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# On the extreme event probability estimation of land ice mass loss



**CCR**  
Center for Computing Research

*Presented by:*

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- ▶ Introduction:
  - ▶ Land ice simulation and UQ big picture;
  - ▶ Extreme event probabilities;
- ▶ Optimization approach;
  - ▶ Computation of the most likely point;
  - ▶ Importance sampling;
- ▶ Numerical results;
  - ▶ Test problem and implementation;
  - ▶ Probabilities and performance;
- ▶ Conclusions and future work.

Rare events with high impacts:

- ▶ Are difficult to quantify with standard approaches; events with probability of  $10^{-3}$  or less require a large number of samples to be evaluated precisely;
- ▶ Can have severe consequences (even if they are rare); large tsunamis are very rare but can lead to high losses.

When rare events have high impacts, it is important to have a precise estimation of their probabilities.

In the case of land ice simulation, this will allow us to estimate smaller probabilities for different levels of sea level rise due to land ice mass loss.

Stokes equations:

$$\begin{cases} -\nabla \cdot \sigma &= \rho \mathbf{g}, \\ \nabla \cdot \mathbf{u} &= 0 \end{cases}$$

Stress tensor:

$$\sigma = 2\mu \mathbf{D} - p\mathbf{I},$$

where:

$$\mathbf{D}_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Ice behave like a shear thinning fluid  
Used model: Blatter-Pattyn model

Random parameter: basal friction

Quantity of interest: grounding line flux

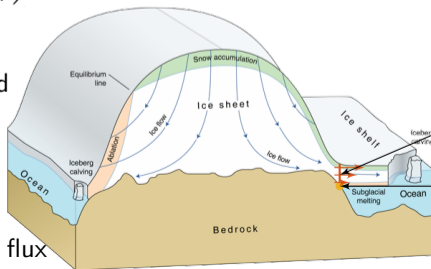
Ice viscosity (dependent on temperature):

$$\mu = \frac{1}{2} A(T) |\mathbf{D}(\mathbf{u})|^{\frac{1}{n}-1}, \quad n \geq 1.$$

Sliding boundary condition at ice bed:

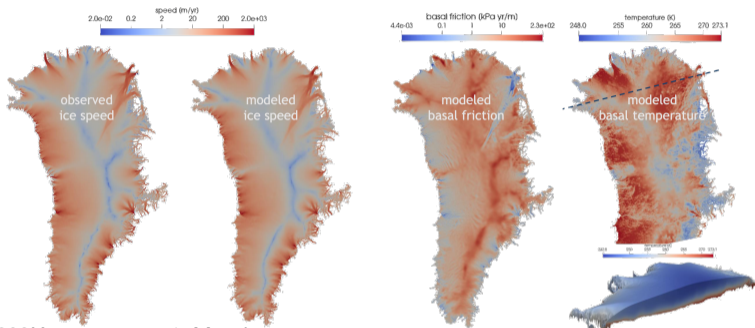
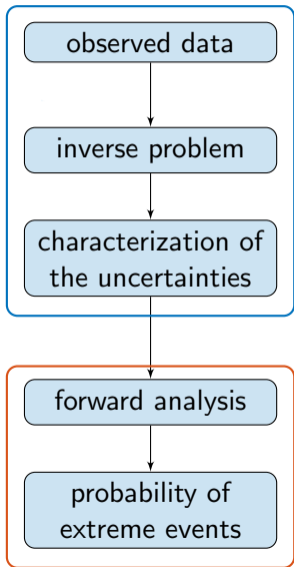
$$\begin{cases} \mathbf{u} \cdot \mathbf{n} &= 0, \\ (\sigma \mathbf{n})_{\parallel} &= \beta \mathbf{u} \end{cases}$$

where  $\beta$  is the basal friction between the ice and the bedrock.



Grounding line flux

Grounding line



300K parameters, 14M unknowns.

Initialization: 10 hours on 2k cores on NERSC Cori (Haswell),

The optimization is constrained by the coupled velocity-temperature solvers.

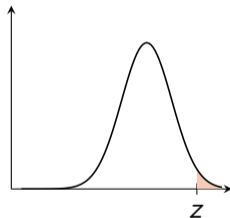
The work presented in this talk.

Given a  $n$ -variate Gaussian variable  $\theta \sim \mathcal{N}_n(0, \mathbf{I})$  and a  $F$  parameter-to-event map (involved PDE solve), quantity of interest:

$$F : \theta \sim \mathcal{N}_n(0, \mathbf{I}) \rightarrow \mathbb{R}$$



$$\xrightarrow{F(\theta)}$$



Target: Estimate the measure of extreme event sets for  $z \gg 0$   
 $\Omega(z) := \{\theta : F(\theta) \geq z\}$  i.e., compute  $\mathbb{P}(F(\theta) \geq z)$  when  $\mathbb{P}(F(\theta) \geq z) \ll 1$ .

The used strategy relies on finding  $\theta^*(z)$  the most likely point above the threshold which can be computed as follows:

$$\theta^*(z) = \arg \min_{\theta \in \Omega(z)} I(\theta),$$

where  $I(\theta) = \frac{1}{2} \|\theta\|^2$  for  $\theta \sim \mathcal{N}_n(0, I)$  as  $pdf(\theta) = c \exp(-I(\theta))$ .

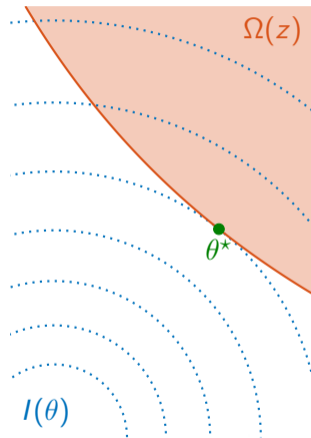
Then, the probability can be approximated as follows:

$$\mathbb{P}(F(\theta) \geq z) \approx C_0(z) \exp(-I(\theta^*(z))), \quad \text{as } z \rightarrow \infty,$$

where  $C_0(z)$  is a sub-exponential prefactor.

The method relies on 2 steps:

- ▶ Compute the most likely point  $\theta^*$ ,
- ▶ Compute the prefactor  $C_0(F(\theta^*))$ .



Under some assumptions, the minimizer over  $\Omega(z)$  is reached on  $\partial\Omega(z)$  and the inequality constraint is now active:

$$\theta^*(z) = \arg \min_{\theta \in \partial\Omega(z)} I(\theta).$$

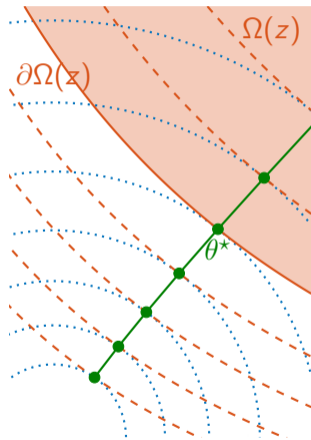
We used a quadratic penalty method as follows:

$$\theta^*(z) = \arg \min_{\theta} I(\theta) + \alpha (F(\theta) - z)^2,$$

where  $\alpha$  is a penalty weight that should be large enough such that  $F(\theta) \approx z$ .

The used strategy is the following:

- ▶ Select an increasing sequence  $z_1, \dots, z_m$  of quantity of interest,
- ▶ For a given  $z_i$ , solve the corresponding optimization problem using  $\theta^*(z_{i-1})$  as the initial guess,
- ▶ Deduce the sequence  $\theta^*(z_1), \dots, \theta^*(z_m)$ .



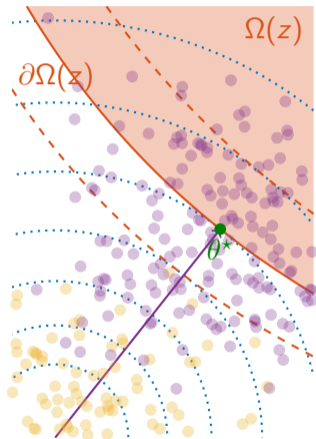
One strategy is to use an Importance Sampling (IS) strategy:

- ▶ to draw  $N$  random samples  $\theta_1, \dots, \theta_N$  from the initial distribution,
- ▶ for a given value of  $z_i$ :
  - ▶ shift the samples:  $\tilde{\theta}_k = \theta_k + \theta^*(z_i)$ ,
  - ▶ evaluate  $F$  for all the  $N$  shifted samples,
  - ▶ evaluate:

$$P_N^{IS}(z) = e^{-I(\theta^*)} \frac{1}{N} \sum_{k=1}^N \left[ \mathbb{1}_{\Omega(z)}(\tilde{\theta}_k) \exp \left( - \left( \tilde{\theta}_k - \theta^* \right)^\top \theta^* \right) \right],$$

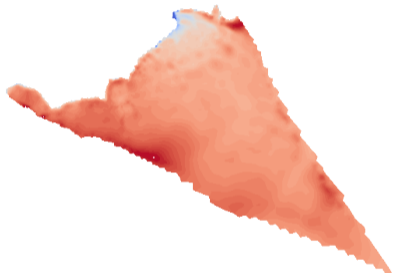
where  $z$  can be different from  $F(\theta^*)$ .

- ▶ Advantages:  $\mathbb{E} [P_N^{IS}(z)] = P(z)$  and variance in  $1/N$ ,
- ▶ Challenges: this approach requires  $N \times m$  evaluations of  $F$  where  $m$  is the number of  $z$  values.

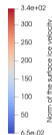
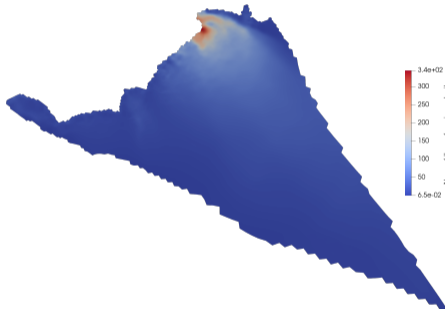


Mean case:

Basal friction:



Surface ice velocity:



Courtesy of  
T. Hillebrand.

The velocity is faster if the friction is smaller, the quantity of interest is the flux at the grounding line; we expect the extreme events to be associated to smaller basal friction values.

- ▶ Random parameter: the basal friction represented using a log-normal random field and a KL expansion with 24 modes:

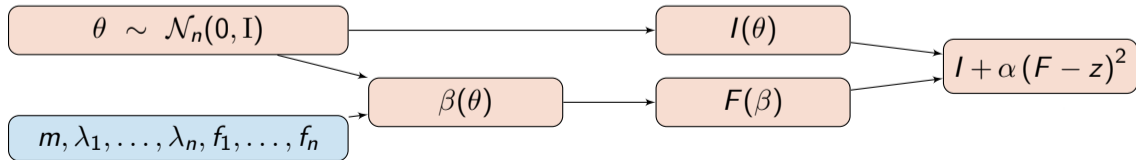
$$\beta(\mathbf{x}, \theta) = e^{\log(m(\mathbf{x})) + \sum_{i=0}^n \sqrt{\lambda_i} \varepsilon_i(\theta) f_i(\mathbf{x})},$$

$$\int_{\Omega} \text{Cov}(\mathbf{x}, \mathbf{x}') f_i(\mathbf{x}') d\mathbf{x}' = \lambda_i f_i(\mathbf{x}),$$

$$\text{Cov}(\mathbf{x}, \mathbf{x}') = \sigma e^{-\frac{\|\mathbf{x} - \mathbf{x}'\|}{\ell}},$$

with  $\sigma = 0.1$ ,  $\ell = 50$  km, and  $n = 24$ .

- ▶ Quantity of interest: flux at the grounding line.
- ▶ PDE: steady state first order Stokes equation, Blatter-Pattyn model.

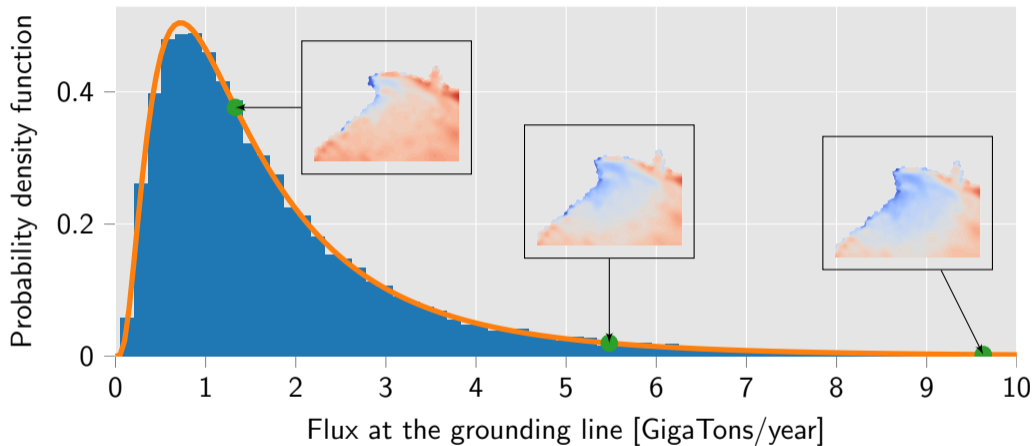


- ▶ FE software: Albany,
- ▶ PDE constrain optimizer: algorithm: trust region, software: ROL,
- ▶ Non-linear solver: algorithm: Newton solver, software: NOX,
- ▶ Linear solver: algorithm: GMRES, software: Belos,
- ▶ Preconditioner: algorithm: Schwarz, software: FROSch,
- ▶ First and second derivative computation: algorithm: automatic differentiation (AD), software: Sacado,
- ▶ Reduced Hessian and Gradient vector product computed using ROL and AD.



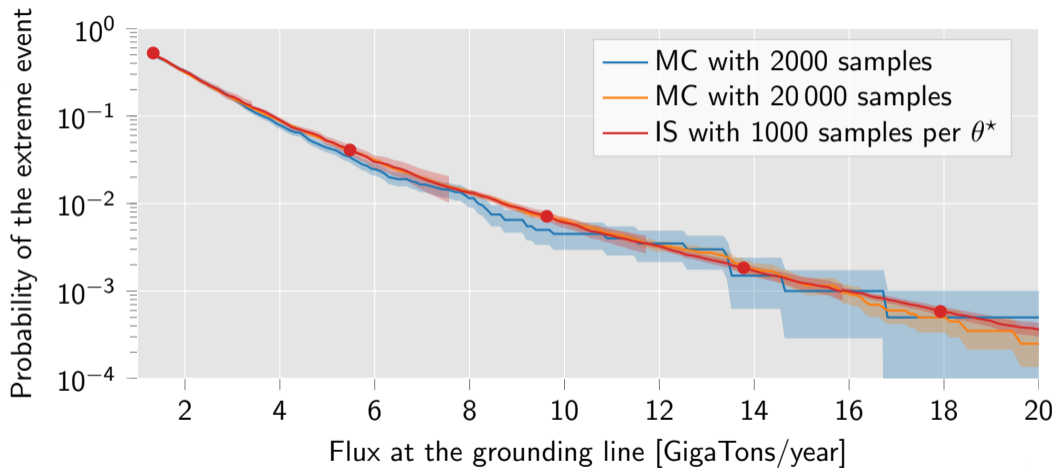
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Histogram of 20 000 samples, the orange curve is a log-normal distribution that fits the data the best.

## Probability of the extreme events



For smaller values of  $z$ , all the methods are consistent. When increasing  $z$ , the MC approach needs more and more samples to be consistent with the importance sampling.

Average cost per simulation (measured on Skylake):

	Wall-clock time	Relative cost
To compute $F(\theta)$	$c_F = 13 \text{ sec}$	1
To compute $\theta^*(z)$	$c_{\theta^*} = 2257 \text{ sec}$	174

Expected cost per method:

	$\mathbb{E}[c]$	$\mathbb{E}[c/c_F]$
Monte Carlo	$N_{MC} c_F$	$N_{MC}$
Importance Sampling per $\theta^*$	$c_{\theta^*} + N_{IS} c_F$	$174 + N_{IS}$

Comparison of Monte Carlo with  $N_{MC} = 20\,000$  and importance sampling with  $N_{IS} = 1000$ :

$$\frac{N_{MC}}{174 + N_{IS}} = \frac{20\,000}{1174} = 17,$$

the importance sampling is 17 times faster.

The more extreme the event, the more samples will be required for the Monte Carlo method and the more efficient the importance sampling will be compared to standard Monte Carlo.



## Conclusions:

- ▶ Discussion of the usage of optimization strategies to compute the probability of extreme events,
- ▶ Discussion of the implementation using open source libraries and software,
- ▶ Computation of the extreme event probabilities of high fluxes at the grounding line of the Humboldt glacier,
- ▶ Performance comparison of the proposed approach with the standard Monte Carlo method.

## Future work:

- ▶ Consider solving the constrained problem instead of using a quadratic penalty method,
- ▶ Move towards transient analysis,
- ▶ Consider larger problems,
- ▶ Use characterization of the uncertainties computed using the inverse problem,
- ▶ Deduce probability of extreme sea level rise due to land ice mass loss for the future.