Structure-preserving model reduction for marginally stable LTI systems

Liqian Peng and Kevin Carlberg

Sandia National Laboratories

2017 West Coast ROM Workshop
Berkeley, CA
November 16, 2017

boratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LL
Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE

Table of Contents

Classical model reduction methods

Most classical model-reduction methodologies were originally developed for asymptotically stable LTI systems

Balanced truncation (Moore 81),
Hankel norm approximation (Glover 84)
Optimal $\mathcal{H}_2$ approximation (Gugercin et al. 08)
Galerkin projection exploiting inner-product structure (Rowley et al. 04)

Although many well-known model reduction methods can be directly applied to systems with purely imaginary poles, they do not guarantee stability.

POD–Galerkin (Holmes et al. 12)
Balanced POD (Rowley et al. 05)
Moment matching (Bai 02, Freund 03)
Shift-reduce-shift-back (Yang et al. 93)

Stability-preserving model reduction methods

A priori a stability-preserving model reduction framework.

An energy-based inner product (Rowley et al. 04, Barone et al. 09, Kalashnikova et al. 10)
Lagrangian structure (Lall et al. 03, Carlberg et al. 12, Carlberg et al. 15)
Symplectic structure (Peng and Mohseni 16, Afkham and Hesthaven 17)
Port-Hamiltonian structure (van der Schaft and Oeloff 90, Scherpen and van der Schaft 08, Polyuga and van der Schaft 10, Gugercin et al. 12)

A posteriori stabilization step to stabilize an unstable ROM.

Optimization-based eigenvalue reassignment (Kalashnikova et al. 14)
Minimal subspace rotation (Bond and Daniel 08, Amsallem and Farhat 12)
Viscosity(Aubry et al. 88, Podvin et al. 88, Delville et al. 99)
Penalty term (Cazemier et al. 98)
Calibrate POD coefficients (Couplet et al. 05, Kalb et al. 07)

Specific contributions of this work

1. A novel structure-preserving model reduction method for marginally stable LTI systems.
2. Analysis that demonstrates that any pure marginally stable system is equivalent to a generalized Hamiltonian system with marginal stability.
3. A general symplectic-projection framework with symplectic balancing.
4. A geometric framework that enables a unified analysis and comparison of inner-product and symplectic projection.

[1] L. Peng and K. Carlberg, *Structure-preserving model reduction for marginally stable LTI systems*, (2017). http://arXiv:1704.04009.

- Full-order model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

(A, B, C): $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{q \times n}$.

- Full-order autonomous system:

$$\dot{x} = Ax \tag{2}$$

- Reduced-order model:

$$\begin{aligned} \dot{z} &= \tilde{A}z + \tilde{B}u \\ y &= \tilde{C}z \end{aligned} \tag{3}$$

$(\tilde{A}, \tilde{B}, \tilde{C})$: $\tilde{A}{:=}\Psi^\tau A\Phi \in \mathbb{R}^{k \times k}$, $\tilde{B}{:=}\Psi^\tau B \in \mathbb{R}^{k \times p}$, $\tilde{C}{:=}C\Phi \in \mathbb{R}^{q \times k}$, $k \ll n$.

- Reduced-order autonomous system:

$$\dot{z} = \tilde{A}z \tag{4}$$

- If the original system is marginally stable and A has a full rank, there exists a nonsingular matrix T such that

$$A = T \begin{bmatrix} A_s & 0 \\ 0 & A_m \end{bmatrix} T^{-1}, \tag{1}$$

where $\lambda(A_s) < 0$ and $\lambda(A_m) = 0$.

- With $x = T \begin{bmatrix} x_s^\tau & x_m^\tau \end{bmatrix}^\tau$, we obtain a decoupled LTI system

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} x_s \\ x_m \end{bmatrix} &= \begin{bmatrix} A_s & 0 \\ 0 & A_m \end{bmatrix} \begin{bmatrix} x_s \\ x_m \end{bmatrix} + \begin{bmatrix} B_s \\ B_m \end{bmatrix} u \\ y &= \begin{bmatrix} C_s & C_m \end{bmatrix} \begin{bmatrix} x_s \\ x_m \end{bmatrix}, \end{aligned} \tag{2}$$

where $T^{-1}B = \begin{bmatrix} B_s^\tau & B_m^\tau \end{bmatrix}^\tau$ and $CT = \begin{bmatrix} C_s & C_m \end{bmatrix}$.

Algorithm 1 Structure-preserving model reduction for marginally stable LTI systems

Input: A marginally stable LTI system (A, B, C).
Output: Reduced-order systems $(\tilde{A}, \tilde{B}, \tilde{C})$.

1: Decompose the original LTI system into an asymptotically stable subsystem (A_s, B_s, C_s) and a marginally stable subsystem (A_m, B_m, C_m).
2: Apply inner-product projection to construct the low-order asymptotically stable system $\tilde{A}_s = \Psi_s^\tau A_s \Phi_s$, $\tilde{B}_s = \Psi_s^\tau B_s$, $\tilde{C}_s = C_s \Phi_s$.
3: Apply symplectic projection to construct the low-order marginally stable system $\tilde{A}_m = \Psi_m^\tau A_m \Phi_m$, $\tilde{B}_m = \Psi_m^\tau B_m$, $\tilde{C}_m = C_m \Phi_m$.
4: Construct the reduced-order system $\tilde{A} = \mathrm{diag}(\tilde{A}_s, \tilde{A}_m)$, $\tilde{B} = \begin{bmatrix} \tilde{B}_s^\tau & \tilde{B}_m \end{bmatrix}^\tau$, and $\tilde{C} = \begin{bmatrix} \tilde{C}_s & \tilde{C}_m \end{bmatrix}$.

Inner-product reduction v. symplectic reduction

	Asymptotically stable subsystem	Marginally stable subsystem
Autonomous system	$\dot{x} = Ax$ with $\lambda(A) < 0$	$\dot{x} = Ax$ with $\lambda(A) = 0$
Original space	Inner-product space	Symplectic space
Projection	Inner-product projection	Symplectic projection
Reduced space	Inner-product space	Symplectic space
Reduced autonomous system	$\dot{z} = \tilde{A}z$ $\tilde{A} = \Psi^{\tau} A \Phi$ with $\lambda(\tilde{A}) < 0$	$\dot{z} = \tilde{A}z$ $\tilde{A} = \Psi^{\tau} A \Phi$ with $\lambda(\tilde{A}) = 0$
Structure-preserving	Lyapunov inequality	Hamiltonian property
Energy property of reduced system	Strictly monotonically decreasing	Energy conservation

[1] For notational simplicity, we omit the subscripts s and m.

Definition (Pure marginal stability)

An LTI system (A, B, C) is pure marginally stable, if A is nonsingular and diagonalizable, and has a *purely imaginary spectrum*.

Definition (Hamiltonian)

An LTI system (A, B, C) is Hamiltonian if its corresponding autonomous system is given by

$$\dot{x} = J\nabla_x H(x) = JLx, \tag{1}$$

where $J \in \mathrm{SS}(2n)$ and $L \in \mathbb{R}^{2n\times 2n}$ is symmetric. The matrix L defines the (quadratic) Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R},\ x \mapsto \frac{1}{2}x^\tau Lx$.

Theorem (Contribution of this work)

The following conditions are equivalent:

1. (A, B, C) *is pure marginally stable.*
2. (A, B, C) *is Hamiltonian and marginally stable.*

Proof.

$(2) \Rightarrow (1)$:

- Since A is Hamiltonian, if λ is an eigenvalue of A, $-\lambda$ is also an eigenvalue of A. Thus, the eigenvalues of A are *purely imaginary*.
- Because A is marginally stable, every Jordan block for purely imaginary eigenvalues must has dimension 1×1. Thus, A is *diagonalizable*.

$(1) \Rightarrow (2)$:

- Let λ be an eigenvalue of A. Then λ is a root of the characteristic polynomial $\det(\lambda I_{2n} - A) = 0$. Because A is a real matrix, A contains eigenvalues of the form $\{\pm i\beta_1, \ldots, \pm i\beta_n\}$.
- We can construct a nonsingular matrix $G \in \mathbb{R}^{2n\times 2n}$ such that $G^{-1}AG = J_{2n}L_0$, where $L_0 = \mathrm{diag}(\beta, \beta)$ and $\beta = \mathrm{diag}(\beta_1, \ldots, \beta_n)$.
- We can prove that A is *Hamiltonian* if and only if there exits a nonsingular matrix $G \in \mathbb{R}^{2n\times 2n}$ such that $G^{-1}AG = J_{2n}L_0$.

□

Definition (Symplectic space)

Let $\mathbb{V}$ denote a vector space. A symplectic form $\Omega : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is a skew-symmetric, nondegenerate, bilinear function on the vector space $\mathbb{V}$. The pair $(\mathbb{V}, \Omega)$ is called a symplectic vector space.

Let $(\mathbb{V}, \Omega)$ and $(\mathbb{W}, \Pi)$ be two symplectic vector spaces with coordinate representations $(\mathbb{R}^{2n}, J_\Omega)$ and $(\mathbb{R}^{2k}, J_\Pi)$, respectively, $\dim(\mathbb{V}) = 2n$, $\dim(\mathbb{W}) = 2k$, and $k \leq n$.

Definition (Symplectic lift, Peng and Mohseni 16)

A symplectic lift is a linear mapping $\phi : (\mathbb{W}, \Pi) \to (\mathbb{V}, \Omega)$ that preserves symplectic structure:

$$\Pi(\hat{z}_1, \hat{z}_2) = \Omega(\phi(\hat{z}_1), \phi(\hat{z}_2)), \quad \forall \hat{z}_1, \hat{z}_2 \in \mathbb{W}. \tag{1}$$

In coordinate space, the symplectic lift can be expressed as $\phi(\hat{z}) = \Phi z$, $\forall z \in \mathbb{R}^{2k}$, where (1) implies that $\Phi \in \mathbb{R}^{2n \times 2k}$ satisfies

$$(\Phi z_1)^\tau J_\Omega (\Phi z_2) = z_1^\tau J_\Pi z_2 \quad \forall z_1, z_2 \in \mathbb{R}^k. \tag{2}$$

This is equivalent to $\Phi^\tau J_\Omega \Phi = J_\Pi$. For convenience, we write $\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi)$.

Definition (Symplectic projection, Peng and Mohseni 16)

Let $\phi : (\mathbb{W}, \Pi) \to (\mathbb{V}, \Omega)$ be a symplectic lift. The adjoint of ϕ is the linear mapping $\psi : (\mathbb{V}, \Omega) \to (\mathbb{W}, \Pi)$ satisfying

$$\Pi(\psi(\hat{x}), \hat{z}) = \Omega(\hat{x}, \phi(\hat{z})), \quad \forall \hat{z} \in \mathbb{W}, \ \hat{x} \in \mathbb{V}. \tag{3}$$

We say ψ is the symplectic projection induced by ϕ.

In coordinate space, the symplectic projection can be expressed as $\psi(\hat{x}) = \Psi^\tau x$, $\forall x \in \mathbb{R}^{2n}$, where (3) implies that $\Psi \in \mathbb{R}^{2n \times 2k}$ satisfies

$$\Psi J_\Pi = J_\Omega \Phi, \tag{4}$$

from which it follows that

$$\Psi = J_\Omega \Phi J_\Pi^{-1}. \tag{5}$$

It can be verified that Ψ^τ is a left inverse of Φ, as

$$\Psi^\tau \Phi = (J_\Omega \Phi J_\Pi^{-1})^\tau \Phi = J_\Pi^{-1}(\Phi^\tau J_\Omega \Phi) = J_\Pi^{-1} J_\Pi = I_{2k}, \tag{6}$$

which implies that $\psi \circ \phi$ is the identity map on $\mathbb{W}$.

Definition (Symplectic projection of systems, Peng and Mohseni 16)

A reduced-order model $(\tilde{A}, \tilde{B}, \tilde{C})$ with $\tilde{A} = \Psi^\tau A\Phi$, $\tilde{B} = \Psi^\tau B$, and $\tilde{C} = C\Phi$ is constructed by a symplectic projection if $\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi)$ and $\Psi = J_\Omega \Phi J_\Pi^{-1}$, where $J_\Omega \in \mathrm{SS}(2n)$ and $J_\Pi \in \mathrm{SS}(2k)$.

Lemma (Symplectic structure preservation)

If the original LTI system (A, B, C) is Hamiltonian and the reduced-order model is constructed by symplectic projection with $J_\Omega = -J^{-1}$, then the reduced-order model $(\tilde{A}, \tilde{B}, \tilde{C})$ remains Hamiltonian.

Proof.

Because $A = -J_\Omega^{-1} L$ and $\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi)$, we have that

$$\tilde{A} = \Psi^\tau A\Phi = (J_\Pi^{-1}\Phi^\tau J_\Omega)(-J_\Omega^{-1} L)\Phi = -J_\Pi^{-1}(\Phi^\tau L\Phi).$$

Because $J_\Pi \in \mathrm{SS}(2k)$, $-J_\Pi^{-1} \in \mathrm{SS}(2k)$. Define $\tilde{L} = \Phi^\tau L\Phi \in \mathbb{R}^{2k\times 2k}$. Because L is symmetric and nonsingular, so is $\tilde{L}$. □

Theorem (Contribution of this work)

Suppose the original system (A, B, C) *is pure marginally stable, i.e.,* $A = JL$ *with* $J \in \mathrm{SS}(2n)$ *and* $L \in \mathrm{SPD}(2n)$. *Then the reduced system* $(\tilde{A}, \tilde{B}, \tilde{C})$ *constructed by symplectic projection with* $J_\Omega = -J^{-1}$ *and any* $J_\Pi \in \mathrm{SS}(2k)$ *remains pure marginally stable.*

Proof.

- The reduced system matrix $\tilde{A}$ constructed by symplectic projection can be written as $\tilde{A} = -J_\Pi^{-1}\tilde{L}$ with $\tilde{L} = \Phi^\tau L \Phi$ and $\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi)$.
- Because $L \in \mathrm{SPD}(2n)$, we have $\tilde{L} \in \mathrm{SPD}(2k)$.
- Let $\tilde{H} : z \mapsto \frac{1}{2} z^\tau \tilde{L} z$ denote the Hamiltonian function of the reduced system $\dot{z} = \tilde{A}z$. So z is bounded.
- Because the reduced system is also linear, it is marginally stable.
- Since $\tilde{A}$ is a generalized Hamiltonian matrix with marginal stability, $\tilde{A}$ is pure marginally stable.

□

Definition (Symplectic balancing)

Given any $\Xi, \Xi' \in \mathrm{SPD}(n)$, $J_\Omega \in \mathrm{SS}(2n)$, and $G \in \mathrm{Sp}(J_\Omega, J_{2n})$, the trial and test bases characterizing a symplectic balancing correspond to

$$\Phi = G\mathrm{diag}(\bar{\Phi}, \bar{\Psi}) \quad \text{and} \quad \Psi = G^{-\tau}\mathrm{diag}(\bar{\Psi}, \bar{\Phi}), \tag{7}$$

where basis matrices $(\bar{\Psi}, \bar{\Phi})$ characterize an inner-product balancing on matrices Ξ and Ξ', i.e.,

$$\bar{\Phi} = SV_1\Sigma_1^{-1/2} \quad \text{and} \quad \bar{\Psi} = RU_1\Sigma_1^{-1/2}. \tag{8}$$

Here, $\Xi = RR^\tau$, $\Xi' = SS^\tau$, and $R^\tau S = U\Sigma V^\tau$ is the singular value decomposition.

Lemma (Properties of symplectic balancing)

The test and trial (full-system) basis matrices (Ψ, Φ) *balance* $M = G^{-\tau}\mathrm{diag}(\Xi, \Xi')G^{-1}$ *and* $M' = G\mathrm{diag}(\Xi', \Xi)G^\tau$*, i.e.,* $\Phi \in O(M, \mathrm{diag}(\Sigma_1, \Sigma_1))$ *and* $\Psi \in O(M', \mathrm{diag}(\Sigma_1, \Sigma_1))$.

Proposed algorithms for constructing an inner-product projection that preserves asymptotically stability

	Method 1 (symplectic balancing)	Method 2	Method 3
Input	$\Xi, \Xi' \in \mathrm{SPD}(n)$, $J_\Omega \in \mathrm{SS}(2n)$, G satisfying $J = GJ_{2n}G^\tau$	$\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi)$, $J_\Pi \in \mathrm{SS}(2k)$, $J_\Omega \in \mathrm{SS}(2n)$	$\Phi_0 \in \mathrm{Sp}(J_{2n}, J_{2k})$, $J_\Pi \in \mathrm{SS}(2k)$, $J_\Omega \in \mathrm{SS}(2n)$
Output	$J_\Pi \in \mathrm{SS}(2k)$, $\Phi \in \mathrm{Sp}(J_\Omega, J_{2k})$, $\Psi \in \mathrm{Sp}(J_{\Omega'}, J_{2k})$	$\Psi \in \mathbb{R}^{2n \times 2k}$	$\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi)$, $\Psi \in \mathbb{R}^{2n \times 2k}$
Algorithm	1. Compute symmetric factorization $\Xi = RR^\tau$, $\Xi' = SS^\tau$ 2. Compute SVD $R^\tau S = U\Sigma V^\tau$ 3. $\bar{\Phi} = SV_1\Sigma_1^{-1/2}$, $\bar{\Psi} = RU_1\Sigma_1^{-1/2}$ 4. $\Phi = G\mathrm{diag}(\bar{\Phi}, \bar{\Psi})$, $\Psi = G^{-\tau}\mathrm{diag}(\bar{\Psi}, \bar{\Phi})$ 5. $J_\Pi = J_{2k}$	1. $\Psi = J_\Omega \Phi J_\Pi^{-1}$	1. Compute $G \in \mathrm{Sp}(J_\Omega, J_{2n})$ 2. Compute $\tilde{G} \in \mathrm{Sp}(J_\Pi, J_{2k})$ 3. $\Phi = G\Phi_0\tilde{G}^{-1}$ 4. $\Psi = J_\Omega \Phi J_\Pi^{-1}$

Definition (Inner-product lift)

Let $(\mathbb{W}, \Pi)$ and $(\mathbb{V}, \Omega)$ be two inner-product spaces and $\dim(\mathbb{W}) \leq \dim(\mathbb{V})$. An inner-product lift is a linear mapping $\phi : \mathbb{W} \to \mathbb{V}$ that preserves inner-product structure: $\langle \hat{z}_1, \hat{z}_2 \rangle_{\mathbb{W}} = \langle \phi(\hat{z}_1), \phi(\hat{z}_2) \rangle_{\mathbb{V}}, \quad \forall \hat{z}_1, \hat{z}_2 \in \mathbb{W}$.

Definition (Inner-product projection)

Let $\phi : \mathbb{W} \to \mathbb{V}$ be an inner-product lift. The adjoint of ϕ is the linear mapping $\psi : \mathbb{V} \to \mathbb{W}$ satisfying $\langle \psi(\hat{x}), \hat{z} \rangle_{\mathbb{W}} = \langle \hat{x}, \phi(\hat{z}) \rangle_{\mathbb{V}}, \quad \forall \hat{z} \in \mathbb{W}, \ \hat{x} \in \mathbb{V}$. We say ψ is the inner-product projection induced by ϕ.

In coordinate space, $\mathbb{V}$ and $\mathbb{W}$ can be represented by $(\mathbb{R}^n, M)$ and $(\mathbb{R}^k, N)$ respectively. This inner-product lift can be expressed as $\phi(\hat{z}) = \Phi z, \quad \forall z \in \mathbb{R}^k$. Then, that $\Phi \in \mathbb{R}^{n \times k}$ satisfies

$$\Phi^\tau M \Phi = N \tag{1}$$

For convenience, we write $\Phi \in O(M, N)$. The induced inner-product projection can be expressed as $\psi(\hat{x}) = \Psi^\tau x, \quad \forall x \in \mathbb{R}^n$. Then,

$$\Psi = M \Phi N^{-1}. \tag{2}$$

Definition (Inner-product projection of systems)

A reduced-order model $(\tilde{A}, \tilde{B}, \tilde{C})$ with $\tilde{A} = \Psi^\tau A \Phi$, $\tilde{B} = \Psi^\tau B$, and $\tilde{C} = C\Phi$ is constructed by an inner-product projection if $\Phi \in O(M, N)$, $\Psi = M\Phi N^{-1}$, where $M \in \mathrm{SPD}(n)$ and $N \in \mathrm{SPD}(k)$.

Lemma (Rowley et al. 04)

If the original LTI system (A, B, C) has a Lyapunov matrix Θ satisfying $A^\tau \Theta + \Theta A \prec 0$ and the reduced-order model is constructed by inner-product projection with $M = \Theta$, then the reduced-order model $(\tilde{A}, \tilde{B}, \tilde{C})$ is asymptotically stable with Lyapunov matrix N.

Existing algorithms for computing test and trial basis matrices

	POD–Galerkin	Balanced truncation	Balanced POD	Shift-reduce-shift-back
Input	Snapshot matrix X	(A, B, C)	Primal snapshots S and Dual snapshots R	(A, B, C) Shift margin μ
Output	Ψ, $\Phi \in O(I_n, I_k)$.	$\Phi \in O(W_o, \Sigma_1)$, $\Psi \in O(W_c, \Sigma_1)$	$\Phi \in O(\hat{W}_o, \Sigma_1)$; $\Psi \in O(\hat{W}_c, \Sigma_1)$.	$\Phi \in O(W_o^\mu, \Sigma_1)$, $\Psi \in O(W_c^\mu, \Sigma_1)$
Algorithm	1. Compute SVD $X = U\Sigma V^\tau$. 2. $\Psi = \Phi = U_1$.	1. Compute W_o and W_c by the Lyapunov equation 2. Compute symmetric factorization $W_c = SS^\tau$, $W_o = RR^\tau$. 3. Compute SVD $R^\tau S = U\Sigma V^\tau$. 4. $\Phi = SV_1\Sigma_1^{-1/2}$. 5. $\Psi = RU_1\Sigma_1^{-1/2}$.	1. Compute SVD $R^\tau S = U\Sigma V^\tau$ 2. $\Phi = SV_1\Sigma_1^{-1/2}$ 3. $\Psi = RU_1\Sigma_1^{-1/2}$.	1. Compute W_o^μ and W_c^μ by the Lyapunov equation 2. Compute symmetric factorization $W_c^\mu = SS^\tau$, $W_o^\mu = RR^\tau$. 3. Compute SVD $R^\tau S = U\Sigma V^\tau$. 4. $\Phi = SV_1\Sigma_1^{-1/2}$. 5. $\Psi = RU_1\Sigma_1^{-1/2}$.

Proposed algorithms for constructing an inner-product projection that preserves asymptotically stability

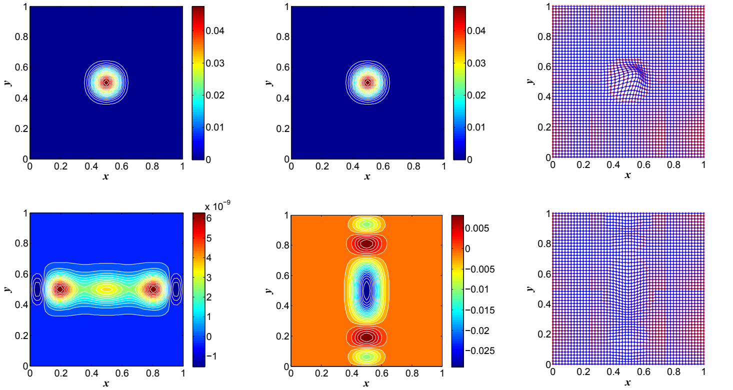
	Method 1 (inner-product balancing)	Method 2	Method 3
Input	$\Xi, \Xi' \in \mathrm{SPD}(n)$ with $\Xi = \Theta$ or $\Xi' = \Theta'$ satisfying the Lyapunov equation	$\Phi \in \mathbb{R}^{n\times k}$, Θ satisfying satisfying the Lyapunov equation	$\Phi_0 \in O(M_0, N_0)$, $N_0, N \in \mathrm{SPD}(k)$, $M_0 \in \mathrm{SPD}(n)$, Θ satisfying the Lyapunov equation
Output	$M \in \mathrm{SPD}(n)$, $N \in \mathrm{SPD}(k)$, $\Phi \in O(M,N)$, $\Psi \in O(M',N)$	$M \in \mathrm{SPD}(n)$, $N \in \mathrm{SPD}(k)$, $\Psi \in \mathbb{R}^{n\times k}$	$M \in \mathrm{SPD}(n)$, $\Phi \in O(M,N)$, $\Psi \in \mathbb{R}^{n\times k}$
Algorithm	1. Compute symmetric factorization $\Xi = RR^\tau$, $\Xi' = SS^\tau$ 2. Compute SVD $R^\tau S = U\Sigma V^\tau$ 3. $\bar\Phi = SV_1\Sigma_1^{-1/2}$ 4. $\bar\Psi = RU_1\Sigma_1^{-1/2}$ 5. $M = \Xi$, $M' = \Xi'$, $N = \Sigma_1$	1. $M = \Theta$ 2. $N = \Phi^\tau M\Phi$ 3. $\Psi = M\Phi N^{-1}$	1. Set $M = \Theta$ 2. Construct $G \in O(M, M_0)$ 3. Construct $\tilde G \in O(N, N_0)$ 4. $\Phi = G\Phi_0\tilde G^{-1}$ 5. $\Psi = M\Phi N^{-1}$

Inner-product reduction v. symplectic reduction

	Asymptotically stable subsystem	Marginally stable subsystem
Original space	Inner-product space: $(\mathbb{R}^n, M)$ with $M \in \mathrm{SPD}(n)$	Symplectic space: $(\mathbb{R}^m, J_\Omega)$ with $J_\Omega \in \mathrm{SS}(m)$
Autonomous system	$\dot{x} = Ax$ with $\lambda(A) < 0$	$\dot{x} = Ax$ with $\lambda(A) = 0$
Key property of full system	Lyapunov inequality: $A^\tau M + MA \prec 0$	Hamiltonian property: $A^\tau J_\Omega + J_\Omega A = 0$
Energy property of full system	$\frac{d}{dt}\left(\frac{1}{2}x^\tau Mx\right) < 0$	$\frac{d}{dt}\left(\frac{1}{2}x^\tau Lx\right) = 0$
Reduced space	Inner-product space: $(\mathbb{R}^k, N)$ with $N \in \mathrm{SPD}(k)$	Symplectic space: $(\mathbb{R}^k, J_\Pi)$ with $J_\Pi \in \mathrm{SS}(k)$
Projection	Inner-product projection	Symplectic projection
Trial basis matrix	$\Phi \in O(M, N) : \Phi^\tau M \Phi = N$	$\Phi \in \mathrm{Sp}(J_\Omega, J_\Pi) : \Phi^\tau J_\Omega \Phi = J_\Pi$
Test basis matrix	$\Psi = M\Phi N^{-1} \in \mathbb{R}^{n \times k}$	$\Psi = J_\Omega \Phi J_\Pi^{-1} \in \mathbb{R}^{m \times k}$
Reduced autonomous system	$\dot{z} = \tilde{A}z$ $\tilde{A} = \Psi^\tau A \Phi$ with $\lambda(\tilde{A}) < 0$	$\dot{z} = \tilde{A}z$ $\tilde{A} = \Psi^\tau A \Phi$ with $\lambda(\tilde{A}) = 0$
Key property of reduced system	Lyapunov inequality: $\tilde{A}^\tau N + N\tilde{A} \prec 0$	Hamiltonian property: $\tilde{A}^\tau J_\Pi + J_\Pi \tilde{A} = 0$
Energy property of reduced system	$\frac{d}{dt}\left(\frac{1}{2}z^\tau Nz\right) < 0$	$\frac{d}{dt}\left(\frac{1}{2}z^\tau \tilde{L}z\right) = 0$ with $\tilde{A} = -J_\Pi^{-1}\tilde{L}$

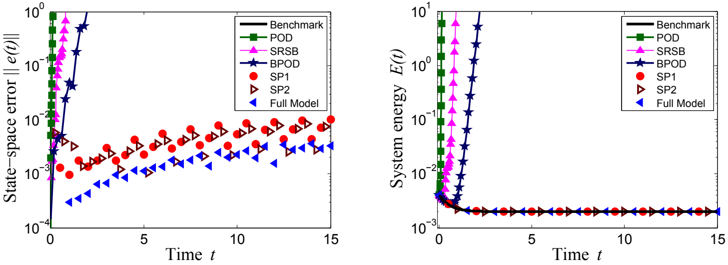
2D mass–spring system ($n = 2 \times 51^2$)

$$
\begin{aligned}
m\ddot{u}_{i,j} &= k_x(u_{i+1,j} + u_{i-1,j} - 2u_{i,j}) - 2b\dot{u}_{i,j}, \\
m\ddot{v}_{i,j} &= k_y(v_{i,j+1} + v_{i,j-1} - 2v_{i,j}),
\end{aligned}
\qquad (1)
$$



Comparison of different model-reduction methods for reduced dimension $k = 40$.

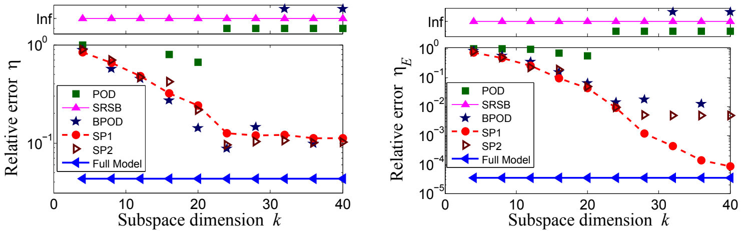
	POD	SRSB	BPOD	SP1	SP2	Full-order model
Number of unstable modes	8	16	18	0	0	0
Instability margin $\max(\mathrm{Re}(\lambda))$	50.480	10.586	3.695	0	0	0
Marginal-stability preservation	No	No	No	Yes	Yes	Yes
Relative state-space error η	$+\infty$	$+\infty$	$+\infty$	0.11156	0.10214	0.04358
Relative system-energy error η_E	$+\infty$	$+\infty$	$+\infty$	8.6868×10^{-5}	4.8843×10^{-3}	3.413×10^{-5}
Infinite-time energy	$+\infty$	$+\infty$	$+\infty$	1.9958×10^{-3}	1.9959×10^{-3}	1.9959×10^{-3}



(a) The evolution of the state-space error $\|e(t)\| = \|x(t) - \hat{x}(t)\|$

(b) The evolution of the system energy $E(t)$

Figure: The evolution of the state-space error $\|e(t)\| = \|x(t) - \hat{x}(t)\|$ and system energy $E(t)$ for all tested methods and reduced dimension $k = 40$.



(a) Relative state-space error η versus subspace dimension k

(b) Relative system-energy error η_E versus subspace dimension k

Figure: Method performance as a function of reduced dimension k.

Conclusions

- We propose a structure-preserving model reduction for marginally stable linear time-invariant (LTI) systems
- The method decomposes a marginally stable LTI system into an asymptotically stable subsystem and a pure marginally stable subsystem
- The pure marginally stable subsystem is Hamiltonian.
- Symplectic projection preserves pure marginal stability of this subsystem.
- Symplectic balancing method is proposed to reduce this subsystem.
- The accuracy, stability, and energy preservation of the proposed method is demonstrated through the 2D mass–spring system.
- The offline complexity of this method is $O(n^3)$, which is the same as balanced truncation. We will continue our work to reduce the offline complexity.

Acknowledgments

The authors thank Mohan Sarovar for his invaluable input and contributions to this work.
Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.