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Time-integration of coupled-component Earth system models via discontinuous-Galerkin-in-time methods

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Introduction

Realistic Earth system models are typically composed of several component models (e.g. atmosphere and ocean) coupled across an interface. Each model component is usually governed by a system of partial differential equations (PDEs) that, when coupled across an interface, results in a monolithic differential-algebraic equation (DAE). While this monolithic approach results in a succinct mathematical formulation, numerical modeling is complicated by physical processes in separate model components that evolve on disparate time-scales. Therefore, traditional time-stepping approaches for such models have been "bottom-up" - an optimized time-stepping method is derived for each component model and a coupling procedure is then derived that links these two methods in a (hopefully) stable and accurate way. An alternative is a "top-down" approach - starting from the coupled DAE problem we derive time-stepping methods that are stable and accurate for the monolithic formulation. We present a methodology for deriving and analyzing methods arising from either the top-down or bottom-up approach by casting time-stepping methods for coupled problems in a discontinuous Galerkin (DG) framework.

Model problem

Let Ω_1, Ω_2 be two open domains with common interface Γ and consider the following advection-diffusion model problem:

$$\begin{cases} \dot{u}_j = \mathcal{L}_j(u_j) + f_j \text{ on } \Omega_j \times (0, t_f] \\ u_j = 0 \text{ on } \Gamma_j \times (0, t_f] \\ T(u_1, u_2, F_{\Gamma,1}, F_{\Gamma,2}) = 0 \text{ on } \Gamma \times (0, t_f] \\ u_j(x, 0) = u_j^0(x) \text{ on } \Gamma_2 \end{cases}$$

$\dot{u}_j = \frac{d}{dt} u_j$, f_j - forcing functions

$F_{\Gamma,j}$ - oriented fluxes

$\mathcal{L}_j = \nabla \cdot (\nu_j \nabla u_j - s_j u_j)$

$\nu_j > 0$, s_j - steady advection field

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - C \begin{bmatrix} F_{\Gamma,1} \\ F_{\Gamma,2} \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$B, C \in \mathbb{R}^{2\times 2}$, g_1, g_2 - prescribed forcings

We assume flux conservation:

$$F_{\Gamma,1} + F_{\Gamma,2} = 0 \text{ a.e. on } \Gamma \times (0, t_f]$$

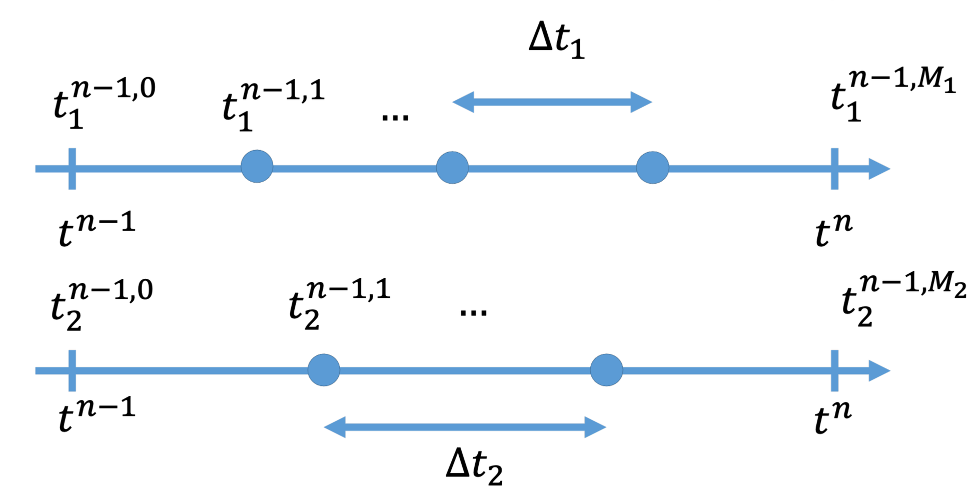
This general form incorporates both jump-type and Robin boundary conditions
and is useful for analysis.

For suitably defined finite element spaces U_1, U_2, U_Γ, the spatially discrete problems is to find functions $u_j: [0, t_f] \to U_j$, $u_{\Gamma,j}: [0, t_f] \to U_\Gamma$ and fluxes $F_{\Gamma,j}: [0, t_f] \to U_\Gamma$ so that:

$$\begin{cases} (\dot{u}_j, v_j) = -L(u_j, v_j) - (F_{\Gamma,j}, v_j)_\Gamma + (f_j, v_j) \quad \forall u_j \in U_j, t \in [0, t_f] \\ (u_j(0), v_j) = (u_j^0, v_j) \quad \forall v_j \in U_j \\ (u_{\Gamma,j}, \mu_j)_\Gamma = (u_j, \mu_j)_\Gamma \quad \forall \mu_j \in U_\Gamma, t \in (0, t_f] \end{cases}$$

DG in time formulation

We want method that can handle different sub-steps $\Delta t_1, \Delta t_2 > 0$ for each model component. To ensure accuracy at the coupled model time-step $\Delta t > 0$ we enforce that there exists integers $M_1, M_2 > 0$ so that $M_1 \Delta t_1 = \Delta t = M_2 \Delta t_2$. We illustrate this in the following diagram:



Associated to this sub-stepping structure are intervals $I^n = (t^{n-1}, t^n)$ and $I_j^{n,k} = (t_j^{n,k-1}, t_j^{n,k})$ where $t^n = t^{n-1} + \Delta t$ and $t_j^{n,k} = t_j^{n,k-1} + \Delta t_j$. Our DG approach is to first define spaces U_j^q and V^q to be the spaces of all U_j and U_Γ valued functions whose restriction on each interval $I_j^{n,k}$ and I^n is a polynomial of order q in time. Second, time-stepping is defined from these spaces by finding $\tilde{u}_j \in U_j^q, \tilde{U}_j^{n,k} \in U_j, \tilde{u}_{\Gamma,j} \in V^{r_j}, \tilde{F}_{\Gamma,j} \in V^{r_j}$ so that:

$$\begin{cases} \sum_{k=1}^{M_j} (\tilde{U}_j^{n,k}, \tilde{v}_j^-(t_j^{n,k})) - ((\tilde{u}_j, \dot{\tilde{v}}_j))^n - \sum_{k=1}^{M_j} (\tilde{U}_j^{n,k-1}, \tilde{v}_j^+(t_j^{n,k-1})) \\ \qquad = -L(\tilde{u}_j, \tilde{v}_j)^n - ((\tilde{F}_{\Gamma,j}, \tilde{v}_j))_\Gamma^n + ((f_j, \tilde{v}_j))^n, \\ \qquad \forall \tilde{v}_j \in U_j^{q+1-n_s}, \\ (\tilde{U}_j^{1,0}, v_j) = (u_j^0, v_j), \quad \forall v_j \in U_j \\ ((\tilde{u}_{\Gamma,j}, \mu_j))_\Gamma^n = ((\tilde{u}_j, \mu_j))_\Gamma^n, \quad \forall \mu_j \in V^{r_j} \end{cases}$$

where $\tilde{u}_j|_{I_j^{n,k}} \approx u_j|_{I_j^{n,k}}, \quad \tilde{U}_j^{n,k} \approx u_j|_{t=t_j^{n,k}}, \quad \tilde{u}_{\Gamma,j}|_{I^n} \approx u_{\Gamma,j}|_{I^n}, \quad \tilde{F}_{\Gamma,j}|_{I^n} \approx F_{\Gamma,j}|_{I^n}$

This formulation includes standard multirate Runge-Kutta and linear multistep methods as special cases. The properties and convergence of this method are investigate in (Connors and Sockwell, 2021).

Results

We show results for a fully coupled, multirate Crank-Nicholson method.

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