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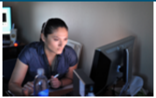
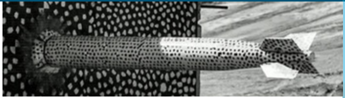
Sandia National Laboratories

Block Preconditioning for Magnetic Confinement Fusion Relevant Resistive MHD Simulations





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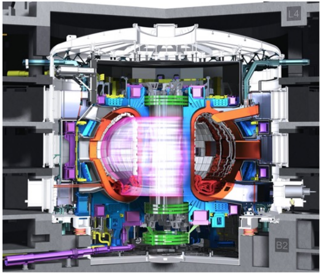
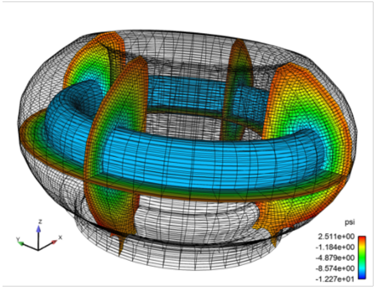
Tokamak Simulation

- Achieve temperatures of 100M deg K (6x Sun temp.)
- Energy confinement times $\mathcal{O}(1-10)$ min.



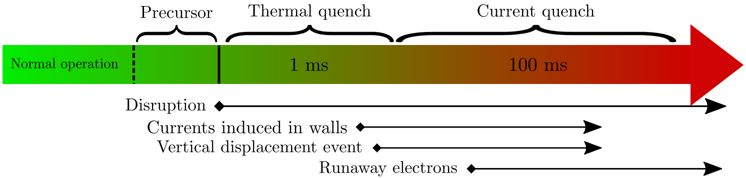


- Plasma disruptions can cause a breakdown of the magnetic field surface structure
  - loss of plasma confinement, plasma interacts with wall
  - huge thermal energy loss (thermal quench)
  - possible discharge of very large electrical currents (20MA) into structure
- ITER can sustain only a limited number of significant disruptions/instabilities

A vertical displacement event

Definition

Disruption event in Tokamak devices with sudden loss of plasma confinement and vertical movement towards wall.



1. Fast temperature drop $\Longrightarrow$ change in MHD equilibrium, $\mathbf{j} \times \mathbf{B} \approx 0$ $\Longrightarrow$ loss of vertical position of control.
2. Temperature drop $\Longrightarrow$ resistivity increase $\Longrightarrow$ plasma current drop + ohmic to runaway current conversion.
3. Plasma current drop $\Longrightarrow$ magnetic field rearrangement, i.e. VDE.
4. VDE $\Longrightarrow$ Induce large electromagnetic force in the walls with halo current.

Tokamak Vertical Disruption Event Simulation

Compressible visco-resistive MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left[(\rho \mathbf{u} \otimes \mathbf{u}) + pI - \frac{2}{3}\mu(\nabla \cdot \mathbf{u})I - \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right] - \mathbf{j} \times \mathbf{B} = \mathbf{0}, \tag{2}$$

$$\rho c_v \frac{\partial T}{\partial t} + \rho c_v \mathbf{u} \cdot \nabla T - \nabla \cdot (\kappa \nabla T) - p(\nabla \cdot \mathbf{u}) + \eta \|\mathbf{j}\|^2 + \gamma [T - T_0]_+ = 0, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left[\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u} - \frac{\eta}{\mu_0}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \psi \right] = \mathbf{0}, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

plus appropriate boundary conditions.

Discretization

- First order cG.
- VMS (convective & saddle point stabilization).
- DCO on equation (1) & (2).
- Lagrange multiplier for $\nabla \cdot \mathbf{B} = 0$.

Block Linear System

Newton linearized stabilized finite element discretization

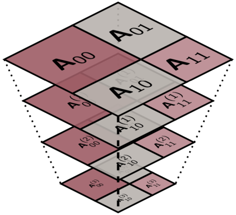
$$\begin{bmatrix} \mathbf{F_B} & \mathcal{B}_\mathbf{B}{}^T & \mathbf{Y}_{\mathrm{nst}} \\ \mathcal{B}_\mathbf{B} & \mathbf{L_r} & \mathbf{C}_{\mathrm{nst}} \\ \mathbf{Z} & & \mathbf{F}_{\mathrm{nst}} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{B} \\ \Delta \lambda \\ \Delta \mathbf{u}_{nst} \end{bmatrix} = \begin{bmatrix} -r_\mathbf{B} \\ -r_\lambda \\ -r_{\mathbf{u}_{nst}} \end{bmatrix}$$

- $\mathbf{F_B}$ - Magnetics terms
- $\mathbf{L_r}$ - Lagrange multiplier, VMS stabilization laplacian
- $\mathbf{F}_{\mathrm{nst}}$ - Momentum, Density, and Temperature terms

$$\mathbf{u}_{nst} = \begin{bmatrix} \mathbf{u} \\ \rho \\ T \end{bmatrix}$$

Results - Monolithic AMG

- Monolithic AMG preconditioned GMRES
  - Deal with elliptic diffusion operator stiffness
  - Not intended to deal with off-diagonal Alfven wave physics
- Relaxation: proc. based domain decomposition Schwarz
  - overlap 1 with ILU subsolve
- Increasing time step size, up to a multiple of Alfven CFL, $\mathrm{CFL}_a^{\max}$
  - $\mathrm{CFL}_a = \lambda\, \mathrm{d}t/h < \mathrm{CFL}_a^{\max}$
  - $\lambda = |\mathbf{u}| + |\mathbf{u}_A|$
  - $\mathbf{u}_A = |\mathbf{B}|/\sqrt{\rho\mu_0}$
- 663,984 dofs, 144 mpi ranks
- Linear solve to 10^{-12}, ensure correct physics



Results - Monolithic AMG

CFL_a^{max}	Linear Its. per non-Lin It.	Setup time per non-Lin It.	Solve time per non-Lin It.	Total Linear Time (Seup + Solve)
50	28.89	2.44	1.94	1909.15
100	75.01	2.43	4.97	2118.48
200	221.46	2.43	16.93	4493.42
400	236.16	2.45	18.34	5928.57

- Increasing iteration/solve time.
- Linear solve stagnates before we reach target CFL timescales
- Detrimental for scaling with mesh size

Operator Splitting

Approximate factor of 3×3 system into two 2×2 systems.

$$\mathcal{M}_{Split} = \begin{bmatrix} \mathbf{F_B} & & \mathbf{Y}_{nst} \\ & \mathbf{I} & \\ \mathbf{Z} & & \mathbf{F}_{nst} \end{bmatrix} \begin{bmatrix} \mathbf{F_B}^{-1} & & \\ & \mathbf{I} & \mathbf{C}_{nst} \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F_B} & \mathcal{B}_\mathbf{B}{}^T & \\ \mathcal{B}_\mathbf{B} & \mathbf{L_r} & \\ & & \mathbf{I} \end{bmatrix}$$

- Groups magnetics and solenoid constraint
- Groups interaction between Lorenz force and convective term of magnetics
  - Develop Alfven wave propagation mode (fast hyperbolic time scale)

$$\mathcal{M}_{Split} = \begin{bmatrix} \mathbf{F_B} & \mathcal{B}_\mathbf{B}{}^T & \mathbf{Y}_{nst} \\ \mathcal{B}_\mathbf{B} & \mathbf{L_r} & \mathbf{C}_{nst} \\ \mathbf{Z} & \mathbf{Z}\,\mathbf{F_B}^{-1}\,\mathcal{B}_\mathbf{B}{}^T & \mathbf{F}_{nst} \end{bmatrix} \approx \mathcal{J} = \begin{bmatrix} \mathbf{F_B} & \mathcal{B}_\mathbf{B}{}^T & \mathbf{Y}_{nst} \\ \mathcal{B}_\mathbf{B} & \mathbf{L_r} & \mathbf{C}_{nst} \\ \mathbf{Z} & & \mathbf{F}_{nst} \end{bmatrix}$$

- Structural perturbation $\mathbf{Z}\,\mathbf{F_B}^{-1}\,\mathcal{B}_\mathbf{B}{}^T$ term is "small"

Cyr, Shadid, Tuminaro, Pawlowski, Chacón, *A new approximate block factorization preconditioner for two-dimensional incompressible (reduced) resistive mhd*, 2013.

Operator Splitting - Implementation

Block LU decomposition

$$\mathcal{M}_{Split}^{-1} \approx \left(\begin{bmatrix} \mathbf{F_B} & & \mathbf{Y}_{\mathrm{nst}} \\ & \mathbf{I} & \\ \mathbf{Z} & & \mathbf{F}_{\mathrm{nst}} \end{bmatrix} \begin{bmatrix} \mathbf{F_B}^{-1} & & \\ & \mathbf{I} & \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F_B} & \mathcal{B}_\mathbf{B}{}^{T} & \\ \mathcal{B}_\mathbf{B} & \mathbf{L_r} & \\ & & \mathbf{I} \end{bmatrix} \right)^{-1}$$

$$\approx \left(\begin{bmatrix} \mathbf{I} & & \mathbf{Y}_{\mathrm{nst}} \\ & \mathbf{I} & \\ & & \mathbf{F}_{\mathrm{nst}} - \mathbf{Z}\,\mathbf{F_B}^{-1}\,\mathbf{Y}_{\mathrm{nst}} \end{bmatrix} \begin{bmatrix} \mathbf{F_B} & \mathcal{B}_\mathbf{B}{}^{T} & \\ \mathcal{B}_\mathbf{B} & \mathbf{L_r} & \\ & & \mathbf{I} \end{bmatrix} \right)^{-1}$$

Murphy, Golub, Wathen, *A note on preconditioning for indefinite linear systems*, 2000.

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Operator Splitting - Implementation

Block LU decomposition

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$$\approx \left(\begin{bmatrix} \mathbf{I} & & \mathbf{Y}_{nst} \\ & \mathbf{I} & \\ & & \mathbf{F}_{nst} - \mathbf{Z}\,\mathbf{F_B}^{-1}\,\mathbf{Y}_{nst} \end{bmatrix} \begin{bmatrix} \mathbf{I} & & \\ \mathcal{B}_{\mathbf{B}}\,\mathbf{F_B}^{-1} & \mathbf{I} & \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F_B} & \mathcal{B}_{\mathbf{B}}{}^{T} & \\ & \mathbf{L_r} - \mathcal{B}_{\mathbf{B}}\,\mathbf{F_B}^{-1}\,\mathcal{B}_{\mathbf{B}}{}^{T} & \\ & & \mathbf{I} \end{bmatrix} \right)^{-1}$$

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Operator Splitting - Inverses

$$\mathcal{M}_{Split}^{-1} \approx \left(\begin{bmatrix} \mathbf{I} & & \mathbf{Y}_{\text{nst}} \\ & \mathbf{I} & \\ & & S_{nst} \end{bmatrix} \begin{bmatrix} \mathbf{I} & & \\ \mathcal{B}_{\mathbf{B}} \, \mathbf{F}_{\mathbf{B}}^{-1} & \mathbf{I} & \\ & & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{\mathbf{B}} & \mathcal{B}_{\mathbf{B}}{}^{T} & \\ & S_L & \\ & & \mathbf{I} \end{bmatrix} \right)^{-1}$$

SIMPLE-type Schur complement approximation

$$\mathbf{F}_{\text{nst}} - \mathbf{Z}\,\mathbf{F}_{\mathbf{B}}^{-1}\,\mathbf{Y}_{\text{nst}} \approx S_{nst} := \mathbf{F}_{\text{nst}} - \mathbf{Z}(\text{absrowsum}(\mathbf{F}_{\mathbf{B}}))^{-1}\,\mathbf{Y}_{\text{nst}}$$

$$\mathbf{L}_{\mathbf{r}} - \mathcal{B}_{\mathbf{B}}\,\mathbf{F}_{\mathbf{B}}^{-1}\,\mathcal{B}_{\mathbf{B}}{}^{T} \approx S_L := \mathbf{L}_{\mathbf{r}} - \mathcal{B}_{\mathbf{B}}(\text{absrowsum}(\mathbf{F}_{\mathbf{B}}))^{-1}\,\mathcal{B}_{\mathbf{B}}{}^{T}$$

Need to compute the inverses for S_{nst}, S_L, and $\mathbf{F}_{\mathbf{B}}$.

- S_{nst} is the primary Alfven term

Cyr, Shadid, Tuminaro, Pawlowski, Chacón, *A new approximate block factorization preconditioner for two-dimensional incompressible (reduced) resistive mhd*, 2013.

Results - Operator Splitting Block Precond.

Operator Splitting Block Preconditioner

- Preconditioned GMRES, using the Operator Splitting Block Preconditioner
- Inverses (S_{nst}, S_L, and $\mathbf{F_B}$) computed with AMG
  - Relaxation: proc. based domain decomposition Schwarz, overlap 1 with ILU subsolve
- Increasing time step size, up to a multiple of Alfven CFL, $\text{CFL}_a^{\text{max}}$
  - $\text{CFL}_a = \lambda \, \text{d}t/h < \text{CFL}_a^{\text{max}}$
  - $\lambda = |\mathbf{u}| + |\mathbf{u}_A|$
  - $\mathbf{u}_A = |\mathbf{B}|/\sqrt{\rho\mu_0}$
- 663,984 dofs, 144 mpi ranks
- Linear solve to 10^{-12}, ensure correct physics

Results - Operator Splitting Block Precond.

CFL_a^{max}	Linear Its. per non-Lin It.	Setup time per non-Lin It.	Solve time per non-Lin It.	Total Linear Time (Seup + Solve)
50	26.99	9.85	4.40	6622.13
100	28.28	9.83	4.59	4400.53
200	30.60	9.84	4.93	3239.33
400	47.87	9.84	7.56	3290.29

- Better CFL scaling than monolithic AMG approach
- ILU with Schwarz overlap 1 is expensive to setup
- S_{nst} has a large stencil size (5 times more nnz/row than $\mathbf{F}_{\mathrm{nst}}$)

Results - Subsolve options - Reduced stencil

- Use sparsity structure of $\mathbf{F}_{\text{nst}}$ for approx ILU of $S_{nst} := \mathbf{F}_{\text{nst}} - \mathbf{Z}(\text{absrowsum}(\mathbf{F}_{\mathbf{B}}))^{-1}\,\mathbf{Y}_{\text{nst}}$

$\text{CFL}_a^{\max}$	Linear Its. per non-Lin It.	Setup time per non-Lin It.	Solve time per non-Lin It.	Total Linear Time (Seup + Solve)
50	32.27	1.15	2.28	2139.18
100	51.62	1.15	3.49	1750.87
200	65.94	1.15	4.46	1464.23
400	131.87	1.15	9.89	2268.03

- Significantly reduced setup time
- Scales less well with CFL

Results - Subsolve options - ILUT

- Use ILUT to drop small values.
- Threshold=0.1

CFL_a^{max}	Linear Its. per non-Lin It.	Setup time per non-Lin It.	Solve time per non-Lin It.	Total Linear Time (Seup + Solve)
50	26.99	6.04	4.66	5061.41
100	28.21	6.19	4.90	3441.40
200	28.18	6.11	4.88	2459.12
400	27.91	6.03	4.83	2116.38

- The "best" scaling with CFL

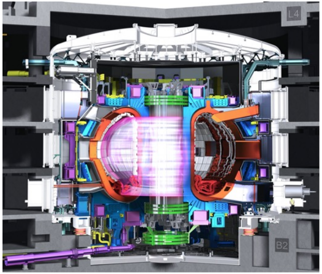
Results - Operator Splitting - Long Run

- ILUT with threshold=0.1 for S_{nst}
- Enforce max $\text{CFL}_{\mathbf{u}} \leq 1$

CFL_a^{max}	Num. Timestep	Linear Its. per non-Lin It.	Setup Time	Solve Time	Total Time
50	1764	25.48	25250.70	16628.80	41879.50
100	908	29.16	13283.60	9835.80	23119.40
200	490	34.27	7367.84	6293.74	13661.58
400	288	42.86	4594.03	4838.48	9432.51
600	222	49.33	3680.76	4435.00	8115.76
800	196	84.73	3310.57	6864.51	10175.08
1000	186	136.23	3169.22	10972.20	14141.42
1600	180	138.63	3190.51	11202.20	14392.71

Future Work

- Refine mesh size
  - Resolve elliptic diffusion operator
  - Weak scaling
- Include off-diagonal flow/constraint coupling $\mathbf{C}_{nst}$ in block preconditioner
- Subblock solves and Relaxation
  - S_{nst}
  - S_L, $\mathbf{F_B}$
- Heterogenous domain
  - Model magnetics outside of the plasma region



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