A Note on Thermal History Kernel for Unsteady Heat Transfer of a Spherical Particle

S. Balachandar * and M.Y. Ha †

Abstract

When a particle is subjected to an unsteady ambient flow, in terms of either time-dependent relative velocity or time-dependent temperature difference, the net heat transfer from the particle cannot be calculated based on the quasi-steady heat transfer correlation alone. Due to unsteady evolution of the thermal boundary layer, there is also a history contribution to heat transfer. The history contribution to heat transfer is expressed as a convolution integral of past evolution of temperature difference between the particle and the surrounding. While Basset history force and its finite Reynolds number extension have been well studied, similar understanding of unsteady heat transfer and thermal history kernel is lacking. We use existing particle-resolved simulation results of [2] to develop a finite Peclet number thermal history kernel, which when used with the convolution integral is demonstrated to accurately predict unsteady heat transfer over a range of Peclet numbers and particle-to-fluid heat capacity ratio.

1 Introduction

In many applications, heat transfer between a particle and its surrounding is in the context of significant unsteadiness. For example, a particle suddenly injected into a hot (or cold) stream will undergo unsteady heat transfer as it is heated (or cooled). In a turbulent flow, even a particle that has been in the system for a long time will undergo unsteady heat transfer as it moves between different regions of the turbulent flow. In the above examples, the term *unsteady* refers to the fact that as heat is being exchanged between the particle and the surrounding fluid, the condition under which the transfer is occurring is changing. This change can be in the form of either the relative velocity changing in magnitude or direction, or the temperature difference between the particle and the ambient changing over time. The purpose of this paper is to address the *intrinsic effect* of unsteadiness on the rate of heat transfer. If the unsteady effects are unimportant, then at each instance the heat transfer rate can be taken to be the same as that would exist had the problem been held steady at the instantaneous relative velocity and temperature difference - this heat transfer can be termed the *quasi-steady contribution.* If the actual heat transfer rate deviates from the quasi-steady contribution, then the difference is the intrinsic effect of unsteadiness and can be termed the *unsteady contribution.*

In the modeling of momentum exchange between a particle and the surrounding fluid, the unsteady effects appear as three different force contributions in addition to the quasi-steady force:

*Corresponding Author, Professor, University of Florida, bala1s@ufl.edu
†Mechanical Engineering, Pusan National University, Pusan, S. Korea

(i) the undisturbed flow force (or the stress-divergence), (ii) the added-mass force and (iii) the Basset history force. The undisturbed flow force is the simplest of the three and it arises even in the absence of the particle and it is the momentum exchange between the volume occupied by the particle and the surrounding fluid. The added-mass force is due to the inviscid perturbation flow induced by the no-penetration condition on the surface of the particle, and the inviscid perturbation flow develops rapidly on the acoustic timescale [24, 25]. Thus, the added-mass force depends on the instantaneous relative acceleration between the particle and the surrounding fluid. The Basset history force is due to the viscous perturbation flow induced by the no-slip condition on the surface of the particle and the perturbation develops on vorticity diffusional timescale. Thus, the Basset history force depends on the past history of relative acceleration and is represented as a convolution integral [14, 19, 26].

In the modeling of heat exchange between a particle and the surrounding fluid, the unsteady effects appear only as two additional contributions to quasi-steady heat transfer: (i) undisturbed flow and (ii) history heat transfer [10, 22]. The undisturbed flow heat transfer arises even in the absence of the particle and it represents the thermal exchange between the volume occupied by the particle and the surrounding fluid. The history heat transfer is analogous to the Basset history force and it is due to the perturbation thermal field induced by the thermal boundary condition imposed on the surface of the particle. This perturbation thermal field develops on thermal diffusion timescale and depends on the past history of temperature difference between the particle and the surrounding fluid. As a result of this memory effect, the history heat transfer appears as a convolution integral. There is no added-mass term for heat transfer, since there is no analog of no-penetration condition for the temperature field.

The undisturbed flow heat transfer is given by $m_f c_f DT_f/Dt$, where m_f is the mass of the fluid displaced by the particle, c_f the specific heat capacity of the fluid, and DT_f/Dt the total derivative of the fluid temperature at the particle location. The above model for undisturbed flow heat transfer is exact even under nonlinear conditions of finite Reynolds number. In contrast, an accurate model of history contribution to heat transfer has been rigorously derived only in the linear limit of zero Peclet number [10, 22]. Their derivation mirrored the analytical approach pursued by [5, 6, 19, 23] for the history force in the limit of zero Reynolds number. In the zero Reynolds and Peclet number limit, both the force history and the thermal history kernels decay as $1/\sqrt{t}$. In the case of history force, finite Reynolds number extension has been pursued by a number of researchers [13, 17, 18, 20]. The general agreement is that the $1/\sqrt{t}$ decay of the kernel is appropriate at small time and at larger time the kernel undergoes a faster decay, whose precise form somewhat depends on the precise nature of relative acceleration or deceleration [11]. Nevertheless, force history kernels, as a function time, applicable at finite Re, have been proposed and successfully used in force calculation [20]. A corresponding finite Peclet number extension to the thermal history kernel is still lacking. Without such a finite Peclet number kernel we cannot reliably calculate the unsteady heat transfer of a particle at finite Peclet numbers. Obtaining a finite Peclet number thermal history kernel is the focus of this paper.

The work of Balachandar and Ha [2] is of relevance to the present work. They considered the problem of unsteady heat transfer at finite Peclet number using particle-resolved numerical simulations. Three different problems were considered: (i) unsteady heat transfer and free thermal evolution of a spherical particle suddenly injected into a uniform flow of different temperature, (ii) heat transfer from a particle in a uniform flow with a sudden change in particle temperature, and (iii) a particle subjected to a uniform flow of oscillatory temperature variation. All three problems considered finite Peclet number effect with the focus on unsteady heat transfer. Here we will use the results of the first set of particle-resolved simulations to obtain a close form expression for the thermal history kernel. We will then use this kernel to predict the unsteady heat transfer in the context of oscillatory temperature variation and compare the results with the third set of particle-resolved simulation results.

2 Conditions of Unsteadiness

We start with the following question: under what conditions heat transfer between a particle and the surrounding can be treated as quasi-steady? If these conditions are not satisfied, then the unsteady contribution is significant and must be included in the heat transfer analysis. To answer this question we define the following timescales, and a comparison of these timescales can be used to evaluate the importance of unsteadiness. There are three primary ways by which the conditions of heat transfer changes: (i) change in particle temperature, (ii) change in particle velocity and (iii) change in ambient flow temperature or velocity seen by the particle.

The thermal timescale of the particle is given by $\tau_T = d_p^2/(6\gamma\kappa_f \mathrm{Nu})$ [3, 16], where d_p is the particle diameter, $\gamma = \rho_f c_f/(\rho_p c_p)$ the fluid-to-particle specific heat capacity ratio, κ_f the thermal diffusivity of the fluid and Nu the Nusselt number. Thermal timescale represents the time over which the difference in temperature between the particle and the surrounding fluid decreases by a factor of $1/e$ or becomes 36.8% of the initial value [4, 26]. Provided the ambient fluid temperature and the relative velocity between the particle and the surrounding remain the same, τ_T is the timescale on which the conditions of heat transfer changes.

Instead, if the ambient fluid velocity and the temperature difference between the particle and the surrounding are held independent of time, then unsteadiness can only be due to the acceleration (or deceleration) of the particle towards the surrounding fluid velocity. This momentum timescale of the particle is given by $\tau_V = \rho d_p^2/(18\nu_f\,\Phi)$, where ρ is particle-to-fluid density ratio, ν_f the kinematic viscosity of the fluid and Φ the finite Reynold number correction to Stokes drag. The momentum timescale represents the time over which the difference in velocity between the particle and the surrounding fluid decreases by a factor of approximately $1/e$.

As the third source of unsteadiness we consider either the velocity or temperature of the surrounding fluid seen by the particle changing on the timescale τ_f. While τ_T and τ_V are determined by the thermodynamic and transport properties of the fluid and the particle, τ_f is dependent on the nature of the flow in which the particle is immersed in, and therefore must be carefully evaluated in any given problem. We now define the timescale of unsteadiness to be the smallest of the three as

$$\tau_{un} = \min\{\tau_T,\, \tau_V,\, \tau_f\}\,, \tag{1}$$

since the smallest timescale will dictate the unsteady nature of heat transfer.

This timescale of unsteadiness must be compared with both the advection timescale d_p/U and the diffusion timescales d_p^2/κ_f and d_p^2/ν_f of the flow, where U is the relative velocity between the particle and the flow. For Peclet and Reynolds numbers $\mathrm{Pe} = d_pU/\kappa_f, \mathrm{Re} = d_pU/\nu_f > 1$, the diffusion timescales are larger than the advection timescale, while for $\mathrm{Pe}, \mathrm{Re} < 1$ the advection timescale is larger than the other two. In order for heat transfer to be considered quasi-steady (i.e., to ignore unsteady effects), τ_{un} must be much larger than the advection and diffusion timescales.

	(Pe, Re) < 1	(Pe, Pr) > 1	(Re, $1/\,$Pr) > 1
$\tau_T < (\tau_V,\, \tau_f)$	$\dfrac{\mathrm{Pe}}{6\,\gamma\,\mathrm{Nu}} \gg 1$	$\dfrac{1}{6\,\gamma\,\mathrm{Nu}} \gg 1$	$\dfrac{\mathrm{Pr}}{6\,\gamma\,\mathrm{Nu}} \gg 1$
$\tau_V < (\tau_T,\, \tau_f)$	$\dfrac{\rho\mathrm{Re}}{18\,\Phi} \gg 1$	$\dfrac{\rho}{18\ \mathrm{Pr}\ \Phi} \gg 1$	$\dfrac{\rho}{18\,\Phi} \gg 1$
$\tau_f < (\tau_T,\, \tau_V)$	$\dfrac{\tau_f\, U}{d_p} \gg 1$	$\dfrac{\tau_f\, \kappa_f}{d_p^2} \gg 1$	$\dfrac{\tau_f\, \nu_f}{d_p^2} \gg 1$

Table 1: The rows represent conditions where τ_T, τ_V and τ_f are smaller than the other two, while the three columns represent the largest of convective, thermal-diffusion, momentum-diffusion timescales. The corresponding conditions for considering heat transfer to be quasi-steady are given in the table entries. In this work we focus on the case of thermal timescale of the particle being the smallest, which has been highlighted in gray in the table.

I.e., the condition for validity of quasi-steady heat transfer is

$$\tau_{un} \gg \max\left\{\frac{d_p}{U},\, \frac{d_p^2}{\kappa_f},\, \frac{d_p^2}{\nu_f}\right\}\,. \tag{2}$$

Only then, the inertial and diffusional adjustment of the velocity and temperature fields of the fluid can be considered faster than the change in the conditions of heat transfer. The above compactly-expressed condition can be more explicitly expressed as the nine different conditions given in Table 1.

In this work we will primarily consider unsteadiness in the simpler context of a particle subjected to a steady ambient stream of uniform velocity at a constant temperature different from that of the particle. The relative velocity between the particle and the ambient will be held fixed and the temperature of the particle is allowed to evolve towards that of the ambient stream. Thus, we consider the case $\tau_T < (\tau_V, \tau_f)$. In this limit, for quasi-steady approximation to be valid we require the volumetric heat capacity of the fluid to be significantly smaller than that of the particle (i.e., the ratio γ must be substantially smaller than the smallest of $1/(6\mathrm{Nu})$, $\mathrm{Pe}/(6\mathrm{Nu})$ and $\mathrm{Pr}\,/(6\mathrm{Nu})$).

3 Unsteady Heat Transfer Model

Consider a particle of diameter d_p freely moving with time-dependent velocity $\mathbf{V}(t)$ and temperature $T_p(t)$ in an ambient fluid of velocity $\mathbf{u}(t)$ and temperature $T_f(t)$. If the particle velocity differs from the fluid velocity (i.e., $\mathbf{V} - \mathbf{u} \neq 0$), then the particle will experience a drag force and the particle velocity will evolve towards the fluid velocity. Similarly, if the particle temperature differs from the fluid temperature (i.e., $T_p - T_f \neq 0$), due to heat transfer, the particle temperature will evolve towards the fluid temperature. The time evolution of particle velocity is given by the Basset-Boussinesq-Oseen equation of motion [5, 6, 19, 23]. The corresponding equation of thermal evolution of the particle is given by [10, 21, 22]

$$m_p\, c_p \frac{dT_p}{dt} = \underbrace{m_f\, c_f \left[\frac{DT_f}{Dt}\right]_{@}}_{q_{un}} + \underbrace{2\pi d_p k_f \left([T_f]_{@} - T_p\right)}_{q_{qs}} + \underbrace{d_p^2 \sqrt{\pi \kappa_f} \rho_f c_f \int_0^{\infty} [K_{T0}]_{\xi} \left[\frac{d(T_f - T_p)}{dt}\right]_{@(t-\xi)} d\xi}_{q_{hi}}\,, \tag{3}$$

where m_p is the mass of the particle, c_p the specific heat capacity of the particle, k_f the thermal conductivity of the fluid and ρ_f the density of the fluid. On the right hand side, the three terms correspond to undisturbed flow heat transfer q_{un}, quasi-steady heat transfer q_{qs} which depends on the instantaneous temperature difference between the particle and the surrounding, and the history heat transfer q_{hi}, which is the quantity of interest here. In the above, $K_{T0} = 1/\sqrt{\xi}$ is the thermal history kernel, and the argument $d(T_f - T_p)/dt$ in the convolution integral is evaluated at time $(t - \xi)$ and at the particle location as indicated by the subscript $@(t - \xi)$. The notation $[\cdot]_{@}$ indicates the fluid quantity being evaluated at the particle location and at time t, while $[\cdot]_{\xi}$ indicates the kernel being evaluated at time ξ.

The above rigorous expression was derived under two important assumptions: (i) It is appropriate only in the zero Peclet limit. At finite Peclet number both the quasi-steady heat transfer and history kernel will depend on the value of Peclet number. (ii) The particle diameter d_p is assumed to be much smaller than the length scales of the ambient flow. As a result, fluid-related quantities such as T_f, DT_f/Dt and dT_f/dt are taken to be spatially uniform on the scale of the particle and evaluated at the particle center. The latter assumption has been relaxed by [22] following the approach of Maxey and Riley [19] to obtain unsteady heat transfer from a particle of finite size, in the zero Peclet number limit. In contrast, the assumption of linearity is hard to relax, since a rigorous nonlinear solution is not possible. Nevertheless, an empirical expression for unsteady heat transfer at finite Peclet number, for a finite-sized particle can be expressed as

$$m_p\, c_p \frac{dT_p}{dt} = \underbrace{m_f\, c_f \overline{\frac{DT_f}{Dt}}^{V}}_{q_{un}} + \underbrace{\pi d_p k_f \mathrm{Nu} \left(\overline{T_f}^{S} - T_p\right)}_{q_{qs}} + \underbrace{d_p^2 \sqrt{\pi \kappa_f} \rho_f c_f \int_0^{\infty} [K_T]_{\xi} \left[\frac{d(\overline{T_f}^{S} - T_p)}{dt}\right]_{t-\xi} d\xi}_{q_{hi}}\,. \tag{4}$$

Three important differences from (3) can be observed. First, the fluid related quantities are not evaluated at the center of the particle. Since the particle is not restricted to be small anymore, T_f and its derivatives vary over the volume occupied by the particle. Therefore, in the above equation, the fluid quantities are evaluated as averages over the particle surface or volume, and indicated by the notations $\overline{(\cdot)}^{S}$ and $\overline{(\cdot)}^{V}$. Note that in the small Peclet number limit, we can approximate $\overline{T_f}^{S} \approx [T_f]_{@} + (d_p^2/6)\left[\nabla^2 T_f\right]_{@}$ and this is the form presented in [22]. Second, quasi-steady heat transfer is a function of both Reynolds and Peclet numbers and this functional dependence appears through the Nusselt number Nu(Re, Pr). In the zero Peclet number limit, Nu = 2 and quasi-steady heat transfer reduces to that in (3). Third, the thermal history kernel K_T will be not only a function of time, but also depend on the Peclet number.

3.1 Nu and K_T for Small Pe

In this subsection we will consider analytical solutions for small Peclect number (i.e., for Pe $\ll$ 1). Acrivos & Taylor [1] solved the problem of steady heat transfer from a spherical particle using singular perturbation and obtained the following result

$$\textbf{Small}\text{−Pe}\ \textbf{limit}: \qquad \text{Nu} = 2 + \frac{1}{2}\text{Pe} + \frac{1}{4}\text{Pe}^2 \ln(\text{Pe}) + O(\text{Pe}^3)\,. \tag{5}$$

The above solution correctly approaches the conduction value of Nu = 2 as Pe → 0. The above expression has been verified to be accurate for Pe in the range from 0 to 0.7 [22].

The small-Pe form of the thermal history kernel has been established by considering the exact solution of the temperature field around a spherical particle subjected to Stokes flow. The particle and the surrounding fluid are initially at the same temperature but at $t = 0$ the temperature of the particle is suddenly decreased and thereafter the temperature difference is maintained at $T_f - T_p = \Delta T$. From the exact solution of the governing equations, the non-dimensional heat transfer to the particle is obtained as

$$\frac{m_p\,c_p}{\pi d_p k_f \Delta T}\frac{dT_p}{dt} = \underbrace{2 + \frac{\text{Pe}}{2} + \frac{1}{4}\text{Pe}^2 \ln(\text{Pe})}_{q_{qs}/(\pi d_p k_f \Delta T)=\text{Nu}} + \underbrace{\frac{2\sqrt{\text{Pe}}}{\sqrt{\pi\,\tilde{t}}}\exp\left(-\frac{\text{Pe}\,\tilde{t}}{16}\right) - \frac{\text{Pe}}{2}\,\text{erfc}\left(\frac{\sqrt{\text{Pe}\,\tilde{t}}}{4}\right)}_{q_{hi}/(\pi d_p k_f \Delta T)}, \tag{6}$$

where erfc is the complementary error function and $\tilde{t} = tU/d_p$ is the non-dimensional time. We compare the above to the three heat transfer contributions given on the right hand side of the unsteady model given in (4). Since the ambient flow is at constant temperature, the undisturbed flow heat transfer is zero (i.e., $q_{un} = 0$). The other two contributions are clearly identified above. When non-dimensionalized by $(\pi d_p k_f \Delta T)$ the quasi-steady contribution to heat transfer is simply the Nusselt number Nu. In the present problem, for $t > 0$, the temperature difference between the ambient and the particle remains a constant and as a result the quasi-steady heat transfer is time independent. Immediately after the change in particle temperature, unsteady effects will contribute to q_{hi} and in non-dimensional terms this contribution can be obtained by performing the convolution integral in (4). In evaluating the integral, we can set $[d(T_f - T_p)/dt]_{t-\xi} = \Delta T\delta(t-\xi)$, where δ is the delta function. We then obtain the third term on the right hand side of (4) to be

$$\frac{q_{hi}}{\pi d_p k_f \Delta T} = \sqrt{\frac{\text{Pe}}{\pi}}\tilde{K}_T\,, \tag{7}$$

where $\tilde{K}_T = d_p K_T/U$ is the non-dimensional kernel. Comparing the above with (6) we obtain the thermal history kernel to be

$$\textbf{Small}\text{−Pe}\ \textbf{limit}: \qquad \tilde{K}_T(\tilde{t}) = \frac{2}{\sqrt{\tilde{t}}}\exp\left(-\frac{\text{Pe}\,\tilde{t}}{16}\right) - \frac{\sqrt{\pi\text{Pe}}}{2}\,\text{erfc}\left(\frac{\sqrt{\text{Pe}\,\tilde{t}}}{4}\right)\,. \tag{8}$$

For a wide range of Pe it can be verified that the first term on the right is much larger than the second term. Henceforth, we will ignore the second term and simply consider the small-Pe thermal history kernel to be given by the first term.

3.2 Nu and K_T for Finite Re, Pe

We first consider empirical correlations of Nusselt number obtained from experiments performed under steady conditions at finite values of Reynolds and Prandtl numbers, where $\text{Pr} = \nu_f/\kappa_f$ (Note

Pe = Re Pr). The two most popular finite-Re Nusselt number correlations are [27, 29]

$$\mathrm{Nu} = 2 + (0.4\mathrm{Re}^{1/2} + 0.06\mathrm{Re}^{2/3})\,\mathrm{Pr}^{0.4} \quad \text{and} \quad \mathrm{Nu} = 2 + 0.6\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}\,. \tag{9}$$

For a fluid of Pr = 0.7, particle-resolved simulations over a wide range of Reynolds number from 10 to 500 [2] yielded Nusselt number values that were in between the above two correlations. This illustrates the level of uncertainty in the prediction of quasi-steady heat transfer.

Here we will establish the finite-Pe thermal history kernel by considering the problem of heat transfer from a spherical particle that is subjected to a steady uniform flow. The particle and the surrounding fluid are initially at the same temperature, but at $t = 0$ the temperature of the particle is suddenly decreased. However, the temperature of the particle is allowed to freely evolve towards the constant ambient fluid temperature. No analytic solution is possible at finite Reynolds number and therefore we will consider the particle-resolved simulations of [2].

For a wide range of Reynolds number Re and heat capacity ratio γ, Balachandar and Ha [2] observed the heat transfer coefficient to start at a large value and rapidly decrease to a constant value. Correspondingly, the particle temperature initially increases more rapidly, but quickly settles to an exponential approach to the surrounding constant fluid temperature. The time evolution of the non-dimensional particle temperature can be expressed as

$$\frac{T_f - T_p(t)}{\Delta T} = \exp\left(-\frac{6\gamma\,\mathrm{Nu}_{eff}\,\tilde{t}}{\mathrm{Pe}}\right)\,, \tag{10}$$

where Nu_{eff} is the heat transfer coefficient expressed as an effective Nusselt number. Here Nu_{eff} is defined such that it accounts for both the quasi-steady and the unsteady heat transfer effects. Therefore it is termed the effective Nusselt number. It was noted that Nu_{eff} was lower than the corresponding quasi-state Nu and the reduction is due to unsteady heat transfer. Substituting the above equation into (4) and simplifying we obtain the relation

$$\mathrm{Nu}_{eff} = \mathrm{Nu} - \frac{6\gamma\,\mathrm{Nu}_{eff}}{\sqrt{\pi}\,\mathrm{Pe}} \int_0^{\tilde{t}} \tilde{K}_T(\xi)\exp\left(\frac{6\gamma\,\mathrm{Nu}_{eff}\,\tilde{\xi}}{\mathrm{Pe}}\right) d\tilde{\xi}\,. \tag{11}$$

The above relation holds only after the initial transient dies out, when the effective heat transfer coefficient settles to the constant value of Nu_{eff}. Thus, for $\tilde{t}$ greater than the initial transient period, the convolution integral on the right hand side converges to a time independent value (this must be so, since the other two terms are time independent). This convergence to a constant value is made possible by the rapid decay of the thermal history kernel. Motivated by the form of the thermal history kernel in the small-Pe limit, we propose the following finite-Pe thermal history kernel (here finite-Pe indicated Peclect numbers larger than unity)

$$\textbf{Finite}{-}\mathrm{Pe}: \qquad \tilde{K}_T(\tilde{t}) = \frac{2}{\sqrt{\tilde{t}}}\exp(-b\,\tilde{t})\,. \tag{12}$$

By comparing the above with (8) it is clear that $b = \mathrm{Pe}/16$ for small Pe, but the dependence of b on Pe is to be determined for finite values of Pe. Substituting the above kernel into (11) and carrying out the integral yields an error function solution, which for $\tilde{t} \gg \mathrm{Pe}/(b\,\mathrm{Pe} - 6\gamma\,\mathrm{Nu}_{eff})$ becomes time independent. In this limit, we obtain the following expression

$$\frac{6\gamma\,\mathrm{Nu}_{eff}}{\mathrm{Pe}}\left[1 + \frac{24\gamma\,\mathrm{Nu}_{eff}}{(\mathrm{Nu}_{eff} - \mathrm{Nu})^2}\right] \tag{13}$$

for the value of b. Thus, for each combination of Pe and γ, the corresponding values of Nu_{eff} and Nu obtained from the particle-resolved simulations of [2] can be used to obtain the value of b. These results are plotted as symbols in Figure 1 for Pe ranging from 7 to 350 and for $\gamma = 0.004$, 0.02 and 0.1 plotted in red, blue and green. Although the value of b evaluated from the simulations show some variation with γ, the primary dependence seems to be on Pe. Based on the simulation results and on the analytical small-Pe behavior given in (8), we propose the following curve fit for b:

$$b = (1.63 - 0.92\,\mathrm{erf}(0.017(\mathrm{Pe} - 80)))\left[1 - 0.4\exp\left(-\frac{\mathrm{Pe}}{16}\right) - 0.6\exp\left(-\frac{\mathrm{Pe}^2}{30}\right)\right]. \tag{14}$$

The above expression for b when applied in (12) will yield a thermal history kernel that remains appropriate over a wide range of Peclet numbers. For large Peclet number, the term within the square parenthesis becomes 1 and the value of b is simply given by the expression within the first parenthesis. On the other hand, for $\mathrm{Pe} \ll 1$ the Taylor series expansion of the above correctly approaches the limit $b \to 1/16$.

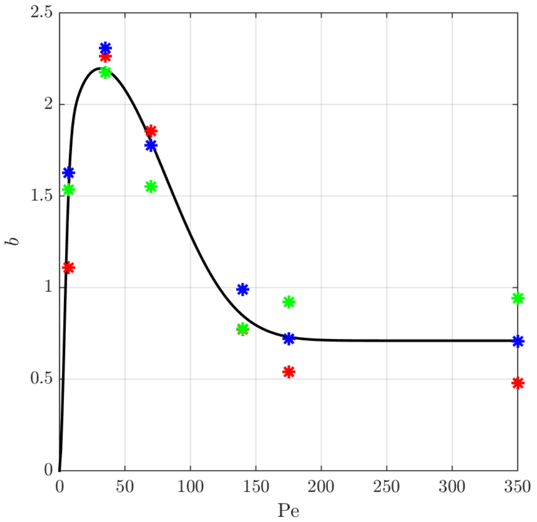

Figure 1: The value of the exponent b for varying combinations of Pe and $\gamma = 0.004$, 0.02 and 0.1 are plotted in red, blue and green symbols. The black line shows the curve fit presented in (14)

3.3 Oscillatory Heat Transfer Problem

As a simple application of the above finite Peclet number thermal history kernel we will test it in the problem of oscillatory particle temperature. In this problem, the ambient fluid velocity U and temperature T_f were held steady. The particle is held stationary and its temperature is varied in a sinusoidal manner. However, at all times the particle temperature is maintained lower than that of the ambient fluid. This problem was studied by [2] with particle-resolved simulations for varying Peclet number and for varying frequency of oscillation. Here we will predict time variation of heat transfer using the unsteady heat transfer model and compare against the results of particle-resolved simulations.

The temperature of the particle was varied as $T_p(t) = T_f - \Delta T + \Delta T \alpha e^{-\iota\tilde{\omega}\tilde{t}}$, where $om\tilde{e}ga = \omega d_p/U$ is the nondimensional frequency. Here the amplitude α is smaller than unity and thus particle temperature remains smaller than the ambient fluid. Substituting the ambient fluid and particle temperatures into the right hand side of (4) and non-dimensionalizing by $\pi d_p k_f \Delta T$ we obtain each of the three terms to be as follows: $\tilde{q}_{un} = 0$, $\tilde{q}_{qs} = \mathrm{Nu}(1 - \alpha e^{-\iota\tilde{\omega}\tilde{t}})$, and

$$\tilde{q}_{hi} = \hat{q}_{hi} e^{-\iota\tilde{\omega}\tilde{t}} = \left[\frac{2\alpha\sqrt{\mathrm{Pe}}}{\sqrt{b^2 + \tilde{\omega}^2}} \, \iota\tilde{\omega} \, \sqrt{b + \iota\tilde{\omega}} \right] e^{-\iota\tilde{\omega}\tilde{t}} \,. \tag{15}$$

In performing the convolution integral, the thermal history kernel has been taken to be (12). The real and imaginary parts of the Fourier coefficients (i.e., the right hand side within the square parenthesis) is plotted in Figure 2 as a function of $\tilde{\omega}$ for few different Peclet numbers. The agreement with the results presented in [2] is very good, thus providing support for the finite-Pe thermal history kernel. Also plotted in the figure are the power-laws for small and large values of $\tilde{\omega}$. For small oscillation frequency (i.e., for $\tilde{\omega} \ll b$), the above solution can be expanded as

$$\hat{q}_{hi} = 2\alpha\sqrt{\mathrm{Pe}} \left[-\frac{\tilde{\omega}^2}{2b^{3/2}} + \iota \frac{\tilde{\omega}}{b^{1/2}} \right] . \tag{16}$$

For higher oscillation frequency (i.e., for $\tilde{\omega} \gg b$), a Taylor series expansion yields

$$\hat{q}_{hi} = \alpha\sqrt{2\,\mathrm{Pe}\,\tilde{\omega}}(-1 + \iota) \,. \tag{17}$$

From the figure it can be observed that at high frequency oscillation the results for different Pe collapse when plotted as a function of $\mathrm{Pe}\,\tilde{\omega}$, consistent with the scaling given in (17). At low frequency, simultaneous collapse of both the real and imaginary pats is not possible according to the scaling (16). These power-law behaviors were precisely the same observed in the particle-resolved simulations.

4 Conclusions

In multiphase flow applications, unsteadiness is the norm. At the level of an individual particle, the velocity and temperature of the ambient flow relative to that of the particle will in general vary in time. If the timescale of variation is faster than the inertial and diffusional timescales on which the flow adjust, then unsteadiness will affect both the force on the particle and the heat transfer. On the other hand, if the timescale on which the relative velocity and temperature vary are slower, then the force and heat transfer on the particle can be calculated using quasi-steady assumption.

Under unsteady conditions, the force on a particle includes both the added-mass and Basset history contributions, which represent the inviscid (potential) and viscous effects of unsteadiness. The added-mass force is typically parameterized in terms of an added-mass coefficient and the Basset history force is expressed as a convolution integral with a force history kernel as the weighting function. In the case of unsteady heat transfer, unsteadiness contributes to history heat transfer, which is again modeled as a convolution integral with a thermal history kernel as the weighting function. Despite the complex nature of the history force, accurate models of the force history kernel have been developed that apply for a range of acceleration/deceleration conditions and Reynolds numbers. This paper presented an accurate thermal history kernel (see equations (12)

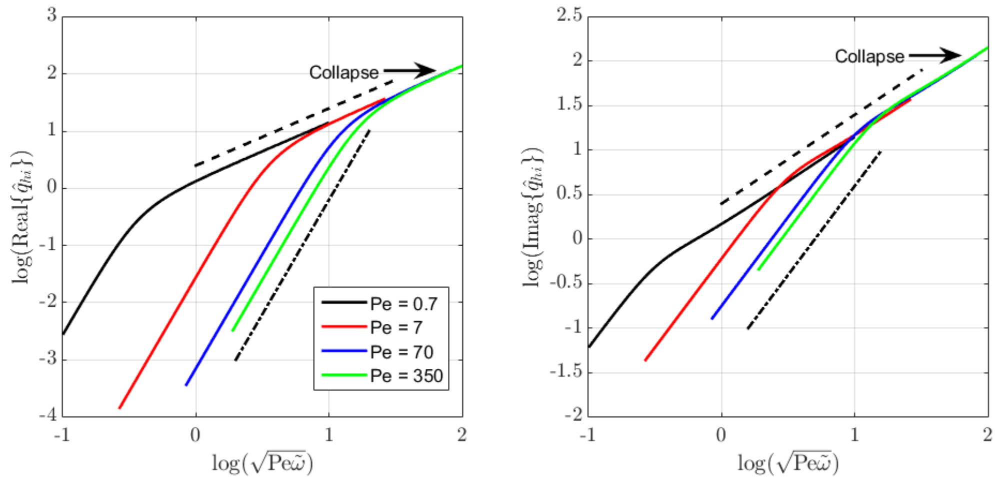

Figure 2: (a) Real part of $\hat{q}_{hi}$ as a function of $\sqrt{\mathrm{Pe}\tilde{\omega}}$ for four different values of Pe. Also plotted are lines of $(\mathrm{Pe}\tilde{\omega})^2$ and $(\mathrm{Pe}\tilde{\omega})^{1/2}$ as dash-dot and dash lines. (b) Imaginary part of $\hat{q}_{hi}$ as a function of $\sqrt{\mathrm{Pe}\tilde{\omega}}$. Also plotted are lines of $(\mathrm{Pe}\tilde{\omega})$ and $(\mathrm{Pe}\tilde{\omega})^{1/2}$ as dash-dot and dash lines.

and (14)), which with the convolution integral can be used to reliably predict the unsteady history contribution to heat transfer.

The thermal history kernel was developed using the particle-resolved simulation results of [2]. They considered the free thermal evolution of a particle in a uniform flow where the particle temperature was suddenly lowered below the steady uniform temperature of the ambient. After a brief period of initial transient, the simulation results showed an exponential reduction in the temperature difference and a constant net heat transfer rate. Based on the simulation results, a composite thermal history kernel was developed, which decayed as $1/\sqrt{\tilde{t}}$ at short time and as $e^{-b\tilde{t}}$ at long time, where the exponential decay rate b was obtained as an empirical function of Peclet number. The thermal history kernel was also ensured to follow the correct analytic behavior in the small-Pe limit. The resulting model for history heat transfer was satisfactorily tested against a different set of particle-resolved simulations where the particle temperature was sinusoidally varied for varying Peclect number and oscillation frequency [2]. The results presented above were based on particle-resolved simulations performed for a fluid of $\mathrm{Pr} = 0.7$. Thus, the results are strictly applicable for air and other fluids of comparable Prandtl number. However, since the results are obtained as functions of $\mathrm{Pe} = \mathrm{Re}\,\mathrm{Pr}$, their applicability extends to somewhat higher Prandtl numbers as well. Particle-resolved simulations and present analysis must be repeated for large Prandtl number fluids.

We now consider conditions under which the unsteady heat transfer can be ignored both when the particle temperature is monotonically evolving towards an ambient fluid temperature and when unsteadiness is due to time-periodic (oscillatory) behavior of the particle or ambient fluid temperature. Similar analysis on the importance of Basset history force in evaluation particle velocity has been considered by a number of researchers [?, 8, 9, 12, 15, 28]. In the case of a particle monotonically evolving towards an ambient fluid temperature (similar to a particle accelerating from a stationary state to its final velocity of that of surrounding fluid), it was observed that the effects of quasi-steady and unsteady heat transfers can be combined to define the effective Nusselt number. The importance of unsteady heat transfer can then be evaluated from the ratio (see [2])

$$\frac{\mathrm{Nu}_{qs} - \mathrm{Nu}_{eff}}{\mathrm{Nu}_{qs}} \approx \frac{(10\gamma/\sqrt{\mathrm{Pe}})}{1 + (10\gamma/\sqrt{\mathrm{Pe}})}. \tag{18}$$

Therefore, unsteadiness becomes unimportant when $(10\gamma/\sqrt{\mathrm{Pe}}) \ll 1$, where the ratio $\gamma/\sqrt{\mathrm{Pe}}$ can be considered as a measure of particle thermal Stokes number. On the other hand, in the case when wither particle or ambient fluid temperature is sinusoidally varied, the importance of unsteady heat transfer can be judged from the ratio $|\tilde{q}_{hi}|/|\tilde{q}_{qs}|$. From the high frequency expression given in (17) we estimate the ratio as

$$\frac{2\sqrt{\mathrm{Pe}}\,\tilde{\omega}}{\mathrm{Nu}}. \tag{19}$$

Thus, the importance of unsteadiness increases both with Peclet number and with the non-dimensional frequency of oscillation. This result is in agreement with those of Vojir and Michaelides [28], who observed that for a particle of density ratio 2.7 in oscillatory motion, the Basset history force can be neglected provided $\tilde{\omega}$ is smaller than a threshold (also see similar conclusions by [7, 9].

The limitations of the present unsteady thermal history kernel for heat transfer modeling must be stressed. The kernel is expected to be quite accurate in situations where unsteadiness arises mainly from a time-dependent particle or ambient fluid temperature. In such situations, the Reynolds and Peclet numbers of the flow do not change and only $\left[d(\overline{T_f}^S - T_p)/dt\right]$ varies over time. Particle-resolved simulations of this scenario were used in developing the current thermal history kernel and as a result the model performance will be good under similar conditions of thermal variation. Consider a different scenario where the temperature difference $(\overline{T_f}^S - T_p)$ remains a constant, but the flow Reynolds and Peclet numbers change due to time variation of relative velocity of the surrounding fluid. This unsteadiness will also result in a unsteady heat transfer contribution, which is not addressed by the above model. Modeling of this history effect can be expected to be similar to that for force history modeling and therefore may depend on details of how the flow is varied [11]. Clearly more work is needed to obtain a comprehensive model that encompasses all possible modes of unsteadiness.

5 Acknowledgements

SB would like to thank Prof. Efstathios Michaelides and Prof. Zhigang Feng for their help. SB would also like to thank support from the U.S. Department of Energy, National Nuclear Security Administration, Advanced Simulation and Computing Program, as a Cooperative Agreement under the Predictive Science Academic Alliance Program, under Contract No. DE-NA0002378 and from the Office of Naval Research (ONR) as part of the Multidisciplinary University Research Initiatives (MURI) Program, under grant number N00014-16-1-2617.

Nomenclature

b	history kernel exponent
C_f	specific heat capacity of fluid

C_p	specific heat capacity of particle
d_p	diameter of the particle
e	Euler's number
KT	thermal history kernel
K_{T0}	thermal history kernel in the zero Peclet number limit
m_f	mass of displaced fluid
m_p	mass of the particle
Nu	Nusselt number
Nu_{eff}	effective Nusselt number
Pe	Peclet number
Pr	Prandtl number
q_{un}	undisturbed flow heat transfer
q_{qs}	quasi-steady heat transfer
q_{hi}	unsteady history heat transfer
Re	Reynolds number
T_f	fluid temperature
ΔT	temperature change
t and ξ	time
$\mathbf{u}$	ambient flow velocity
U	ambient flow velocity scale
V	particle velocity
T_p	particle temperature
T_f	fluid temperature
γ	density-weighted specific heat ratio $= \rho_f C_F/(\rho_p C_P)$
ι	$= \sqrt{-1}$
κ_f	thermal diffusivity of the fluid
ω	oscillation frequency
ν_f	kinematic viscity of the fluid
Φ	finite-Re correction of Stokes drag
ρ	particle-to-fluid density ratio
ρ_f	fluid density
ρ_p	particle density
τ_T	thermal time scale of the particle
τ_V	momentum time scale of the particle
τ_f	time scale of variation of ambient fluid velocity or temperature
τ_{un}	time scale of unsteadiness
D/Dt	total derivative
∇	spatial gradient operator
$\overline{(\cdot)}^V$	an overage over the volume of the particle
$\overline{(\cdot)}^V$	an overage over the surface of the particle
$\widetilde{(\cdot)}$	non-dimensional quantity
$\widehat{(\cdot)}$	Fourier coefficient

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