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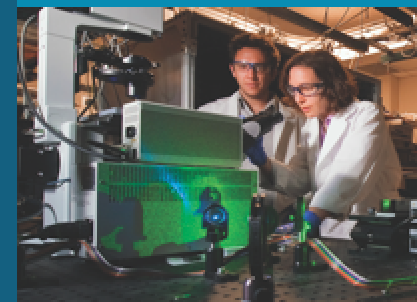
Application of the Level-2 Quantum Lasserre Hierarchy in Quantum Approximation Algorithms



PRESENTED BY

Kevin Thompson

Collaboration with

Ojas Parekh

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2-Local Hamiltonian Problem

- Given a Hamiltonian which is the sum of "local" interactions, what is the smallest/largest eigenvalue?

$$H \leftrightarrow -H$$

- Much known about the complexity of solving exactly, little known about approximating them (no PCP)

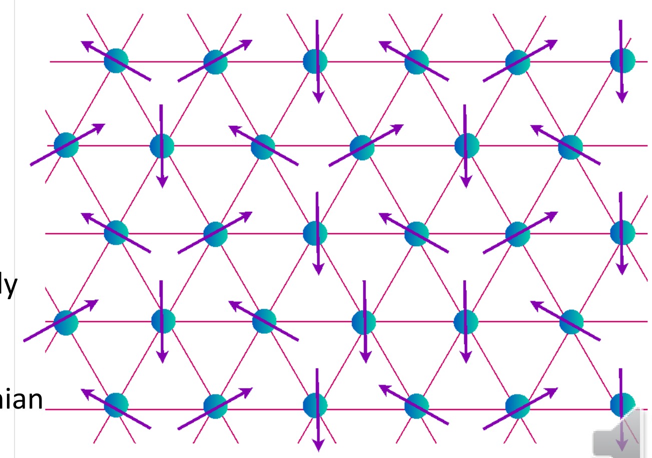
Motivation

The Heisenberg model is fundamental for describing quantum magnetism, superconductivity, and charge density waves. Beyond 1 dimension, the properties of the anti-ferromagnetic Heisenberg model are notoriously difficult to analyze.

- QMA-hard, so we can't exactly solve these problems how well can we expect to approximate them?

- Is nature also approximating them?

Anti-ferromagnetic Heisenberg model: roughly neighboring quantum particles aim to align in opposite directions. This kind of Hamiltonian appears, for example, as an effective Hamiltonian for so-called Mott insulators.
[Image: Sachdev, arxiv:1203.4565]



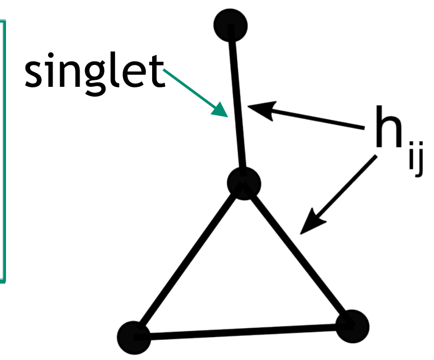
Quantum Max cut

$$I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, X=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, Y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, Z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

Notation- Let σ_i for $\sigma \in \{X,Y,Z\}$ be Pauli σ on qubit i, i.e. $Z_2 = I \otimes Z \otimes I \ldots$
$Z_1 Z_3 = Z \otimes I \otimes Z \otimes I \ldots$

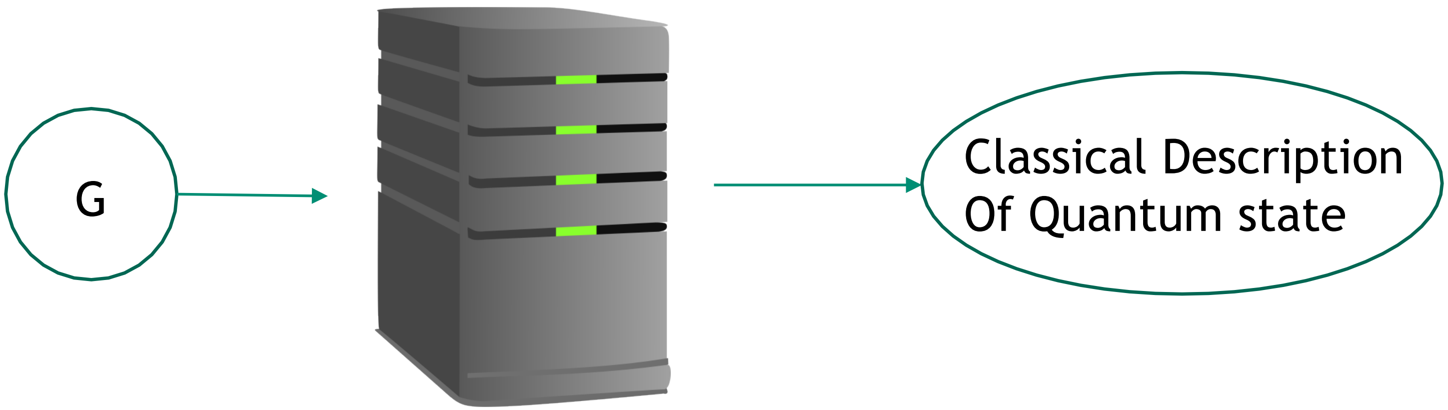
Quantum Max Cut (QMC)- Define $h_{ij} = 1/4(I - X_i X_j - Y_i Y_j - Z_i Z_j)$. Given graph G=(V, E), find:

$$\lambda_{max}\left(H(G)\right) \quad \text{where} \quad H(G) = \sum_{ij \in E} h_{ij}$$



- "How close" to the singlet on each edge?

Approximation Algorithms and Ansatze



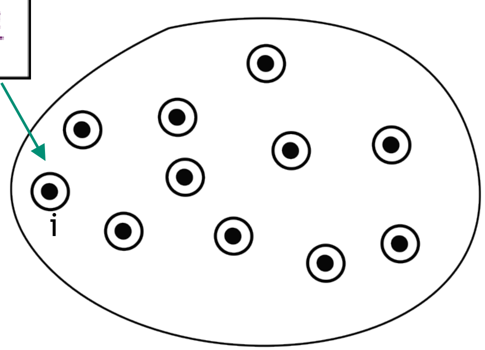
Approx. Alg.- $\min_G \frac{Alg(G)}{\lambda_{max}(H(G))} \geq \alpha$ ← Runs in poly time in n, provable guarantee independent of instance

- Unlike classical Max-cut not clear what kind of description is best
- Two ansatze of interest:

Product State Ansatz

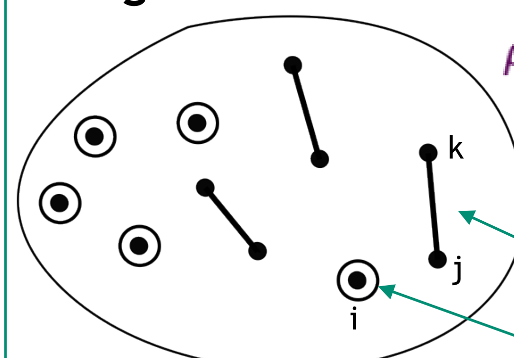
$$\rho_i = \frac{I + \alpha_i X_i + \beta_i Y_i + \gamma_i Z_i}{2}$$

$= \prod$



Singlets+Product States

$$\rho = \prod_i \rho_i \cdot \prod_{jk} \rho_{jk}$$

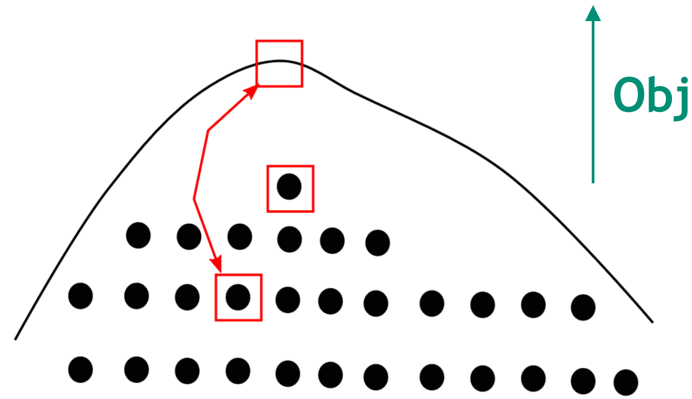


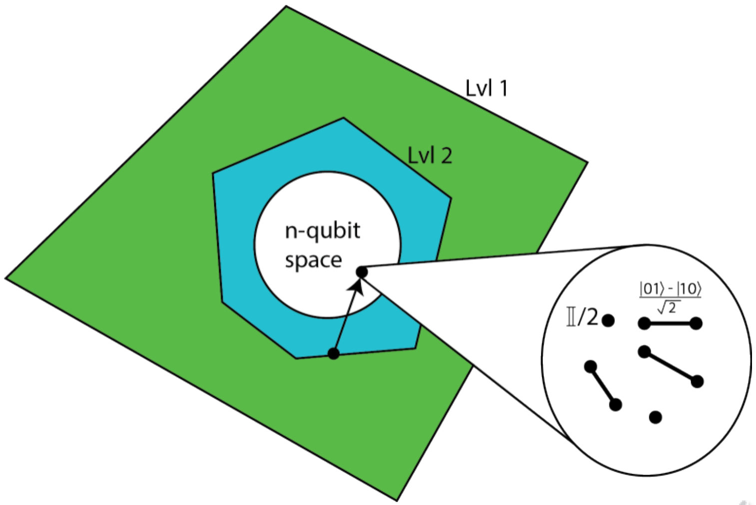
$$\rho_{jk} = \frac{I - X_j X_k - Y_j Y_k - Z_j Z_k}{4}$$

$$\rho_i = \frac{I + \alpha_i X_i + \beta_i Y_i + \gamma_i Z_i}{2}$$

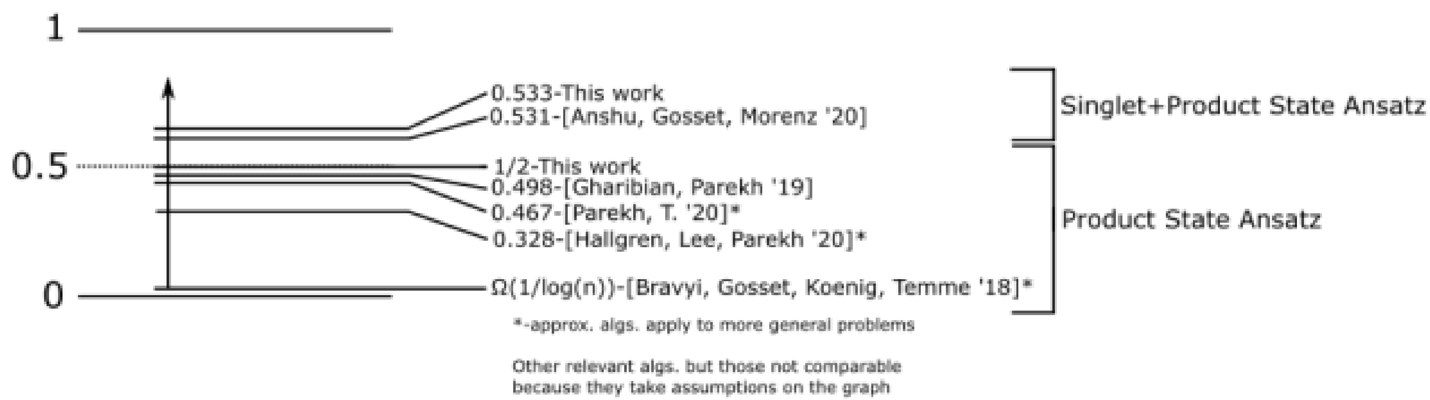
Rounding Strategy

- These problems are hard because the domain is complicated
- We can “relax” the domain to a larger domain that is easier to optimize over
- Generally this is an LP or SDP
- Use the relaxed solution to construct point in the domain (i.e. round your decimal number to ± 1)
- Bound objective **loss** incurred by rounding
- In quantum picture, optimize over “unphysical” pseudo-density matrices on a classical computer
- Same idea for rounding: use the unphysical state to construct a physical state and worry about the objective loss.





Previous Work



- Features to note:
  - All except AGM20 produce product states
  - Generic states (i.e. max energy states of QMA-complete) are highly entangled. **There is great need for additional "non-product" state algorithms.**
- In particular for product states (PS) $\alpha \leq 1/2$. Why? If $\vec{\eta}$ and $\vec{\zeta}$ are Bloch vectors, then:

$$\mathrm{Tr}\left(\rho_{\vec{\eta}} \otimes \rho_{\vec{\zeta}} \frac{\mathbb{I} - X\otimes X - Y \otimes Y - Z \otimes Z}{4}\right) = \frac{1 - \vec{\eta}\cdot\vec{\zeta}}{4}$$

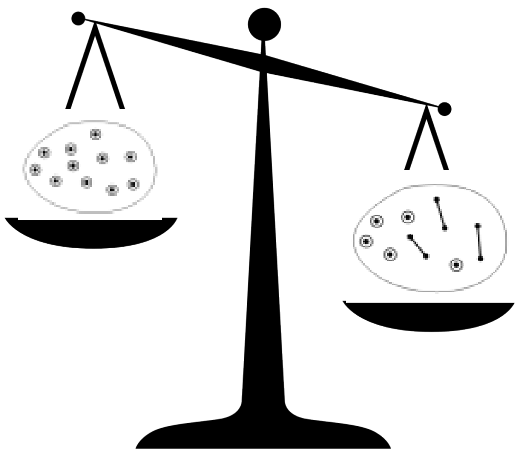
(Maximized when $\vec{\eta} = -\vec{\zeta}$ **)**

Approximation factor must hold for **all instances**

- On the other hand, there is evidence [Anshu, Gosset, Morenz '20] that these small graphs like the example above are "blocking" a good approx. factor with product states. Give $0.546 - O(1/|E|)$ PS algorithm.

Previous Work (cont.)

- Indeed, [Brandão, Harrow '16] suggests that for very dense graphs the optimal state is very nearly a product state
- Hence, it's reasonable to expect **any** algorithm which achieves a good approximation factor "looks like" a product state rounding algorithm for dense G
- [Anshu, Gosset, Morenz '20] achieves an approximation factor better than ½ by trading off between product state and entangled state
- **So, finding good product state algorithms can improve performance of entangled algorithms**
- That being said, finding entangled algorithms is the frontier, **we do both!**



Our Contribution

- Discussing the contents of two papers arxiv: 2105.05698 arxiv: ???

- Main contribution is new techniques

Result #1- We construct a polynomial-time classical algorithm achieving approx. factor 0.533 for QMC.

- Also construct algorithm which achieves the **optimal** approx. factor for product states

Result #2- We construct a polynomial-time classical algorithm achieving approx. factor 1/2 for QMC using product state ansatz.

- **Not** achieving optimal product state on every instance (NP-hard), alg. has best approx. factor with respect to **all** instances.

- Also achieve a “fine-tuned” version of [Brandão, Harrow ’16] restricted to QMC:

Result #3- If the graph G has min degree d, we achieve $\alpha(d)$-approx. where, e.g.

$\alpha(3) = 0.557, \quad \alpha(4) = 0.574$

Moment Matrix Description of Quantum States

State on n qubits

$$\langle\psi| \in \mathbb{C}^{2^n} \Rightarrow \mathbb{C}^{4^n \times 2^n} \ni V := \begin{bmatrix} \langle\psi| \\ \langle\psi|X_1 \\ \langle\psi|Y_1 \\ \vdots \\ \langle\psi|Z_n \\ \langle\psi|X_1X_2 \\ \vdots \\ \langle\psi|Z_1\ldots Z_n \end{bmatrix}$$

$$M := VV^\dagger = \begin{array}{c} \\ \mathbb{I} \\ X_1 \\ Y_1 \\ \vdots \end{array} \overset{\begin{array}{ccc} \mathbb{I} & X_1 & \cdots \end{array}}{\begin{bmatrix} \langle\psi|\psi\rangle & \boxed{\langle\psi|X_1|\psi\rangle} & \\ \boxed{\langle\psi|X_1|\psi\rangle} & \langle\psi|X_1^2|\psi\rangle & \\ \langle\psi|Y_1|\psi\rangle & \langle\psi|Y_1X_1|\psi\rangle & \\ & & \ddots \end{bmatrix}} \in \mathbb{C}^{4^n \times 4^n} \quad (Hermitian)$$

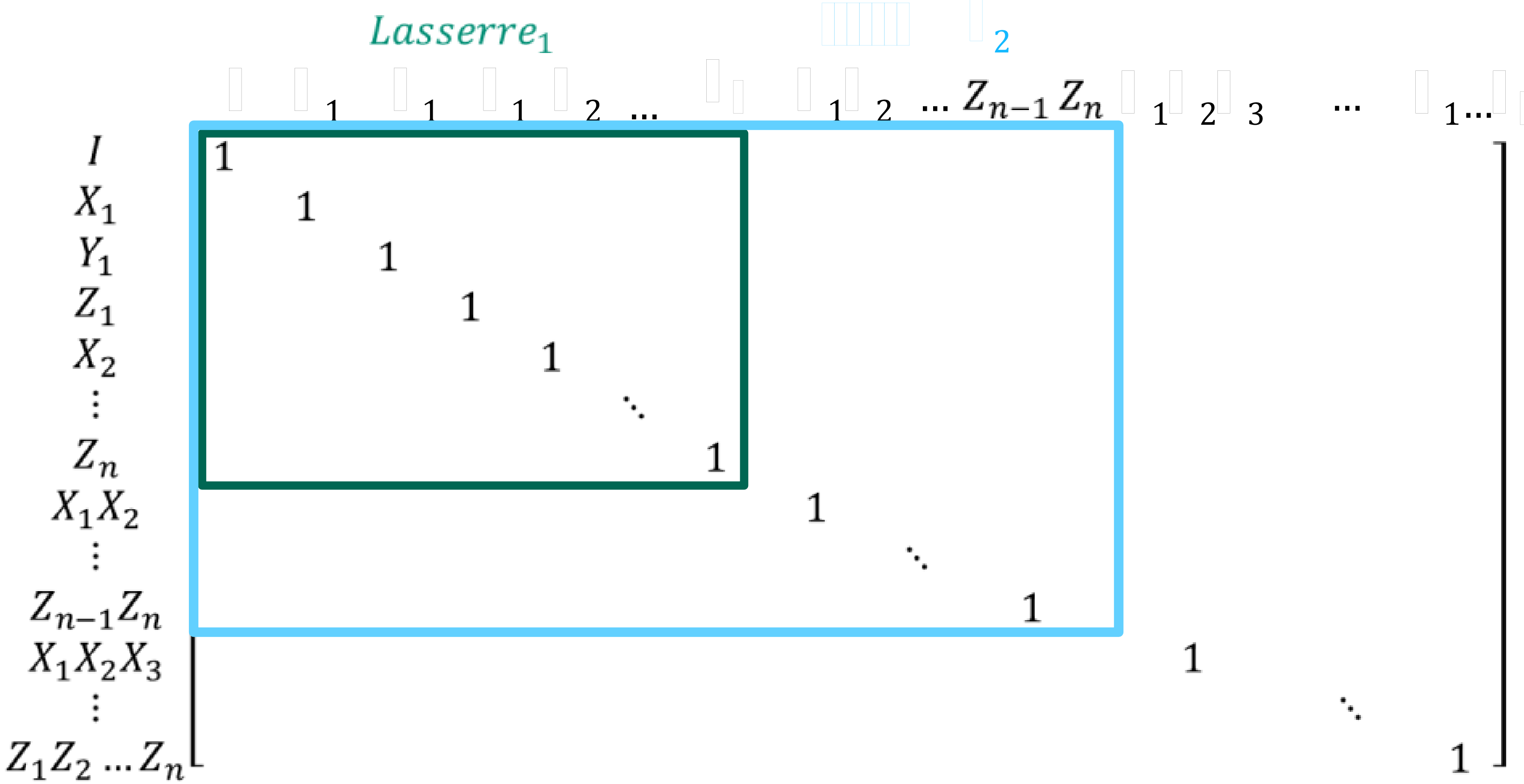
(The two boxed entries are marked "=".)

- Properties
  - Matrix entries encode the expectation values of **all** Pauli observables.
  - $M \succeq 0$ since it has a Gram vectors by it's definition.
  - Many of the entries would be the same up to phase → redundant description of state

Relaxing the Moment Matrix

[Lasserre '01]
[Pironio, Navascués, Acìn, '10]

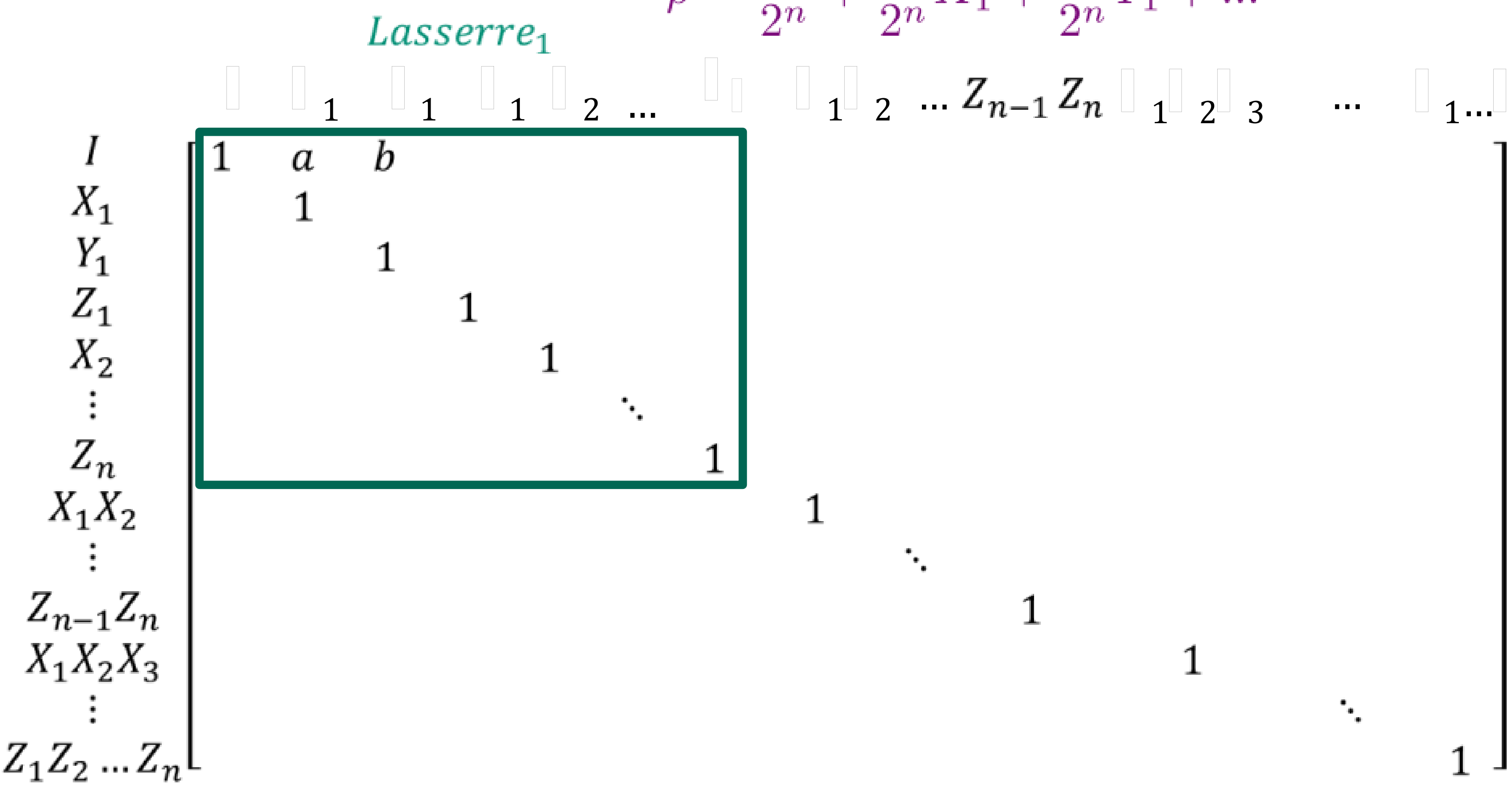
- Try to solve the Local Hamiltonian by optimizing over moment matrices. Problem is this is exponentially large. Optimize over submatrix?
  - Still guaranteed PSD
  - Satisfies all equality constraints the submatrix intersects with
  - Relaxation because submatrix likely not embeddable



Pseudo-Density matrices

- Pseudo-density $\tilde{\rho}$ is density matrix with Pauli statistics matching submatrix of moment matrix
- Can optimize over since local expectations **defined** by classical SDP

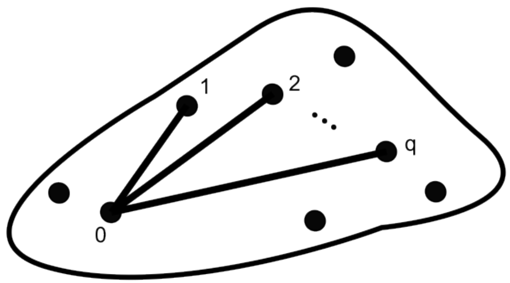
$$\tilde{\rho} = \frac{\mathbb{I}}{2^n} + \frac{a}{2^n} X_1 + \frac{b}{2^n} Y_1 + \dots$$



Relaxation Quality

Star Bound [Lieb, Mattis, '62] [Anshu, Gosset, Morenz, '20]

Monogamy of Entanglement

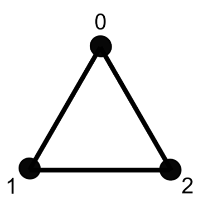


$$\sum_{j=1}^{m} Tr(\rho h_{0j}) \leq \frac{q+1}{2}$$

- *Lasserre*$_1$ gets q
- *Lasserre*$_2$ gets $(q+1)/2$

- Relatively simple bound (sufficient for our entangled rounding algorithm) not fully capturing allowed edge overlaps for a quantum state:
  - I.e. $q=2: \mathrm{Tr}(h_{01}\rho)=1 \Rightarrow \mathrm{Tr}(h_{02}\rho) \leq 1/2$, but for physical state: $\mathrm{Tr}(\rho h_{01})=1 \Rightarrow \mathrm{Tr}(\rho h_{02})=1/4$
- Demonstrate a stronger inequality for the optimal PS rounding paper:

Triangle Bound $\mathrm{Lasserre}_2$ satisfies:



$$\mu_{01} = \mathrm{Tr}(\tilde{\rho} h_{01})$$
$$\mu_{02} = \mathrm{Tr}(\tilde{\rho} h_{02})$$
$$\mu_{12} = \mathrm{Tr}(\tilde{\rho} h_{12})$$

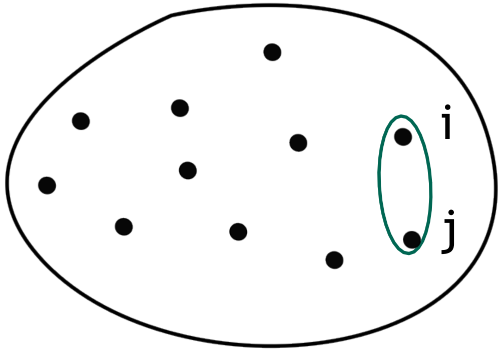
$$0 \leq \mu_{01}+\mu_{02}+\mu_{12} \leq 3/2$$
$$4(\mu_{01}^2+\mu_{02}^2+\mu_{12}^2) - 8(\mu_{01}\mu_{02}+\mu_{01}\mu_{12} + \mu_{02}\mu_{12}) \leq 0$$

$$\mu_{01}=1 \Rightarrow \mu_{02}=1/4$$

- We think these constraints are fully capturing the allowed values on a triangle!

Rounding Algorithm

- PS rounding algorithm and singlet+PS rounding algorithm follow similar meta-algorithm, with different "building blocks"



$$\mu_{ij} = \mathrm{Tr}(\tilde{\rho} h_{ij})$$

$0 \leq \mu_{ij} \leq 1$, if $\mu_{ij} \approx 1$ then $Lasserre_2$ "thinks" that edge should be a singlet.

Overall idea- Find the edges $Lasserre_2$ "thinks" should be a singlet, take care to get good objective value on these edges

Meta-Algorithm

1. Solve $Lasserre_2$ to get submatrix of M
2. Initialize $L = \{\}$
3. For all ij calculate μ_{ij}. If $\mu_{ij} > \gamma$ add ij to L.
4. Find Maximum matching M on L.
5. Consider two states
    1. Take optimal state on M, something standard on the rest
    2. PS rounding from [GP 19']
6. Take whichever has better objective.

Block 1: Threshold

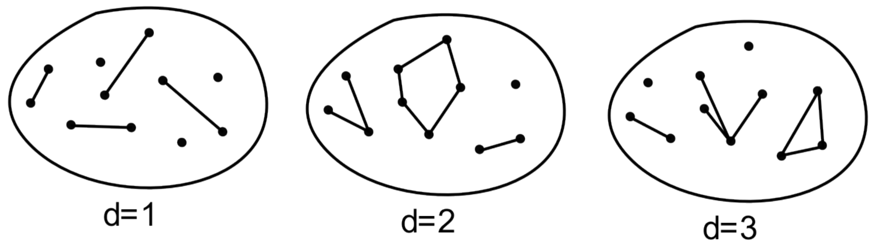
Block 2: Handling large edges

Block 3: Handling qubits outside M

Rounding Algorithm (cont.)

Block 1

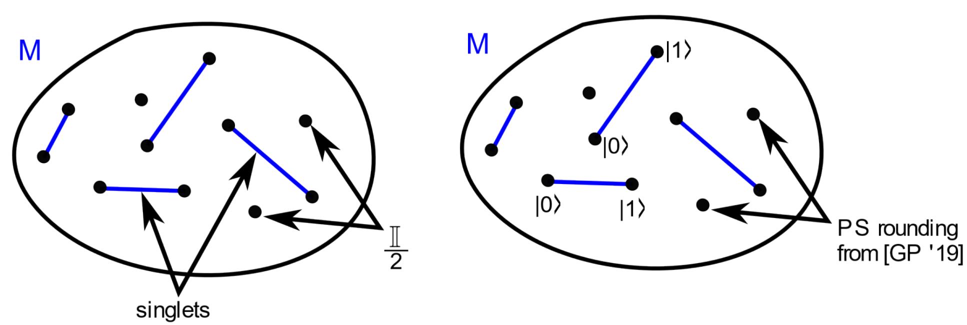
- Star/Triangle bounds say that large edges must be adjacent to small edges ⇒ set L forms a subgraph of small degree
  - Threshold controls degree of subgraph



d=1 for PS rounding
d=2 for Entangled

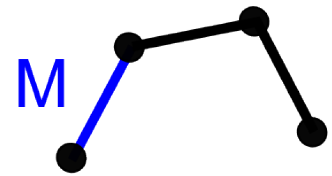
  - Why set them differently? Technical reasons
  - Tradeoff in d:
    - d is too small ⇒ product state rounding bad
    - d is too large ⇒ matching is bad

Block 2/Block 3

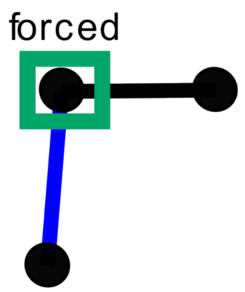


Analysis

- Three kinds of edges
  - Edges in M
  - Edges adjacent to M
  - Edges not adjacent to M



- We want to show algorithm has good performance on all types
  - Edges in M are easy: state specifically constructed to be optimal
  - Edges not adjacent to M:
    - Looking at $(\mathbb{I}/2) \otimes (\mathbb{I}/2)$ or at state produced by [GP'19]
    - Performance of [GP'19] is well understood
  - Edges adjacent to M we have little control over the objective value
    - We don't need to do well on these edges because of Triangle/Star Bounds!
    - This is where the Triangle bound really shines, it says that the adjacent edge value is quite small



- Additional proof techniques
  - Symmeterization over transformations
  - "Sum of Squares" proofs

Implications

- Demonstrated that $Lasserre_2$ satisfies physically motivated constraints, possibly opening the door to additional approximation algorithms.
  - "low-order" quasi-description of a state can look "entangled"
  - Demonstrate explicit gap in "representational power" of different levels of Lasserre

- Classical approximation algorithms follow a standard "meta" algorithm,
  1. Solve SDP
  2. Use solution to round to feasible point
- Only other known algorithm which produces entangled ansatz [Anshu, Gosset, Morenz '20] does not follow this format
  - By bringing in the meta algorithm we have opened the door to using the rich background of classical techniques for combinatorial opt.

- Also give computational evidence that our product state rounding is optimal for a much more general class of Hamiltonians

Open Questions

- Likely only scratching the surface of the power of $Lasserre_2$
  - What other kinds of graphs is $Lasserre_2$ exact on?
  - Are moments subject to other monogamy inequalities?
  - Can these be used to further improve approx. factor?
- More generally, what kind of physical constraints are present in $Lasserre_k$ for $k = O(1)$?
- Can we find new monogamy of entanglement inequalities for quantum states by looking at $Lasserre$?
- Singlets + product state still *locally* entangled. Can we get more entangled ansatz? i.e. tensor network states?
- Genuinely quantum Approximation algorithms? i.e. alg requires quantum computer and produces quantum state

Thank you!

Questions?