Closure theory for high-collisionality multi-ion plasmas

Jeong-Young Ji

*Department of Physics, Utah State University, Logan, Utah 84322, USA**

(Dated: July 20, 2023)

Abstract

A general formalism is developed to construct and solve a system of linearized moment equations for parallel and perpendicular closures in high-collisionality plasmas. It is applicable for multiple ion species with arbitrary masses, temperatures, charges, and densities. The convergence of closure coefficients is evaluated by increasing the number of moments from 2 to 32 for scalar, vector, and rank-2 tensor moments. As an example, the complete set of closure coefficients for a deuterium-carbon plasma over the entire Hall parameter range is presented. The closure coefficients at various temperature ratios show that one-temperature closure coefficients can differ significantly from two-temperature coefficients.

* j.ji@usu.edu

I. INTRODUCTION

Due to its low dimensionality, a fluid model of a plasma has a significant advantage over a kinetic model, especially as the number of plasma species increases. A fluid model consists of fluid variables and the governing equations that describe their behavior. Since the governing equations involve additional moments referred to as closures, a fluid model requires closure relations that connect the additional moments to the fluid variables.

A straightforward closure scheme is to expand the fluid system by including the evolution equations of the closure moments (See Refs. [1–6] and references therein). Explicit formulas of writing an infinite hierarchy of moment equations are developed in Refs. [7, 8]. The fluid system can be further expanded if it is computationally feasible, however, the computational effort increases as the number of moments increases and the advantage of low dimensionality may be lost when the number of fluid moments becomes comparable to the number of velocity space grid points required to solve the kinetic equation. On the other hand, the asymptotic closure scheme (see p. 216, [9]) allows for the approximate solution of the added moment equations for closures instead of numerically advancing the full equations. This closure scheme provides analytic quantitative closures that are rigorously accurate up to the expansion order of a small parameter, if such a parameter exists.

In the five-moment fluid model, closure quantities: the heat flux density $\mathbf{h}_a$, viscosity $\boldsymbol{\pi}_a$, friction force density $\mathbf{R}_a$, and collisional heating Q_a, are expressed in terms of fluid quantities: density n_a, temperature T_a, and flow velocity $\mathbf{V}_a$, where the subscript a denotes a plasma species. The closure calculations are based on a linear response theory, where closure relations are determined by closure coefficients that connect the closures to drives: the temperature gradient ∇T_a, temperature difference $T_{ab} = T_a - T_b$, velocity gradient $\nabla \mathbf{V}_a$, and flow-velocity difference $\mathbf{V}_{ab} = \mathbf{V}_a - \mathbf{V}_b$. For high collisionality [Knudsen number $\epsilon \sim$ (mean free path)/(macroscopic scale length) $\ll 1$], closure relations for the two-fluid (electrons and ions) five-moment system are developed in Refs. [10, 11] and improved in Refs. [12, 13]. For multiple ion species, additional moment equations are added to the fluid system to construct systems with up to 29 moments [3–6]. Braginskii type of closures for two or several ion species with *disparate* masses are obtained in Refs. [14–16] when one ion mass is much larger than the others. Parallel closure coefficients are obtained for multiple ion species with *general* masses in Refs. [17, 18].

Due to its simplicity, the calculation of collisional moments is often performed assuming a single temperature for multiple ion species [5, 17, 19]. The choice of one single equilibrium temperature seems acceptable because the system approaches thermal equilibrium at that temperature. However, when the equilibration time is much larger than the time scale of interest, or when different temperatures are maintained by external heating sources, the closure coefficients calculated from one-temperature collision coefficients can be significantly different from those calculated from two-temperature collision coefficients. Additionally, the temperature used for normalizing the velocity variable in the moment expansion should be appropriate. The moment expansion will not converge if the temperature used, say T_0, is less than half the species temperature, say $T_b/2$. This can be understood from $\exp(-mv^2/2T_b) = w\chi$. With the weight function $w = \exp(-mv^2/2T_0)$, the function $\chi = \exp(-mv^2/2T_b + mv^2/2T_0)$ is *not* square-integrable if $T_0 < T_b/2$, and the polynomial expansion of χ fails to converge. As a result, using the equilibrium temperature $T_0 = (n_aT_a + n_bT_b)/(n_a + n_b)$ in the moment expansion of species b is acceptable only if $T_0 > T_b/2$. This condition can be easily violated, for example, when dealing with a minor species b ($n_b \ll n_a$) if $T_b > 2T_a$, as $T_b/2 > T_a \approx T_0$. The 21-moment multi-temperature closure coefficients are calculated in Ref. [18, 20], albeit for *parallel* closures only.

In this work, we develop a general formulation to derive closures for high-collisionality electron-multi-ion plasmas. We provide a general method for calculating parallel and *perpendicular* closures that are valid for arbitrary masses, temperatures, charges, and densities in a general magnetic field. Additionally, we verify the *convergence* of the closure coefficients by increasing the number of moments up to 32 scalars (1 independent variable), 32 vectors (3), and 32 rank-2 tensors (5), which corresponds to the 293 [$= 5+32\times(1+3+5)$] moment system for each species.

The general moment equations for closures are constructed in Sec. II. The formulation to derive closures for multi-ion plasmas is developed for electrons in Sec. III and for ions in Sec. IV. In Sec. V, we apply the formulation to a two-ion plasma and present the complete set of closure coefficients for a deuterium-carbon plasma. Sec. VI is devoted to discussion.

II. FLUID AND CLOSURE MOMENT EQUATIONS

For a plasma with multiple ion species, a distribution function f_a with $a = \mathrm{e}$ (electrons), 1 (first ion species), 2 (second ion species), etc. is governed by the kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \nabla f_a + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b), \tag{1}$$

where $C(f_a, f_b)$ is the Landau collision operator. In the five-moment model, the density n_a, flow velocity $\mathbf{V}_a$, and temperature T_a are defined from the distribution function f_a as: $n_a = \int d\mathbf{v} f_a$, $\mathbf{V}_a = n_a^{-1} \int d\mathbf{v} \mathbf{v} f_a$, and $T_a = n_a^{-1} \int d\mathbf{v} (m_a w_a^2/3) f_a$, where $\mathbf{w}_a = \mathbf{v} - \mathbf{V}_a$, the random velocity. Their evolution equations are obtained by taking the corresponding moments of the kinetic equation (1):

$$d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0, \tag{2}$$

$$m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a, \tag{3}$$

$$\frac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a, \tag{4}$$

where $d_a = \partial/\partial t + \mathbf{V}_a \cdot \nabla$. The system of fluid equations (2)–(4) needs to be closed by expressing the heat flux density $\mathbf{h}_a$, viscosity $\boldsymbol{\pi}_a$, collisional heating Q_a, and friction force density $\mathbf{R}_a$ in terms of n_a, $\mathbf{V}_a$, and T_a. Here the closure quantities are defined as

$$\mathbf{h}_a = \int d\mathbf{v} \tfrac{1}{2} m_a w_a^2 \mathbf{w}_a f_a, \tag{5}$$

$$\boldsymbol{\pi}_a = \int d\mathbf{v} m_a (\mathbf{w}_a \mathbf{w}_a - \tfrac{1}{3} w_a^2 \mathsf{I}) f_a, \tag{6}$$

$$Q_a = \int d\mathbf{v} \tfrac{1}{2} m_a w_a^2 \sum_b C(f_a, f_b), \tag{7}$$

and

$$\mathbf{R}_a = \int d\mathbf{v} m_a \mathbf{w}_a \sum_b C(f_a, f_b). \tag{8}$$

In order to obtain the closure relations for the system of fluid equations (2)–(4), we construct and solve a system of moment equations for higher order (non-Maxwellian) moments where Maxwellian moments appear as source (drive) terms of the system. For the moment equations we expand a distribution function as

$$f_a = f_a^{\mathrm{M}} \sum_{lq} \hat{\mathsf{p}}_a^{lq} \cdot \mathsf{m}_a^{lq}, \tag{9}$$

where $f_a^{\rm M} = \left(n_a/\pi^{3/2}v_{Ta}^3\right)e^{-c_a^2}$ (a Maxwellian distribution), $v_{Ta} = \sqrt{2T_a/m_a}$, $\mathbf{c}_a = (\mathbf{v} - \mathbf{V}_a)/v_{Ta}$, and

$$\hat{\mathsf{p}}_a^{lq} = \frac{1}{\sqrt{\sigma_{lq}}}\mathsf{P}^l(\mathbf{c}_a)L_q^{(l+1/2)}(c_a^2). \tag{10}$$

Here P^l is a symmetric traceless rank-l tensor, $L_q^{(l+1/2)}$ is an associated Laguerre polynomial, and the normalization constants are

$$\sigma_{lq} = \sigma_l\lambda_{lq},\ \sigma_l = \frac{l!}{(2l+1)!!},\ \lambda_{lq} = \frac{(l+q+1/2)!}{q!(1/2)!}. \tag{11}$$

The moments are the expansion coefficients in Eq. (9),

$$\mathsf{m}_a^{lq} = n_a^{-1}\int d\mathbf{v}\hat{\mathsf{p}}_a^{lq}f_a. \tag{12}$$

The closure calculations in this work are based on a linear response theory that closure quantities are linearly connected to drive terms ∇T_a, $\nabla\mathbf{V}_a$, $T_{ab} = T_a - T_b$, and $\mathbf{V}_{ab} = \mathbf{V}_a - \mathbf{V}_b$, which are assumed to be order of ϵ [$\mathcal{O}(\epsilon)$]. The exact collisional moments are calculated in Ref. [21]. Noting that $\mathsf{m}_a^{lq} \sim \mathcal{O}(\epsilon)$, the collisional moments up to $\mathcal{O}(\epsilon)$ can be written as

$$\begin{aligned}\int d\mathbf{v}\hat{\mathsf{p}}_a^{lp}C\left(f_a^{\rm M}\hat{\mathsf{p}}_a^{lq}\cdot\mathsf{m}_a^{lq}, f_b^{\rm M}\right) &= \frac{n_a}{\tau_{ab}}a_{ab}^{lpq}\mathsf{m}_a^{lq},\\ \int d\mathbf{v}\hat{\mathsf{p}}_a^{lp}C\left(f_a^{\rm M}, f_b^{\rm M}\hat{\mathsf{p}}_b^{lq}\cdot\mathsf{m}_b^{lq}\right) &= \frac{n_a}{\tau_{ab}}b_{ab}^{lpq}\mathsf{m}_b^{lq},\end{aligned} \tag{13}$$

where a_{ab}^{lpq} and b_{ab}^{lpq} are related to A_{ab}^{lpq} and B_{ab}^{lpq} of Ref. [7]: $a_{ab}^{lpq} = A_{ab}^{lpq}\tau_{ab}/n_a\sqrt{\lambda_{lp}\lambda_{lq}}$ and $b_{ab}^{lpq} = B_{ab}^{lpq}\tau_{ab}/n_a\sqrt{\lambda_{lp}\lambda_{lq}}$. Here the collision time is defined as $\tau_{ab} = 6\pi^{3/2}\varepsilon_0^2 m_aT_av_{Ta}/n_bq_a^2q_b^2\ln\Lambda_{ab}$, and the Coulomb logarithm, $\ln\Lambda_{ab}$, is defined by $\Lambda_{ab} = 12\pi\varepsilon_0(m_aT_b + m_bT_a)\lambda_{\rm D}/|q_aq_b|(m_a + m_b)$ with $\lambda_{\rm D} = \left(\sum_a n_aq_a^2/\varepsilon_0T_a\right)^{-1/2}$ being the Debye length. Several lowest order coefficients a_{ab}^{lpq} and b_{ab}^{lpq} necessary for closure calculations are given in Appendix A. The collisional heating (7) and friction (8) up to $\mathcal{O}(\epsilon)$ are

$$Q_a = \sum_b\left(Q_{ab}^{\rm M} + Q_{ab}^{\rm N}\right),\ Q_{ab}^{\rm M} = \frac{n_a}{\tau_{ab}}\frac{3X_{ab}^{3/2}}{\mu_{ab}}T_{ba}, \tag{14}$$

$$\mathbf{R}_a = \sum_b\left(\mathbf{R}_{ab}^{\rm M} + \mathbf{R}_{ab}^{\rm N}\right),\ \mathbf{R}_{ab}^{\rm M} = \frac{m_an_a}{\tau_{ab}}X_{ab}^{3/2}\frac{1+\mu_{ab}}{\mu_{ab}}\mathbf{V}_{ba}, \tag{15}$$

where $X_{ab} = (1 + \theta_{ab}/\mu_{ab})^{-1}$, $\mu_{ab} = m_b/m_a$, $\theta_{ab} = T_b/T_a$. Here the non-Maxwellian contributions, $Q_{ab}^{\rm N}$ and $\mathbf{R}_{ab}^{\rm N}$, can be written as

$$Q_{ab}^{\rm N} = -\sqrt{\frac{3}{2}}\frac{n_aT_a}{\tau_{ab}}\sum_{q=2}(a_{ab}^{01q}\mathsf{m}_a^{0q} + b_{ab}^{01q}\mathsf{m}_b^{0q}), \tag{16}$$

$$\mathbf{R}_{ab}^{\rm N} = \frac{m_an_av_{Ta}}{\sqrt{2}\tau_{ab}}\sum_{q=1}(a_{ab}^{10q}\mathsf{m}_a^{1q} + b_{ab}^{10q}\mathsf{m}_b^{1q}). \tag{17}$$

Here the coefficients can be explicitly written as

$$a_{ab}^{01q} = \sqrt{\frac{3\,(q-1/2)!}{(q+1/2)q!(-1/2)!}} X_{ab}^{q+3/2} \frac{\mu_{ab} - (2q+1)\theta_{ab}\mu_{ab} - 2q\theta_{ab}}{\mu_{ab}^2}, \tag{18}$$

$$b_{ab}^{01q} = \sqrt{\frac{3\,(q-1/2)!}{(q+1/2)q!(-1/2)!}} X_{ab}^{q+3/2} \left(\frac{\theta_{ab}}{\mu_{ab}}\right)^q \frac{2q\mu_{ab} - \theta_{ab} + 2q + 1}{\mu_{ab}}, \tag{19}$$

$$a_{ab}^{10q} = -\sqrt{\frac{3\,(q+1/2)!}{(q+3/2)q!(-1/2)!}} X_{ab}^{q+3/2} \frac{1+\mu_{ab}}{\mu_{ab}}, \tag{20}$$

$$b_{ab}^{10q} = \sqrt{\frac{3\,(q+1/2)!}{(q+3/2)q!(-1/2)!}} X_{ab}^{q+3/2} \frac{1+\mu_{ab}}{\mu_{ab}} \left(\frac{\theta_{ab}}{\mu_{ab}}\right)^{q+1/2}. \tag{21}$$

For high collisionality, the $\partial f_a^{\rm N}/\partial t$, $\mathbf{v}\cdot\nabla f_a^{\rm N}$, and $(q_a/m_a)\mathbf{E}\cdot\partial_{\mathbf{v}} f_a^{\rm N}$ terms may be assumed to be $\mathcal{O}(\epsilon^2)$ and ignored, where $f_a^{\rm N} = f_a - f_a^{\rm M}$ is the non-Maxwellian part of a distribution f_a. Then the equations for non-Maxwellian moments $(l,p) \notin {\rm M} = \{(0,0),(0,1),(1,0)\}$ can be written as

$$n_a \Omega_a \mathbf{b} \breve{\times} \mathsf{m}_a^{lp} = \sum_{b,q} \frac{n_a}{\tau_{ab}} \left(a_{ab}^{lpq} \mathsf{m}_a^{lq} + b_{ab}^{lpq} \mathsf{m}_b^{lq} \right) + \mathcal{G}_a^{lp}. \tag{22}$$

Here $\Omega_a = q_a B/m_a$ is the cyclotron frequency, and $\breve{\times}$ denotes the generalized cross product defined for a rank-l tensor m^l: $\mathbf{b}\breve{\times}\mathsf{m}^0 = 0$ for a scalar m^0, $\mathbf{b}\breve{\times}\mathsf{m}^1 = \mathbf{b}\times\mathsf{m}^1$ for a vector m^1, and

$$(\mathbf{b}\breve{\times}\mathsf{m}^l)_{\alpha\beta\cdots\gamma} = \epsilon_{\alpha\mu\nu} b_\mu m^l_{\nu\beta\cdots\gamma} + \epsilon_{\beta\mu\nu} b_\mu m^l_{\alpha\nu\cdots\gamma} + \cdots + \epsilon_{\gamma\mu\nu} b_\mu m^l_{\alpha\beta\cdots\nu} \tag{23}$$

for a rank $l \geq 2$ tensor, where the summation convention is understood for repeated indices. The drive terms are defined by

$$\mathcal{G}_a^{lp} = \int d\mathbf{v}\,\hat{\mathsf{p}}_a^{lp} [-\mathbf{v}\cdot\nabla f_a^{\rm M} + \sum_b C(f_a^{\rm M}, f_b^{\rm M})]. \tag{24}$$

The nonvanishing drive terms are

$$\mathcal{G}_a^{0p} = n_a \sum_{b\neq a} \frac{a_{ab}^{0p0}}{\tau_{ab}} = n_a \sum_{b\neq a} \frac{\alpha_{ab}^p}{\tau_{ab}} \frac{T_{ab}}{T_a}, \tag{25}$$

$$\mathcal{G}_a^{1p} = \delta_{p1} \frac{\sqrt{5}}{2} n_a v_{Ta} \nabla \ln T_a + \sqrt{2} n_a \sum_{b\neq a} \frac{b_{ab}^{1p0}}{\tau_{ab}} \frac{\mathbf{V}_{ba}}{v_{Tb}}, \tag{26}$$

$$\mathcal{G}_a^{20} = -\frac{1}{\sqrt{2}} n_a \mathsf{W}_a, \tag{27}$$

where $(\mathsf{W})_{\alpha\beta} = \partial V_\beta/\partial x_\alpha + \partial V_\alpha/\partial x_\beta - (2/3)\delta_{\alpha\beta}\nabla\cdot\mathbf{V}$.

The collision coefficients in the $\mathcal{G}_a^{0p}$ drive are written as

$$a_{ab}^{0p0} = \alpha_{ab}^{p}\frac{T_{ab}}{T_a},\ \alpha_{ab}^{p} = \sqrt{\frac{p\left(p-\frac{1}{2}\right)!}{\left(p+1/2\right)\left(p-1\right)!(1/2)!}}\frac{3X_{ab}^{p+1/2}}{\mu_{ab}}. \tag{28}$$

As the $\mathcal{G}_a^{0p}$ drives are proportional to T_{ab}, the temperature relaxation rate in Eq. (14) will be modified by Q_{ab}^{N}. This modification is negligible in electron-ion plasmas due to the small mass-ratio of electrons to ions. The coefficients of the linearized collisional moments in the $\mathcal{G}_a^{1p}$ drive can be written as

$$b_{ab}^{1p0} = \sqrt{\frac{3(p+1/2)!}{(2p+3)p!(1/2)!}}X_{ab}^{p+\frac{3}{2}}\frac{v_{Tb}}{v_{Ta}}\frac{(\mu_{ab}-2p\theta_{ab}+2p+1)}{\mu_{ab}}. \tag{29}$$

Once Eq. (32) has been solved for the moments, the closure quantities can be obtained from the relations to the moments:

$$\mathbf{h}_a = -\frac{\sqrt{5}}{2}n_aT_av_{Ta}\mathsf{m}_a^{11}, \tag{30}$$

$$\boldsymbol{\pi}_a = \sqrt{2}n_aT_a\mathsf{m}_a^{20}, \tag{31}$$

Eq. (16), and Eq. (17).

When solving Eq. (22) for each l, we consider a truncated system of equations for K moments,

$$\mathsf{m}_a^0 = \begin{pmatrix} \mathsf{m}_a^{02} \\ \mathsf{m}_a^{03} \\ \vdots \\ \mathsf{m}_a^{0,K+1} \end{pmatrix},\ \mathsf{m}_a^1 = \begin{pmatrix} \mathsf{m}_a^{11} \\ \mathsf{m}_a^{12} \\ \vdots \\ \mathsf{m}_a^{1K} \end{pmatrix},\ \mathsf{m}_a^{l\geq 2} = \begin{pmatrix} \mathsf{m}_a^{l0} \\ \mathsf{m}_a^{l1} \\ \vdots \\ \mathsf{m}_a^{l,K-1} \end{pmatrix}.$$

Multiplying $\tau_{aa}n_a^{-1}$ and defining $r_a = \Omega_a\tau_{aa}$, $c_{aa}^l = a_{aa}^l + b_{aa}^l$, $z_{ab} = \tau_{aa}/\tau_{ab}$, and $\mathsf{g}_a^l = \tau_{aa}n_a^{-1}\mathcal{G}_a^l$, we rewrite Eq. (22) in matrix form,

$$r_a\mathbf{b}\breve{\times}\mathsf{m}_a^l = \left(c_{aa}^l + \sum_{b\neq a} z_{ab}a_{ab}^l\right)\mathsf{m}_a^l + \sum_{b\neq a} z_{ab}b_{ab}^l\mathsf{m}_b^l + \mathsf{g}_a^l, \tag{32}$$

where g_a^l is a K-dimensional column vector, and a_{ab}^l, b_{ab}^l, and c_{aa}^l are $K\times K$ matrices whose elements are a_{ab}^{lpq}, b_{ab}^{lpq}, and c_{aa}^{lpq}, respectively. The moment indices for the matrix elements are

$$p, q = \begin{cases} 2, 3, \cdots, K+1, & l = 0, \\ 1, 2, \cdots, K, & l = 1, \\ 0, 1, \cdots, K-1, & l \geq 2. \end{cases}$$

Since $\mathsf{g}_a^l = 0$ for $l \geq 3$, the solutions are trivial

$$\mathsf{m}_a^l = 0 \text{ for } l \geq 3. \tag{33}$$

For $l = 0$, we can solve Eq. (32) inverting the collision matrix. For $l = 1$ and 2, we solve Eq. (32) adopting the geometric method developed in [13, 22]. We define operators $\mathsf{b}_A \, (A = \|, \times, \perp)$ acting on a vector $\mathbf{V}$ to define

$$\begin{aligned} \mathbf{V}_\| &= \mathsf{b}_\| \mathbf{V} = \mathbf{b}\mathbf{b} \cdot \mathbf{V}, \\ \mathbf{V}_\perp &= \mathsf{b}_\perp \mathbf{V} = -\mathbf{b} \times (\mathbf{b} \times \mathbf{V}), \\ \mathbf{V}_\times &= \mathsf{b}_\times \mathbf{V} = \mathbf{b} \times \mathbf{V}. \end{aligned}$$

Similarly, we define $\mathsf{b}_{AB} \, (A,\, B = \|, \times, \perp)$ acting on a rank 2 tensor W to define

$$\mathsf{W}_{AB} = \mathsf{b}_{AB} \mathsf{W} = \frac{1}{2}(\mathsf{b}_A \otimes \mathsf{b}_B + \mathsf{b}_B \otimes \mathsf{b}_A)\mathsf{W}.$$

We further define

$$\mathsf{W}_\pm = \frac{1}{2}(\mathsf{W}_{\perp\perp} \pm \mathsf{W}_{\times\times}) \tag{34}$$

for a rank-2 tensor W. Then Braginskii's W^i $(i = 0, 1, 2, 3, 4)$ [10, 11] can be written as

$$\begin{aligned} \mathsf{W}^0 &= \mathsf{W}_{\|\|} + \frac{1}{2}(\mathsf{W}_{\times\times} + \mathsf{W}_{\perp\perp}), \\ \mathsf{W}^1 &= \frac{1}{2}(\mathsf{W}_{\perp\perp} - \mathsf{W}_{\times\times}), \\ \mathsf{W}^2 &= 2\mathsf{W}_{\|\perp}, \\ \mathsf{W}^3 &= \mathsf{W}_{\times\perp}, \\ \mathsf{W}^4 &= 2\mathsf{W}_{\|\times}, \end{aligned} \tag{35}$$

and any rank-2 tensor can be decomposed as

$$\mathsf{W} = \mathsf{W}_{\|\|} + 2\mathsf{W}_{\|\perp} + \mathsf{W}_{\perp\perp} = \mathsf{W}^0 + \mathsf{W}^1 + \mathsf{W}^2. \tag{36}$$

III. ELECTRON CLOSURES FOR A PLASMA WITH MULTIPLE ION SPECIES

For a = e (electrons) and $b = j$ (an ion species), the small-mass-ratio approximation is adopted for collision coefficients [8]. Then $b^l_{\mathrm{e}j}$ can be ignored and Eq. (32) is written as

$$r_{\mathrm{e}}\mathbf{b}\check{\times}\mathsf{m}^l_{\mathrm{e}} = c^l_{\mathrm{e}}\mathsf{m}^l_{\mathrm{e}} + \mathsf{g}^l_{\mathrm{e}}, \tag{37}$$

where

$$c^l_{\mathrm{e}} = c^l_{\mathrm{ee}} + \sum_j z_{\mathrm{e}j} a^l_{\mathrm{ei}}, \tag{38}$$

and drive terms are

$$\mathsf{g}^{1p}_{\mathrm{e}} = \delta_{p1}\frac{\sqrt{5}}{2}\frac{\tau_{\mathrm{ee}}v_{T\mathrm{e}}}{T_{\mathrm{e}}}\nabla T_{\mathrm{e}} + \sqrt{2}\sum_j z_{\mathrm{e}j} a^{1p0}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{e}j}}{v_{T\mathrm{e}}}, \tag{39}$$

$$\mathsf{g}^{2p}_{\mathrm{e}} = -\,\delta_{p0}\frac{1}{\sqrt{2}}\tau_{\mathrm{ee}}\mathsf{W}_{\mathrm{e}}, \tag{40}$$

with $\sum_j$ denoting a sum over ion species j (here and hereafter the subscript i, j, k, and $l = 1, 2, \cdots, S$ will be used for ion species). The collision matrices $c^{l=1,2}_{\mathrm{ee}}$ and $a^{l=1,2}_{\mathrm{ei}}$ for $K = 3$ are listed in Eqs. (28) of Ref. [13].

The $\mathcal{G}^{0p}_{\mathrm{e}}$ drive, Eq. (25), is not considered because the collision coefficients $a^{01q}_{\mathrm{e}j} \sim \mathcal{O}(\mu_{j\mathrm{e}})$, $b^{01q}_{\mathrm{e}j} \sim \mathcal{O}(\mu^q_{j\mathrm{e}})$, and $a^{0p0}_{\mathrm{e}j} \sim \mathcal{O}(\mu_{j\mathrm{e}})$ where $\mu_{j\mathrm{e}} = m_{\mathrm{e}}/m_j$ [see Eqs. (18), (19), and (28) with a = e and $b = j$ (an ion species)] . Therefore $Q^{\mathrm{N}}_{\mathrm{e}j} \sim \mathcal{O}(\mu^2_{j\mathrm{e}})$ is negligible compared to $Q^{\mathrm{M}}_{\mathrm{e}j} \sim \mathcal{O}(\mu_{j\mathrm{e}})$ in Eq. (14) (see also Fig. 1 and related remarks). Then the collisional heating of electrons are written as

$$Q_{\mathrm{e}} = \sum_j Q_{\mathrm{e}j},\ Q_{\mathrm{e}j} = \frac{n_{\mathrm{e}}}{\tau_{\mathrm{e}j}}\frac{3}{\mu_{\mathrm{e}j}}T_{j\mathrm{e}}. \tag{41}$$

Defining an effective ion mass m_{i}, temperature T_{i}, and collision time τ_{ei} by

$$m^{-1}_{\mathrm{i}} = \frac{\sum_j \tau^{-1}_{\mathrm{e}j} m^{-1}_j}{\sum_j \tau^{-1}_{\mathrm{e}j}}, \tag{42}$$

$$T_{\mathrm{i}} = \frac{\sum_j \tau^{-1}_{\mathrm{e}j} m^{-1}_j T_j}{\sum_j \tau^{-1}_{\mathrm{e}j} m^{-1}_j}, \tag{43}$$

and

$$\tau^{-1}_{\mathrm{ei}} = \sum_j \tau^{-1}_{\mathrm{e}j} = Z\tau^{-1}_{\mathrm{ee}}, \tag{44}$$

the collisional heating of electrons becomes that of a single ion species

$$Q_\mathrm{e} = 3\frac{n_\mathrm{e}}{\tau_\mathrm{ei}}\frac{m_\mathrm{e}}{m_\mathrm{i}}(T_\mathrm{i} - T_\mathrm{e}). \tag{45}$$

Note that a^l_ei in Eq. (38) is independent of ion species j as a consequence of the small-mass-ratio approximation. Defining an effective ion charge number Z for multiple ion species,

$$Z = \sum_j z_{\mathrm{e}j} = \sum_j \frac{\tau_\mathrm{ee}}{\tau_{\mathrm{e}j}} = \sum_j Z_j^2 \frac{n_j \ln \Lambda_{\mathrm{e}j}}{n_\mathrm{e} \ln \Lambda_\mathrm{ee}}, \tag{46}$$

and an effective ion flow velocity (the collision-frequency weighted average of ion flow velocities) $\mathbf{V}_\mathrm{i}$,

$$\mathbf{V}_\mathrm{i} = \frac{\sum_j \tau_{\mathrm{e}j}^{-1}\mathbf{V}_j}{\sum_j \tau_{\mathrm{e}j}^{-1}} = \sum_j \frac{z_{\mathrm{e}j}}{Z}\mathbf{V}_j = \tau_\mathrm{ei}\sum_j \tau_{\mathrm{e}j}^{-1}\mathbf{V}_j, \tag{47}$$

Eqs. (38) and (39) become the collision matrix and the drive of a *single* ion species Z,

$$c^l_\mathrm{e} = c^l_\mathrm{ee} + Z a^l_\mathrm{ei}, \tag{48}$$

$$\mathsf{g}^{1p}_\mathrm{e} = \delta_{p1}\frac{\sqrt{5}}{2}\frac{\tau_\mathrm{ee} v_{T\mathrm{e}}}{T_\mathrm{e}}\nabla T_\mathrm{e} + \sqrt{2} Z a^{1p0}_\mathrm{ei}\frac{\mathbf{V}_\mathrm{ei}}{v_{T\mathrm{e}}}, \tag{49}$$

the same as Eqs. (32a) and (31a) of Ref. [13] (up to the density factor). Therefore the electron closure relations become those of a single ion species:

$$\mathbf{R}_\mathrm{e} = \frac{m_\mathrm{e} n_\mathrm{e}}{\tau_\mathrm{ei}}(-\hat{\alpha}_\parallel \mathbf{V}_{\mathrm{ei}\parallel} - \hat{\alpha}_\perp \mathbf{V}_{\mathrm{ei}\perp} + \hat{\alpha}_\times \mathbf{V}_{\mathrm{ei}\times}) + n_\mathrm{e}(-\hat{\beta}_\parallel \nabla_\parallel T_\mathrm{e} - \hat{\beta}_\perp \nabla_\perp T_\mathrm{e} - \hat{\beta}_\times \nabla_\times T_\mathrm{e}), \tag{50}$$

$$\mathbf{h}_\mathrm{e} = n_\mathrm{e} T_\mathrm{e}(\hat{\beta}_\parallel \mathbf{V}_{\mathrm{ei}\parallel} + \hat{\beta}_\perp \mathbf{V}_{\mathrm{ei}\perp} + \hat{\beta}_\times \mathbf{V}_{\mathrm{ei}\times}) + \frac{n_\mathrm{e} T_\mathrm{e} \tau_\mathrm{ee}}{m_\mathrm{e}}(-\hat{\kappa}^\mathrm{e}_\parallel \nabla_\parallel T_\mathrm{e} - \hat{\kappa}^\mathrm{e}_\perp \nabla_\perp T_\mathrm{e} - \hat{\kappa}^\mathrm{e}_\times \nabla_\times T_\mathrm{e}). \tag{51}$$

The closure coefficients can be obtained from Eq. (83) with Table III in Ref. [13]. The electron friction due to ion species j, $\mathbf{R}_{\mathrm{e}j}$, can be calculated as

$$\mathbf{R}_{\mathrm{e}j} = \frac{z_{\mathrm{e}j}}{Z}\mathbf{R}_\mathrm{e}. \tag{52}$$

For $l = 2$, the moment equation (37) with Eqs. (38) and (40) is exactly the same as that of an electron-ion plasma for a *single* ion species Z and so are the electron closure relations:

$$\boldsymbol{\pi}_\mathrm{e} = -\eta^0_\mathrm{e}\mathsf{W}^0_\mathrm{e} - \eta^1_\mathrm{e}\mathsf{W}^1_\mathrm{e} - \eta^2_\mathrm{e}\mathsf{W}^2_\mathrm{e} - \eta^3_\mathrm{e}\mathsf{W}^3_\mathrm{e} - \eta^4_\mathrm{e}\mathsf{W}^4_\mathrm{e}. \tag{53}$$

The closure coefficients can be obtained from Eq. (84) with Table IV in Ref. [13]. Therein the superscript and subscript appear switched.

IV. ION CLOSURES FOR MULTIPLE ION SPECIES

For an ion species i, the ion-electron collision coefficients b_{ie} can be ignored by the small-mass-ratio approximation based on $\mu = m_e/m_i \ll 1$, but b_{ij} for another ion species j should be kept, which makes the ion equations coupled to each other. For clarity, the moment index will be denoted by a superscript, and the species index by a subscript when they appear together. A system of coupled equations for S ion species, $i, j = 1, 2, \cdots, S$, can be written in matrix form:

$$\mathtt{R}\mathbf{b}\breve{\times}\mathtt{M}^l = \mathtt{C}^l\mathtt{M}^l + \mathtt{G}^l, \tag{54}$$

where $\mathtt{M}^l$ and $\mathtt{G}^l$ are KS dimensional column vectors, $\mathtt{C}^l$ is a $KS \times KS$ dimensional matrix:

$$\mathtt{M}^l = \begin{pmatrix} \mathsf{m}_1^l \\ \mathsf{m}_2^l \\ \vdots \\ \mathsf{m}_S^l \end{pmatrix}, \ \mathtt{C}^l = \begin{pmatrix} \mathtt{C}_{11}^l & \mathtt{C}_{12}^l & \cdots & \mathtt{C}_{1S}^l \\ \mathtt{C}_{21}^l & \mathtt{C}_{22}^l & \cdots & \mathtt{C}_{2S}^l \\ \vdots & \vdots & \ddots & \vdots \\ \mathtt{C}_{S1}^l & \mathtt{C}_{S2}^l & \vdots & \mathtt{C}_{SS}^l \end{pmatrix}, \tag{55}$$

$\mathtt{G}^l$ is defined in a similar manner to $\mathtt{M}^l$, and $\mathtt{R} = \mathrm{diag.}\,(r_1 1_K, r_2 1_K, r_3 1_K, \cdots, r_S 1_K)$ with 1_K being a K-dimensional identity matrix. Here the $K \times K$ block matrices are defined as

$$\mathtt{C}_{ij}^l = \begin{cases} c_{ii}^l + z_{ie}a_{ie}^l + \sum_{k\neq i} z_{ik}a_{ik}^l, & i = j \\ z_{ij}b_{ij}^l, & i \neq j. \end{cases} \tag{56}$$

The elements of column vector $\mathtt{G}^l$ are, for $l = 0$ and $p = 2, 3, \cdots, K+1$,

$$\mathtt{G}^0_{(i-1)*K+p-1} = \mathsf{g}_i^{0p} = \sum_{b=0}^{S} z_{ib}\alpha_{ib}^p \frac{T_{ib}}{T_i}, \tag{57}$$

for $l = 1$ and $p = 1, 2, \cdots, K$,

$$\mathtt{G}^1_{(i-1)*K+p} = \mathsf{g}_i^{1p} = \delta_{p1}\frac{\sqrt{5}}{2}\frac{\tau_{ii}v_{Ti}}{T_i}\nabla T_i + \sqrt{2}\sum_{b=0}^{S} z_{ib}b_{ib}^{1p0}\frac{\mathbf{V}_{bi}}{v_{Tb}}, \tag{58}$$

for $l = 2$ and $p = 0, 1, \cdots, K-1$,

$$\mathtt{G}^2_{(i-1)*K+p+1} = \mathsf{g}_i^{2p} = -\delta_{p0}\frac{\tau_{ii}}{\sqrt{2}}\mathsf{W}_i, \tag{59}$$

where the subscript A in $\mathtt{G}_A^l$ denotes the row number of a column vector $\mathtt{G}^l$.

A. $l = 0$ scalar moments

For $l = 0$, solving

$$0 = \mathtt{C}^0\mathtt{M}^0 + \mathtt{G}^0, \tag{60}$$

we have the solution

$$\mathtt{M}^0 = \mathtt{X}^0\mathtt{G}^0,\ \mathtt{X}^0 = -\left(\mathtt{C}^0\right)^{-1}. \tag{61}$$

Using Eq. (57), the moment element can be written as

$$\mathsf{m}_i^{0p} = \sum_j \sum_{q=2} \mathtt{X}_{ij}^{0pq}\mathsf{g}_j^{0q} = \sum_j \sum_{q=2} \mathtt{X}_{ij}^{0pq} \sum_{k\neq j} z_{jk}\alpha_{jk}^q \frac{T_{jk}}{T_j}, \tag{62}$$

where the electron term has been ignored because $\alpha_{j\mathrm{e}}^q = \mathcal{O}(\mu_{j\mathrm{e}}^{q-1/2})$ with $q \geq 2$.

The collisional heating is decomposed into $Q_{i\mathrm{e}}$ due to electrons and Q_{ij} due to ions j,

$$Q_i = Q_{i\mathrm{e}} + \sum_j Q_{ij}, \tag{63}$$

where $Q_{i\mathrm{e}} = -Q_{\mathrm{e}i}$ can be obtained from Eq. (41) and Q_{ij} can be written from Eqs. (14), (16), (62), and (57) as

$$Q_{ij} = -\frac{n_i}{\tau_{ij}} \sum_{k<l} \hat{\gamma}_{ijkl} T_{kl}, \tag{64}$$

with the dimensionless coefficients

$$\hat{\gamma}_{ijkl} = \sqrt{\frac{3}{2}}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\alpha_{kl}^1 + \gamma_{ijkl}^{\mathrm{N}}, \tag{65}$$

where

$$\gamma_{ijkl}^{\mathrm{N}} = \sum_{p,q=2}\left[\sqrt{\frac{3}{2}}(a_{ij}^{01p}\mathtt{X}_{ik}^{0pq} + b_{ij}^{01p}\mathtt{X}_{jk}^{0pq}) z_{kl}\alpha_{kl}^q \frac{T_i}{T_k} - (k \leftrightarrow l)\right] \tag{66}$$

and the symbol $(k \leftrightarrow l)$ denotes a repetition of the previous term with k and l interchanged. Note that (i) the replacement of $\sum_{k<l}$ with $(1/2)\sum_{k,l}$ or $\sum_{k>l}$ in Eq. (64) yields the same results because $\hat{\gamma}_{ijkl} = -\hat{\gamma}_{ijlk}$, and $T_{kl} = -T_{lk}$ (ii) $Q_{ij} + Q_{ji} = 0$, the energy conservation, is verified by $\hat{\gamma}_{ijkl} + \hat{\gamma}_{jikl} m_i T_i v_{Ti}/m_j T_j v_{Tj} = 0$, and $m_j n_j T_j v_{Tj}/\tau_{ji} = m_i n_i T_i v_{Ti}/\tau_{ij}$. It may be convenient to express

$$Q_i = Q_{i\mathrm{e}} - \frac{n_i}{\tau_{ii}} \sum_j \hat{\gamma}_{ij} T_j, \tag{67}$$

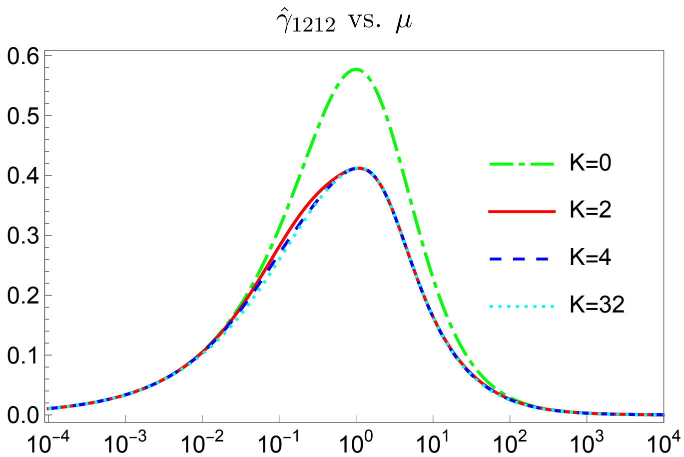

Figure 1. Collisional heating coefficients of ion species 1 ($m_1 = 1\mathrm{u}$, $T_1 = T_\mathrm{e}$, $Z_1 = 1$, and $n_1 = 0.5n_\mathrm{e}$) due to ion species 2 ($m_2 = \mu m_1$, $T_2 = 2T_1$, $Z_2 = 1$, and $n_2 = n_1$). The coefficients $\hat{\gamma}_{1212}$ vs. mass ratio μ are depicted for various numbers of moments: $K = 0$ (Maxwellian only, green, dash-dotted), 2 (red, solid), 4 (blue, dashed), and 32 (cyan, dotted).

where $\hat{\gamma}_{ij} = \sum_{k,l} z_{ik}\hat{\gamma}_{ikjl}$. For $S > 3$, this expression reduces $S(S-1)/2$ temperature difference terms to S temperature terms.

Figure 1 shows that the modification of Q_{ij}^{M} ($K = 0$) by Q_{ij}^{N} is substantial for comparable masses ($\mu \approx 1$). The deviation of Q_{ij}^{M} from the $K = 32$ calculation of Q_{ij} is about 40% at $\mu = 1$. The deviations are ignorable for $\mu \gg 1$ and $\mu \ll 1$. This explains that $Q_{\mathrm{ei}}^{\mathrm{N}}$ and $Q_{\mathrm{ie}}^{\mathrm{N}}$ are ignorable in electron-ion plasmas. Figure 1 also displays the convergence behavior as the number of moments increases. The $K = 2$ calculation is already a good approximation compared to the $K = 32$ calculation. The deviations of $K = 2$ and $K = 4$ calculations, respectively, from $K = 32$ are at most 8.5% at $\mu \approx 0.08$ and 4.8% at $\mu \approx 0.04$ and only 0.1% and 0.01% at $\mu = 1$.

B. $l = 1$ vector moments

For $l = 1$, we solve

$$\mathtt{R}\mathbf{b} \times \mathtt{M}^1 = \mathtt{C}^1\mathtt{M}^1 + \mathtt{G}^1. \tag{68}$$

Applying $\mathtt{b}_\parallel$ yields

$$\mathtt{C}^1\mathtt{M}^1_\parallel = -\mathtt{G}^1_\parallel \tag{69}$$

and the solution is

$$\mathtt{M}^1_{\parallel} = -\left(\mathtt{C}^1\right)^{-1}\mathtt{G}^1_{\parallel}. \tag{70}$$

Applying $\mathsf{b}_{\perp}$ and $\mathsf{b}_{\times}$ yields

$$\begin{bmatrix} \mathtt{C}^1 & -\mathtt{R} \\ \mathtt{R} & \mathtt{C}^1 \end{bmatrix} \begin{bmatrix} \mathtt{M}^1_{\perp} \\ \mathtt{M}^1_{\times} \end{bmatrix} = - \begin{bmatrix} \mathtt{G}^1_{\perp} \\ \mathtt{G}^1_{\times} \end{bmatrix}, \tag{71}$$

and the solution is

$$\begin{bmatrix} \mathtt{M}^1_{\perp} \\ \mathtt{M}^1_{\times} \end{bmatrix} = -\left(\mathtt{D}^{1,1}\right)^{-1} \begin{bmatrix} \mathtt{RC}^1\mathtt{R}^{-1} & \mathtt{R} \\ -\mathtt{R} & \mathtt{RC}^1\mathtt{R}^{-1} \end{bmatrix} \begin{bmatrix} \mathtt{G}^1_{\perp} \\ \mathtt{G}^1_{\times} \end{bmatrix}. \tag{72}$$

From Eqs. (70) and (72), the solution $\mathtt{M}^1 = \mathtt{M}^1_{\parallel} + \mathtt{M}^1_{\perp}$ is

$$\mathtt{M}^1 = -\left(\mathtt{C}^1\right)^{-1}\mathtt{G}^1_{\parallel} - \left(\mathtt{D}^{1,1}\right)^{-1}\left(\mathtt{RC}^1\mathtt{R}^{-1}\mathtt{G}^1_{\perp} + \mathtt{RG}^1_{\times}\right), \tag{73}$$

where

$$\mathtt{D}^{l,m} = \mathtt{RC}^l\mathtt{R}^{-1}\mathtt{C}^l + (m\mathtt{R})^2. \tag{74}$$

Then the m_i^{1p} moment can be read from Eq. (73),

$$\mathsf{m}_i^{1p} = \sum_{j,q}\left(\mathtt{X}^{1pq}_{\parallel ij}\mathsf{g}^{1q}_{j\parallel} + \mathtt{X}^{1pq}_{\perp ij}\mathsf{g}^{1q}_{j\perp} - \mathtt{X}^{1pq}_{\times ij}\mathsf{g}^{1q}_{j\times}\right), \tag{75}$$

where

$$\begin{aligned} \mathtt{X}^1_{\parallel} &= -\left(\mathtt{C}^1\right)^{-1}, \\ \mathtt{X}^1_{\perp} &= -\left(\mathtt{D}^{1,1}\right)^{-1}\mathtt{RC}^1\mathtt{R}^{-1}, \\ \mathtt{X}^1_{\times} &= \left(\mathtt{D}^{1,1}\right)^{-1}\mathtt{R}, \end{aligned} \tag{76}$$

and $\mathtt{X}^{1pq}_{Aij}$ is the $[(i-1)K+p]$th row and $[(j-1)K+q]$th column of matrix $\mathtt{X}^1_A$. The ion heat flux density can be obtained from Eqs. (30), (75), and (58):

$$\begin{aligned} \mathbf{h}_i = &\sum_j \frac{n_i T_i \tau_{jj}}{m_j}\left(-\hat{\kappa}_{\parallel ij}\nabla_{\parallel}T_j - \hat{\kappa}_{\perp ij}\nabla_{\perp}T_j + \hat{\kappa}_{\times ij}\nabla_{\times}T_j\right) \\ &+ n_i T_i \sum_{j<k}\left(\hat{\beta}^{TV}_{\parallel ijk}\mathbf{V}_{jk\parallel} + \hat{\beta}^{TV}_{\perp ijk}\mathbf{V}_{jk\perp} - \hat{\beta}^{TV}_{\times ijk}\mathbf{V}_{jk\times}\right), \end{aligned} \tag{77}$$

where

$$\hat{\kappa}_{Aij} = \frac{5}{2}\frac{v_{Ti}}{v_{Tj}}\mathtt{X}^{111}_{Aij}, \tag{78}$$

$$\hat{\beta}^{TV}_{Aijk} = \sqrt{\frac{5}{2}}\sum_q \left[\frac{v_{Ti}}{v_{Tk}}\mathtt{X}^{11q}_{Aij}z_{jk}b^{1q0}_{jk} - (j \leftrightarrow k)\right]. \tag{79}$$

The friction force density is decomposed into $\mathbf{R}_{ie}$ due to electrons and $\mathbf{R}_{ij}$ due to ions j:

$$\mathbf{R}_i = \mathbf{R}_{ie} + \sum_j \mathbf{R}_{ij}, \tag{80}$$

where $\mathbf{R}_{ie} = -\mathbf{R}_{ei}$ can be obtained from Eq. (52), and $\mathbf{R}_{ij}$ can be obtained from Eqs. (15), (17), (75), and (58):

$$\begin{aligned}\mathbf{R}_{ij} =& \sum_k n_k \left(-\hat{\beta}^{VT}_{\parallel ijk}\nabla_\parallel T_k - \hat{\beta}^{VT}_{\perp ijk}\nabla_\perp T_k + \hat{\beta}^{VT}_{\times ijk}\nabla_\times T_k\right) \\ &- \sum_{k<l}\frac{m_i n_i}{\tau_{ij}}\left(\hat{\alpha}_{\parallel ijkl}\mathbf{V}_{kl\parallel} + \hat{\alpha}_{\perp ijkl}\mathbf{V}_{kl\perp} + \hat{\alpha}_{\times ijkl}\mathbf{V}_{kl\times}\right),\end{aligned} \tag{81}$$

with the dimensionless coefficients

$$\hat{\beta}^{VT}_{Aijk} = -\sum_{p=1}\sqrt{\frac{5}{2}}\frac{z_{ki}}{z_{ik}}\frac{T_k}{T_i}z_{ij}(a^{10p}_{ij}\mathtt{X}^{1p1}_{Aik} + b^{10p}_{ij}\mathtt{X}^{1p1}_{Ajk}), \tag{82}$$

$$\hat{\alpha}_{Aijkl} = -\delta_A a^{100}_{ij}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) + (2\delta_A - 1)\alpha^{\mathrm{N}}_{Aijkl}, \tag{83}$$

where $\delta_\parallel = \delta_\perp = 1$, $\delta_\times = 0$, and

$$\alpha^{\mathrm{N}}_{Aijkl} = \sum_{p,q=1}\left[\left(a^{10p}_{ij}\mathtt{X}^{1pq}_{Aik} + b^{10p}_{ij}\mathtt{X}^{1pq}_{Ajk}\right)z_{kl}b^{1q0}_{kl}\frac{v_{Ti}}{v_{Tl}} - (k \leftrightarrow l)\right]. \tag{84}$$

Note that (i) the replacement of $\sum_{k<l}$ with $(1/2)\sum_{k,l}$ or $\sum_{k>l}$ yields the same results since $\hat{\alpha}_{Aijkl} = -\hat{\alpha}_{Aijlk}$, and $\mathbf{V}_{klA} = -\mathbf{V}_{lkA}$ (ii) $\mathbf{R}_{ij} + \mathbf{R}_{ji} = 0$, the momentum conservation, is verified by $\hat{\beta}^{VT}_{Aijk} + \hat{\beta}^{VT}_{Ajik} = 0$, $\hat{\alpha}_{Aijkl} + \hat{\alpha}_{Ajikl}T_i v_{Ti}/T_j v_{Tj} = 0$, and $m_j n_j T_j v_{Tj}/\tau_{ji} = m_i n_i T_i v_{Ti}/\tau_{ij}$. It may be convenient to write

$$\begin{aligned}\mathbf{h}_i =& \sum_j \frac{n_i T_i \tau_{jj}}{m_j}\left(-\hat{\kappa}_{\parallel ij}\nabla_\parallel T_j - \hat{\kappa}_{\perp ij}\nabla_\perp T_j + \hat{\kappa}_{\times ij}\nabla_\times T_j\right) \\ &+ n_i T_i \sum_j \left(\hat{\beta}^{TV}_{\parallel ij}\mathbf{V}_{j\parallel} + \hat{\beta}^{TV}_{\perp ij}\mathbf{V}_{j\perp} - \hat{\beta}^{TV}_{\times ij}\mathbf{V}_{j\times}\right),\end{aligned} \tag{85}$$

$$\begin{aligned}\mathbf{R}_i =& \mathbf{R}_{ie} + \sum_j n_j \left(-\hat{\beta}^{VT}_{\parallel ij}\nabla_\parallel T_j - \hat{\beta}^{VT}_{\perp ij}\nabla_\perp T_j + \hat{\beta}^{VT}_{\times ij}\nabla_\times T_j\right) \\ &- \frac{m_i n_i}{\tau_{ii}}\sum_j \left(\hat{\alpha}_{\parallel ij}\mathbf{V}_{j\parallel} + \hat{\alpha}_{\perp ij}\mathbf{V}_{j\perp} + \hat{\alpha}_{\times ij}\mathbf{V}_{j\times}\right),\end{aligned} \tag{86}$$

where $\hat{\beta}^{TV}_{Aij} = \sum_k \hat{\beta}^{TV}_{Aijk}$, $\hat{\beta}^{VT}_{Aij} = \sum_k \hat{\beta}^{VT}_{Aikj}$, and $\hat{\alpha}_{Aij} = \sum_{k,l} z_{ik}\hat{\alpha}_{Aikjl}$. Particularly for more than three ion species ($S > 3$), these equations reduce $S(S-1)/2$ terms of relative flow velocities to S terms of flow velocities.

C. $l = 2$ tensor moments

The moment equation (54) for $l = 2$ becomes

$$\mathtt{R}\left(\mathbf{b} \times \mathtt{M}^2 - \mathtt{M}^2 \times \mathbf{b}\right) = \mathtt{C}^2\mathtt{M}^2 + \mathtt{G}^2. \tag{87}$$

Applying $\mathsf{b}_{\parallel\perp}$and $\mathsf{b}_{\parallel\times}$ on Eq. (87) yields [the same form as Eq. (71)]

$$\begin{bmatrix} \mathtt{C}^2 & -\mathtt{R} \\ \mathtt{R} & \mathtt{C}^2 \end{bmatrix} \begin{bmatrix} \mathtt{M}^2_{\parallel\perp} \\ \mathtt{M}^2_{\parallel\times} \end{bmatrix} = - \begin{bmatrix} \mathtt{G}^2_{\parallel\perp} \\ \mathtt{G}^2_{\parallel\times} \end{bmatrix}, \tag{88}$$

which is solved for

$$\mathtt{M}^2_{\parallel\perp} = -(\mathtt{D}^{2,1})^{-1}\left(\mathtt{R}\mathtt{C}^2\mathtt{R}^{-1}\mathtt{G}^2_{\parallel\perp} + \mathtt{R}\mathtt{G}^2_{\parallel\times}\right). \tag{89}$$

Applying $\mathsf{b}_{\perp\perp}$, $\mathsf{b}_{\times\perp}$, and $\mathsf{b}_{\times\times}$ on Eq. (87) yields

$$2\mathtt{R}\mathtt{M}^2_{\times\perp} = \mathtt{C}^2\mathtt{M}^2_{\perp\perp} + \mathtt{G}^2_{\perp\perp}, \tag{90}$$

$$\mathtt{R}(-\mathtt{M}^2_{\perp\perp} + \mathtt{M}^2_{\times\times}) = \mathtt{C}^2\mathtt{M}^2_{\times\perp} + \mathtt{G}^2_{\times\perp}, \tag{91}$$

$$-2\mathtt{R}\mathtt{M}^2_{\times\perp} = \mathtt{C}^2\mathtt{M}^2_{\times\times} + \mathtt{G}^2_{\times\times}. \tag{92}$$

Applying $\mathsf{b}_{\parallel\parallel}$ yields

$$0 = \mathtt{C}^2\mathtt{M}^2_{\parallel\parallel} + \mathtt{G}^2_{\parallel\parallel},$$

and its solution is

$$\mathtt{M}^2_{\parallel\parallel} = -(\mathtt{C}^2)^{-1}\mathtt{G}^2_{\parallel\parallel}. \tag{93}$$

With the definitions (34), the system of equations (90)-(92) can be reduced to

$$\mathtt{C}^2\mathtt{M}^2_{+} = -\mathtt{G}^2_{+} \tag{94}$$

and

$$\begin{bmatrix} \mathtt{C}^2 & -2\mathtt{R} \\ 2\mathtt{R} & \mathtt{C}^2 \end{bmatrix} \begin{bmatrix} \mathtt{M}^2_{-} \\ \mathtt{M}^2_{\times\perp} \end{bmatrix} = - \begin{bmatrix} \mathtt{G}^2_{-} \\ \mathtt{G}^2_{\times\perp} \end{bmatrix}. \tag{95}$$

The solutions are

$$\mathtt{M}^2_+ = -(\mathtt{C}^2)^{-1}\mathtt{G}^2_+, \tag{96}$$

$$\mathtt{M}^2_- = -(\mathtt{D}^{2,2})^{-1}(\mathtt{R}\mathtt{C}^2\mathtt{R}^{-1}\mathtt{G}^2_- + 2\mathtt{R}\mathtt{G}^2_{\times\perp}), \tag{97}$$

and, from Eq. (34), $\mathtt{M}^2_{\perp\perp} = \mathtt{M}_+ + \mathtt{M}_-$ becomes

$$\mathtt{M}^2_{\perp\perp} = -(\mathtt{C}^2)^{-1}\mathtt{G}^2_+ - (\mathtt{D}^{2,2})^{-1}(\mathtt{R}\mathtt{C}^2\mathtt{R}^{-1}\mathtt{G}^2_- + 2\mathtt{R}\mathtt{G}^2_{\times\perp}). \tag{98}$$

Using the definitions (35), we can write $\mathtt{M}^2 = \mathtt{M}^2_{\parallel\parallel} + 2\mathtt{M}^2_{\parallel\perp} + \mathtt{M}^2_{\perp\perp}$ with the help of Eqs. (93), (89), and (98):

$$\mathtt{M}^2 = \mathtt{X}^2_0\mathtt{G}^2_0 + \mathtt{X}^2_1\mathtt{G}^2_1 + \mathtt{X}^2_2\mathtt{G}^2_2 - \mathtt{X}^2_3\mathtt{G}^2_3 - \mathtt{X}^2_4\mathtt{G}^2_4,$$

where

$$\mathtt{X}^2_0 = -(\mathtt{C}^2)^{-1},$$

$$\mathtt{X}^2_1 = -(\mathtt{D}^{2,2})^{-1}\mathtt{R}\mathtt{C}^2\mathtt{R}^{-1},$$

$$\mathtt{X}^2_2 = -(\mathtt{D}^{2,1})^{-1}\mathtt{R}\mathtt{C}^2\mathtt{R}^{-1},$$

$$\mathtt{X}^2_3 = 2(\mathtt{D}^{2,2})^{-1}\mathtt{R},$$

$$\mathtt{X}^2_4 = (\mathtt{D}^{2,1})^{-1}\mathtt{R}.$$

Now we use Eq. (59) and Eq. (31) to write the viscosity tensor

$$\boldsymbol{\pi}_i = -p_i \sum_j \tau_{jj}(\hat{\eta}^0_{ij}\mathsf{W}^0_j + \hat{\eta}^1_{ij}\mathsf{W}^1_j + \hat{\eta}^2_{ij}\mathsf{W}^2_j - \hat{\eta}^3_{ij}\mathsf{W}^3_j - \hat{\eta}^4_{ij}\mathsf{W}^4_j), \tag{99}$$

where

$$\hat{\eta}^A_{ij} = \mathtt{X}^{200}_{A,ij}, \text{ for } A = 0, 1, 2, 3, 4, \tag{100}$$

and $\mathtt{X}^{200}_{Aij}$ is the $[(i-1)K+1]$st row and $[(j-1)K+1]$st column of matrix $\mathtt{X}^2_A$. Note that $\hat{\eta}^0_{ij} = \hat{\eta}^2_{ij}(r_i = 0)$, $\hat{\eta}^1_{ij}(r_i) = \hat{\eta}^2_{ij}(2r_i)$, and $\hat{\eta}^3_{ij}(r_i) = \hat{\eta}^4_{ij}(2r_i)$.

V. EXAMPLE STUDY FOR A PLASMA WITH TWO ION SPECIES

In this section we apply the formulation developed in the previous section to a two-ion system. For the collisional heating, from Eq. (63),

$$Q_1 = Q_{1\mathrm{e}} + Q_{12}, \tag{101}$$

$$Q_2 = Q_{2\mathrm{e}} - Q_{12}, \tag{102}$$

where $Q_{1\mathrm{e}} = -Q_{\mathrm{e}1}$ and $Q_{2\mathrm{e}} = -Q_{\mathrm{e}2}$ can be obtained from Eq. (41), and from Eq. (64),

$$Q_{12} = -\frac{n_1}{\tau_{12}}\hat{\gamma}_{1212}T_{12}. \tag{103}$$

For the heat flux density, it follows from Eq. (77) with $i, j, k = 1, 2$ that

$$\begin{aligned}\mathbf{h}_1 =& \frac{n_1 T_1 \tau_{11}}{m_1}\left(-\hat{\kappa}_{\parallel 11}\nabla_\parallel T_1 - \hat{\kappa}_{\perp 11}\nabla_\perp T_1 + \hat{\kappa}_{\times 11}\nabla_\times T_1\right)\\ &+ \frac{n_1 T_1 \tau_{22}}{m_2}\left(-\hat{\kappa}_{\parallel 12}\nabla_\parallel T_2 - \hat{\kappa}_{\perp 12}\nabla_\perp T_2 + \hat{\kappa}_{\times 12}\nabla_\times T_2\right)\\ &+ n_1 T_1\left(\hat{\beta}^{TV}_{\parallel 112}\mathbf{V}_{12\parallel} + \hat{\beta}^{TV}_{\perp 112}\mathbf{V}_{12\perp} - \hat{\beta}^{TV}_{\times 112}\mathbf{V}_{12\times}\right)\end{aligned} \tag{104}$$

and

$$\begin{aligned}\mathbf{h}_2 =& \frac{n_2 T_2 \tau_{11}}{m_2}\left(-\hat{\kappa}_{\parallel 21}\nabla_\parallel T_1 - \hat{\kappa}_{\perp 21}\nabla_\perp T_1 + \hat{\kappa}_{\times 21}\nabla_\times T_1\right)\\ &+ \frac{n_2 T_2 \tau_{22}}{m_2}\left(-\hat{\kappa}_{\parallel 22}\nabla_\parallel T_2 - \hat{\kappa}_{\perp 22}\nabla_\perp T_2 + \hat{\kappa}_{\times 22}\nabla_\times T_2\right)\\ &+ n_2 T_2\left(\hat{\beta}^{TV}_{\parallel 221}\mathbf{V}_{21\parallel} + \hat{\beta}^{TV}_{\perp 221}\mathbf{V}_{21\perp} - \hat{\beta}^{TV}_{\times 221}\mathbf{V}_{21\times}\right),\end{aligned} \tag{105}$$

where we have used $\hat{\beta}^{TV}_{A221} = -\hat{\beta}^{TV}_{A212}$ and $\mathbf{V}_{21A} = -\mathbf{V}_{12A}$. For the friction force density, from Eq. (80),

$$\mathbf{R}_1 = \mathbf{R}_{1\mathrm{e}} + \mathbf{R}_{12} \tag{106}$$

and

$$\mathbf{R}_2 = \mathbf{R}_{2\mathrm{e}} - \mathbf{R}_{12}, \tag{107}$$

where $\mathbf{R}_{1\mathrm{e}} = -\mathbf{R}_{\mathrm{e}1}$ and $\mathbf{R}_{2\mathrm{e}} = -\mathbf{R}_{\mathrm{e}2}$ can be obtained from Eqs. (50) and (52), and from Eq. (81),

$$\begin{aligned}\mathbf{R}_{12} =& n_1\left(-\hat{\beta}^{VT}_{\parallel 121}\nabla_\parallel T_1 - \hat{\beta}^{VT}_{\perp 121}\nabla_\perp T_1 + \hat{\beta}^{VT}_{\times 121}\nabla_\times T_1\right)\\ &+ n_2\left(-\hat{\beta}^{VT}_{\parallel 122}\nabla_\parallel T_2 - \hat{\beta}^{VT}_{\perp 122}\nabla_\perp T_2 + \hat{\beta}^{VT}_{\times 122}\nabla_\times T_2\right)\\ &- \frac{m_1 n_1}{\tau_{12}}\left(\hat{\alpha}_{\parallel 1212}\mathbf{V}_{12\parallel} + \hat{\alpha}_{\perp 1212}\mathbf{V}_{12\perp} + \hat{\alpha}_{\times 1212}\mathbf{V}_{12\times}\right).\end{aligned} \tag{108}$$

Finally, for the viscosity, it follows from Eq. (99) that

$$\begin{aligned}\boldsymbol{\pi}_1 =& p_1\tau_{11}(-\hat{\eta}^0_{11}\mathsf{W}^0_1 - \hat{\eta}^1_{11}\mathsf{W}^1_1 - \hat{\eta}^2_{11}\mathsf{W}^2_1 + \hat{\eta}^3_{11}\mathsf{W}^3_1 + \hat{\eta}^4_{11}\mathsf{W}^4_1)\\ &+ p_1\tau_{22}(-\hat{\eta}^0_{12}\mathsf{W}^0_2 - \hat{\eta}^1_{12}\mathsf{W}^1_2 - \hat{\eta}^2_{12}\mathsf{W}^2_2 + \hat{\eta}^3_{12}\mathsf{W}^3_2 + \hat{\eta}^4_{12}\mathsf{W}^4_2),\end{aligned}$$

and

$$\boldsymbol{\pi}_2 = p_2\tau_{11}(-\hat{\eta}^0_{21}\mathsf{W}^0_1 - \hat{\eta}^1_{21}\mathsf{W}^1_1 - \hat{\eta}^2_{21}\mathsf{W}^2_1 + \hat{\eta}^3_{21}\mathsf{W}^3_1 + \hat{\eta}^4_{21}\mathsf{W}^4_1)$$
$$+ p_2\tau_{22}(-\hat{\eta}^0_{22}\mathsf{W}^0_2 - \hat{\eta}^1_{22}\mathsf{W}^1_2 - \hat{\eta}^2_{22}\mathsf{W}^2_2 + \hat{\eta}^3_{22}\mathsf{W}^3_2 + \hat{\eta}^4_{22}\mathsf{W}^4_2).$$

For a deuterium-carbon (D-C) plasma with $m_\mathrm{D} = 2.014\mathrm{u}$, $m_\mathrm{C} = 12.00\mathrm{u}$, $n_\mathrm{C} = 0.06n_\mathrm{e}$, $n_\mathrm{D} = 0.64n_\mathrm{e}$, $T_\mathrm{D} = 1.5T_\mathrm{e}$, $T_\mathrm{C} = 1.8T_\mathrm{e}$, and $\ln\Lambda_\mathrm{ee} = 17$, the closure coefficients are calculated for $K = 2, 4, 8, 16$, and 32. Ion involving Coulomb logarithms are calculated from $\ln\Lambda_{\mathrm{e}i} = \ln\Lambda_\mathrm{ee} - \ln Z_i = \ln\Lambda_{i\mathrm{e}}$, $\ln\Lambda_{ii} = \ln\Lambda_\mathrm{ee} + \ln(T_i/T_\mathrm{e}Z_i^2)$, and $\ln\Lambda_{ij} = \ln\Lambda_{ii} + \ln[Z_i(m_j/m_i + T_j/T_i)/Z_j(m_j/m_i + 1)]$. The parameter z_{ab} can be calculated from the collision time $\tau_{ab} = 6\pi^{3/2}\varepsilon_0^2 m_a T_a v_{Ta}/n_b q_a^2 q_b^2 \ln\Lambda_{ab}$: $z_{ab} = n_b Z_b^2 \ln\Lambda_{ab}/n_a Z_a^2 \ln\Lambda_{aa}$.

For $l = 0$, the collisional heating coefficient $\hat{\gamma}_{1212} = 0.213551$ and $\hat{\gamma}_{2121} = 0.685228$ which are substantially reduced by the non-Maxwellian contribution from the Maxwellian contribution $\hat{\gamma}^{K=0}_{1212} = 0.382356$ and $\hat{\gamma}^{K=0}_{2121} = 1.22688$, respectively. The coefficients quickly converge as the number of moments increases: $\hat{\gamma}^{K=2}_{1212} = 0.213475$, $\hat{\gamma}^{K=4}_{1212} = 0.213582$, $\hat{\gamma}^{K=8}_{1212} = 0.213562$, $\hat{\gamma}^{K=16}_{1212} = 0.213551$, and $\hat{\gamma}^{K=32}_{1212} = 0.213551$; $\hat{\gamma}^{K=2}_{2121} = 0.684982$, $\hat{\gamma}^{K=4}_{2121} = 0.685327$, $\hat{\gamma}^{K=8}_{2121} = 0.685263$, $\hat{\gamma}^{K=16}_{2121} = 0.685228$, and $\hat{\gamma}^{K=32}_{2121} = 0.685228$.

For $l = 1$, the dimensionless closure coefficients are depicted in Figs. 2-3. As shown in Eqs. (104), (105), and (108), the heat flux and friction of ion species 1 and 2 are determined by temperature gradients of the ion species 1 and 2 and the relative flow velocity. Figure 2 shows the coefficients for the heat flux and friction of ion species 1 due to ∇T_1 and $\mathbf{V}_{12}$. Figure 3 shows the coefficients due to ∇T_2. Figure 4 shows the coefficients of the ion species 2 due to ∇T_1, ∇T_2, and $\mathbf{V}_{21}$. Note that the closure coefficients of ion species 2 are plotted against r_1 which can be converted to r_2 by $r_2/r_1 = \Omega_2\tau_{22}/\Omega_1\tau_{11} = 0.05803$ for the given parameters.

For $l = 2$, we may introduce the geometric notation $\hat{\eta}_\parallel$, $\hat{\eta}_\perp$, and $\hat{\eta}_\times$ for the viscosity coefficients: $\hat{\eta}^0 = \hat{\eta}_\parallel = \hat{\eta}_\perp(r_1 = 0)$, $\hat{\eta}^1 = \hat{\eta}_\perp(2r_1)$, $\hat{\eta}^2 = \hat{\eta}_\perp$, $\hat{\eta}^3 = \hat{\eta}_\times(2r_1)$, and $\hat{\eta}^4 = \hat{\eta}_\times$. As shown in Eq. (99), the viscosity closures of ion species 1 and 2 are determined by the gradients of flow velocity of the ion species 1 and 2. Figures 5 and 6 show the dimensionless viscosity closure coefficients of the ion species 1 and 2, respectively.

The convergence study is performed by increasing the number of moments, $K = 2, 4, 8, 16$, and 32. The $K = 4$ calculations are good approximations and large percentage errors appear

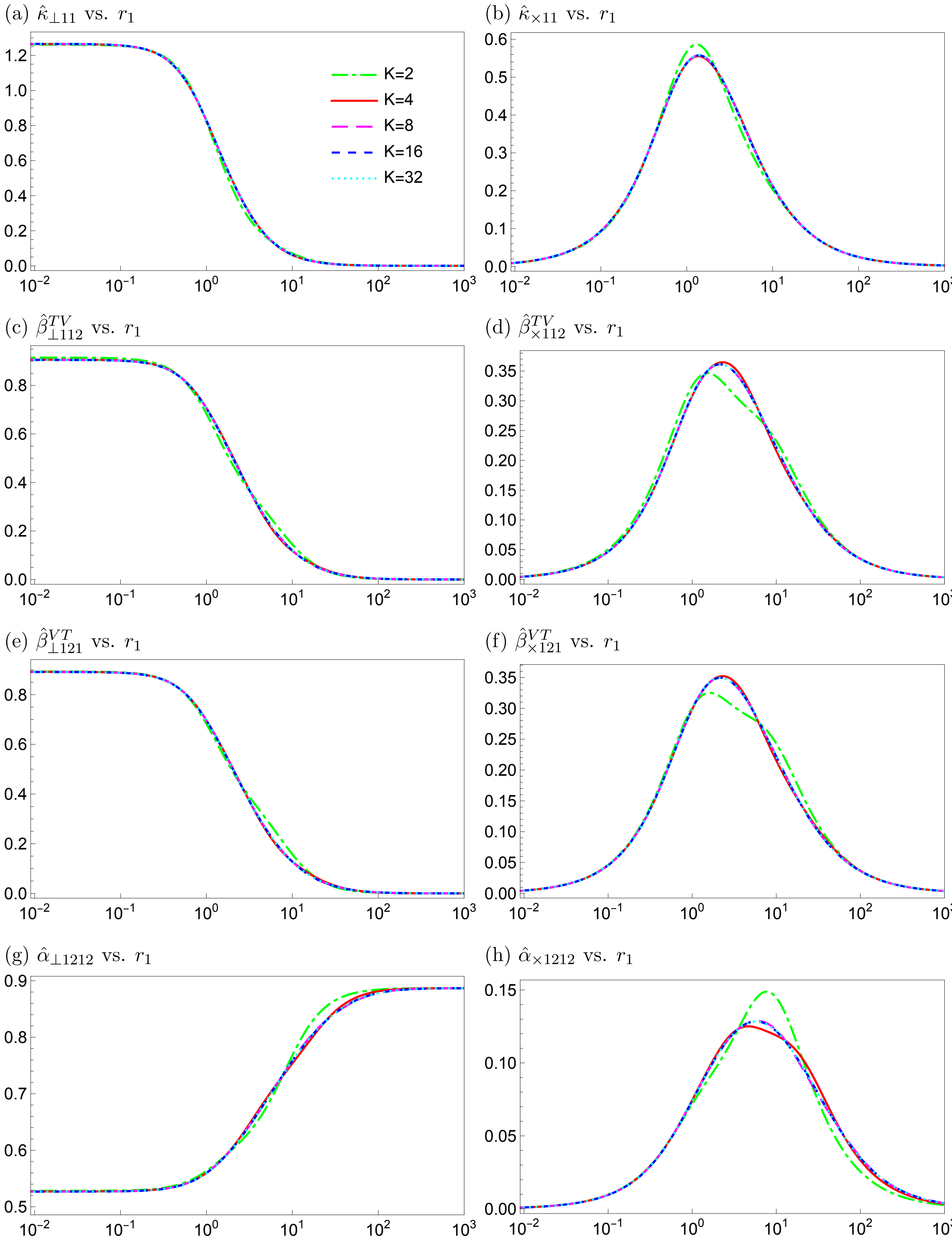

Figure 2. Heat flux and friction closure coefficients of ion species 1 due to ∇T_1 and $\mathbf{V}_{12}$. The coefficients are presented for various number of moments: $K = 2$ (green, dash-dotted), 3 (red, solid), 8 (magenta, long-dashed), 16 (blue, dashed), and 32 (cyan, dotted) throughout figures 2-6.

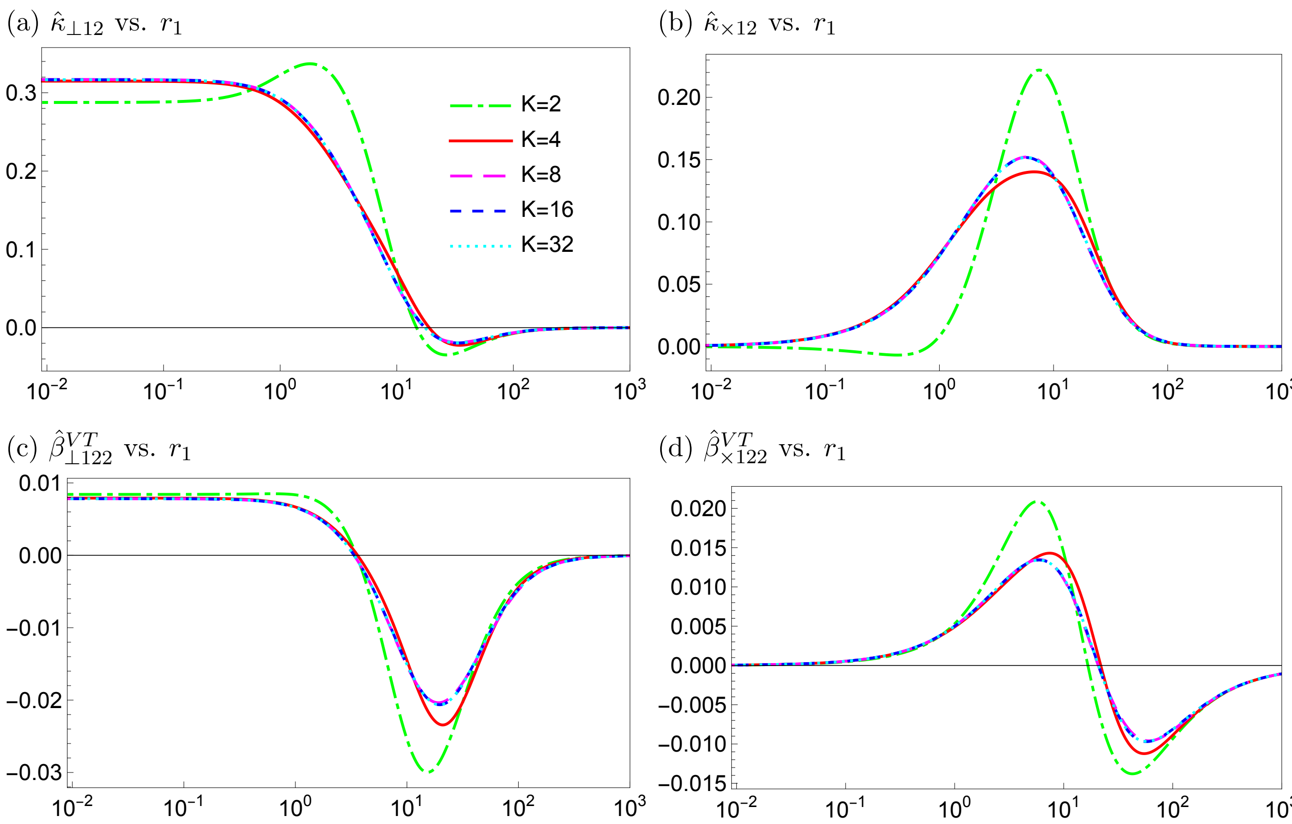

Figure 3. Heat flux and friction closure coefficients of ion species 1 due to ∇T_2.

when the coefficient values cross the zero line where the coefficient values are small enough not to affect the physical results. The change of coefficients from $K = 16$ to 32 calculations is at most 0.29% for $\hat{\beta}^{TV}_{\times 22}$ and much smaller for other coefficients when the absolute value of coefficient is greater than 0.01. For the example parameters, the $K = 16$ coefficients can be considered as practically exact ones.

Finally, we discuss the importance of the two-temperature formulation of calculating closure coefficients. Figures 7 and 8 display the thermal conductivity and viscosity coefficients for various temperature ratios. As evident from these figures, the closure coefficients for different temperatures can differ significantly from those for $T_2/T_1 = 1$, one-temperature calculation. For instance, when $T_2/T_1 = 0.8$ (a difference of only 20%), $\hat{\kappa}_{\perp 12}$ obtained from $T_2/T_1 = 1$ is overestimated by a factor of 2. The errors of one-temperature calculation can be attributed to the temperature-ratio dependence of collision coefficients, combined with the collision-time ratio $z_{ab} = \tau_{aa}/\tau_{ab}$, which is sensitive to the density ratio.

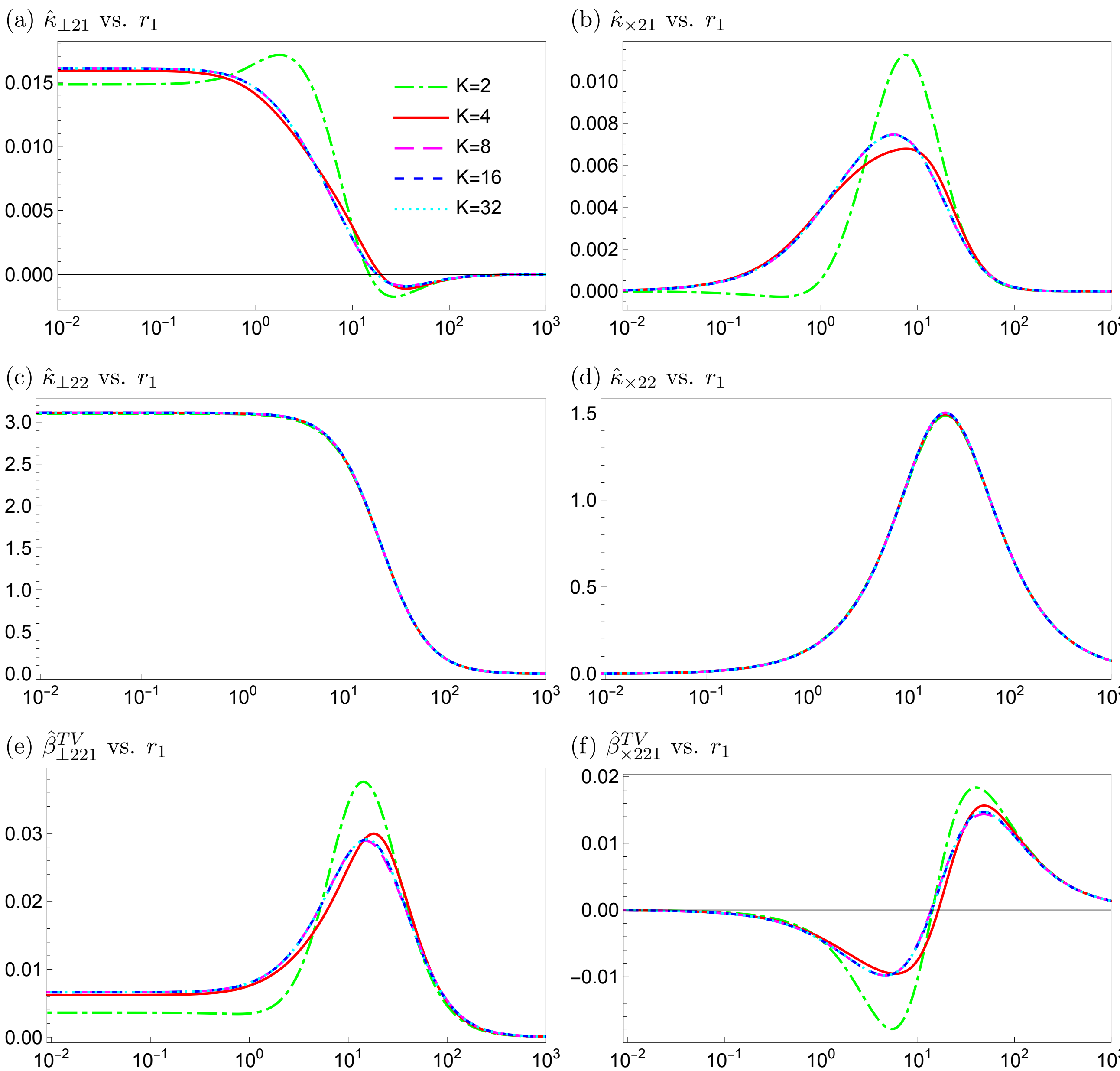

Figure 4. Heat flux and friction closure coefficients of ion species 2.

VI. DISCUSSION

A general method for calculating closure coefficients for high-collisionality multi-ion plasmas has been presented. The necessary collision coefficients for $K = 4$ (corresponding to the 41 moment model) calculations are presented in the appendix. Even higher order collision coefficients necessary for more accurate closure coefficients can be calculated from explicit formulas derived in Ref. [7]. Note that the collision coefficients obtained from the Landau

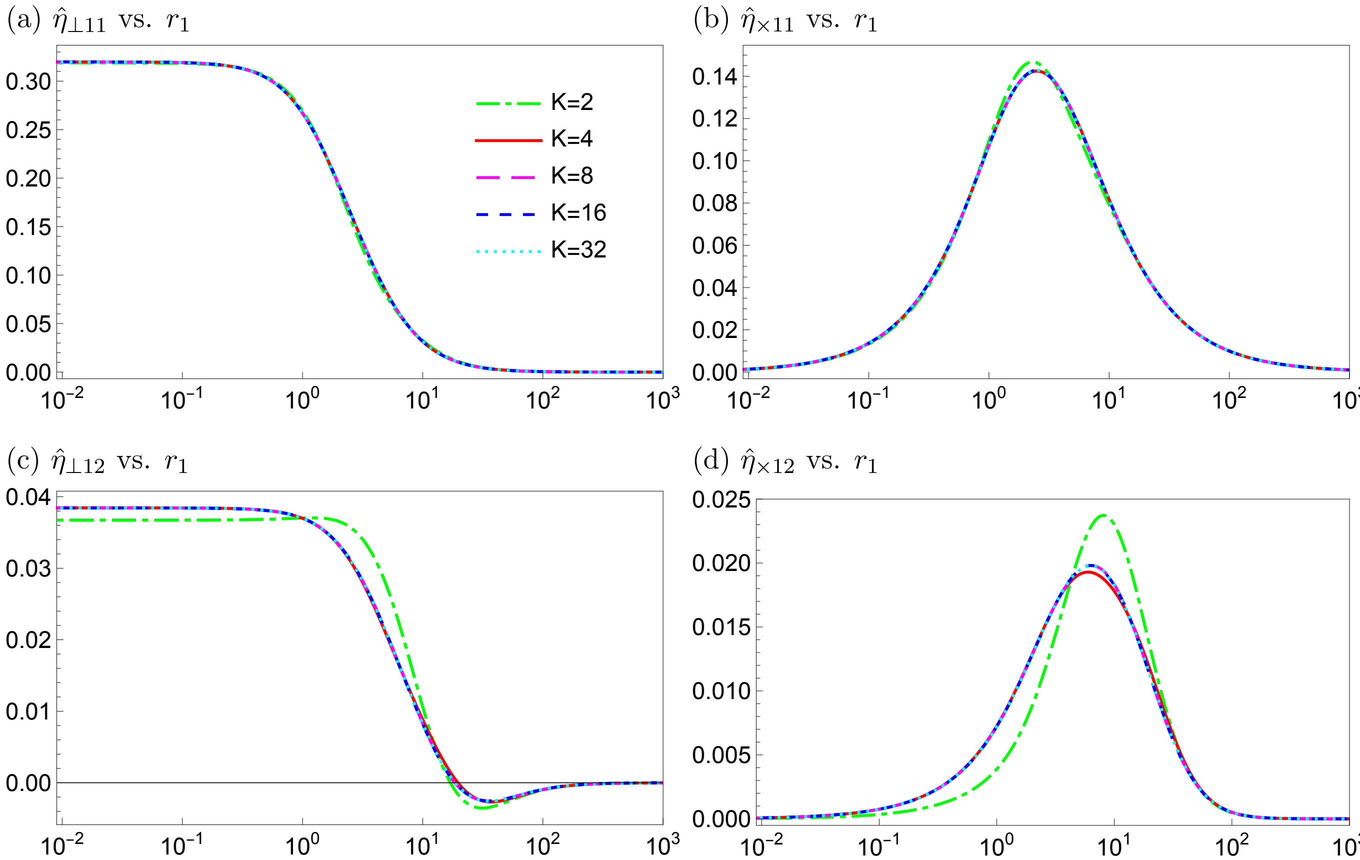

Figure 5. Viscosity closure coefficients of ion species 1 due to ion species 1 and 2 ($\hat{\eta}_\perp = \hat{\eta}^2$ and $\hat{\eta}_\times = \hat{\eta}^4$).

collision operator in this work are slightly different from those obtained from the Boltzmann operator [23]. The formulation developed here is useful for a wide range of weakly coupled plasmas where the Landau operator is valid. Although the formulation may produce analytic results for the closure relations of plasmas with given parameters, explicit expressions of $K = 4$ calculations are too complex to be written out. For practical applications one may use collision coefficients of Appendix A to calculate the collision coefficients as described in this work.

The convergence analysis in Sec. V for the example calculations shows that $K = 8$ calculations produce nearly converged closure coefficients. However, in some parameter ranges, convergence is slow and requires $K = 16$ or $K = 32$ calculations for accurate results. Although high K calculations are useful for theoretical analyses, performing those calculations at every time step can be computationally inefficient in numerical simulations of fluid equations. To overcome this, it is necessary to investigate the convergence behavior of the

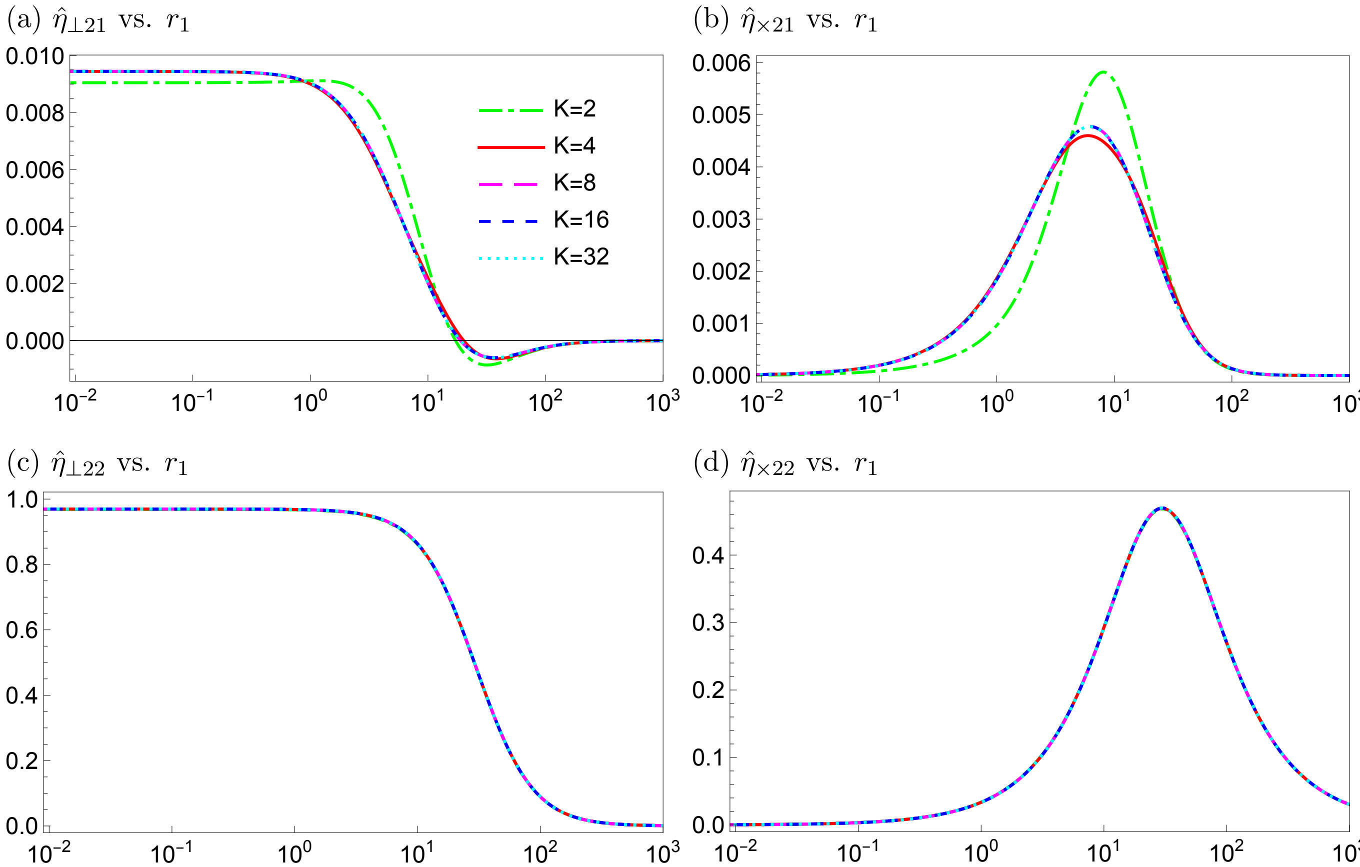

Figure 6. Viscosity closure coefficients of ion species 2.

coefficients across the desired parameter range and develop fitting functions for the convergent closure coefficients. These fitting formulas can then be conveniently used in numerical simulations, avoiding the need for time-consuming closure coefficient calculations at every time step. The development of fitting formulas for convergent closures is ongoing and will be presented in the near future.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available upon request from the author.

ACKNOWLEDGMENTS

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Science under Award Numbers DE-SC0022048 and DE-FG02-

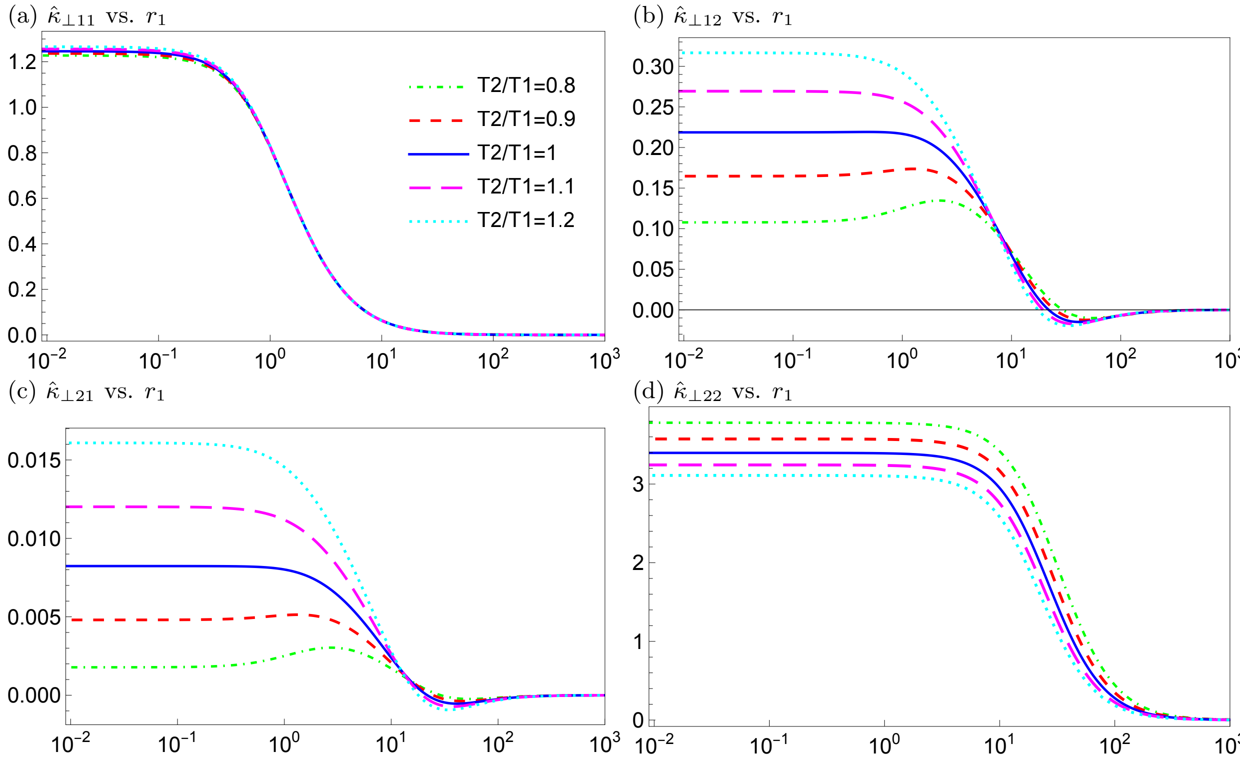

Figure 7. Perpendicular thermal conductivity closure coefficients of two ion species. The coefficients are presented for various temperature ratios: $T_2/T_1 = 0.8$ (green, dash-dotted), 0.9 (red, dashed), 1 (blue, solid), 1.1 (magenta, long-dashed), and 1.2 (cyan, dotted) in figures 7 and 8.

04ER54746. The author would like to thank Dr. Min Uk Lee for reading the manuscript and finding typos.

Appendix A: Collision coefficients

The collision coefficients can be calculated from formulas presented in Ref. [7]. Define a^{lpq} and b^{lpq} by

$$a_{ab}^{lpq} = \frac{3X_{ab}^{l+p+q+1/2}}{\sqrt{\lambda_{lp}\lambda_{lq}}}\mathsf{a}^{lpq}, \tag{A1}$$

$$b_{ab}^{lpq} = \frac{3X_{ab}^{l+p+q+1/2}}{\sqrt{\lambda_{lp}\lambda_{lq}}}\left(\frac{\theta}{\mu}\right)^{l/2+q}\mathsf{b}^{lpq}, \tag{A2}$$

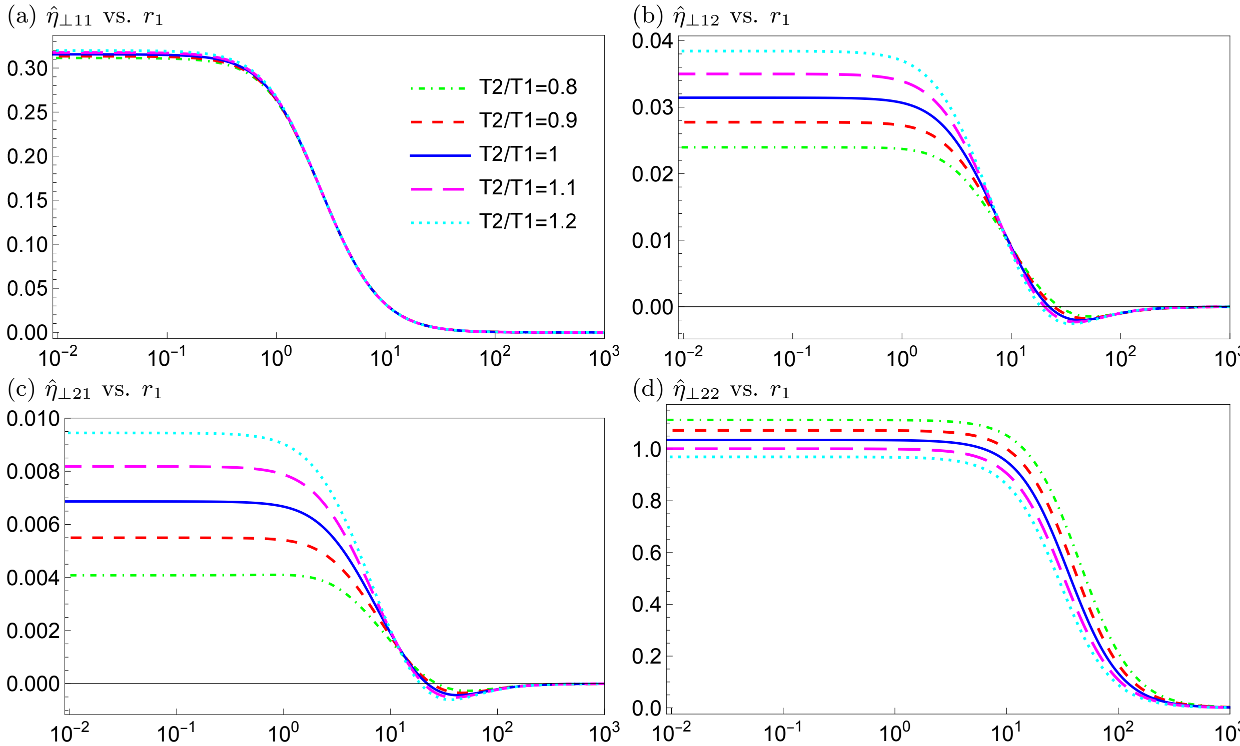

Figure 8. Perpendicular viscosity closure coefficients ($\hat{\eta}_{\perp ij} = \hat{\eta}^2_{ij}$) of two ion species for various temperature ratios.

where $X_{ab} = (1+\theta/\mu)^{-1}$, $\theta = T_b/T_a$, and $\mu = m_b/m_a$. The necessary coefficients for $K = 4$ closure calculations are as follows:

$$
\begin{aligned}
\mathsf{a}^{010} =& \frac{1}{\mu} - \frac{\theta}{\mu}, \\
\mathsf{a}^{011} =& -\frac{\theta}{\mu^2} - \frac{3\theta}{2\mu} + \frac{1}{2\mu}, \\
\mathsf{a}^{012} =& -\frac{3\theta}{2\mu^2} - \frac{15\theta}{8\mu} + \frac{3}{8\mu}, \\
\mathsf{a}^{013} =& -\frac{15\theta}{8\mu^2} - \frac{35\theta}{16\mu} + \frac{5}{16\mu}, \\
\mathsf{a}^{014} =& -\frac{35\theta}{16\mu^2} - \frac{315\theta}{128\mu} + \frac{35}{128\mu}, \\
\mathsf{a}^{015} =& -\frac{315\theta}{128\mu^2} - \frac{693\theta}{256\mu} + \frac{63}{256\mu},
\end{aligned}
$$

$$\mathsf{a}^{020} = \frac{3}{2\mu} - \frac{3\theta}{2\mu},$$

$$\mathsf{a}^{021} = -\frac{5\theta^3}{2\mu^3} + \frac{5\theta^2}{2\mu^3} - \frac{2\theta^2}{\mu^2} + \frac{\theta}{2\mu^2} - \frac{13\theta}{4\mu} + \frac{7}{4\mu},$$

$$\mathsf{a}^{022} = -\frac{5\theta^3}{2\mu^4} - \frac{21\theta^3}{4\mu^3} + \frac{3\theta^2}{4\mu^3} - \frac{3\theta^2}{\mu^2} - \frac{9\theta}{4\mu^2} - \frac{69\theta}{16\mu} + \frac{17}{16\mu},$$

$$\mathsf{a}^{023} = -\frac{21\theta^3}{4\mu^4} - \frac{135\theta^3}{16\mu^3} + \frac{3\theta^2}{16\mu^3} - \frac{15\theta^2}{4\mu^2} - \frac{57\theta}{16\mu^2} - \frac{165\theta}{32\mu} + \frac{27}{32\mu},$$

$$\mathsf{a}^{024} = -\frac{135\theta^3}{16\mu^4} - \frac{385\theta^3}{32\mu^3} - \frac{5\theta^2}{32\mu^3} - \frac{35\theta^2}{8\mu^2} - \frac{145\theta}{32\mu^2} - \frac{1505\theta}{256\mu} + \frac{185}{256\mu},$$

$$\mathsf{a}^{025} = -\frac{385\theta^3}{32\mu^4} - \frac{4095\theta^3}{256\mu^3} - \frac{105\theta^2}{256\mu^3} - \frac{315\theta^2}{64\mu^2} - \frac{1365\theta}{256\mu^2} - \frac{3339\theta}{512\mu} + \frac{329}{512\mu},$$

$$\mathsf{a}^{030} = \frac{15}{8\mu} - \frac{15\theta}{8\mu},$$

$$\mathsf{a}^{031} = -\frac{21\theta^3}{4\mu^3} + \frac{21\theta^2}{4\mu^3} - \frac{3\theta^2}{\mu^2} + \frac{9\theta}{8\mu^2} - \frac{69\theta}{16\mu} + \frac{39}{16\mu},$$

$$\mathsf{a}^{032} = -\frac{35\theta^5}{8\mu^5} + \frac{35\theta^4}{8\mu^5} - \frac{7\theta^4}{\mu^4} + \frac{7\theta^3}{4\mu^4} - \frac{153\theta^3}{8\mu^3} + \frac{87\theta^2}{8\mu^3} - \frac{17\theta^2}{2\mu^2} + \frac{19\theta}{16\mu^2} - \frac{433\theta}{64\mu} + \frac{157}{64\mu},$$

$$\begin{aligned}\mathsf{a}^{033} = &-\frac{35\theta^5}{8\mu^6} - \frac{189\theta^5}{16\mu^5} + \frac{7\theta^4}{16\mu^5} - \frac{27\theta^4}{2\mu^4} - \frac{101\theta^3}{8\mu^4} - \frac{1023\theta^3}{32\mu^3} + \frac{139\theta^2}{32\mu^3} - \frac{93\theta^2}{8\mu^2} - \frac{233\theta}{64\mu^2}\\ &- \frac{1077\theta}{128\mu} + \frac{211}{128\mu},\end{aligned}$$

$$\begin{aligned}\mathsf{a}^{034} = &-\frac{189\theta^5}{16\mu^6} - \frac{1485\theta^5}{64\mu^5} - \frac{135\theta^4}{64\mu^5} - \frac{165\theta^4}{8\mu^4} - \frac{795\theta^3}{32\mu^4} - \frac{2925\theta^3}{64\mu^3} + \frac{135\theta^2}{64\mu^3}\\ &- \frac{225\theta^2}{16\mu^2} - \frac{765\theta}{128\mu^2} - \frac{10005\theta}{1024\mu} + \frac{1389}{1024\mu},\end{aligned}$$

$$\begin{aligned}\mathsf{a}^{035} = &-\frac{1485\theta^5}{64\mu^6} - \frac{5005\theta^5}{128\mu^5} - \frac{605\theta^4}{128\mu^5} - \frac{455\theta^4}{16\mu^4} - \frac{2425\theta^3}{64\mu^4} - \frac{30975\theta^3}{512\mu^3} + \frac{375\theta^2}{512\mu^3}\\ &- \frac{2065\theta^2}{128\mu^2} - \frac{7885\theta}{1024\mu^2} - \frac{224335\theta}{2048\mu} + \frac{2425}{2048\mu},\end{aligned}$$

$$
\begin{aligned}
\mathsf{a}^{040} =& \frac{35}{16\mu} - \frac{35\theta}{16\mu}, \\
\mathsf{a}^{041} =& -\frac{135\theta^3}{16\mu^3} + \frac{135\theta^2}{16\mu^3} - \frac{15\theta^2}{4\mu^2} + \frac{25\theta}{16\mu^2} - \frac{165\theta}{32\mu} + \frac{95}{32\mu}, \\
\mathsf{a}^{042} =& -\frac{189\theta^5}{16\mu^5} + \frac{189\theta^4}{16\mu^5} - \frac{27\theta^4}{2\mu^4} + \frac{81\theta^3}{16\mu^4} - \frac{1023\theta^3}{32\mu^3} + \frac{633\theta^2}{32\mu^3} - \frac{93\theta^2}{8\mu^2} + \frac{87\theta}{32\mu^2} \\
& - \frac{1077\theta}{128\mu} + \frac{417}{128\mu}, \\
\mathsf{a}^{043} =& -\frac{105\theta^7}{16\mu^7} + \frac{105\theta^6}{16\mu^7} - \frac{63\theta^6}{4\mu^6} + \frac{63\theta^5}{16\mu^6} - \frac{1935\theta^5}{32\mu^5} + \frac{1125\theta^4}{32\mu^5} - \frac{215\theta^4}{4\mu^4} + \frac{265\theta^3}{32\mu^4} \\
& - \frac{9165\theta^3}{128\mu^3} + \frac{3585\theta^2}{128\mu^3} - \frac{705\theta^2}{32\mu^2} + \frac{255\theta}{128\mu^2} - \frac{29957\theta}{256\mu} + \frac{803}{256\mu}, \\
\mathsf{a}^{044} =& -\frac{105\theta^7}{16\mu^8} - \frac{693\theta^7}{32\mu^7} - \frac{21\theta^6}{32\mu^7} - \frac{297\theta^6}{8\mu^6} - \frac{1251\theta^5}{32\mu^6} - \frac{15675\theta^5}{128\mu^5} + \frac{1055\theta^4}{128\mu^5} - \frac{1425\theta^4}{16\mu^4} \\
& - \frac{4645\theta^3}{128\mu^4} - \frac{27315\theta^3}{256\mu^3} + \frac{3345\theta^2}{256\mu^3} - \frac{1821\theta^2}{64\mu^2} - \frac{1313\theta}{256\mu^2} - \frac{282257\theta}{2048\mu} + \frac{4601}{2048\mu}, \\
\mathsf{a}^{045} =& -\frac{693\theta^7}{32\mu^8} - \frac{6435\theta^7}{128\mu^7} - \frac{1089\theta^6}{128\mu^7} - \frac{2145\theta^6}{32\mu^6} - \frac{11847\theta^5}{128\mu^6} - \frac{52455\theta^5}{256\mu^5} - \frac{1695\theta^4}{256\mu^5} - \frac{4035\theta^4}{32\mu^4} \\
& - \frac{17835\theta^3}{256\mu^4} - \frac{292485\theta^3}{2048\mu^3} + \frac{15693\theta^2}{2048\mu^3} - \frac{17205\theta^2}{512\mu^2} - \frac{177709\theta}{2048\mu^2} - \frac{642275\theta}{4096\mu} + \frac{7761}{4096\mu},
\end{aligned}
$$

$$\mathsf{a}^{050} = \frac{315}{128\mu} - \frac{315\theta}{128\mu},$$

$$\mathsf{a}^{051} = -\frac{385\theta^3}{32\mu^3} + \frac{385\theta^2}{32\mu^3} - \frac{35\theta^2}{8\mu^2} + \frac{245\theta}{128\mu^2} - \frac{1505\theta}{256\mu} + \frac{875}{256\mu},$$

$$\mathsf{a}^{052} = -\frac{1485\theta^5}{64\mu^5} + \frac{1485\theta^4}{64\mu^5} - \frac{165\theta^4}{8\mu^4} + \frac{275\theta^3}{32\mu^4} - \frac{2925\theta^3}{64\mu^3} + \frac{1875\theta^2}{64\mu^3} - \frac{225\theta^2}{16\mu^2} + \frac{975\theta}{256\mu^2} - \frac{10005\theta}{1024\mu} + \frac{3985}{1024\mu},$$

$$\mathsf{a}^{053} = -\frac{693\theta^7}{32\mu^7} + \frac{693\theta^6}{32\mu^7} - \frac{297\theta^6}{8\mu^6} + \frac{891\theta^5}{64\mu^6} - \frac{15675\theta^5}{128\mu^5} + \frac{10065\theta^4}{128\mu^5} - \frac{1425\theta^4}{16\mu^4} + \frac{1455\theta^3}{64\mu^4} - \frac{27315\theta^3}{256\mu^3} + \frac{12015\theta^2}{256\mu^3} - \frac{1821\theta^2}{64\mu^2} + \frac{4731\theta}{1024\mu^2} - \frac{28257\theta}{2048\mu} + \frac{8247}{2048\mu},$$

$$\mathsf{a}^{054} = -\frac{1155\theta^9}{128\mu^9} + \frac{1155\theta^8}{128\mu^9} - \frac{231\theta^8}{8\mu^8} + \frac{231\theta^7}{32\mu^8} - \frac{9141\theta^7}{64\mu^7} + \frac{5379\theta^6}{64\mu^7} - \frac{3047\theta^6}{16\mu^6} + \frac{3949\theta^5}{128\mu^6} - \frac{182195\theta^5}{512\mu^5} + \frac{73895\theta^4}{512\mu^5} - \frac{14015\theta^4}{64\mu^4} + \frac{5945\theta^3}{256\mu^4} - \frac{98209\theta^3}{512\mu^3} + \frac{29011\theta^2}{512\mu^3} - \frac{5777\theta^2}{128\mu^2} + \frac{5903\theta}{2048\mu^2} - \frac{288473\theta}{16384\mu} + \frac{62417}{16384\mu},$$

$$\mathsf{a}^{055} = -\frac{1155\theta^9}{128\mu^{10}} - \frac{9009\theta^9}{256\mu^9} - \frac{693\theta^8}{256\mu^9} - \frac{1287\theta^8}{16\mu^8} - \frac{5841\theta^7}{64\mu^8} - \frac{87087\theta^7}{256\mu^7} + \frac{1771\theta^6}{256\mu^7} - \frac{23751\theta^6}{64\mu^6} - \frac{89691\theta^5}{512\mu^6} - \frac{635355\theta^5}{1024\mu^5} + \frac{46725\theta^4}{1024\mu^5} - \frac{42357\theta^4}{128\mu^4} - \frac{40901\theta^3}{512\mu^4} - \frac{1090467\theta^3}{4096\mu^3} + \frac{119931\theta^2}{4096\mu^3} - \frac{57393\theta^2}{1024\mu^2} - \frac{109641\theta}{16384\mu^2} - \frac{670407\theta}{32768\mu} + \frac{93461}{32768\mu},$$

$$\mathsf{b}^{010} = \frac{1}{\mu} - \frac{\theta}{\mu},$$

$$\mathsf{b}^{011} = -\frac{\theta}{2\mu} + \frac{3}{2\mu} + 1,$$

$$\mathsf{b}^{012} = -\frac{3\theta}{8\mu} + \frac{15}{8\mu} + \frac{3}{2},$$

$$\mathsf{b}^{013} = -\frac{5\theta}{16\mu} + \frac{35}{16\mu} + \frac{15}{8},$$

$$\mathsf{b}^{014} = -\frac{35\theta}{128\mu} + \frac{315}{128\mu} + \frac{35}{16},$$

$$\mathsf{b}^{015} = -\frac{63\theta}{256\mu} + \frac{693}{256\mu} + \frac{315}{128},$$

$$\mathsf{b}^{020} = \frac{3}{2\mu} - \frac{3\theta}{2\mu},$$
$$\mathsf{b}^{021} = -\frac{9\theta}{4\mu} + \frac{15}{4\mu} + \frac{3}{2},$$
$$\mathsf{b}^{022} = -\frac{45\theta}{16\mu} + \frac{105}{16\mu} + \frac{15}{4},$$
$$\mathsf{b}^{023} = -\frac{105\theta}{32\mu} + \frac{315}{32\mu} + \frac{105}{16},$$
$$\mathsf{b}^{024} = -\frac{945\theta}{256\mu} + \frac{3465}{256\mu} + \frac{315}{32},$$
$$\mathsf{b}^{025} = -\frac{2079\theta}{512\mu} + \frac{9009}{512\mu} + \frac{3465}{256},$$

$$\mathsf{b}^{030} = \frac{15}{8\mu} - \frac{15\theta}{8\mu},$$
$$\mathsf{b}^{031} = -\frac{75\theta}{16\mu} + \frac{105}{16\mu} + \frac{15}{8},$$
$$\mathsf{b}^{032} = -\frac{525\theta}{64\mu} + \frac{945}{64\mu} + \frac{105}{16},$$
$$\mathsf{b}^{033} = -\frac{1575\theta}{128\mu} + \frac{3465}{128\mu} + \frac{945}{64},$$
$$\mathsf{b}^{034} = -\frac{173325\theta}{1024\mu} + \frac{45045}{1024\mu} + \frac{3465}{128},$$
$$\mathsf{b}^{035} = -\frac{450045\theta}{2048\mu} + \frac{135135}{2048\mu} + \frac{45045}{1024},$$

$$\mathsf{b}^{040} = \frac{35}{16\mu} - \frac{35\theta}{16\mu},$$
$$\mathsf{b}^{041} = -\frac{245\theta}{32\mu} + \frac{315}{32\mu} + \frac{35}{16},$$
$$\mathsf{b}^{042} = -\frac{2205\theta}{128\mu} + \frac{3465}{128\mu} + \frac{315}{32},$$
$$\mathsf{b}^{043} = -\frac{8085\theta}{256\mu} + \frac{15015}{256\mu} + \frac{3465}{128},$$
$$\mathsf{b}^{044} = -\frac{1051105\theta}{2048\mu} + \frac{225225}{2048\mu} + \frac{15015}{256},$$
$$\mathsf{b}^{045} = -\frac{3153315\theta}{4096\mu} + \frac{765765}{4096\mu} + \frac{225225}{2048},$$

$$\mathsf{b}^{050} = \frac{315}{128\mu} - \frac{315\theta}{128\mu},$$
$$\mathsf{b}^{051} = -\frac{2835\theta}{256\mu} + \frac{3465}{256\mu} + \frac{315}{128},$$
$$\mathsf{b}^{052} = -\frac{311185\theta}{1024\mu} + \frac{45045}{1024\mu} + \frac{3465}{256},$$
$$\mathsf{b}^{053} = -\frac{1351135\theta}{2048\mu} + \frac{225225}{2048\mu} + \frac{45045}{1024},$$
$$\mathsf{b}^{054} = -\frac{20270025\theta}{16384\mu} + \frac{3828825}{16384\mu} + \frac{225225}{2048},$$
$$\mathsf{b}^{055} = -\frac{68918885\theta}{32768\mu} + \frac{14549535}{32768\mu} + \frac{3828825}{16384},$$

$$\mathsf{a}^{100} = -\frac{1}{2\mu} - \frac{1}{2},$$
$$\mathsf{a}^{101} = -\frac{3}{4\mu} - \frac{3}{4},$$
$$\mathsf{a}^{102} = -\frac{15}{16\mu} - \frac{15}{16},$$
$$\mathsf{a}^{103} = -\frac{35}{32\mu} - \frac{35}{32},$$
$$\mathsf{a}^{104} = -\frac{315}{256\mu} - \frac{315}{256},$$

$$\mathsf{a}^{110} = -\frac{5\theta^2}{2\mu^2} + \frac{5\theta}{2\mu^2} - \frac{\theta}{\mu} + \frac{1}{4\mu} - \frac{3}{4},$$
$$\mathsf{a}^{111} = -\frac{15\theta^2}{4\mu^3} - \frac{13\theta^2}{2\mu^2} + \frac{3\theta}{4\mu^2} - \frac{5\theta}{2\mu} - \frac{9}{8\mu} - \frac{13}{8},$$
$$\mathsf{a}^{112} = -\frac{63\theta^2}{8\mu^3} - \frac{177\theta^2}{16\mu^2} + \frac{3\theta}{16\mu^2} - \frac{27\theta}{8\mu} - \frac{57}{32\mu} - \frac{69}{32},$$
$$\mathsf{a}^{113} = -\frac{405\theta^2}{32\mu^3} - \frac{65\theta^2}{4\mu^2} - \frac{5\theta}{32\mu^2} - \frac{65\theta}{16\mu} - \frac{145}{64\mu} - \frac{165}{64},$$
$$\mathsf{a}^{114} = -\frac{1155\theta^2}{64\mu^3} - \frac{5635\theta^2}{256\mu^2} - \frac{105\theta}{256\mu^2} - \frac{595\theta}{128\mu} - \frac{1365}{512\mu} - \frac{1505}{512},$$

$$\mathsf{a}^{120} = -\frac{21\theta^2}{4\mu^2} + \frac{21\theta}{4\mu^2} - \frac{3\theta}{2\mu} + \frac{9}{16\mu} - \frac{15}{16},$$

$$\mathsf{a}^{121} = -\frac{35\theta^4}{4\mu^4} + \frac{35\theta^3}{4\mu^4} - \frac{21\theta^3}{2\mu^3} + \frac{21\theta^2}{8\mu^3} - \frac{87\theta^2}{4\mu^2} + \frac{87\theta}{8\mu^2} - \frac{23\theta}{4\mu} + \frac{19}{32\mu} - \frac{69}{32},$$

$$\mathsf{a}^{122} = -\frac{175\theta^4}{16\mu^5} - \frac{413\theta^4}{16\mu^4} + \frac{7\theta^3}{8\mu^4} - \frac{95\theta^3}{4\mu^3} - \frac{303\theta^2}{16\mu^3} - \frac{1329\theta^2}{32\mu^2} + \frac{139\theta}{32\mu^2} - \frac{161\theta}{16\mu} - \frac{233}{128\mu} - \frac{433}{128},$$

$$\mathsf{a}^{123} = -\frac{945\theta^4}{32\mu^5} - \frac{837\theta^4}{16\mu^4} - \frac{135\theta^3}{32\mu^4} - \frac{603\theta^3}{16\mu^3} - \frac{2385\theta^2}{64\mu^3} - \frac{987\theta^2}{16\mu^2} + \frac{135\theta}{64\mu^2} - \frac{411\theta}{32\mu} - \frac{765}{256\mu} - \frac{1077}{256},$$

$$\mathsf{a}^{124} = -\frac{7425\theta^4}{128\mu^5} - \frac{11495\theta^4}{128\mu^4} - \frac{605\theta^3}{64\mu^4} - \frac{1695\theta^3}{32\mu^3} - \frac{7275\theta^2}{128\mu^3} - \frac{42675\theta^2}{512\mu^2} + \frac{375\theta}{512\mu^2}$$
$$- \frac{3865\theta}{256\mu} - \frac{7885}{2048\mu} - \frac{10005}{2048},$$

$$\mathsf{a}^{130} = -\frac{135\theta^2}{16\mu^2} + \frac{135\theta}{16\mu^2} - \frac{15\theta}{8\mu} + \frac{25}{32\mu} - \frac{35}{32},$$

$$\mathsf{a}^{131} = -\frac{189\theta^4}{8\mu^4} + \frac{189\theta^3}{8\mu^4} - \frac{81\theta^3}{4\mu^3} + \frac{243\theta^2}{32\mu^3} - \frac{579\theta^2}{16\mu^2} + \frac{633\theta}{32\mu^2} - \frac{123\theta}{16\mu} + \frac{87}{64\mu} - \frac{165}{64},$$

$$\mathsf{a}^{132} = -\frac{315\theta^6}{16\mu^6} + \frac{315\theta^5}{16\mu^6} - \frac{315\theta^5}{8\mu^5} + \frac{315\theta^4}{32\mu^5} - \frac{4059\theta^4}{32\mu^4} + \frac{1125\theta^3}{16\mu^4} - \frac{699\theta^3}{8\mu^3} + \frac{795\theta^2}{64\mu^3}$$
$$- \frac{11211\theta^2}{128\mu^2} + \frac{3585\theta}{128\mu^2} - \frac{1077\theta}{64\mu} + \frac{255}{256\mu} - \frac{1077}{256},$$

$$\mathsf{a}^{133} = -\frac{735\theta^6}{32\mu^7} - \frac{273\theta^6}{4\mu^6} - \frac{63\theta^5}{32\mu^6} - \frac{1611\theta^5}{16\mu^5} - \frac{6255\theta^4}{64\mu^5} - \frac{8805\theta^4}{32\mu^4} + \frac{1055\theta^3}{64\mu^4}$$
$$- \frac{5135\theta^3}{32\mu^3} - \frac{13935\theta^2}{256\mu^3} - \frac{285\theta^2}{2\mu^2} + \frac{3345\theta}{256\mu^2} - \frac{3231\theta}{128\mu} - \frac{1313}{512\mu} - \frac{2957}{512},$$

$$\mathsf{a}^{134} = -\frac{4851\theta^6}{64\mu^7} - \frac{20691\theta^6}{128\mu^6} - \frac{3267\theta^5}{128\mu^6} - \frac{119913\theta^5}{64\mu^5} - \frac{59235\theta^4}{256\mu^5} - \frac{120585\theta^4}{256\mu^4} - \frac{1695\theta^3}{128\mu^4}$$
$$- \frac{14955\theta^3}{64\mu^3} - \frac{53505\theta^2}{512\mu^3} - \frac{401745\theta^2}{2048\mu^2} + \frac{156933\theta}{2048\mu^2} - \frac{317773\theta}{1024\mu} - \frac{17709}{4096\mu} - \frac{28257}{4096}$$

$$
\begin{aligned}
\mathsf{a}^{140} =& -\frac{385\theta^2}{32\mu^2}+\frac{385\theta}{32\mu^2}-\frac{35\theta}{16\mu}+\frac{245}{256\mu}-\frac{315}{256},\\
\mathsf{a}^{141} =& -\frac{1485\theta^4}{32\mu^4}+\frac{1485\theta^3}{32\mu^4}-\frac{495\theta^3}{16\mu^3}+\frac{825\theta^2}{64\mu^3}-\frac{1655\theta^2}{32\mu^2}+\frac{1875\theta}{64\mu^2}-\frac{295\theta}{32\mu}+\frac{975}{512\mu}-\frac{1505}{512},\\
\mathsf{a}^{142} =& -\frac{2079\theta^6}{32\mu^6}+\frac{2079\theta^5}{32\mu^6}-\frac{1485\theta^5}{16\mu^5}+\frac{4455\theta^4}{128\mu^5}-\frac{32835\theta^4}{128\mu^4}+\frac{10065\theta^3}{64\mu^4}-\frac{4605\theta^3}{32\mu^3}+\frac{4365\theta^2}{128\mu^3}\\
&-\frac{33165\theta^2}{256\mu^2}+\frac{12015\theta}{256\mu^2}-\frac{2721\theta}{128\mu}+\frac{4731}{2048\mu}-\frac{10005}{2048},\\
\mathsf{a}^{143} =& -\frac{1155\theta^8}{32\mu^8}+\frac{1155\theta^7}{32\mu^8}-\frac{1617\theta^7}{16\mu^7}+\frac{1617\theta^6}{64\mu^7}-\frac{7029\theta^6}{16\mu^6}+\frac{16137\theta^5}{64\mu^6}-\frac{15829\theta^5}{32\mu^5}+\frac{19745\theta^4}{256\mu^5}\\
&-\frac{98935\theta^4}{128\mu^4}+\frac{73895\theta^3}{256\mu^4}-\frac{47745\theta^3}{128\mu^3}+\frac{17835\theta^2}{512\mu^3}-\frac{31381\theta^2}{128\mu^2}+\frac{290011\theta}{512\mu^2}-\frac{9419\theta}{256\mu}+\frac{5903}{4096\mu}\\
&-\frac{28257}{4096},\\
\mathsf{a}^{144} =& -\frac{103395\theta^8}{256\mu^9}-\frac{371191\theta^8}{256\mu^8}-\frac{693\theta^7}{64\mu^8}-\frac{9471\theta^7}{32\mu^7}-\frac{408887\theta^6}{128\mu^7}-\frac{279543\theta^6}{256\mu^6}+\frac{5313\theta^5}{256\mu^6}\\
&-\frac{1309943\theta^5}{128\mu^5}-\frac{4484555\theta^4}{1024\mu^5}-\frac{14529905\theta^4}{1024\mu^4}+\frac{467225\theta^3}{512\mu^4}-\frac{1551101\theta^3}{256\mu^3}-\frac{1227703\theta^2}{1024\mu^3}\\
&-\frac{1483303\theta^2}{4096\mu^2}+\frac{1199931\theta}{4096\mu^2}-\frac{1036609\theta}{2048\mu}-\frac{109641}{32768\mu}-\frac{288473}{32768},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{b}^{100} =& \frac{1}{2\mu}+\frac{1}{2},\\
\mathsf{b}^{101} =& \frac{3}{4\mu}+\frac{3}{4},\\
\mathsf{b}^{102} =& \frac{15}{16\mu}+\frac{15}{16},\\
\mathsf{b}^{103} =& \frac{35}{32\mu}+\frac{35}{32},\\
\mathsf{b}^{104} =& \frac{315}{256\mu}+\frac{315}{256},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{b}^{110} =& -\frac{3\theta}{2\mu}+\frac{9}{4\mu}+\frac{3}{4},\\
\mathsf{b}^{111} =& -\frac{9\theta}{4\mu}+\frac{45}{8\mu}+\frac{27}{8},\\
\mathsf{b}^{112} =& -\frac{45\theta}{16\mu}+\frac{315}{32\mu}+\frac{225}{32},\\
\mathsf{b}^{113} =& -\frac{105\theta}{32\mu}+\frac{945}{64\mu}+\frac{735}{64},\\
\mathsf{b}^{114} =& -\frac{945\theta}{256\mu}+\frac{10395}{512\mu}+\frac{8505}{512},
\end{aligned}
$$

$$\mathsf{b}^{120} = -\frac{15\theta}{4\mu} + \frac{75}{16\mu} + \frac{15}{16},$$
$$\mathsf{b}^{121} = -\frac{75\theta}{8\mu} + \frac{525}{32\mu} + \frac{225}{32},$$
$$\mathsf{b}^{122} = -\frac{525\theta}{32\mu} + \frac{4725}{128\mu} + \frac{2625}{128},$$
$$\mathsf{b}^{123} = -\frac{1575\theta}{64\mu} + \frac{17325}{256\mu} + \frac{11025}{256},$$
$$\mathsf{b}^{124} = -\frac{173250\theta}{512\mu} + \frac{225225}{2048\mu} + \frac{155925}{2048},$$

$$\mathsf{b}^{130} = -\frac{105\theta}{16\mu} + \frac{245}{32\mu} + \frac{35}{32},$$
$$\mathsf{b}^{131} = -\frac{735\theta}{32\mu} + \frac{2205}{64\mu} + \frac{735}{64},$$
$$\mathsf{b}^{132} = -\frac{6615\theta}{128\mu} + \frac{24255}{256\mu} + \frac{11025}{256},$$
$$\mathsf{b}^{133} = -\frac{24255\theta}{256\mu} + \frac{105105}{512\mu} + \frac{56595}{512},$$
$$\mathsf{b}^{134} = -\frac{315315\theta}{2048\mu} + \frac{1576575}{4096\mu} + \frac{945945}{4096},$$

$$\mathsf{b}^{140} = -\frac{315\theta}{32\mu} + \frac{2835}{256\mu} + \frac{315}{256},$$
$$\mathsf{b}^{141} = -\frac{2835\theta}{64\mu} + \frac{31185}{512\mu} + \frac{8505}{512},$$
$$\mathsf{b}^{142} = -\frac{31185\theta}{256\mu} + \frac{405405}{2048\mu} + \frac{155925}{2048},$$
$$\mathsf{b}^{143} = -\frac{135135\theta}{512\mu} + \frac{2027025}{4096\mu} + \frac{945945}{4096},$$
$$\mathsf{b}^{144} = -\frac{2027025\theta}{4096\mu} + \frac{34459425}{32768\mu} + \frac{18243225}{32768},$$

$$\mathsf{a}^{200} = -\frac{5\theta}{2\mu^2} - \frac{3\theta}{\mu} - \frac{1}{\mu} - \frac{3}{2},$$
$$\mathsf{a}^{201} = -\frac{21\theta}{4\mu^2} - \frac{6\theta}{\mu} - \frac{3}{2\mu} - \frac{9}{4},$$
$$\mathsf{a}^{202} = -\frac{135\theta}{16\mu^2} - \frac{75\theta}{8\mu} - \frac{15}{8\mu} - \frac{45}{16},$$
$$\mathsf{a}^{203} = -\frac{385\theta}{32\mu^2} - \frac{105\theta}{8\mu} - \frac{35}{16\mu} - \frac{105}{32},$$

$$
\begin{aligned}
\mathsf{a}^{210} &= -\frac{35\theta^3}{4\mu^3}+\frac{35\theta^2}{4\mu^3}-\frac{7\theta^2}{\mu^2}+\frac{7\theta}{4\mu^2}-\frac{8\theta}{\mu}+\frac{1}{2\mu}-\frac{9}{4},\\
\mathsf{a}^{211} &= -\frac{35\theta^3}{2\mu^4}-\frac{273\theta^3}{8\mu^3}-\frac{21\theta^2}{8\mu^3}-\frac{111\theta^2}{4\mu^2}-\frac{117\theta}{8\mu^2}-\frac{105\theta}{4\mu}-\frac{13}{4\mu}-\frac{51}{8},\\
\mathsf{a}^{212} &= -\frac{189\theta^3}{4\mu^4}-\frac{2349\theta^3}{32\mu^3}-\frac{351\theta^2}{32\mu^3}-\frac{189\theta^2}{4\mu^2}-\frac{891\theta}{32\mu^2}-\frac{333\theta}{8\mu}-\frac{81}{16\mu}-\frac{279}{32},\\
\mathsf{a}^{213} &= -\frac{1485\theta^3}{16\mu^4}-\frac{8305\theta^3}{64\mu^3}-\frac{1265\theta^2}{64\mu^3}-\frac{2185\theta^2}{32\mu^2}-\frac{2665\theta}{64\mu^2}-\frac{1835\theta}{32\mu}-\frac{205}{32\mu}-\frac{675}{64},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{a}^{220} &= -\frac{189\theta^3}{8\mu^3}+\frac{189\theta^2}{8\mu^3}-\frac{27\theta^2}{2\mu^2}+\frac{81\theta}{16\mu^2}-\frac{99\theta}{8\mu}+\frac{9}{8\mu}-\frac{45}{16},\\
\mathsf{a}^{221} &= -\frac{315\theta^5}{8\mu^5}+\frac{315\theta^4}{8\mu^5}-\frac{63\theta^4}{\mu^4}+\frac{63\theta^3}{4\mu^4}-\frac{2511\theta^3}{16\mu^3}+\frac{1161\theta^2}{16\mu^3}-\frac{675\theta^2}{8\mu^2}+\frac{297\theta}{32\mu^2}\\
&\quad-\frac{387\theta}{8\mu}+\frac{27}{16\mu}-\frac{279}{32},\\
\mathsf{a}^{222} &= -\frac{945\theta^5}{16\mu^6}-\frac{2457\theta^5}{16\mu^5}-\frac{189\theta^4}{16\mu^5}-\frac{3213\theta^4}{16\mu^4}-\frac{1323\theta^3}{8\mu^4}-\frac{26163\theta^3}{64\mu^3}-\frac{441\theta^2}{64\mu^3}-\frac{1629\theta^2}{8\mu^2}\\
&\quad-\frac{5733\theta}{128\mu^2}-\frac{6543\theta}{64\mu}-\frac{441}{64\mu}-\frac{2115}{128},\\
\mathsf{a}^{223} &= -\frac{6237\theta^5}{32\mu^6}-\frac{24057\theta^5}{64\mu^5}-\frac{4455\theta^4}{64\mu^5}-\frac{12375\theta^4}{32\mu^4}-\frac{12375\theta^3}{32\mu^4}-\frac{92655\theta^3}{128\mu^3}-\frac{7155\theta^2}{128\mu^3}\\
&\quad-\frac{19845\theta^2}{64\mu^2}-\frac{211195\theta}{256\mu^2}-\frac{585\theta}{4\mu}-\frac{1413}{128\mu}-\frac{5463}{256},
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{a}^{230} &= -\frac{1485\theta^3}{32\mu^3}+\frac{1485\theta^2}{32\mu^3}-\frac{165\theta^2}{8\mu^2}+\frac{275\theta}{32\mu^2}-\frac{135\theta}{8\mu}+\frac{25}{16\mu}-\frac{105}{32},\\
\mathsf{a}^{231} &= -\frac{2079\theta^5}{16\mu^5}+\frac{2079\theta^4}{16\mu^5}-\frac{297\theta^4}{2\mu^4}+\frac{891\theta^3}{16\mu^4}-\frac{20031\theta^3}{64\mu^3}+\frac{10461\theta^2}{64\mu^3}-\frac{4317\theta^2}{32\mu^2}+\frac{1599\theta}{64\mu^2}\\
&\quad-\frac{2163\theta}{32\mu}+\frac{123}{32\mu}-\frac{675}{64},\\
\mathsf{a}^{232} &= -\frac{3465\theta^7}{32\mu^7}+\frac{3465\theta^6}{32\mu^7}-\frac{2079\theta^6}{8\mu^6}+\frac{2079\theta^5}{32\mu^6}-\frac{30591\theta^5}{32\mu^5}+\frac{16335\theta^4}{32\mu^5}-\frac{28875\theta^4}{32\mu^4}\\
&\quad+\frac{4125\theta^3}{32\mu^4}-\frac{277155\theta^3}{256\mu^3}+\frac{77535\theta^2}{256\mu^3}-\frac{13455\theta^2}{32\mu^2}+\frac{7065\theta}{256\mu^2}-\frac{5151\theta}{32\mu}+\frac{471}{128\mu}-\frac{5463}{256},\\
\mathsf{a}^{233} &= -\frac{1155\theta^7}{8\mu^8}-\frac{29799\theta^7}{64\mu^7}-\frac{3003\theta^6}{64\mu^7}-\frac{274423\theta^6}{32\mu^6}-\frac{54153\theta^5}{64\mu^6}-\frac{3350049\theta^5}{128\mu^5}-\frac{5995\theta^4}{128\mu^5}\\
&\quad-\frac{137625\theta^4}{64\mu^4}-\frac{97915\theta^3}{128\mu^4}-\frac{1149855\theta^3}{512\mu^3}+\frac{3945\theta^2}{512\mu^3}-\frac{206571\theta^2}{256\mu^2}-\frac{523309\theta}{512\mu^2}-\frac{713349\theta}{256\mu}\\
&\quad-\frac{3077}{256\mu}-\frac{17331}{512},
\end{aligned}
$$

$$\mathsf{b}^{200} = -\frac{\theta}{2\mu} + \frac{3}{2\mu} + 1,$$
$$\mathsf{b}^{201} = -\frac{3\theta}{4\mu} + \frac{15}{4\mu} + 3,$$
$$\mathsf{b}^{202} = -\frac{15\theta}{16\mu} + \frac{105}{16\mu} + \frac{45}{8},$$
$$\mathsf{b}^{203} = -\frac{35\theta}{32\mu} + \frac{315}{32\mu} + \frac{35}{4},$$

$$\mathsf{b}^{210} = -\frac{9\theta}{2\mu} + \frac{15}{2\mu} + 3,$$
$$\mathsf{b}^{211} = -\frac{45\theta}{4\mu} + \frac{105}{4\mu} + 15,$$
$$\mathsf{b}^{212} = -\frac{315\theta}{16\mu} + \frac{945}{16\mu} + \frac{315}{8},$$
$$\mathsf{b}^{213} = -\frac{945\theta}{32\mu} + \frac{3465}{32\mu} + \frac{315}{4},$$

$$\mathsf{b}^{220} = -\frac{225\theta}{16\mu} + \frac{315}{16\mu} + \frac{45}{8}$$
$$\mathsf{b}^{221} = -\frac{1575\theta}{32\mu} + \frac{2835}{32\mu} + \frac{315}{8},$$
$$\mathsf{b}^{222} = -\frac{141175\theta}{128\mu} + \frac{31185}{128\mu} + \frac{8505}{64},$$
$$\mathsf{b}^{223} = -\frac{519750\theta}{256\mu} + \frac{135135}{256\mu} + \frac{10395}{32},$$

$$\mathsf{b}^{230} = -\frac{245\theta}{8\mu} + \frac{315}{8\mu} + \frac{35}{4},$$
$$\mathsf{b}^{231} = -\frac{2205\theta}{16\mu} + \frac{3465}{16\mu} + \frac{315}{4},$$
$$\mathsf{b}^{232} = -\frac{242555\theta}{64\mu} + \frac{45045}{64\mu} + \frac{10395}{32},$$
$$\mathsf{b}^{233} = -\frac{1051050\theta}{128\mu} + \frac{225225}{128\mu} + \frac{15015}{16}.$$

[1] H. Grad, Comm. Pure and Appl. Math. **2**, 331 (1949).

[2] H. Grad, The physics of Fluids **6**, 147 (1963).

[3] R. Balescu, *Transport Processes in Plasmas* (North-Holland, Amsterdam, 1988), vols. 1 and 2.

[4] R. W. Schunk and A. F. Nagy, *Ionosphere* (Cambridge University Press, Cambridge, 2009), 2nd ed.

[5] V. M. Zhdanov, *Transport Processes in Multicomponent Plasma* (Taylor & Francis, London and New York, 2002).

[6] P. Hunana, T. Passot, E. Khomenko, D. Martínez-Gómez, M. Collados, A. Tenerani, G. P. Zank, Y. Maneva, M. L. Goldstein, and G. M. Webb, The Astrophysical Journal Supplement Series **260**, 26 (2022), URL `https://doi.org/10.3847/1538-4365/ac5044`.

[7] J.-Y. Ji and E. D. Held, Phys. Plasmas **13**, 102103 (2006), URL `http://dx.doi.org/10.1063/1.2356320`.

[8] J.-Y. Ji and E. D. Held, Phys. Plasmas **15**, 102101 (2008), URL `http://dx.doi.org/10.1063/1.2977983`.

[9] R. D. Hazeltine and J. D. Meiss, *Plasma Confinement* (Dover Pub., Inc., New York, 2003).

[10] S. I. Braginskii, Sov. Phys. JETP **6**, 358 (1958).

[11] S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), vol. 1, p. 205.

[12] E. M. Epperlein and M. G. Haines, The Physics of fluids **29**, 1029 (1986).

[13] J.-Y. Ji and E. D. Held, Phys. Plasmas **20**, 042114 (2013), URL `http://dx.doi.org/10.1063/1.4801022`.

J.-Y. Ji and E. D. Held, Phys. Plasmas **23**, 019901 (2016), URL `http://dx.doi.org/10.1063/1.4939473`.

[14] N. A. Bobrova, E. Lazzaro, and P. V. Sasorov, Phys. Plasmas **12**, 022105 (2005).

[15] N. A. Bobrova, A. É. Kocharyan, and P. V. Sasorov, Plasma Physics Reports **33**, 714 (2007), URL `https://doi.org/10.1134/S1063780X07090024`.

[16] I. V. Fomin, N. A. Bobrova, and P. V. Sasorov, Plasma Physics Reports **43**, 521 (2017).

[17] S. O. Makarov, D. P. Coster, V. A. Rozhansky, A. A. Stepanenko, V. M. Zhdanov, E. G. Kaveeva, I. Y. Senichenkov, and X. Bonnin, Physics of Plasmas **28**, 062308 (2021), https://doi.org/10.1063/5.0047618, URL `https://doi.org/10.1063/5.0047618`.

[18] M. Raghunathan, Y. Marandet, H. Bufferand, G. Ciraolo, P. Ghendrih, P. Tamain, and E. Serre, Plasma Physics and Controlled Fusion **64**, 045005 (2022), URL `https://dx.doi.org/10.1088/1361-6587/ac414d`.

[19] V. M. Zhdanov and P. N. Yushmanov, Journal of Applied Mechanics and Technical Physics **21**, 453 (1980), URL https://doi.org/10.1007/BF00916477.

[20] M. Raghunathan, Y. Marandet, H. Bufferand, G. Ciraolo, P. Ghendrih, P. Tamain, and E. Serre, Plasma Physics and Controlled Fusion **63**, 064005 (2021), URL https://dx.doi.org/10.1088/1361-6587/abf670.

[21] J.-Y. Ji, J. A. Spencer, and E. D. Held, Plasma Research Express **2**, 015013 (2020), URL https://doi.org/10.1088/2516-1067/ab7d0f.

[22] J.-Y. Ji and E. D. Held, Phys. Rev. E **82**, 016401 (2010).

[23] J.-Y. Ji, M. U. Lee, E. D. Held, and G. S. Yun, Physics of Plasmas **28**, 072113 (2021), https://doi.org/10.1063/5.0054457, URL https://doi.org/10.1063/5.0054457.