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# Air-Sea Light

## Coupling Atmosphere and Ocean models through the Bulk Condition

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# Background

- The Multilayer Thermal Rotating Shallow Water Equations are used to model weather and climate on an ocean-atmosphere system.
- They model variables of velocities, thicknesses (heights), and temperature (or other variables like buoyancy) of vertically “stacked” layers of fluids with varying densities.
- There is a large discontinuity in the densities at the ocean-atmosphere interface, so we use the *bulk condition*, a homogenization of the boundary layer, to couple the models in an accurate and stable manner [3,10].
- We implement a simplified ocean-atmosphere model, which we call Air-Sea Light, in MATLAB to enable rapid testing and prototyping of various coupling methods for the ocean-atmosphere system.
- This project has been done under the CANGA project at Sandia National Laboratories.

# Hamiltonian Framework

- A Hamiltonian system requires variables, the total energy or Hamiltonian *(H),* and the Poisson tensor *(J)*.
- Consider the total energy *H(h,u,t)* of a system as a function of fluid thickness, which we call “height” *h*, velocity *u*, and time *t* [2,4].
- Let *v* = [*h,u*] so that *H(h,u,t)* = *H(v,t).*
- Then by the chain rule in the sense of the Gateaux derivative and by conservation of energy:

$$\frac{dH(v,t)}{dt} = \left(\frac{\partial v}{\partial t}, \frac{\partial H}{\partial v}\right)_{L^2(\Omega)} = 0.$$

- The pair of *h* and *u* that satisfy this equation must also satisfy

$$\frac{\partial v}{\partial t} = J\frac{\partial H}{\partial v} \ , \ \left(\frac{\partial H}{\partial v}, J\frac{\partial H}{\partial v}\right)_{L^2(\Omega)} = 0.$$

  where *J* is a skew-symmetric matrix operator and must satisfy the Jacobi identity [2,4,6].
- We will use this idea of the Hamiltonian framework to derive the Multilayer Thermal Rotating Shallow Water Equations [2,4].

# Multilayer Thermal Rotating Shallow Water Equations

- The Hamiltonian for the Multilayer Thermal Rotating Shallow Water Equations is:

$$H(h, u, \sigma, t) = \sum_{j=1}^{N} \int_{\Omega} \left( \bar{\rho}_j h_j \frac{u_j^2}{2} + \sigma_j \left( b + \sum_{i=j+1}^{N} h_i + \frac{h_j}{2} \right) \right) d\vec{x}.$$

- where:
  - $N$ is the number of layers.
  - $h_j$ is the layer height at layer $j$.
  - $u_j$ is the velocity at layer $j$.
  - $\sigma_j = g\rho_j h_j$ is the mass-weighted buoyancy at layer $j$ [2].
  - $g$ is the gravity constant.
  - $\rho_j$ is the density at layer $j$.
  - $\bar{\rho}_j$ is the *initial* density at layer $j$ as we are using the Boussinesq approximation.
  - $b$ is the bathymetry which contains the bottom topography of the model.

# Multilayer Thermal Rotating Shallow Water Equations

- Observe that (and these are functional derivatives)

$$\frac{\partial H}{\partial h_j} = \bar{\rho}_j \frac{u_j^2}{2} + \frac{\sigma_j}{2} + \sum_{i=1}^{j-1} \sigma_i, \quad \frac{\partial H}{\partial u_j} = \bar{\rho}_j h_j u_j, \quad \frac{\partial H}{\partial \sigma_j} = b + \sum_{i=j+1}^{N} h_i + \frac{h_j}{2}.$$

- Let $J_j$ be the Poisson tensor at layer $j$ be defined as

$$J_j = \frac{1}{\bar{\rho}_j} \begin{pmatrix} 0 & -\nabla \cdot () & 0 \\ -\nabla() & -q(h_j, u_j) k \times () & -s_j \nabla() \\ 0 & -\nabla \cdot (s_j \quad ) & 0 \end{pmatrix}.$$

- where:
  - $q(h,u) = (k \cdot \nabla \times u + f)/h$ is the potential vorticity.
  - $s_j = g\rho_j$ is the buoyancy at layer $j$ [2].
  - $k$ is the unit normal vector in the Cartesian $z$ direction.
  - $f$ is the Coriolis parameter.
- Let $J = \mathrm{diag}_{j = 1,\ldots,N} J_j$ which is the composite operator for the multilayer equations.
- It can be verified that each $J_j$ and $J$ satisfy the Jacobi identity [2,6].

# Multilayer Thermal Rotating Shallow Water Equations

- Then we can derive the Multilayer Thermal Rotating SWE as follows:

$$\begin{pmatrix} \frac{\partial h_j}{\partial t} \\ \frac{\partial u_j}{\partial t} \\ \frac{\partial \sigma_j}{\partial t} \end{pmatrix} = J_j \begin{pmatrix} \frac{\partial H}{\partial h_j} \\ \frac{\partial H}{\partial u_j} \\ \frac{\partial H}{\partial \sigma_j} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla \cdot () & 0 \\ -\nabla() & -q(h_j, u_j)k \times () & -s_j \nabla() \\ 0 & -\nabla \cdot (s_j \quad ) & 0 \end{pmatrix} \begin{pmatrix} K_j + \frac{\sigma_j}{2\bar{\rho_j}} + \frac{1}{\bar{\rho_j}} \sum_{i=1}^{j-1} \sigma_j \\ h_j u_j \\ \frac{1}{\bar{\rho_j}} \left( b + \sum_{i=j+1}^{N} h_j + \frac{h_j}{2} \right) \end{pmatrix}$$

$$= \begin{pmatrix} -\nabla \cdot h_j u_j \\ -q(h_j, u_j)k \times h_j u_j - \nabla K_j - \frac{1}{\bar{\rho_j}} \nabla \left( \frac{\sigma_j}{2} \right) - \frac{1}{\bar{\rho_j}} \nabla \left( \sum_{i=1}^{j-1} \sigma_j \right) - \frac{\sigma_j}{h_j \bar{\rho_j}} \nabla \left( b + \sum_{i=j+1}^{N} h_i + \frac{h_j}{2} \right) \\ -\nabla \cdot \sigma_j u_j \end{pmatrix}$$

- where $K_j = u_j^2/2$.

# Temperature Equations

- We wish to express our $\sigma_j$ variables in terms of temperature of each layer $T_j$ given in Kelvin (K).
- For the atmosphere, we use the Ideal Gas Law: $p_j = \rho_j R T_j$ , where $p_j = \sum_{i=1}^{j} \sigma_i$ and $R$ = 287 J/(kg K) [8]. Then

$$T_j = \frac{p_j}{R\rho_j} = \frac{\sum_{i=1}^{j} \sigma_i}{R\rho_j} = \frac{gh_j \sum_{i=1}^{j} \sigma_i}{R\sigma_j}.$$

- For the ocean temperature, we use the linear equation of state [7]:

$$\rho_j = \rho_0 - \alpha(T_j - T_0) + \beta(S_j - S_0)$$

- where we let $\alpha$ = 0.255 kg/(m$^3$ °C), β = 0, $T_0$ = 19 °C, and $\rho_0$ = 1026.5 kg/m$^3$ [7,8]. Then (also converting to Kelvin)

$$T_j = \frac{\rho_0}{\alpha} + T_0 - \frac{\rho_j}{\alpha} + 273.1 = \frac{\rho_0}{\alpha} + T_0 - \frac{\sigma_j}{\alpha g h_j} + 273.1.$$

# Modifying Thermal Variable

- The bulk condition (described later) is given in terms of temperatures and velocities through the vertical diffusion terms.
- To account for this, we incorporate a temperature based tracer equation instead of a buoyancy based one.
- This breaks the Hamiltonian structure, however we mainly used it for derivation purposes. Additionally, various forcing terms we add in our model also break the Hamiltonian structure, so this framework has served its purpose at this point and we now only focus on the primitive equations as follows:

$$\frac{\partial h_j}{\partial t} = -\nabla \cdot h_j u_j$$

$$\frac{\partial u_j}{\partial t} = -q(h_j, u_j)k \times h_j u_j - \nabla K_j - \frac{1}{\rho_j}\nabla\left(\frac{\sigma_j}{2}\right) - \frac{1}{\rho_j}\nabla\left(\sum_{i=1}^{j-1}\sigma_j\right) - \frac{\sigma_j}{\bar{\rho}_j h_j}\nabla\left(b + \sum_{i=j+1}^{N} h_i + \frac{h_j}{2}\right)$$

$$\frac{\partial (T_j h_j)}{\partial t} = -\nabla \cdot T_j h_j u_j$$

# Description of Test Data

- Test model domain and data come from the SOMA test case [4,9].
- The horizontal domain is a circle on the sphere with diameter 1.25 x $10^6$ m.
- System has a flat bathymetry *b*.
- We discretize each model in space using the *TRiSK* scheme [5].
- This uses centroidal voronoi tessellations as the mesh on the sphere where the cells are on average 32 km in diameter on the quasi-uniform mesh.
- Grid for each layer has 8521 cells, 25898 edges, and 17378 vertices.
- $h_{\bar{j}}$, $\sigma_j$, and $T_j$ are defined on the cells.
- $u_j$ are defined on the edges.
- $q(h_{\bar{j}},u_j)$ are defined on the vertices.

# Description of Model/Implementation

| Parameter | Value (Air) | Value (Ocean) | Unit | Description |
|---|---|---|---|---|
| $b$ | 0 | -2500 | m | Bottom topography (flat) |
| $u_j$ | 0 | 0 | m/s | Initial velocities for $j = 1, 2, 3$ |
| $\rho_j$ | [0.6599,0.9803,1.225] | [1025;1027;1028] | $\text{kg/m}^3$ | Initial densities |
| $T_j$ | [283.1,293.1,303.1] | From ocean temp eq | K | Initial temperatures |
| $h_j$ | From $T_j$ and $\sigma_j$ | [250,450,1800] | m | Initial heights |
| $\sigma_j$ | $g\rho_j h_j$ | $g\rho_j h_j$ | $\text{kg/(ms}^2)$ | For $j = 1, 2, 3$ |

- We have developed a 3-layer air model and a 3-layer ocean model as described in the table above.
- Observe that some of the variables are “forced” in order to satisfy the various equations of state.
- We couple via the bulk condition the 3 layer air model on to the 3 layer ocean model to make the 6 layer model.
- We couple via the bulk condition layers 3 and 4 of the 6 layer model to create the 2 layer model. However, we make the height of the ocean layer 2500 m instead of 250 m.
- We add wind forcing to the top ocean layer $F_w$ in both models [4].

# Description of Model/Implementation

- Shallow Water Equations implemented using the time-stepping method Runge-Kutta 4 (RK4).
- We add a drag term $F_d$ in the ocean layers as seen in [4].

$$F_d = \frac{c_{\text{drag}}}{h_j}|u_j|u_j, \quad c_{\text{drag}} = 10^{-3}$$

- We also add horizontal smoothing and vertical mixing to the velocity and tracer equations, $D_h u_j$, $D_v u_j$, $D_h T_j h_j$, and $D_v T_j h_j$ [4], defined as:

$$D_h u_j = -\nu_u \Delta_h^2 u_j, \quad D_v u_j = K_j^m \Delta_v u_j$$

$$D_h T_j h_j = \nu_T \Delta_h T_j h_j, \quad D_v T_j h_j = K_j^t \Delta_v T_j h_j$$

- for $j = \{a,o\}$, where the various values are obtained from [7,9].

# Bulk Condition

- The so called *bulk condition* is a set of Robin boundary conditions [3,10] that account for a large jump in the density between layers.

$$\rho_a K_a^m \partial_z u_a (\hat{z} \cdot \hat{n}_a) = \rho_o K_o^m \partial_z u_o (\hat{z} \cdot \hat{n}_o) = \tau \quad \text{on } \Gamma \times [0, \mathcal{T}]$$

$$\rho_a c_a^p K_a^t \partial_z (T_a h_a)(\hat{z} \cdot \hat{n}_a) = \rho_o c_o^p K_o^t \partial_z (T_o h_o)(\hat{z} \cdot \hat{n}_o) = Q \quad \text{on } \Gamma \times [0, \mathcal{T}]$$

$$\tau = \rho_a C_D ||\Delta U|| (u_a - u_o), \quad Q = \rho_a c_a^p C_H ||\Delta U|| (T_a h_a - T_o h_o)$$

| Parameter(s) | Value | Units | Description |
|---|---|---|---|
| $z$ | N/A | m | Positive Cartesian "upwards" direction |
| $\hat{z}$ | 1 | m/s | Unit vector in the $z$ direction |
| $\Gamma$ | N/A | N/A | Air-sea interface |
| $\hat{n}_a$, $\hat{n}_o$ | 1, −1 | m/s | Unit normal vectors w.r.t. $\Gamma$ such that $\hat{n}_a = -\hat{n}_o$ |
| $\tau$ | N/A | kg/(m$^2$s) | Surface wind stress |
| $\mathcal{T}$ | N/A | s | Final time |
| $Q$ | N/A | J/m$^2$ | Heat flux |
| $\rho_a$, $\rho_o$ | N/A | kg/m$^3$ | Densities |
| $u_a$, $u_o$ | N/A | m/s | Horizontal velocities |
| $T_a$, $T_o$ | N/A | K | Temperatures |
| $h_a$, $h_o$ | N/A | m | Layer heights |
| $K_a^m$, $K_o^m$ | $10^{-5}$, $10^{-5}$ | m$^2$/s | Eddy viscosities |
| $K_a^t$, $K_o^t$ | $10^{-4}$, $10^{-4}$ | m$^2$/s | Eddy diffusivities |
| $\nu_u$ | $5 \times 10^{13}$ | m$^4$/s | Scaling constant |
| $\nu_T$ | $10^5$ | m$^2$/s | Scaling constant |
| $c_a^p$, $c_o^p$ | 1000, 4190 | J/(kg K) | Specific heats |
| $C_D$, $C_H$ | $10^{-3}$, $10^{-3}$ | N/A | Friction parameters |
| $\|\|\Delta U\|\|$ | $((u_o - u_a)^2)^{1/2}$ | m/s | Exponents are entry-wise operators |

# Bulk Condition

- For our example these four terms simplify to:

$$\text{Wind forcing for air: } K_a^m \partial_z u_a = -\tau/\rho_a$$

$$\text{Wind forcing for ocean: } K_o^m \partial_z u_o = \tau/\rho_o$$

$$\text{Thermal forcing for air: } K_a^t \partial_z (T_a h_a) = -Q/(\rho_a c_a^p)$$

$$\text{Thermal forcing for ocean: } K_o^t \partial_z (T_o h_o) = Q/(\rho_o c_o^p)$$

- Consider taking the discrete Laplacian over *N* layers of the same type of fluid.
- We have *N* variables, one for each layer, and *N*+1 interfaces, where the top and bottom maybe contain boundary conditions.
- Boundary terms for vertical stress (Laplacian) at air-sea interface are modeled as forcing terms in the adjacent layers.

# Bulk Condition

- Consider $F_i$ and $h_j$ for $i$ = 1,2,...,$N$, the functions and heights at the $N$ layers, and consider the discrete partial derivatives:

$$(\partial_z F)_{i+1/2} = \frac{F_i - F_{i+1}}{\frac{1}{2}(h_i + h_{i+1})}$$

- where $i$ = 1/2,3/2,...,$N$+1/2 are the layer interfaces.
- The discrete Laplacians are:

$$(\partial_z \partial_z F)_i = \frac{\partial_z F_{i-1/2} - \partial_z F_{i+1/2}}{h_i}.$$

- For the two values that do not exist: $\partial_z F_{1/2}$ and $\partial_z F_{N+1/2}$, we replace them with the bulk condition.

# Bulk Condition

- The four terms are then simplified as follows:

$$K_a^m \frac{\partial_z u_a}{h_a} = -\frac{C_D}{h_a}||\Delta U||(u_a - u_o)$$
$$K_o^m \frac{\partial_z u_o}{h_o} = \frac{C_D \rho_a}{h_o \rho_o}||\Delta U||(u_a - u_o)$$

$$K_a^t \frac{\partial_z (T_a h_a)}{h_a} = -\frac{C_H}{h_a}||\Delta U||(T_a h_a - T_o h_o)$$
$$K_o^t \frac{\partial_z (T_o h_o)}{h_o} = \frac{\rho_a c_a^p C_H}{\rho_o h_o c_o^p}||\Delta U||(T_a h_a - T_o h_o)$$

- The additional *1/*$h_a$ and *1/*$h_o$ factors come from the vertical viscosity.
- Putting everything together we get:

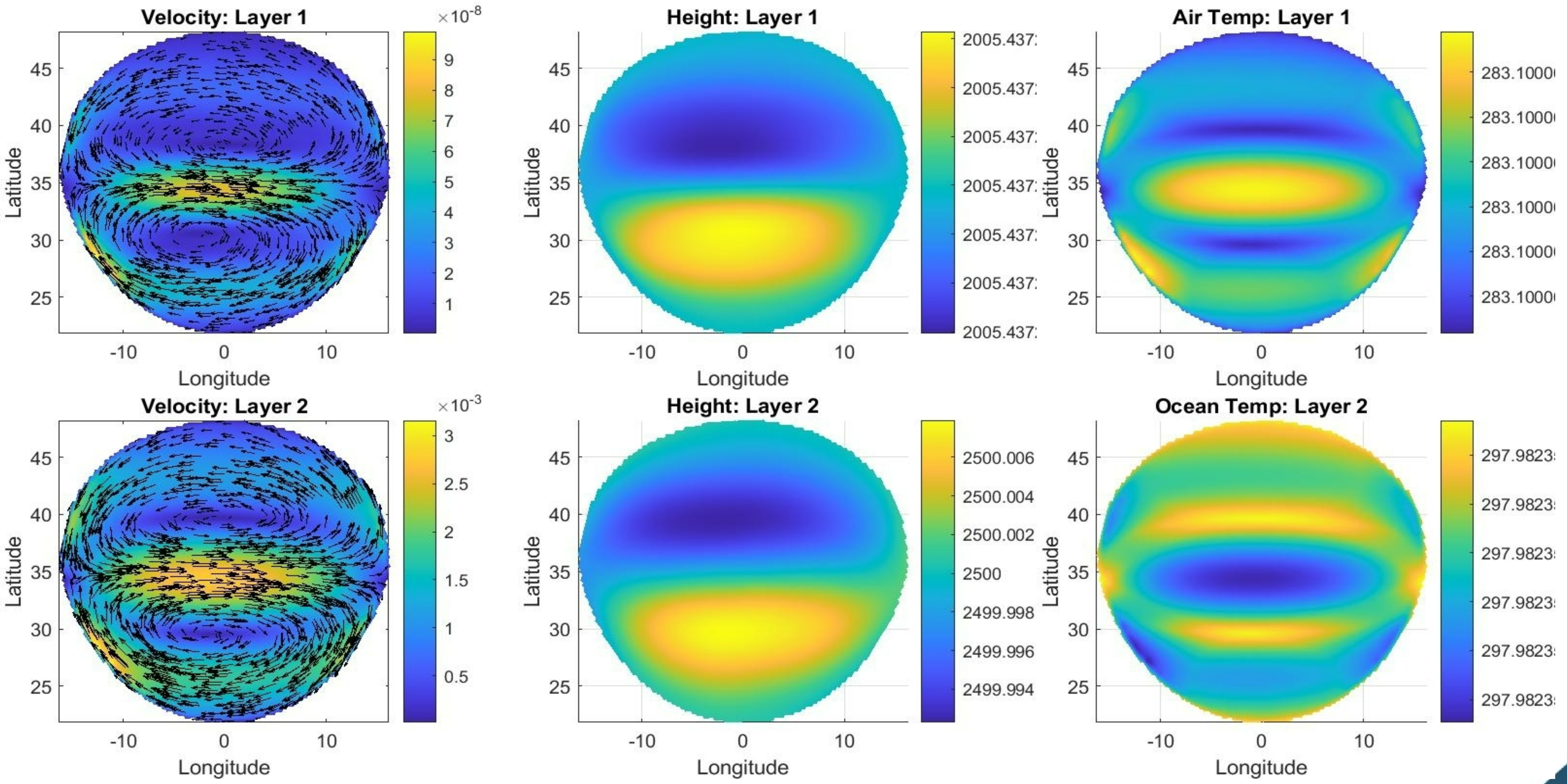
$$\frac{\partial h_j}{\partial t} = -\nabla \cdot h_j u_j$$
$$\frac{\partial u_j}{\partial t} = -q(h_j, u_j)k \times h_j u_j - \nabla K_j - \frac{1}{\rho_j}\nabla\left(\frac{\sigma_j}{2}\right) - \frac{1}{\rho_j}\nabla\left(\sum_{i=1}^{j-1}\sigma_j\right)$$
$$- g\nabla\left(b + \sum_{i=j+1}^{N} h_i + \frac{h_j}{2}\right) + D_h u_j + D_v u_j - F_d + \delta_{o,j}F_w + \delta_{a,j}\frac{\partial_z u_a}{h_a}\bigg|_{\text{top}} + \delta_{o,j}\frac{\partial_z u_o}{h_o}\bigg|_{\text{bottom}}$$
$$\frac{\partial (T_j h_j)}{\partial t} = -\nabla \cdot T_j h_j u_j + D_h T_j h_j + D_v T_j h_j + \delta_{a,j}\frac{\partial_z (T_a h_a)}{h_a}\bigg|_{\text{top}} + \delta_{o,j}\frac{\partial_z (T_o h_o)}{h_o}\bigg|_{\text{bottom}}$$

- where $\delta_{ij}$ is the Kronecker delta.

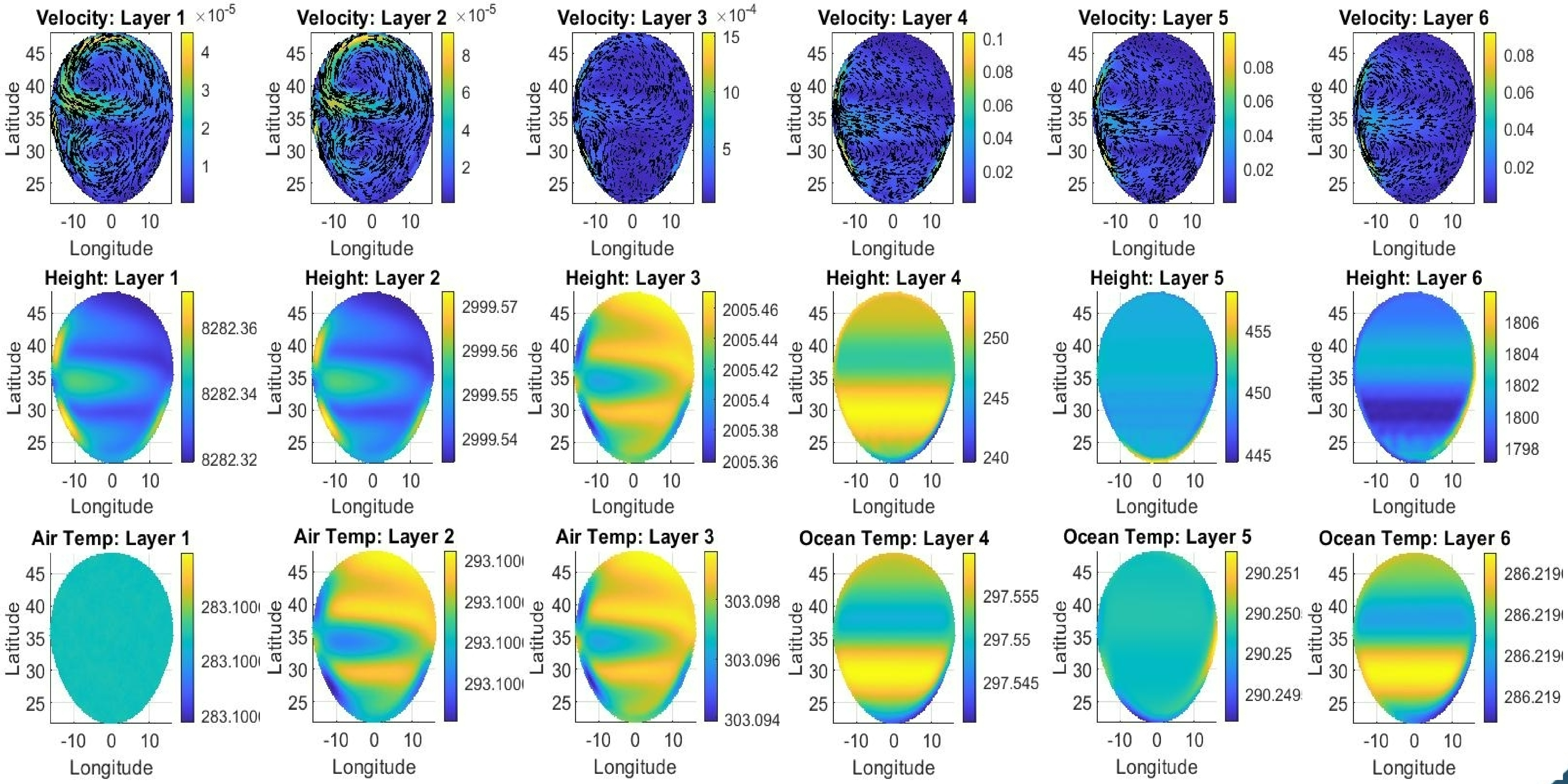
# Results

- We now show some plots of velocities, heights, and temperatures at each layer from the 2 layer model and the 6 layer model.
- In both models, the atmosphere starts at rest and wind forcing $F_w$ is applied to the top ocean layer. The goal is for the ocean layers to create a double gyre in their velocities and induce a similar double gyre in the air layers as a result of the coupling via the bulk condition.
- Both models used a time step of 96 seconds for RK4.
- For the 2 layer model, we show the plots after 1 day.
- For the 6 layer model, we show the plots after 14 days, 270 days and 900 days.
- The parameters in the 6 layer model still need some fine tuning in order for it to work better in the long run.

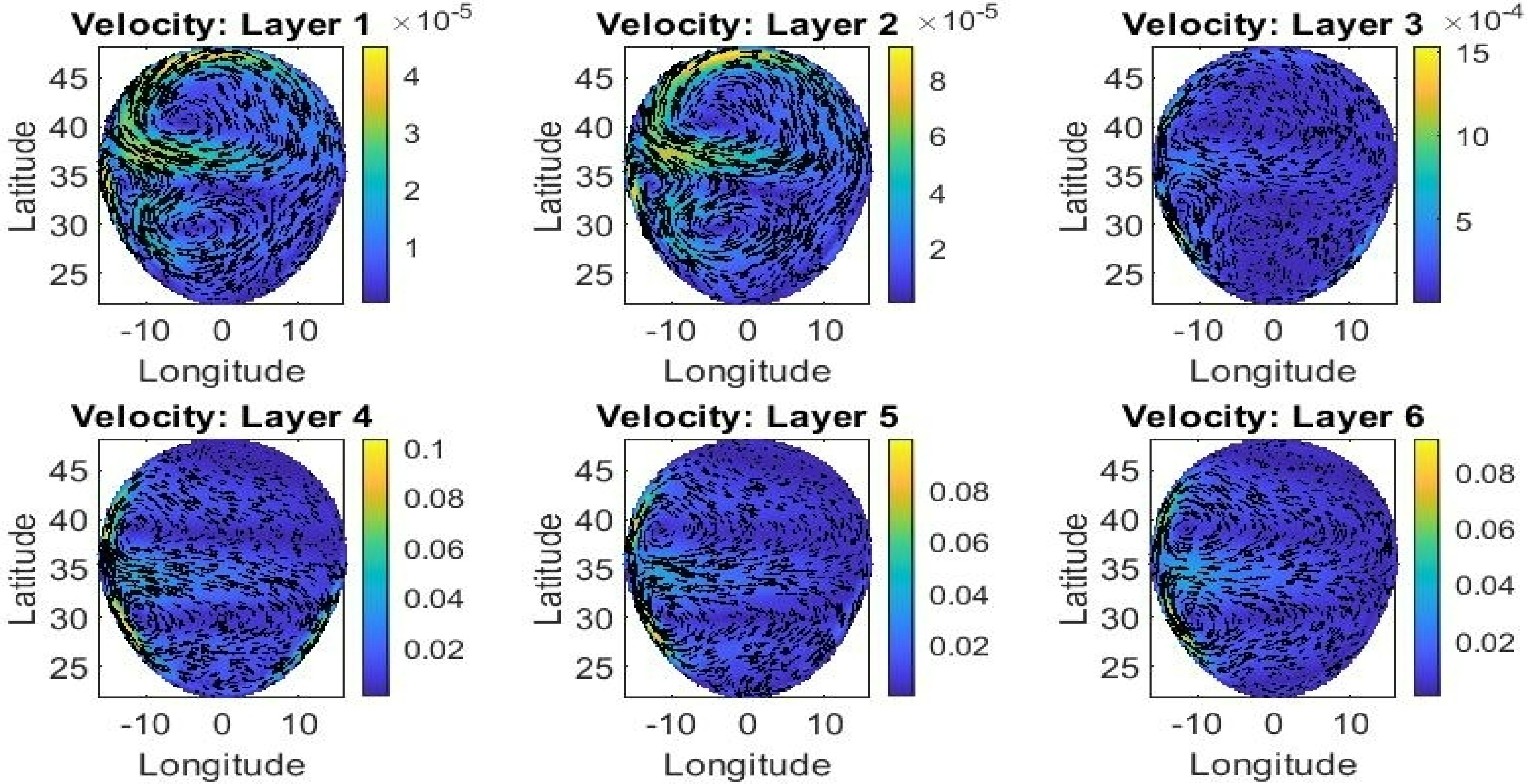
# Results: 2 Layers: 1 Day



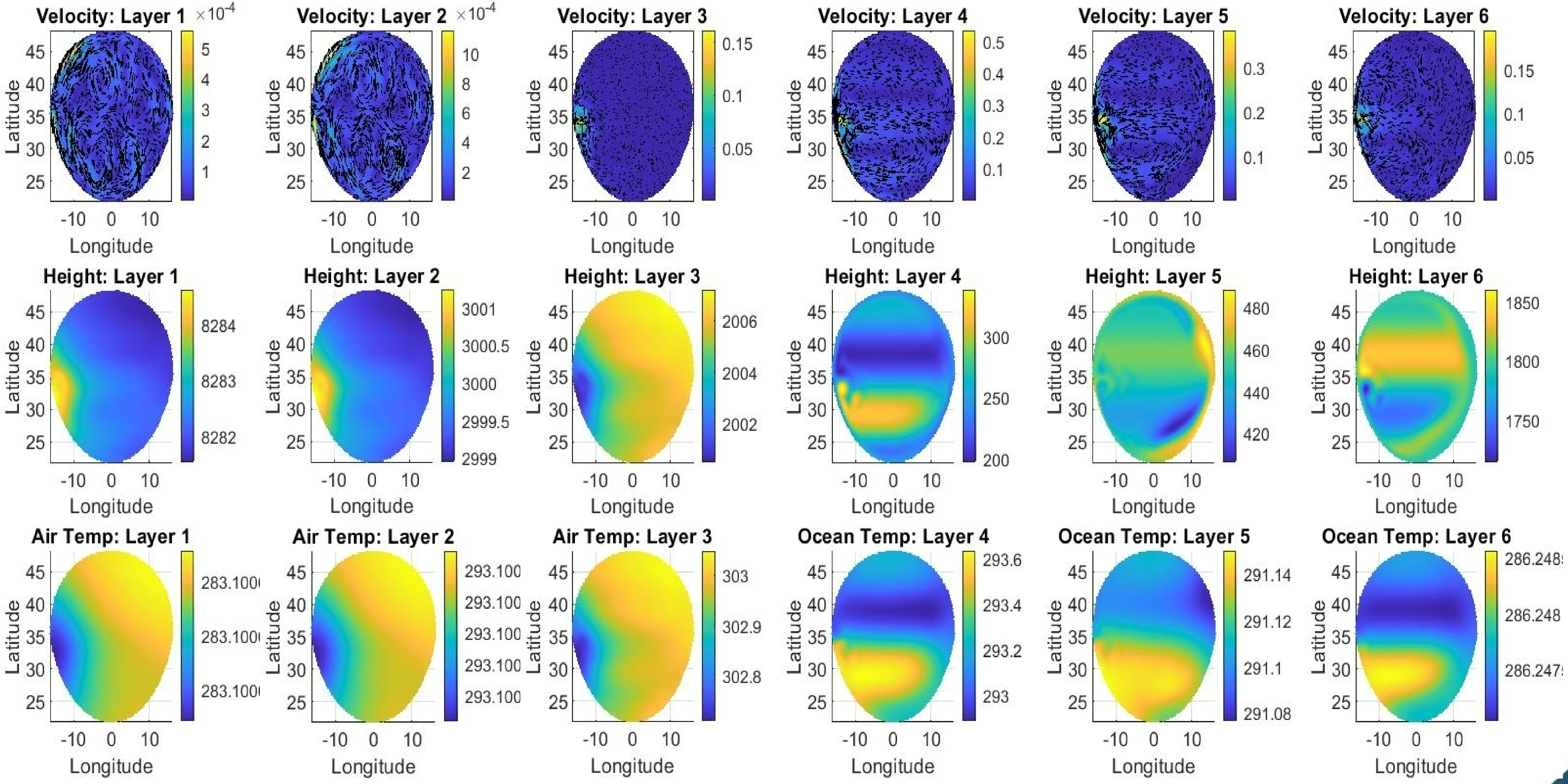
# Results: 6 Layers: 14 Days



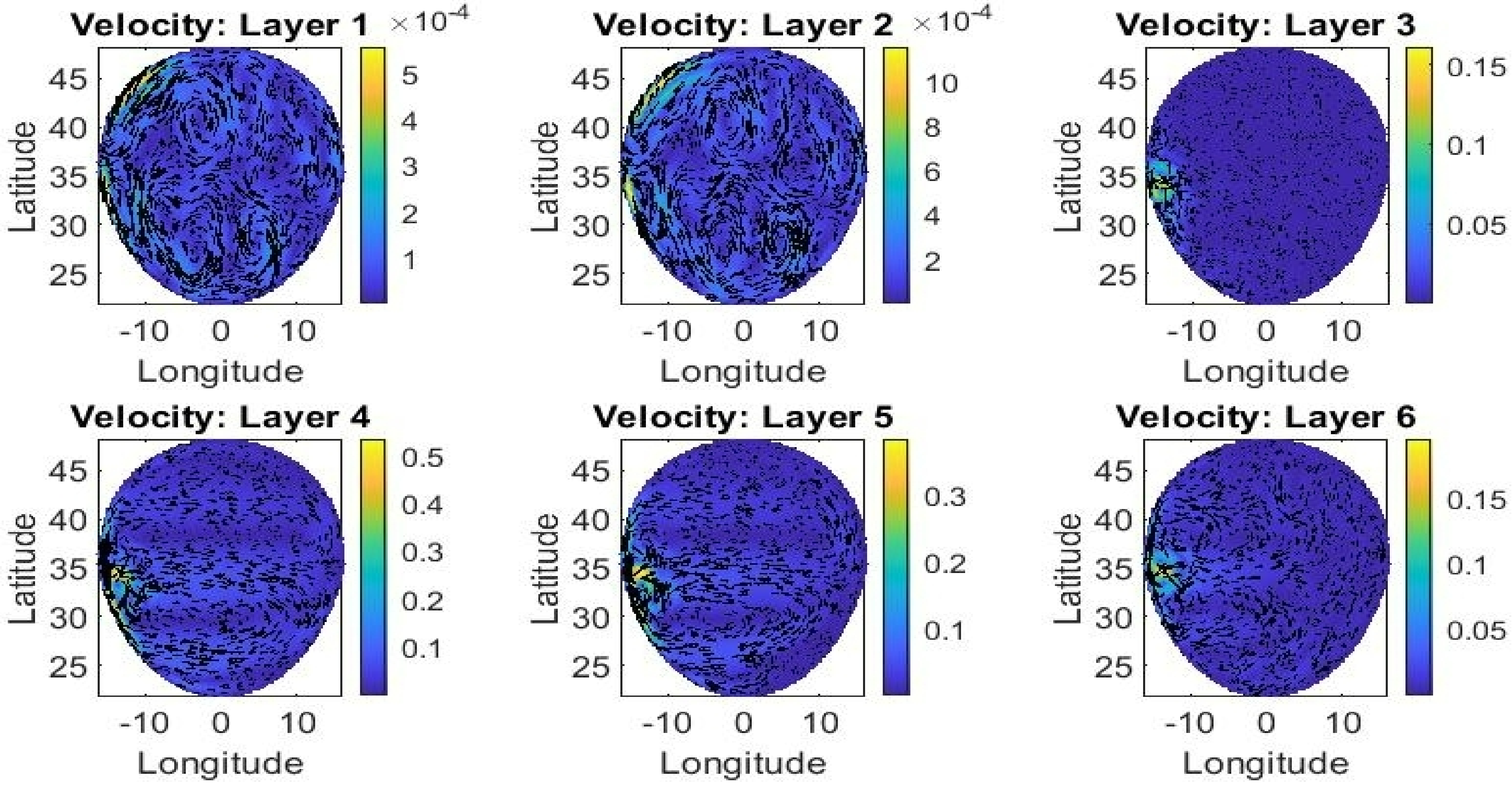
# Results: 6 Layers: 14 Days: Velocities Zoomed In



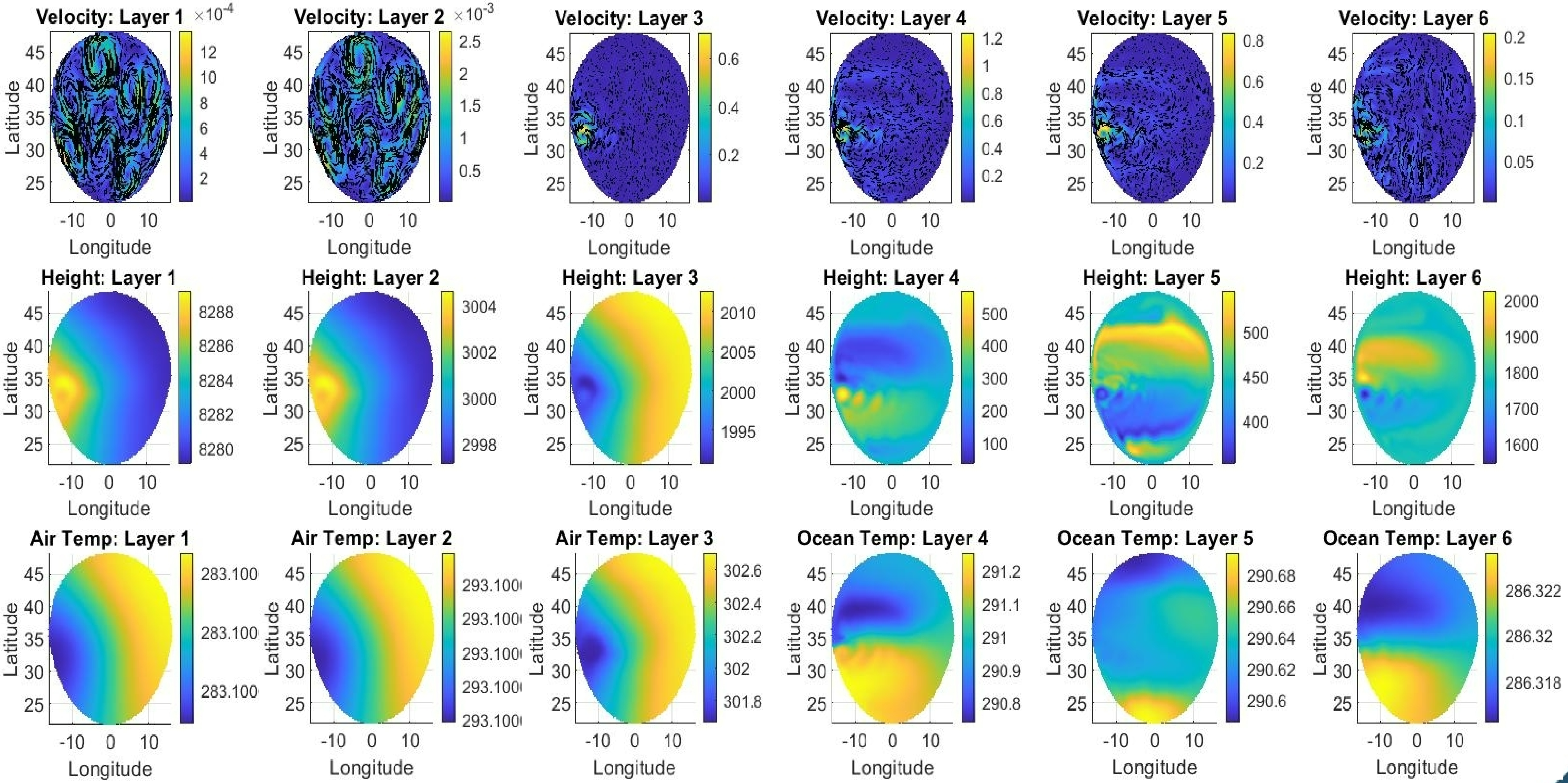
# Results: 6 Layers: 270 Days



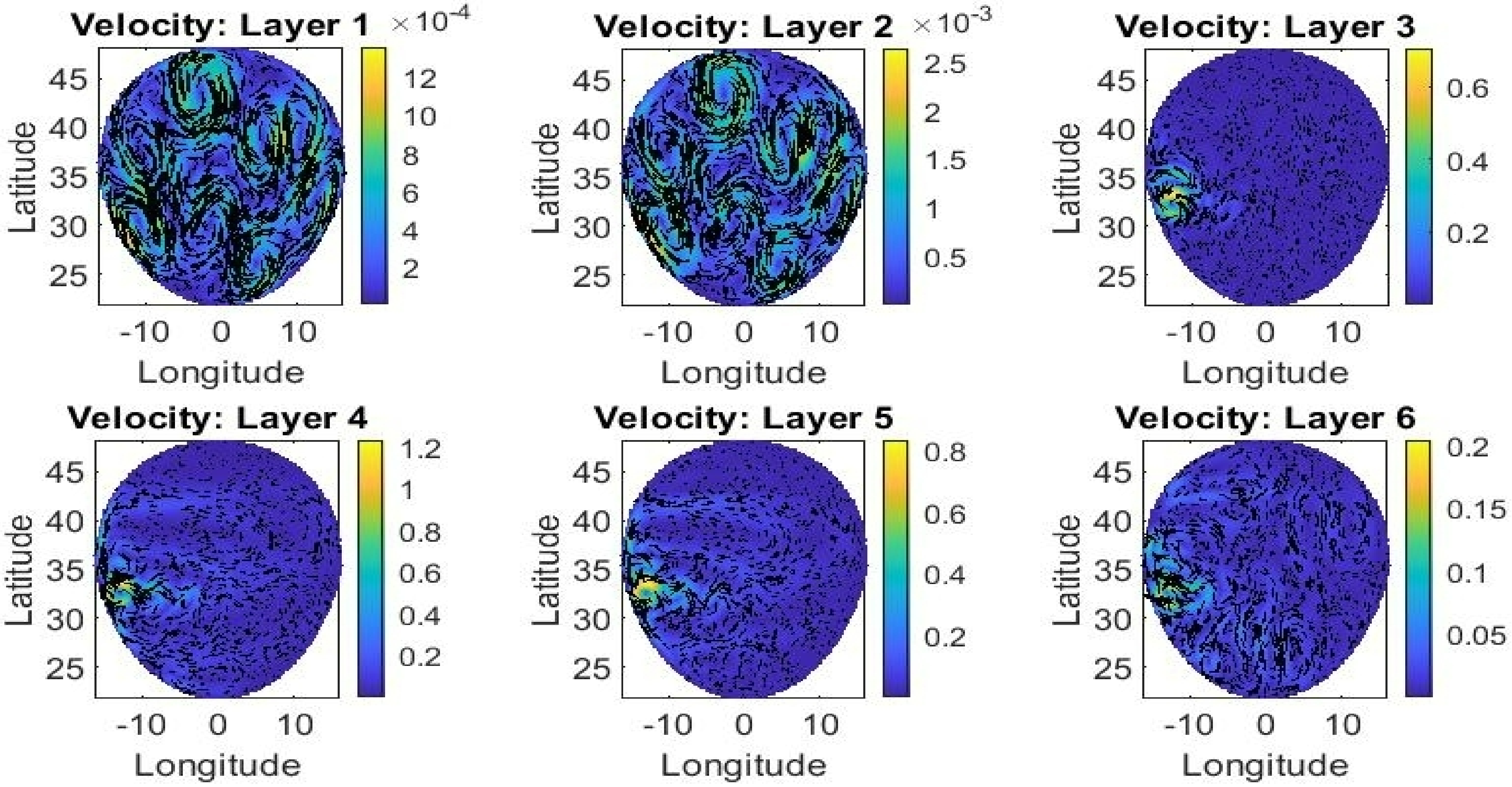
# Results: 6 Layers: 270 Days: Velocities Zoomed In



# Results: 6 Layers: 900 Days



# Results: 6 Layers: 900 Days: Velocities Zoomed In



# Future Work

- We intend to use this code to test new coupling algorithms such as in [1].
- We intend to use the linearized Ideal Gas Law for our air temperature equation of state.
- We wish to fine tune the 6 layer model.
- We intend to modify this code to be compatible with reduced order models (ROMs).
- That is, we can use the data from these tests as “snapshots” to create a ROM basis.
- We wish to modify this code so it can take a ROM basis as an input.
- This will then (hopefully) increase efficiency.

# References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