbol{r}_o, d)]^T \boldsymbol{\Upsilon}^{-1}\boldsymbol{R}(\boldsymbol{r}_o, d)}{\tilde{w}\|\boldsymbol{R}(\boldsymbol{r}_o, d)\|^2_{\Upsilon^{-1}}}$$

$$[\hat{\boldsymbol{r}}_o, \hat{d}] = \underset{\boldsymbol{r}_o, d}{\arg\min}\, c(\boldsymbol{r}_o, d)$$

$$\hat{w} = \tilde{w}(\hat{\boldsymbol{r}}_o, \hat{d}, \hat{\Gamma})$$

$$\hat{\Gamma} = \tilde{\Gamma}(\hat{\boldsymbol{r}}_o, \hat{d}, \hat{w})$$