FERMILAB-PUB-23-102-T (accepted) DOI:10.1016/j.aop.2023.169562

Standard Model in conformal geometry: local vs gauged scale invariance

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Abstract

We discuss comparatively local versus gauged Weyl symmetry beyond Standard Model (SM) and Einstein gravity and their geometric interpretation. The SM and Einstein gravity admit a natural embedding in Weyl *integrable* geometry which is a special limit of Weyl conformal (non-metric) geometry. The theory has a *local* Weyl scale symmetry but no associated gauge boson. Unlike previous models with such symmetry, this embedding is truly minimal i.e. with no additional fields beyond SM and underlying geometry. This theory is compared to a similar minimal embedding of SM and Einstein gravity in Weyl conformal geometry (SMW) which has a full *gauged* scale invariance, with an associated Weyl gauge boson. At large field values, both theories give realistic, Starobinsky-Higgs like inflation. The broken phase of the current model is the decoupling limit of the massive Weyl gauge boson of the broken phase of SMW, while the local scale symmetry of the current model is part of the larger gauged scale symmetry of SMW. Hence, the current theory has a gauge embedding in SMW. Unlike in the SMW, we note that in models with local scale symmetry the associated current is trivial, which is a concern for the physical meaning of this symmetry. Therefore, the SMW is a more fundamental UV completion of SM in a full *gauge theory* of scale invariance that generates Einstein gravity in the (spontaneously) broken phase, as an effective theory.

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1 Introduction

The Standard Model (SM) with the Higgs mass parameter set to zero has a classical global scale symmetry. This could indicate that this symmetry is more fundamental and may play a role in physics beyond SM [1], including gravity. When this symmetry is implemented in physics beyond the SM à la Brans-Dicke [2], (e.g. [3–6]) it can be broken spontaneously and there is a Nambu-Goldstone boson (dilaton ϕ) which generates the Planck scale $M_p \propto \langle\phi^2\rangle$. The constant vev $\langle\phi\rangle$ can be the result of a cosmological evolution in a Friedmann-Robertson-Walker universe [7]. Associated to this symmetry there is a conserved global Weyl current $K_\mu \propto \partial_\mu\phi^2$ [3, 4]. Since the dilaton decouples there are no fifth force constraints [8]. While global symmetries are usually broken by gravitational effects (black holes) [9–11], this might not apply to global scale symmetry which is not compact.

Alternatively, one could consider SM with a *local* (Weyl) scale symmetry or conformal symmetry [12, 13]. Interesting theories of the SM endowed with this symmetry, embedded in a Riemannian geometry, are found in [14, 15]. The action is linear-only in the scalar curvature and Einstein gravity is recovered in the broken phase. To implement this symmetry and to generate spontaneously the Planck scale and the Einstein term, the addition of a new field (dilaton) to the SM spectrum and Einstein gravity is necessary. But unlike in the global case, the associated current to this symmetry is trivial [16, 17], raising concerns about its physical meaning or about its geometric interpretation [18] demanded for a theory including gravity. Further, the new field that generates the Planck scale is usually a ghost (has a negative kinetic term) which some may regard as a concern; however, this field is not physical since it decouples in the (physical) Einstein frame. Questions remain however: can we do better e.g. can we avoid adding "by hand" the otherwise necessary scalar field (ghost/dilaton)? is it possible to avoid the trivial current? can we implement this symmetry in SM embedded in more general gravity theories quadratic in curvature (rather than linear)?

These questions are elegantly answered by applying the successful gauge invariance principle to gravity, just like in the SM. One should thus consider the full, *gauged* Weyl scale symmetry [19–21] in the physics beyond the SM and Einstein gravity. This means that the above local Weyl symmetry is "completed" by the presence of an associated Weyl gauge boson ω_μ. We thus distinguish here between local versus gauged Weyl symmetry, as dictated by the absence or presence of ω_μ, respectively. In fact in the former case ω_μ is simply a "pure gauge" field. When we consider theories with a gauged Weyl symmetry, it can be shown that there exists a non-trivial conserved current J_μ associated to this symmetry [22–24]. As expected, J_μ is now just a Weyl-covariant version of the aforementioned current K_μ of global scale symmetry, hence $J_\mu \propto (\partial_\mu - \alpha\,\omega_\mu)\phi^2$, where α is the Weyl gauge coupling.

Early models with gauged Weyl symmetry were linear-only in scalar curvature ($\tilde{R}$) [18, 26–43] hence they still needed a scalar field be added "by hand" to implement this symmetry and generate Einstein action and Planck scale $M_p \sim \langle\phi\rangle$ from a term $\phi^2\tilde{R}$. Notably, in [27] the Weyl gauge field was shown to become massive after "eating" the dilaton ϕ and decouple.

General actions with gauged Weyl symmetry that were *quadratic* in the curvature were studied in [22, 23], with an action as given by the original gravity theory of Weyl [19–21] where this symmetry was first introduced. The SM with this gauged Weyl symmetry was also studied in this framework in [24] and hereafter this is called SMW. The results of [22–24] show that, even in the absence of matter (SM, etc), this symmetry is spontaneously broken in a Stueckelberg mechanism in which the Weyl gauge boson ω_μ "eats" the would-be Goldstone (dilaton ϕ) and becomes massive. Hence there is no dilaton (ghost) in the spectrum, as expected in a spontaneously broken gauge theory. With ω_μ now massive, it decouples and Einstein gravity is obtained as a "low energy" effective theory and broken phase of a quadratic gravity with gauged Weyl symmetry. This is important because we have: a) an embedding of Einstein gravity into a (quantum) gauge theory of dilatations (which is anomaly-free [25]); b) a general quadratic gravity action and c) the dilaton (Stueckelberg field) is not added "by hand" to implement this symmetry (as in theories linear in R), but is part of the underlying geometry (see later). For more details on this and SMW see [22–25].

There remains unexplored the special case of local (rather than gauged) Weyl symmetry for the SM embedded in quadratic gravity actions with this symmetry. Given the large interest in this symmetry e.g. [14, 15, 44–48] (possibly because its underlying geometry is metric, unlike for the SMW), here we study this special case. We are interested in its *exact* relation to the SM embedded in quadratic gravity with gauged Weyl symmetry (SMW) [24]. The local Weyl symmetry of the current model is part of the larger gauged Weyl symmetry of SMW, while its broken phase will be shown to be the decoupling limit of the massive ω_μ of the broken phase of SMW[1]. Hence the models are closely related, as we detail. Both models have a similar scalar potential and share realistic inflation predictions [51–53]: their tensor-to-scalar ratio r is bounded from above by that of the Starobinsky model.

Both models are naturally formulated in Weyl conformal geometry [19–21] because this geometry has Weyl symmetry built in, i.e. the connection has this symmetry. No knowledge of Weyl geometry is required here - we keep its use to a minimum and emphasize the more familiar gauge theory perspective for this symmetry. The corresponding dilaton/would-be-Goldstone field that generates all the scales (Planck, cosmological constant, m_ω) of both models is part of the Weyl-invariant quadratic term $\sqrt{g}\,\tilde{R}^2$ in the action – hence it has a geometric origin and so do all mass scales it generates [49]. With no new fields added "by hand", our approach to endow the SM with local or gauged Weyl symmetry by embedding it in conformal geometry is natural and truly minimal (which is not the case of models built in (pseudo)Riemannian geometry).

The ultimate question for the SM in quadratic gravity with local Weyl symmetry is whether it can be a UV complete theory. We show that the current associated to this symmetry is vanishing, similar to previous models with this symmetry in Riemannian geometry (that were linear-only in R). This result justifies in our opinion the need for a full, gauged Weyl symmetry as in the SMW where the associated current is non-trivial and conserved. Our comparative study concludes that the SMW is a more fundamental theory, possibly renormalizable[2], that acts as a UV completion of the SM with local scale symmetry, giving a full gauge theory of scale invariance of both Einstein gravity and SM.

The plan of the paper is as follows: in section 2 we review how local Weyl symmetry is implemented in Weyl geometry and how Einstein gravity emerges, without new scalar/matter fields added "by hand" to this end. In Section 3 we consider SM with local Weyl symmetry versus SM with gauged Weyl symmetry - in the former, ω_μ is simply a "pure gauge" field (non-dynamical). We compare the results of the two models, their similarities and differences in the action and predictions for inflation. Our conclusions are followed by an Appendix with details of the Lagrangians of the models, equations of motions and other information.

[1]Correspondingly, the metric/integrable Weyl geometry underlying the local Weyl symmetry model is a special limit of the non-metric Weyl geometry underlying the SMW [49]. In the former, ω_μ is "pure gauge".

[2]Simple power-counting and symmetry arguments support this view, see [24] (section 3), [49] (section 4.6)

2 Local vs gauged Weyl symmetry & Einstein gravity

In this section we briefly review comparatively the local and gauged Weyl symmetry, their geometric interpretation in Weyl geometry and how Einstein gravity is recovered in the broken phase from quadratic gravity with such symmetry, in the absence of matter.

2.1 Weyl conformal geometry and its gravity

Let us first define the symmetry. A *local* Weyl symmetry is the invariance of an action under transformation $\Sigma(x)$ of (1) below together with (2) if scalars ϕ and fermions ψ are also present[3]

$$\hat{g}_{\mu\nu} = \Sigma^q \, g_{\mu\nu}, \quad \sqrt{\hat{g}} = \Sigma^{2\,q}\sqrt{g}, \quad \hat{e}_\mu^a = \Sigma^{q/2} \, e_\mu^a, \tag{1}$$

$$\hat{\phi} = \Sigma^{-q/2} \, \phi, \quad \hat{\psi} = \Sigma^{-3q/4}\psi. \tag{2}$$

The *gauged* Weyl symmetry (or Weyl gauge symmetry) is defined as invariance of the action under (1), (2) and (3) below; eq.(3) is the transformation of an associated Weyl gauge field ω_μ naturally expected from gauge invariance principle, if scale symmetry is indeed gauged:

$$\hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha} \, \partial_\mu \ln \Sigma. \tag{3}$$

Above we denoted by q the Weyl charge of the metric and $\Sigma = \Sigma(x) > 0$ is a positive definite function, α is the Weyl gauge coupling and e_μ^a is the tetrad. Since this is a scale symmetry, not an internal symmetry, there is no i factor in these transformations. In this work we set $q=1$ and the general case is restored by simply rescaling $\alpha \to \alpha q$ in the results. So local Weyl symmetry implicitly assumes that $\omega_\mu = 0$ or that it is a "pure gauge" field.

With this notation, by Weyl geometry[4] we mean a geometry invariant under the above transformations i.e. it is defined by classes of equivalence of the metric and Weyl field, $(g_{\mu\nu}, \omega_\mu)$, related by (1), (3). The definition of Weyl geometry is completed by eq.(4) below for the Weyl connection $\tilde{\Gamma}$, which states that this geometry is non-metric ($\tilde{\nabla}_\mu g_{\alpha\beta} \neq 0$):

$$\tilde{\nabla}_\mu g_{\alpha\beta} = -\alpha \, q \, \omega_\mu \, g_{\alpha\beta}, \quad \text{where} \quad \tilde{\nabla}_\mu g_{\alpha\beta} \equiv \partial_\mu g_{\alpha\beta} - \tilde{\Gamma}_{\alpha\mu}^\rho g_{\rho\beta} - \tilde{\Gamma}_{\beta\mu}^\rho g_{\rho\alpha}. \tag{4}$$

The action of $\tilde{\nabla}$ may be re-written in a "metric geometry" format by a substitution[5]

$$\tilde{\nabla}'_\mu g_{\alpha\beta} = 0, \qquad \tilde{\nabla}' \equiv \tilde{\nabla}\Big|_{\partial_\mu \to \partial_\mu + \alpha \, q \, \omega_\mu}. \tag{5}$$

[3]Our conventions: metric $(+,-,-,-)$, $g = |\det g_{\mu\nu}|$, $R_{\mu\nu\sigma}^\lambda = \partial_\nu \Gamma_{\mu\sigma}^\lambda - \partial_\sigma \Gamma_{\mu\nu}^\lambda + ...$, $R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda$, $R = g^{\mu\nu} R_{\mu\nu}$.

[4]For a very brief guide into Weyl conformal geometry see Appendix A in [24].

[5]Re-writing eq.(4) as in (5) means the theory becomes metric with respect to a new differential operator (see [50] for a discussion). As explained in Appendix A, while in Riemannian geometry the length of a vector is constant under parallel transport, eq.(4) of Weyl geometry means that in general only the relative length (ratio) of arbitrary two vectors (of same Weyl charge) remains invariant. This is consistent with the argument that physics should be independent of the units of length. Actually, the norm of the vector is itself invariant provided that its tangent-space counterpart has vanishing Weyl charge (is invariant), see Appendix A. Finally, if the Weyl gauge boson is "pure gauge" the length of any vector remains constant under parallel transport (the length curvature tensor then vanishes and the geometry is called integrable).

which means that ∂_μ acting on the metric $g_{\alpha\beta}$ is replaced by its Weyl covariant derivative, as expected from a gauge invariance perspective. Then $\tilde\Gamma$ can be found from the Levi-Civita connection (Γ) by the same substitution or by direct calculation from (5), giving (with $q=1$)

$$\tilde\Gamma_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda\Big\vert_{\partial_\mu\to\partial_\mu+\alpha\,\omega_\mu} = \Gamma_{\mu\nu}^\lambda + \frac{\alpha}{2}\,\Big[\delta_\mu^\lambda\;\omega_\nu + \delta_\nu^\lambda\;\omega_\mu - g_{\mu\nu}\,\omega^\lambda\Big], \tag{6}$$

where

$$\Gamma_{\mu\nu}^\lambda = (1/2)\,g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}). \tag{7}$$

So the Weyl connection $\tilde\Gamma$ is completely determined by the metric and ω_μ. Unlike Γ, $\tilde\Gamma$ is gauge invariant i.e. is invariant under (1), (3): the transformation of the metric (in Γ) is compensated by that of ω_μ, leaving $\tilde\Gamma$ invariant. Thus, an action invariant under (1), (2), (3), also has its underlying geometry (i.e. connection $\tilde\Gamma$) invariant. This is very important for the consistency of the action, because the underlying geometry "is" physics[6] so one cannot "separate" it (e.g. ω_μ) from the action itself and so it should have the same symmetry, too. Weyl geometry enables this by construction[7] and this has implications for the scalar curvature. With (6) one computes the scalar curvature $\tilde R$ of Weyl geometry by usual formula in terms of the connection and finds

$$\tilde R = R - 3\,\alpha\,\nabla_\mu\omega^\mu - \frac{3}{2}\alpha^2\,\omega_\mu\,\omega^\mu, \tag{8}$$

$$\tilde C_{\mu\nu\rho\sigma}^2 = C_{\mu\nu\rho\sigma}^2 + \frac{3}{2}\,\alpha^2\,F_{\mu\nu}^2 \tag{9}$$

Here the rhs is in a Riemannian notation, so $\nabla_\mu\omega^\lambda = \partial_\mu\omega^\lambda + \Gamma^\lambda_{\mu\rho}\,\omega^\rho$. Here $\tilde R = R(\tilde\Gamma, g)$ is the scalar curvature of Weyl geometry, $\tilde R = g^{\mu\nu}\tilde R_{\mu\nu}(\tilde\Gamma)$, defined by $\tilde\Gamma$ while R is the scalar curvature of Riemannian case. Also $\tilde C_{\mu\nu\rho\sigma}$ ($C_{\mu\nu\rho\sigma}$) is the Weyl tensor of Weyl (Riemannian) geometry, respectively. What is important here is that $\tilde R$ transforms covariantly under (1), (3), $\hat{\tilde R} = \tilde R/\Sigma$, unlike in the Riemannian case. This is relevant for constructing individual Lagrangian terms invariant under (1), (2), (3) using general covariance and gauge principles.

With the above introduction, we can write the most general gravity action invariant under (1), (3) and defined by Weyl geometry [19–21]

$$\mathcal{L}_1 = \sqrt{g}\,\Big\{\frac{1}{4!\,\xi^2}\tilde R^2 - \frac{1}{\eta^2}\,\tilde C_{\mu\nu\rho\sigma}^2 - \frac{1}{4}\,F_{\mu\nu}^2\Big\}, \qquad 0<\xi,\,\eta<1. \tag{10}$$

which is Weyl's original action. Here $F_{\mu\nu}$ is the field strength of the Weyl gauge field. Since $\tilde\Gamma^\lambda_{\mu\nu} = \tilde\Gamma^\lambda_{\nu\mu}$, then $F_{\mu\nu} = \tilde\nabla_\mu\omega_\nu - \tilde\nabla_\nu\omega_\mu = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, similar to $F_{\mu\nu}$ in flat space-time.

Even in the absence of matter (as above), $\mathcal{L}_1$ is a realistic gauge theory of dilatations, since it has spontaneous breaking (Stueckelberg mechanism) in which ω_μ becomes massive, so it decouples and Einstein gravity is obtained in the broken phase as shown in [22]. For the analysis of $\mathcal{L}_1$ see [22,24]; for convenience we present its equations of motion in Appendix B.1. For the study of the $C^2_{\mu\nu\rho\sigma}$ term which is largely spectator in our analysis see [54, 55].

[6]i.e. ω_μ which defines the connection is physical, being dynamical, see later.
[7]This is unlike in theories with local scale symmetry in Riemannian geometry where Γ is not invariant.

2.2 Weyl integrable geometry and Einstein gravity

Here we are mostly concerned with the special case of local scale symmetry, that corresponds to the case when ω_μ vanishes or is "pure gauge". In such case, obviously

$$F_{\mu\nu} = 0. \tag{11}$$

In fact this is a condition for the underlying geometry (since ω_μ is geometric in origin) and actually defines a special limit of Weyl geometry, called *integrable* geometry. What eq.(11) means is that the length curvature tensor (represented by $F_{\mu\nu}$) of Weyl geometry vanishes. Physically this means there is no kinetic term for ω_μ. An example of such a theory is conformal gravity [55] (where a kinetic term for gauged dilatations is not present). The vanishing of F implies (assuming no topological restrictions, simply connected smooth manifold), that $\omega_\mu \propto \partial_\mu$(scalar field), so ω_μ is "pure gauge". For a suitable Σ one can then obtain $\hat{\omega}_\mu = 0$ which means $\tilde{\Gamma} = \Gamma$, $\tilde{\nabla}_\mu g_{\alpha\beta} = 0$ and the geometry is metric. Weyl integrable geometry is thus associated to local Weyl symmetry while the gauged Weyl symmetry is associated to general Weyl geometry; non-metricity discussed earlier is necessary to have a true gauge theory (i.e. ω_μ dynamical).

With (11), the Lagrangian $\mathcal{L}_1$ becomes

$$\mathcal{L}_1 = \sqrt{g}\,\Big\{\frac{1}{4!\,\xi^2}\tilde{R}^2 - \frac{1}{\eta^2}\,C_{\mu\nu\rho\sigma}^2\Big\}, \qquad 0 < \xi, \eta < 1. \tag{12}$$

Since according to (8), $\tilde{R}^2$ contains Riemannian R^2, $\mathcal{L}_1$ is a higher derivative theory that propagates a spin-zero mode (from R^2), in addition to the graviton. It is easy to "unfold" this higher derivative theory into a second order one and extract this spin-zero mode from $\tilde{R}^2$. To this end, replace $\tilde{R}^2 \to -2\phi^2\tilde{R} - \phi^4$ in $\mathcal{L}_1$, where ϕ is a scalar field [22]

$$\begin{aligned} \mathcal{L}_1 &= \sqrt{g}\,\Big\{\frac{1}{4!\,\xi^2}\Big[-2\phi^2\tilde{R} - \phi^4\Big] - \frac{1}{\eta^2}\,C_{\mu\nu\rho\sigma}^2\Big\}. && (13)\\ &= \sqrt{g}\,\Big\{\frac{1}{4!\,\xi^2}\Big[-2\phi^2\Big(R - 3\alpha\,\nabla_\mu\omega^\mu - \frac{3}{2}\,\alpha^2\omega_\mu\omega^\mu\Big) - \phi^4\Big] - \frac{1}{\eta^2}\,C_{\mu\nu\rho\sigma}^2\Big\}. && (14)\end{aligned}$$

where we used (8). The equation of motion of ϕ has solution $\phi^2 = -\tilde{R}$ which replaced in $\mathcal{L}_1$ recovers (10) which is thus classically equivalent to (13). The equation of motion of ω_μ is

$$\omega_\mu = \frac{1}{\alpha}\,\partial_\mu \ln\phi^2. \tag{15}$$

Using (15) back in $\mathcal{L}_1$

$$\mathcal{L}_1 = \sqrt{g}\,\Big\{-\frac{1}{2\,\xi^2}\Big[\frac{1}{6}\,\phi^2\,R + g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\Big] - \frac{1}{4!\,\xi^2}\,\phi^4 - \frac{1}{\eta^2}\,C_{\mu\nu\rho\sigma}^2\Big\}. \tag{16}$$

This is the gravity action in the absence of matter which has local Weyl invariance left. Note from (16) that the field ϕ is dynamical. Eq.(8) then gives

$$\tilde{R} = R - 6\nabla_\mu\nabla^\mu \ln\phi - 6(\partial_\mu \ln\phi)(\partial^\mu \ln\phi). \tag{17}$$

which relates the Weyl scalar curvature ($\tilde{R}$) to its Riemannian counterpart (R). Note that in $\mathcal{L}_1$ the field ϕ was *not* added by hand but is actually part of the underlying geometry [49]; ϕ comes from the spin zero mode propagated by the $\tilde{R}^2$ term in the original action (which contains R^2) and is part of Weyl scalar curvature. This situation is improved compared to models with local scale symmetry which are linear-only in the curvature, built in Weyl, Riemannian or some other geometry: in these ϕ is necessarily added "by hand" to simultaneously implement this symmetry and generate the Einstein action. Here ϕ is naturally present because $\tilde{R}^2$ is Weyl-covariant.

Assuming that ϕ acquires a vev (at quantum level, etc) and after applying a transformation (1), (2) with $\Sigma = \phi^2/\langle\phi^2\rangle$, then Einstein gravity is generated in the broken phase, as an effective theory[8]

$$\mathcal{L}_1 = \sqrt{g}\,\Big\{ -\frac{1}{2} M_p^2\, R - M_p^2\,\Lambda - \frac{1}{\eta^2}\, C_{\mu\nu\rho\sigma}^2 \Big\}, \qquad \Lambda \equiv \frac{1}{4}\,\langle\phi\rangle^2, \quad M_p^2 \equiv \frac{1}{6\xi^2}\,\langle\phi\rangle^2. \tag{18}$$

We see that ϕ generates both the cosmological constant and the Planck scale, which are thus related, hence the theory predicts that $\Lambda \neq 0$. This suggests an ultraviolet (UV) - infrared (IR) connection between the associated physics at M_p (UV) and at Λ (IR), respectively. The hierarchy $\Lambda \ll M_p^2$ if fixed by one initial classical tuning of the dimensionless $\xi \ll 1$ that fixes M_p, while the vev of ϕ fixes Λ. Since we have the equation of motion $\phi^2 = -\tilde{R}$ (see also Appendix B.1) on the ground state $\langle\phi^2\rangle = -R$, therefore $R = -4\Lambda \neq 0$. Finally, if one considers a Friedmann-Lemaître-Robertson-Walker metric one has that $R = -12H_0^2$ and $\Lambda = 3H_0^2$, where H_0 is the Hubble constant.

Note that Λ is positive; in general this is not obvious in models with local scale symmetry which are linear in the scalar curvature, formulated in either Weyl or Riemannian geometry; in such cases the coefficient of ϕ^4 in the action that generates the cosmological constant is not constrained (as here) and can have any sign/value. We thus see the advantage of the Weyl geometry and its initial *quadratic* action (10) that generates a positive Λ.

To conclude, one can have a local Weyl invariant theory such as Weyl's quadratic action that is associated to integrable geometry and recovers Einstein gravity and predicts a positive Λ. Both M_p and Λ have ultimately a geometric origin [49] because the scalar ϕ that generates them is propagated by $\tilde{R}^2$ which is geometric by nature. No scalar field was added to this purpose. This field decouples in the Einstein gauge (frame) where the symmetry is broken. In the general (non-integrable) Weyl geometry with gauged Weyl symmetry, this scalar field is "eaten" (in a Stueckelberg mechanism) by the Weyl gauge field which becomes massive and subsequently decouples and Einstein gravity is again recovered [22]. For a further discussion of local versus gauged Weyl symmetry and the integrable versus general Weyl geometry see [49]. This ends our review of Weyl geometry, its symmetries and associated Lagrangians.

[8]One may be concerned about the term $C_{\mu\nu\rho\sigma}^2$ in the action which, as a higher derivative, generates a ghost. Since the mass of the ghost is $m \sim \eta M_p$, as long as η is not fine tuned to values $\ll 1$, the ghost is massive, can be integrated out and there is no instability in the theory [56]. In the general (non-integrable) Weyl geometry and SMW, one can actually argue that, since the metric and connection (equivalently, ω_μ) are independent, and treated as such by the variation principle, there is actually no ghost [57].

3 SM in Weyl geometry: local vs gauged scale symmetry

Consider now adding the SM to the previous Lagrangian $\mathcal{L}_1$ in quadratic gravity with local Weyl symmetry, in other words consider the SM in Weyl integrable geometry. This is possible under the assumption of vanishing Higgs mass, in which case the model has a local Weyl symmetry. Such embedding is naturally minimal, without the need for new degrees of freedom beyond those of the SM and Weyl integrable geometry (ϕ and $g_{\mu\nu}$). This is different from models with this symmetry in Riemannian geometry, as we discuss. We study the spontaneous breaking of this symmetry and compare this model to the SMW model [24] obtained by endowing the SM with a gauged Weyl symmetry in general (non-metric) Weyl conformal geometry. For convenience, technical details of the SMW are reviewed in Appendix B.2.

3.1 SM with local Weyl symmetry and integrable geometry

Consider first the SM Higgs sector. The action of the Higgs and gravity is

$$\mathcal{L}_H = \sqrt{g}\Big\{ \frac{\tilde R^2}{4!\,\xi^2} - \frac{\tilde C^2_{\mu\nu\rho\sigma}}{\eta^2} - \frac{\xi_h}{6}\,|H|^2\tilde R + |\tilde D_\mu H\,|^2 - \lambda\,|H|^4\Big\}. \tag{19}$$

As mentioned, in Weyl integrable geometry the length curvature tensor which is actually the field strength of ω_μ is vanishing $F_{\mu\nu}=0$ hence there is no kinetic term for ω_μ. This is the only difference in $\mathcal{L}_H$ and in the total action from the case of SMW, see Appendix B.2 and [24]. This means that locally the Weyl field is "pure gauge", assuming no topological restrictions. Above we introduced

$$\tilde D_\mu H = \big[\partial_\mu - i\mathcal{A}_\mu - (1/2)\,\alpha\,\omega_\mu\big]\,H, \tag{20}$$

$$|\tilde D_\mu H|^2 = |(\partial_\mu - \alpha/2\ \omega_\mu)H|^2 - iH^\dagger\,(\overleftarrow{\partial}_\mu\mathcal{A}^\mu - \mathcal{A}^\mu\partial_\mu)\,H + H^\dagger\mathcal{A}_\mu\mathcal{A}^\mu H, \tag{21}$$

where[9] $\mathcal{A}_\mu = (g/2)\,\vec\tau\cdot\vec A_\mu + (g'/2)\,B_\mu$ with Pauli matrices $\vec\tau$; $\vec A_\mu$, B_μ are the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ gauge bosons; $\mathcal{A}_\mu\mathcal{A}^\mu = (g/2)^2\vec A_\mu.\vec A^\mu + (g\,g'/2)\,(\vec\tau.\vec A_\mu)\,B^\mu$ with g, g' the $\mathrm{SU}(2)_L$ and $\mathrm{U}(1)_Y$ couplings. Each term in $\mathcal{L}_H$ is invariant under (1), (2).

In $\mathcal{L}_H$ we again replace $\tilde R^2 \to -2\phi^2\tilde R - \phi^4$ to find a classically equivalent action; indeed, using the equation of motion of ϕ from the new action and its solution $\phi^2 = -\tilde R$ back in the action, one recovers (19). This implicitly assumes that ϕ is non-vanishing. After this replacement, the initial higher derivative action is "unfolded" to a second order theory; the non-minimal coupling term in (19) is then modified into

$$-\frac{1}{6}\,\xi_h\,|H|^2\,\tilde R \to \frac{-1}{12}\,\Big(\frac{1}{\xi^2}\phi^2 + 2\,\xi_h\,H^\dagger H\Big)\,\tilde R. \tag{22}$$

Then

$$\mathcal{L}_H = \sqrt{g}\,\Big\{\frac{-1}{2}\Big[\frac{1}{6}\theta^2 R + (\partial_\mu\theta)^2 - \frac{\alpha}{2}\,\nabla_\mu(\theta^2\omega^\mu)\Big] + \frac{\alpha^2\,\theta^2}{8}\,\Big[\omega_\mu - \frac{1}{\alpha}\,\nabla_\mu\ln\theta^2\Big]^2 + |\tilde D_\mu H|^2 - V - \frac{C^2_{\mu\nu\rho\sigma}}{\eta^2}\Big\} \tag{23}$$

[9] Also $|\tilde D_\mu H|^2 = |D_\mu H|^2 - (\alpha/2)\omega^\mu\big[\nabla_\mu(H^\dagger H) - (\alpha/2)\,\omega_\mu H^\dagger H\big]$, with D_μ the SM covariant derivative.

with θ the radial direction in the field space of initial ϕ and H:

$$\theta^2 \equiv (1/\xi^2)\ \phi^2 + 2\,\xi_h\, H^\dagger H. \tag{24}$$

Notice that with perturbative ξ and $1/\xi \gg 1$, θ is essentially due to ϕ, for comparable field values of ϕ and H and perturbative ξ_h. Also

$$V = \frac{1}{4!}\Big[24\,\lambda|H|^4 + \xi^2\left(\theta^2 - 2\,\xi_h\,|H|^2\right)\Big]. \tag{25}$$

The equation of motion of ω_μ from $\mathcal{L}_1$ of (23) has locally a solution

$$\omega_\mu = \frac{1}{\alpha}\,\nabla_\mu \ln\big(\theta^2 + 2\,H^\dagger H\big). \tag{26}$$

In the literature, instead of starting with a vanishing length curvature tensor ($F = 0$), as we did, one often assumes that $\omega_\mu = (1/\alpha)\partial_\mu \ln\chi^2$ so it is a "pure gauge" field (ω exact one-form, hence it is closed, $F = 0$); here χ is some real scalar field that is found via its equation of motion, leading to the same solution (26).

Next, we use solution (26) back in $\mathcal{L}_H$. We do this below, in the unitary gauge[10] for the electroweak interaction. Therefore we set $H = (1/\sqrt{2})\,h\,\zeta$, where $\zeta^T \equiv (0,1)$ and $2H^\dagger H = h^2$ where h is the neutral Higgs. Then

$$2H^\dagger \mathcal{A}_\mu \mathcal{A}^\mu H = h^2\,\mathcal{E} \tag{27}$$

with notation (W_μ and Z_μ are the electroweak gauge bosons)

$$\mathcal{E} \equiv (g^2/2)\,W_\mu^+\,W^{-\mu} + [(g^2 + g'^2)/4]\,Z_\mu Z^\mu. \tag{28}$$

For convenience we also replace original higgs h by σ where

$$h^2 = \theta^2\,\sinh^2\frac{\sigma}{\theta} \qquad \Rightarrow \qquad \omega_\mu = \frac{1}{\alpha}\,\nabla_\mu \ln\Big[\theta^2\,\cosh^2\frac{\sigma}{\theta}\Big]. \tag{29}$$

σ will be the actual canonical Higgs field. Using eqs.(26) to (29) back in action (23) then

$$\mathcal{L}_H = \sqrt{g}\,\Big\{\frac{-1}{2}\Big[\frac{\theta^2}{6}R + (\partial_\mu\theta)^2\Big] + \frac{\theta^2}{2}\,\partial_\mu\Big(\frac{\sigma}{\theta}\Big)\,\partial^\mu\Big(\frac{\sigma}{\theta}\Big) + \frac{1}{2}\,\mathcal{E}\,\theta^2\sinh^2\frac{\sigma}{\theta} - \frac{1}{\eta^2}\,C^2_{\mu\nu\rho\sigma} - V\Big\}, \tag{30}$$

where

$$V = \frac{1}{4!}\,\theta^4\Big\{6\lambda\,\sinh^4\frac{\sigma}{\theta} + \xi^2\,\Big[1 - \xi_h\,\sinh^2\frac{\sigma}{\theta}\Big]^2\Big\}. \tag{31}$$

[10] We could still proceed with a general gauge, then extra terms would appear in (30), from the rhs of eq.(21). In addition, in eq.(30) one would replace $\sigma/\theta \to \text{arcsinh}(\sqrt{2H^\dagger H}/\theta)$, etc.

Since θ has a non-vanishing vev (this follows from the initial assumption $\langle\phi\rangle \neq 0$, that allowed the linearisation of $\tilde{R}^2$ term), we can now fix the gauge of the Weyl local scale symmetry. We choose a gauge where

$$\langle\theta^2\rangle = 6\,M_p^2 \tag{32}$$

where M_p is the Planck scale and θ is defined in (24). In this gauge the kinetic terms in $\mathcal{L}_H$ are then canonical and the radial direction field θ actually decouples from the spectrum. In the case of SM with gauged Weyl symmetry embedded in general Weyl conformal symmetry where ω_μ is actually dynamical, the field θ is "eaten" à la Stueckelberg by ω_μ which then becomes massive [22] (see also [24]). This is the main difference from the local Weyl symmetry case discussed here.

The action becomes

$$\mathcal{L}_H = \sqrt{g}\,\Big\{\frac{-1}{2}\,M_p^2 R + \frac{1}{2}\,(\partial_\mu\sigma)^2 + m_W^2(\sigma) W_\mu^+ W^{-\mu} + \frac{m_Z^2(\sigma)}{2}\,Z_\mu Z^\mu - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 - V\Big\} \tag{33}$$

Hence Einstein action is obtained in the broken phase of local Weyl symmetry. Here we introduced the notation

$$m_W^2(\sigma) \;\equiv\; \frac{3}{2}\,M_p^2\,g^2 \sinh^2\frac{\sigma}{M_p\sqrt{6}} = \frac{1}{4}\,g^2\,\sigma^2\Big[1 + \frac{\sigma^2}{18\,M_p^2} + \cdots\Big] \tag{34}$$

$$m_Z^2(\sigma) \;\equiv\; \frac{3}{2}\,(g^2 + g'^2) \sinh^2\frac{\sigma}{M_p\sqrt{6}} = \frac{1}{4}\,(g^2 + g'^2)\,\sigma^2\,\Big[1 + \frac{\sigma^2}{18M_p^2} + \cdots\Big] \tag{35}$$

for the higgs-dependent "masses" of $W^\pm$, Z bosons and finally

$$V \;=\; \frac{3}{2}\,M_p^4\,\Big\{6\lambda\,\sinh^4\frac{\sigma}{M_p\sqrt{6}} + \xi^2\,\Big(1 - \xi_h\,\sinh^2\frac{\sigma}{M_p\sqrt{6}}\Big)^2\Big\}. \tag{36}$$

$$\;=\; \frac{1}{4}\,\Big[\lambda - \frac{1}{9}\,\xi_h\,\xi^2 + \frac{1}{6}\,\xi_h^2\,\xi^2\Big]\sigma^4 - \frac{1}{2}\,\xi_h\,\xi^2 M_p^2\,\sigma^2 + \frac{3}{2}\xi^2\,M_p^2 + \mathcal{O}(\sigma^6/M_p^2). \tag{37}$$

An expansion was made for $\sigma \ll M_p$, to obtain a SM-like Higgs potential. The potential in (36) is similar to that in SMW [24] which has an enlarged, gauged scale symmetry and where it was analysed in detail. Differences remain however in the Higgs sector, as discussed shortly. The above expansions show the emergence of higher dimensional operators corrections in the EW sector of SM, suppressed by M_p. We also find (see also Appendix B.3 and B.2)

$$\Lambda = \frac{1}{4}\langle\phi^2\rangle = \frac{\xi^2\langle\theta\rangle^2}{4\,(1 + \xi^2\,\xi_h^2/(6\lambda)} \approx \frac{\xi^2\langle\theta^2\rangle}{4} + \mathcal{O}(\xi^4\xi_h^2/\lambda). \tag{38}$$

Note that again Λ and M_p have a common origin being $\propto \langle\theta\rangle$. A hierarchy $\Lambda \ll M_p$ is then due to the fact that gravity is ultraweak $\xi \ll 1$, where ξ is the coupling of the $\tilde{R}^2$ term.

Consider next the SM gauge sector. The SM gauge bosons action is invariant under transformation (1), since the gauge bosons do not transform. One way to see this is that they enter under the corresponding covariant derivative acting on a field charged under it and should transform (have same weight) as ∂_μ acting on that field; since coordinates are kept fixed under (1), the gauge fields do not transform under (1) i.e. have a vanishing Weyl charge. Then their kinetic terms are similar to those in the SM in pseudo-Riemannian geometry since the Weyl connection is symmetric: $(F_{\mu\nu})_{SM}$ involves the difference $\tilde\nabla_\mu A_\nu - \tilde\nabla_\nu A_\mu$, where A_μ denotes a SM gauge boson and since $\tilde\nabla_\mu A_\nu = \partial_\mu A_\nu - \tilde\Gamma^\rho_{\mu\nu} A_\rho$ for a symmetric $\tilde\Gamma^\rho_{\mu\nu} = \tilde\Gamma^\rho_{\nu\mu}$ then $\tilde\Gamma^\rho_{\mu\nu}$ (and ω_μ) cancels out in $F_{\mu\nu}$ which then has a form as in flat space-time. Thus the gauge kinetic term does not depend on the Weyl connection and is similar to that in pseudo-Riemannian geometry:

$$\mathcal{L}_g = - \sum_{\text{groups}} \frac{\sqrt{g}}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{39}$$

where the sum is over the SM gauge group factors: $SU(3) \times SU(2)_L \times U(1)_Y$.

Finally, consider the SM fermions sector. According to (2) the fermions have non-zero Weyl weight. However, the fermions action is Weyl vector field independent [28] (for a review [24]) and is invariant under transformation (1), (2). This invariance is due to a cancellation in the Dirac action of the dependences on ω_μ between that of the Weyl-covariant derivative acting on fermions and that of the spin connection part of this derivative. Since the fermions action is independent of ω_μ in Weyl geometry, this remains true in our particular integrable case of ω_μ of (26). So the fermions action is similar to the (pseudo)Riemannian case

$$\mathcal{L}_f = \frac{1}{2} \sqrt{g}\, \overline{\psi}\, i\, \gamma^a\, e_a^\mu \left[\partial_\mu - i g\, \vec{T} \vec{\hat{A}}_\mu - i\, Y g' \hat{B}_\mu + \frac{1}{2} s_\mu^{\ ab}\, \sigma_{ab} \right] \psi + h.c., \tag{40}$$

with the usual quantum numbers of fermions under SM group and with $\vec{T} = \vec{\tau}/2$. The spin connection has the usual form of Riemannian geometry

$$s_\mu^{ab} = -e^{\lambda\, b} \left(\partial_\mu\, e_\lambda^a - \Gamma_{\mu\lambda}^\nu\, e_\nu^a \right). \tag{41}$$

One can check that $\mathcal{L}_f$ is invariant under (1). Regarding the Yukawa interaction $\mathcal{L}_Y$ it is similar to that in flat space-time uplifted to curved space-time and is also invariant under (1), (2), as seen from Weyl charges of the fields involved and of $\sqrt{g}$ [24].

The SM Lagrangian in Weyl integrable geometry is then given by

$$\mathcal{L} = \mathcal{L}_H + \mathcal{L}_f + \mathcal{L}_g + \mathcal{L}_Y. \tag{42}$$

The only difference between our case and SMW [24] is thus manifest in the Higgs sector $\mathcal{L}_H$.

In conclusion, the SM action changes very little when this is endowed with local scale symmetry in quadratic gravity in the context of Weyl integrable geometry. No new degrees of freedom beyond those of the SM and underlying geometry were added, the Einstein action is easily recovered in the broken phase while in the EW sector the higher dimensional operators generated are suppressed by M_p. Thus, the SM admits a truly minimal and natural embedding in Weyl integrable geometry, which is interesting.

3.2 Inflation predictions

Here we comment on an immediate phenomenological prediction. The model can have successful inflation based on the potential in eq.(36). The potential is similar to that discussed in [51, 52] see also [22–24, 53], for which inflation predictions were already studied in detail, hence we can use below these results.

Although the potential depends on the Higgs field only, the inflation mechanism is of Starobinsky-Higgs type. This is understood as follows: successful inflation requires that the second term in eq.(36) dominates and gives the leading (flat) contribution during inflation. This contribution is actually that due to the field ϕ, as seen by comparing eqs.(24) and (25), with ϕ due to the $\tilde{R}^2$ term in the action. This explains the expectation for inflation predictions similar to those in Starobinsky model. One can show that the potential in (36) gives a dependence of the tensor-to-scalar ratio (r) on the spectral index n_s [52, 53] which is

$$r = 2(1-n_s)^2 - \frac{16}{3}\,\xi_h^2 + \mathcal{O}(\xi_h^3). \tag{43}$$

The term dependent on ξ_h is due to the non-minimal coupling of the Higgs field and its effect can be ignored for $\xi_h < 10^{-3}$ since then its correction to r is too small to be measured. In such case the above relation is similar to that in Starobinsky inflation, where $r = 3(1-n_s)^2$. The value of r in Starobinsky inflation is an upper bound that is saturated in the present model for $\xi_h \to 0$. For larger $\xi_h \sim 10^{-3} - 10^{-2}$, a smaller r is obtained. Numerically, for $n_s = 0.9670 \pm 0.0037$ at 68% CL then [51–53] (see also [58])

$$0.00257 \leq r \leq 0.00303, \tag{44}$$

This result demands $\lambda \ll \xi_h^2\,\xi^2$ which can be respected for small enough λ. Such value for r is within the reach of the new generation of CMB experiments: CMB-S4, LiteBIRD, PICO, PIXIE [59–64].

3.3 Comparison to SMW

Let us now compare our model to the SM with gauged scale symmetry [24] (SMW) in general Weyl geometry. The details of the SMW action and its equations of motion are presented in Appendix B.2, for convenience. The difference between the action of the SM with local Weyl symmetry and the SMW is manifest in the Higgs and gravity sectors ($\mathcal{L}_H$) and is given by the absence of $F_{\mu\nu}^2$ in the initial Lagrangian eq.(19) of the current model. Let us then compare the broken and symmetric phases of the final actions of the two theories.

In the broken phase, the Einstein action is naturally recovered in both cases; the difference is that in the SMW action there are additional terms $\Delta\mathcal{L}$ present [24], compared to action (42) of the case with local Weyl symmetry, with

$$\begin{aligned}\Delta\mathcal{L} &= \sqrt{g}\,\frac{3}{4}\,M_p^2\,\alpha^2\,\omega_\mu\,\omega^\mu\Big[1+\sinh^2\frac{\sigma}{M_p\sqrt{6}}\Big] - \sqrt{g}\,\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} &(45)\\ &= \sqrt{g}\,\Big\{\frac{3}{4}\,\alpha^2\,M_p^2\,\omega_\mu\omega^\mu + \frac{1}{8}\,\alpha^2\,\sigma^2\,\omega_\mu\,\omega^\mu + \mathcal{O}(1/M_p^2)\Big\} - \sqrt{g}\,\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} &(46)\end{aligned}$$

where $F_{\mu\nu}$ is the field strength of ω_μ. The mass term for ω_μ follows from a Stueckelberg mechanism in which ω_μ has eaten the θ field to become massive [22]. $\Delta\mathcal{L}$ contains an important new Higgs coupling $\sigma^2\,\omega_\mu\omega^\mu$ absent in our current model. This coupling comes from the Higgs kinetic term $|\tilde{D}_\mu H|^2$. These are the main differences between the current model and SMW. In SMW there can also be a gauge kinetic mixing term between ω_μ and the SM $U(1)_Y$ field, not shown here; however, such mixing is strongly constrained from its correction to the Z boson mass [24].

We see that the *broken phase* of local Weyl symmetry of our model is indeed recovered from that of SMW [24] by simply decoupling the massive Weyl vector field in the action. For a large enough mass m_ω of ω_μ, the correction $\Delta\mathcal{L}$ is suppressed and one is left with the action of the present case. Naively, one would think that m_ω is very large (near M_p) but in fact m_ω can be significantly lower. Actually, the current lower bound on m_ω which sets the so-called "non-metricity scale" is of few TeV only [65]. This bound is found by demanding that non-metricity corrections (i.e. due to ω_μ) to the Bhabha scattering cross section be within the current error of this cross section; this bound is found by using a Dirac action that involves non-metricity (ω_μ) [65] which mediates this process; it must be said, however, that in Weyl geometry there is actually no coupling of SM fermions to vectorial non-metricity (ω_μ) [28] (see also [24]), hence this non-metricity lower bound can actually be evaded and be lower than few TeV. This deserves further study.

Regarding the symmetric phases, the local Weyl symmetry of the current model is naturally enlarged to a Weyl gauge symmetry in SMW [24] that brings in the ω_μ gauge field, see eqs.(1), (2), (3). The SMW with gauged Weyl symmetry seems then a more general and physical UV completion of the SM and Einstein gravity into a *gauge theory* of scale invariance than the current model with local Weyl symmetry only (no ω_μ). To understand why this is so, note that in the SMW there is an associated conserved current [22], see eq.(B-29)

$$\nabla^\mu J_\mu = 0, \qquad J_\mu = \frac{\alpha}{4}\left(\nabla_\mu - \alpha\,\omega_\mu\right) K, \qquad K \equiv \theta^2 + h^2. \tag{47}$$

This relation is a generalisation of an onshell current conservation in *global* scale invariant theories [3,4,7,8]. In our current model, however, ω_μ is that of eq.(26), which when used in (47) gives $J_\mu = 0$. This means that the current associated with a local Weyl symmetry is trivial and the charge is vanishing if ω_μ is not dynamical. This seems a general problem with models with local Weyl symmetry [16,17], regardless of the underlying geometry considered (Weyl or Riemannian) where this symmetry is implemented. This questions if local Weyl symmetry is physical and if it can be a symmetry of a fundamental, UV-complete theory.

In the light of the above arguments, we conclude that SMW is a more fundamental UV completion of SM and Einstein gravity than its version with local Weyl symmetry discussed here. SMW provides such UV completion in a full gauge theory of scale invariance[11]. Correspondingly, the (non-metric) Weyl conformal geometry underlying the SMW is more fundamental than its (metric) integrable version underlying the current model, that is conformal to Riemannian geometry of Einstein gravity. And like any gauge symmetry, the gauged Weyl symmetry of SMW must remain valid at quantum level. Loop corrections thus require a regularisation/renormalization that respects this symmetry [66] (also more recent [67,68]). In this way the symmetry is maintained at quantum level and one avoids the Weyl anomaly, see [25] for an update. The UV completion of SM and Einstein gravity can thus be realised by a (quantum) gauged Weyl invariant theory. This deserves further study.

[11]It remains to be seen if it is also renormalizable as in quadratic gravity [69], see [24,49] for some arguments.

4 Conclusions

Since the SM with the Higgs mass parameter set to zero has global scale symmetry, we explored the consequence of making this symmetry local and including gravity, using the gauge invariance principle. We studied the SM in quadratic gravity with *local* Weyl symmetry which is of strong interest in gravity theories, and compared it to SM in quadratic gravity with a *gauged* Weyl symmetry (SMW) studied previously; these models are distinguished by the absence or presence of an associated Weyl gauge boson of dilatations (ω_μ), respectively.

We showed that SM and Einstein gravity admit a natural embedding in so-called Weyl integrable geometry, with no new fields needed beyond the SM and this geometry. This geometry is a special limit of Weyl conformal geometry of vanishing length curvature tensor ($F_{\mu\nu} = 0$). Physically, this means that $\omega_\mu = 0$ or is "pure gauge" (non-dynamical), as in conformal gravity. Hence, the theory has local Weyl invariance, but there is no gauge boson. The theory is quadratic in the scalar curvature, so is more general than previous theories of SM with local Weyl symmetry in (pseudo)Riemannian geometry which were linear in scalar curvature; these required a scalar field ϕ (beyond SM) be added "by hand" to implement this symmetry and generate Einstein gravity from $\phi^2 R$. The situation also differs from the SMW which has the larger, *gauged* Weyl symmetry in general Weyl geometry, where ω_μ is dynamical, but then the geometry becomes non-metric.

The local Weyl symmetry of our theory is spontaneously broken to Einstein action. The mass scales (Planck scale and cosmological constant Λ) have geometric origin, being generated by a scalar field ϕ arising from the $\tilde{R}^2$ term in the action. Λ is positive due to the quadratic nature of the action. This situation is similar to the SMW where, in addition, $\ln \phi$ is further "eaten" by the Weyl gauge boson ω_μ which then becomes a massive Proca field (by Stueckelberg mechanism) and subsequently decouples. Therefore, after decoupling, the broken phase of SMW recovers the broken phase of local Weyl symmetry of our current model, as we verified.

The hierarchy of the scales $\Lambda \ll M_p$ is generated by one classical tuning of the dimensionless coupling ξ of the $\tilde{R}^2$ term in the action. One may expect this hierarchy remain stable at a quantum level, due to local Weyl symmetry. Inflation is Starobinsky-like, with the inflaton role essentially played by the same ϕ coming from the $\tilde{R}^2$ term, as in SMW, and the scalar potential is similar in both theories. The tensor-to-scalar ratio r is then similar to that in SMW, $r \sim 3 \times 10^{-3}$, with an upper bound equal to the Starobinsky model value, saturated for a vanishing Higgs non-minimal coupling. The present theory differs however from the SMW in the Higgs sector: SMW predicts a direct coupling of the Higgs σ to the Weyl gauge boson $\alpha^2 \, \omega_\mu \omega^\mu \sigma^2$, the implications of which are yet to be explored.

There is one issue that seems common to all models beyond SM and Einstein gravity with local Weyl symmetry, regardless of their underlying geometry: the associated current of this symmetry vanishes. This raises concerns on the physical meaning of such symmetry. The gauge invariance principle tells us that one should actually implement the full gauged Weyl symmetry. The SMW with its (larger) gauged Weyl symmetry and (non-trivial) conserved current is therefore more fundamental. In conclusion, SMW is a good UV completion of the current theory and of SM, giving an (anomaly-free) *gauge theory* of scale invariance that generates Einstein gravity in its broken phase, as an effective theory.

Appendix

A Weyl geometry and parallel transport

We review here the parallel transport in Weyl geometry and its integrable limit. As stated in the text, Weyl geometry is represented by classes of equivalence of $(g_{\mu\nu}, \omega_\mu)$ related by (A-1). If present, scalars ϕ and fermions ψ also transform

$$\hat g_{\mu\nu} = \Sigma^q \, g_{\mu\nu}, \quad \sqrt{\hat g} = \Sigma^{2\,q}\sqrt{g}, \quad \hat\omega_\mu = \omega_\mu - \frac{1}{\alpha}\,\partial_\mu \ln\Sigma, \quad \hat e^a_\mu = \Sigma^{q/2}\, e^a_\mu, \tag{A-1}$$

$$\hat\phi = \Sigma^{-q/2}\,\phi, \quad \hat\psi = \Sigma^{-3q/4}\psi. \tag{A-2}$$

Here q is the Weyl charge of the metric (in the text we set $q = 1$). Weyl geometry is non-metric which means $\tilde\nabla_\mu g_{\alpha\beta} \neq 0$, or more exactly:

$$\tilde\nabla_\mu g_{\alpha\beta} = -\alpha\, q\, \omega_\mu\, g_{\alpha\beta}, \tag{A-3}$$

where $\tilde\nabla$ is defined by the Weyl connection $\tilde\Gamma$

$$\tilde\nabla_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - \tilde\Gamma^\rho_{\alpha\mu} g_{\rho\beta} - \tilde\Gamma^\rho_{\beta\mu} g_{\rho\alpha}. \tag{A-4}$$

Eq.(A-3) may be written as

$$\tilde\nabla'_\mu g_{\alpha\beta} = 0, \qquad \tilde\nabla' \equiv \tilde\nabla\Big\vert_{\partial_\mu\to\partial_\mu+\alpha\, q\,\omega_\mu}. \tag{A-5}$$

Therefore the (symmetric) Weyl connection $\tilde\Gamma$ is found from the Levi-Civita connection (Γ) in which one makes the same substitution: $\tilde\Gamma = \Gamma\vert_{\partial_\lambda\to\partial_\lambda+\alpha\, q\,\omega_\lambda}$. This gives

$$\tilde\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + (q/2)\,\alpha\,\Big[\delta^\lambda_\mu\;\omega_\nu + \delta^\lambda_\nu\;\omega_\mu - g_{\mu\nu}\,\omega^\lambda\Big]. \tag{A-6}$$

Consider now an arbitrary vector u^μ of some non-vanishing Weyl charge ($z_u \neq 0$):

$$\hat u^\mu = \Sigma^{z_u/2} u^\mu \tag{A-7}$$

The parallel transport of a constant vector (in a Weyl-covariant sense) is defined by

$$\frac{D\,u^\mu}{d\tau} = 0, \quad \text{where} \quad D \equiv dx^\lambda\, D_\lambda, \quad D_\lambda\, u^\mu = \tilde\nabla_\lambda u^\mu\Big\vert_{\partial_\lambda\to\partial_\lambda+(z_u/2)\,\alpha\,\omega_\lambda}, \tag{A-8}$$

with

$$\tilde\nabla_\lambda u^\mu = \partial_\lambda u^\mu + \tilde\Gamma^\mu_{\lambda\rho}\, u^\rho, \tag{A-9}$$

and $x = x(\tau)$. Then from (A-8) the differential variation of the vector is

$$d\,u^\mu = dx^\lambda\,\partial_\lambda u^\mu = -dx^\lambda\,\Big[(z_u/2)\,\alpha\,\omega_\lambda\, u^\mu + \tilde\Gamma^\mu_{\lambda\rho}\, u^\rho\Big], \tag{A-10}$$

Under the parallel transport along a curve, a product $\langle u,v\rangle = u^\mu\, v^\nu\, g_{\mu\nu}$ of vectors u, v changes, too, so using (A-3) and (A-10) applied to u, v vectors, one finds

$$d\langle u,v\rangle = d\left[u^\mu\, v^\nu\, g_{\mu\nu}\right] = dx^\lambda \left[\tilde\nabla_\lambda g_{\mu\nu} - \alpha\, \omega_\lambda\, g_{\mu\nu}(z_u+z_v)/2\right] u^\mu u^\nu. \qquad \text{(A-11)}$$

$$= -\alpha\, dx^\lambda\, \omega_\lambda \left[q + (z_u+z_v)/2\right] \langle u,v\rangle \qquad \text{(A-12)}$$

For the norm of a vector

$$d \ln |u|^2 = dx^\lambda\, \omega_\lambda\, (-\alpha)\, (q+z_u), \qquad \text{(A-13)}$$

or, integrating this along a path $\gamma(\tau)$:

$$|u|^2 = |u_0|^2\, e^{-\alpha\, (q+z_u)\, \int_{\gamma(\tau)} \omega_\lambda dx^\lambda}. \qquad \text{(A-14)}$$

u and u_0 are the values at the end points of the path. Hence, using (A-1) we see that, in general, the integral and the norm of the vectors are path-dependent in Weyl geometry (in Riemannian case $\omega_\lambda = 0$ and the norm is invariant for any z_u and any γ).

However, for any tangent space vector $u^a = e^a_\mu u^\mu$ that is invariant i.e. it has a vanishing charge, then $z_u/2 + q/2 = 0$ (since e^a_μ has charge $q/2$), and with this the norm of u^μ itself in eq.(A-14) is actually invariant $|u| = |u_0|$ under parallel transport for any γ.

Finally, the ratio of the norms of two vectors of same but arbitrary Weyl weight is also invariant under the parallel transport, for same γ. This is seen by using (A-13)

$$d\, \ln \frac{|u|^2}{|v|^2} = (-\alpha)\, (z_u - z_v)\, \omega_\lambda\, dx^\lambda, \qquad \text{(A-15)}$$

which vanishes if $z_u = z_v$: the relative length is then invariant.

In Weyl integrable geometry $\omega_\lambda = \partial_\lambda$(scalar-field) and if the path is closed then the integral vanishes and the norm is invariant $|u| = |u_0|$, (and path-independent between 2 points). This result was used in the past to favour gravity theories based on Weyl integrable geometry (instead of general non-metric Weyl geometry) since it meant that there was no second clock effect: the theory is metric, then rods' length and clocks' rates would not change. This was to avoid long-held criticisms [19] directed at gravity theories based on (non-metric) Weyl conformal geometry.

Such criticisms were misleading for (non-metric) Weyl geometry since they implicitly assumed that ω_λ was a massless gauge field. Actually, in the symmetric phase there is no second clock effect since one cannot define a clock rate in the absence of a mass scale (forbidden by the symmetry of the action). To test the second clock effect and compare to an experiment in a gauge theory of scale invariance built in a general Weyl geometry, one must first *fix the gauge* of this symmetry. The gauge is naturally fixed since ϕ (which is part of $\tilde R^2$ geometric term in the action) and has a non-zero vev, is "eaten" in a Stueckelberg mechanism by ω_λ which thus becomes massive and decouples [22], and the symmetry is spontaneously broken. In this broken phase non-metricity effects due to ω_μ are strongly suppressed by a large enough mass of ω_μ (the current lower bound on m_ω is only few TeV [65]!) and the mentioned criticisms are avoided. This breaking is geometric in nature [22] and needs no scalar fields be added "by hand" to this purpose. Hence the absence of second clock effect is general, regardless of the matter content of the theory embedded in Weyl geometry.

B SM in Weyl geometry and its integrable limit

We discuss the equations of motion of Weyl quadratic gravity first in the absence, then in the presence of the SM, for the case with gauged Weyl symmetry then take the limit of local scale symmetry (integrable geometry case).

B.1 Weyl quadratic gravity

The review here is based on [24] and gives the equations of motion for an action with gauged Weyl symmetry, which is similar to action (13), (14) (without the $C^2_{\mu\nu\rho\sigma}$ term), but has in addition a kinetic term for ω_μ. We then examine what happens if this term is absent (integrable case, ω_μ "pure gauge"). Consider then

$$\mathcal{L} = \sqrt{g}\,\Big\{\frac{1}{4!}\,\frac{1}{\xi^2}\,\tilde R^2-\frac{1}{4}\,F_{\mu\nu}^2\Big\} \tag{B-1}$$

$$= \sqrt{g}\,\Big\{-\frac{1}{12}\,\frac{\phi^2}{\xi^2}\,\Big[R-3\alpha\nabla_\mu\omega^\mu-\frac{3}{2}\alpha^2\,\omega_\mu\omega^\mu\Big]-\frac{1}{4!}\,\frac{\phi^4}{\xi^2}-\frac{1}{4}\,F_{\mu\nu}^2\Big\}. \tag{B-2}$$

Eq.(8) was used and $\tilde R^2$ was linearised with the aid of the newly introduced field ϕ, as in the text. After Stueckelberg mechanism [22, 24]

$$\mathcal{L}=\sqrt{g}\,\Big\{-\frac{1}{2}\,M_p^2\,R-\Lambda\,M_p^2-\frac{1}{4}\,F_{\mu\nu}^2+\frac{3}{4}\,\alpha^2\,M_p^2\,\omega_\mu\omega^\mu\Big\},\quad \Lambda\equiv\frac{\langle\phi\rangle^2}{4},\quad M_p^2\equiv\frac{\langle\phi\rangle^2}{6\xi^2}. \tag{B-3}$$

Using the notation:

$$\mathcal{K}=\frac{\phi^2}{\xi^2},\qquad \mathcal{V}=\frac{1}{4!}\,\frac{\phi^4}{\xi^2}, \tag{B-4}$$

the variation of (B-2) with respect to the metric gives

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} &= \frac{1}{12}\Big\{-\mathcal{K}\Big[R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,R\Big]-\Big[g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\Big]\mathcal{K}\\ &+ \frac{3\alpha^2}{2}\mathcal{K}\Big[\omega_\mu\,\omega_\nu-\frac{1}{2}\,g_{\mu\nu}\,\omega^\rho\,\omega_\rho\Big]-\frac{3\,\alpha}{2}\Big[\omega_\nu\nabla_\mu+\omega_\mu\nabla_\nu-g_{\mu\nu}\,\omega^\rho\nabla_\rho\Big]\,\mathcal{K}\Big\}\\ &- \frac{1}{2}\,\Big[g^{\alpha\beta}\,F_{\mu\alpha}F_{\nu\beta}-\frac{1}{4}\,g_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta}\Big]+\frac{1}{2}g^{\mu\nu}\mathcal{V}=0.\end{aligned} \tag{B-5}$$

Taking the trace

$$\frac{1}{12}\,\Big(\mathcal{K}\,R+3\alpha\,\omega^\mu\,\nabla_\mu\mathcal{K}-\frac{3\alpha^2}{2}\,\mathcal{K}\omega_\mu\omega^\mu-3\Box\mathcal{K}\Big)+2\mathcal{V}=0. \tag{B-6}$$

Further, the equation of motion of ϕ that is obtained from the above Lagrangian is

$$\frac{1}{12\,\xi^2}\,\phi^2\Big(\tilde R+\phi^2\Big)=0 \tag{B-7}$$

This is actually known ($\phi^2=-\tilde R$), since it was introduced to linearise the $\tilde R^2$ term. On the ground state $\langle\phi\rangle^2 = -R$ or $R = -4\Lambda$, which is also found from the above trace equation.

For a Friedmann-Robertson-Walker metric one has $R = -12H_0^2$ and equation $R = -4\Lambda$ then gives that in Weyl quadratic gravity

$$\Lambda = 3H_0^2. \tag{B-8}$$

The equation of motion of ω_μ can be written as

$$J^\mu + \nabla_\rho F^{\rho\mu} = 0, \qquad \text{where} \qquad J^\mu \equiv -\frac{\alpha}{4} g^{\mu\nu}\big(\nabla_\nu - \alpha\omega_\nu\big)\,\mathcal{K} \tag{B-9}$$

Using the anti-symmetry in indices of the field strength F, by applying ∇_μ on (B-9) then

$$\nabla_\mu J^\mu = 0. \tag{B-10}$$

Therefore, for a dynamical ω_μ we have a non-trivial conserved current J_μ. This is also seen by subtracting (B-7) from (B-6): one obtains again eq.(B-10) with J^μ as in (B-9).

Finally, consider the integrable geometry case, when ω_μ is non-dynamical in (B-1), ($F_{\mu\nu} = 0$), then the equation of motion of ω_μ is

$$\omega_\mu = \frac{1}{\alpha}\partial_\mu \ln \mathcal{K}. \tag{B-11}$$

Using this back in J^μ one see that J^μ vanishes. Thus the associated current to the symmetry is trivial and the symmetry is then "fake" [16, 17], raising doubts on its physical meaning, in this case. Hence, in the absence of matter, the current is non-trivial only for a dynamical Weyl field, showing the importance of fully gauging the scale symmetry.

B.2 SM in Weyl geometry (SMW)

Here we give details of the SM in Weyl conformal geometry (SMW); this information is complementary to that in [22, 24] and shows the Lagrangian, the equations of motion and conserved current. We then take the limit of integrable geometry studied in the text.

As shown in Section 2 of [24], the SMW action for fermions and gauge bosons is similar to that of SM in Riemannian geometry, but it differs significantly in the Higgs and gravity sectors. A gauge kinetic mixing SM hypercharge - ω_μ may also be present in SMW. Hence, the part of the SMW Lagrangian of Higgs + gravity + hypercharge sectors is [24]

$$\mathcal{L}_H = \sqrt{g}\,\Big\{ \frac{\tilde R^2}{4!\,\xi^2} - \frac{\xi_h}{6}\,|H|^2\tilde R + |\tilde D_\mu H\,|^2 - \lambda\,|H|^4 - \frac{1}{4}\Big(F_{\mu\nu}^{\,2} + 2\sin\chi\,F_{\mu\nu}\,F_Y^{\mu\nu} + F_{Y\,\mu\nu}^{\,2}\Big) - \frac{1}{\eta^2}\tilde C_{\mu\nu\rho\sigma}^2\Big\}. \tag{B-12}$$

which is invariant under (1), (2), (3) and with the Weyl-covariant derivative

$$\tilde D_\mu H = \big[\partial_\mu - i\mathcal{A}_\mu - (1/2)\,\alpha\,\omega_\mu\big]\,H, \tag{B-13}$$

$$|\tilde D_\mu H|^2 = |D_\mu H|^2 - \frac{\alpha}{2}\,\omega^\mu\,\Big[\nabla_\mu(H^\dagger H) - \frac{\alpha}{2}\,\omega_\mu H^\dagger H\Big]. \tag{B-14}$$

where $D_\mu H$ is the SM derivative of the Higgs. $F_{\mu\nu}$ ($F_{Y\,\mu\nu}$) is the Weyl vector (hypercharge) field strength, respectively. As in the text, we linearise the $\tilde R^2$ term with the aid of a scalar field ϕ by replacing in $\mathcal{L}_H$: $\tilde{R}^2 \to -2\phi^2 \tilde{R} - \phi^4$. The equation of motion of ϕ has solution $\phi^2 = -\tilde{R}$ which when used back in $\mathcal{L}_H$ recovers (B-12). Up to a total derivative $\mathcal{L}_H$ becomes

$$\mathcal{L}_H = \sqrt{g}\Big\{ -\frac{1}{12}\theta^2 R - \frac{\alpha}{4}\omega^\mu \nabla_\mu K + \frac{\alpha^2}{8} K\, \omega_\mu \omega^\mu + |D_\mu H|^2 - V - \frac{1}{4}\mathcal{F}^2_{\mu\nu} - \frac{1}{\eta^2} C^2_{\mu\nu\rho\sigma} \Big\} \quad \text{(B-15)}$$

where θ is the radial direction in the field space of initial fields (ϕ, H), and

$$\theta^2 \equiv \frac{1}{\xi^2}\phi^2 + 2\xi_h\, H^\dagger H, \qquad K \equiv \theta^2 + 2H^\dagger H. \quad \text{(B-16)}$$

$$V = \lambda |H|^4 + \frac{1}{24\xi^2}\,\phi^4 = \lambda |H|^4 + \frac{\xi^2}{24}\,(\theta^2 - 2\xi_h\, H^\dagger H)^2. \quad \text{(B-17)}$$

$$\mathcal{F}^2_{\mu\nu} \equiv \frac{1}{\gamma^2} F^2_{\mu\nu} + 2\sin\chi\, F_{\mu\nu}\, F^{\mu\nu}_Y + F^2_{Y\,\mu\nu}, \qquad \frac{1}{\gamma^2} \equiv 1 + \frac{6\alpha^2}{\eta^2} > 1. \quad \text{(B-18)}$$

Below we consider the equations of motion in a unitary gauge $H = 1/(\sqrt{2})\, h\, \zeta^T$, $\zeta = (0, 1)$. We shall also ignore the Weyl term $\tilde{C}^2_{\mu\nu\rho\sigma}$ and hence $\gamma = 1$; this term would not contribute to the trace equation below; this term was studied extensively in [54]. In the unitary gauge

$$\mathcal{L}_H = \sqrt{g}\,\Big\{ -\frac{1}{12}\theta^2 R - \frac{\alpha}{4}\omega^\mu \nabla_\mu K + \frac{\alpha^2}{8} K\, \omega_\mu \omega^\mu + \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}\, h^2\, \mathcal{E}_{\mu\nu}\, g^{\mu\nu} - V - \frac{1}{4}\,\mathcal{F}^2_{\mu\nu} \Big\} \quad \text{(B-19)}$$

where

$$\theta^2 \equiv \frac{1}{\xi^2}\phi^2 + \xi_h\, h^2, \qquad K \equiv \theta^2 + h^2 \quad \text{(B-20)}$$

$$V = \frac{1}{4}\lambda\, h^4 + \frac{\xi^2}{24}\,(\theta^2 - \xi_h\, h^2)^2. \quad \text{(B-21)}$$

$$\mathcal{F}^2_{\mu\nu} \equiv F^2_{\mu\nu} + 2\sin\chi\, F_{\mu\nu}\, F^{\mu\nu}_Y + F^2_{Y\,\mu\nu}, \quad \text{(B-22)}$$

$$\mathcal{E}_{\mu\nu} \equiv \frac{g^2}{2}\, W^+_\mu\, W^-_\nu + \frac{g^2 + g'^2}{4}\, Z_\mu Z_\nu. \quad \text{(B-23)}$$

The action is Weyl gauge invariant. The equation of motion for $g^{\mu\nu}$ from $\mathcal{L}_H$ gives

$$\begin{aligned}
\frac{1}{\sqrt{g}}\frac{\delta \mathcal{L}_H}{\delta g^{\mu\nu}} &= \frac{1}{12}\Big\{ -\theta^2 \Big[R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R\Big] - \Big[g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\Big]\theta^2 \\
&+ \frac{3\alpha^2}{2} K \Big[\omega_\mu\, \omega_\nu - \frac{1}{2}\, g_{\mu\nu}\, \omega^\rho\, \omega_\rho\Big] - \frac{3\,\alpha}{2}\Big[\omega_\nu \nabla_\mu + \omega_\mu \nabla_\nu - g_{\mu\nu}\, \omega^\rho \nabla_\rho\Big]\, K\Big\} \\
&+ \frac{1}{2}\,\partial_\mu h \partial_\nu h - \frac{1}{4}\, g_{\mu\nu} (\partial_\alpha h)^2 + \frac{1}{2}\, h^2\, \Big[\mathcal{E}_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, \mathcal{E}_{\alpha\beta}\, g^{\alpha\beta}\Big] + \frac{1}{2}\, g_{\mu\nu}\, V \\
&- \frac{1}{2}\Big\{\Big[g^{\alpha\beta}\, F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4}\, g_{\mu\nu}\, F_{\alpha\beta}\, F^{\alpha\beta}\Big] + (F \leftrightarrow F^Y) \\
&+ \sin\chi\Big[g^{\rho\sigma}\big(F_{\mu\rho} F^Y_{\nu\sigma} + F_{\nu\sigma} F^Y_{\mu\rho}\big) - \frac{1}{2}\, g_{\mu\nu} F^{\alpha\beta} F^Y_{\alpha\beta}\Big]\Big\} = 0. \quad \text{(B-24)}
\end{aligned}$$

The last two lines are 1/2 times the stress energy tensor of the Weyl and hypercharge gauge fields, including their kinetic mixing. Note that for the full SM action the rhs of this equation should contain a similar contribution for SU(2) and SU(3) gauge bosons and a term $(1/2)T^{\psi}_{\mu\nu}$ which accounts for the stress-energy tensor of SM fermions, which are neglected here.

Taking the trace of (B-24):

$$\frac{1}{12}\,\theta^2 R - \frac{1}{4}\Box\theta^2 + \frac{\alpha}{4}\omega^\alpha\nabla_\alpha K - \frac{\alpha^2}{8}\,K\omega_\alpha\omega^\alpha - \frac{1}{2}\,(\partial_\alpha h)^2 - \frac{1}{2}\,h^2\mathcal{E}_{\alpha\beta}g^{\alpha\beta} + 2V = 0. \quad \text{(B-25)}$$

Another form of this equation making obvious its invariance under Weyl gauge symmetry is:

$$\frac{1}{12}\theta^2\tilde R - \frac{1}{4}\nabla^\mu\Big[(\nabla_\mu - \alpha\omega_\mu)\theta^2\Big] - \frac{1}{2}\Big[\Big(\nabla_\mu - \frac{\alpha}{2}\omega_\mu\Big)h\Big]^2 - \frac{1}{2}\,h^2\mathcal{E}_{\mu\nu}\,g^{\mu\nu} + 2V = 0. \quad \text{(B-26)}$$

with $\tilde R$ as in (8). After multiplying this equation by $\sqrt{g}$, each term in this equation is Weyl gauge invariant, hence the whole equation is invariant, as it should.

Further, the equation of motion of ϕ that is obtained from the above Lagrangian is

$$\phi^2\tilde R + \phi^4 = 0 \quad \text{(B-27)}$$

This is actually known ($\phi^2 = -\tilde R$), since it was used to linearise the $\tilde R^2$ term.

The equation of motion of ω_μ can be written as

$$J^\mu + \nabla_\rho\big(F^{\rho\mu} + \sin\chi\, F_Y^{\rho\mu}\big) = 0, \qquad \text{where} \qquad J^\mu \equiv -\frac{\alpha}{4}g^{\mu\nu}\big(\nabla_\nu - \alpha\omega_\nu\big)\,K \quad \text{(B-28)}$$

Using the anti-symmetry in indices of the Weyl and hypercharge field strengths F and F_Y, by applying the operator ∇_μ on the left equation we find that the current J^μ is conserved

$$\nabla_\mu J^\mu = 0. \quad \text{(B-29)}$$

This result was used in the text, eq.(47). For a further study of J_μ see Appendix A in [49].

There is also an equation of motion of h, but that brings no new information beyond that of "trace" eq.(B-26), eq.(B-27) and eq.(B-29) of current conservation, because it is actually a linear combination of these; hence we replace the equation of h by the much simpler (B-29). Finally, on the ground state, eqs.(B-26) and (B-27) give (with $\langle\phi\rangle^2 \neq 0$) that

$$\langle h\rangle^2 = \xi_h/(6\lambda)\,\langle\phi\rangle^2 \quad \text{(B-30)}$$

discussed in [24] (eq.46). Using (B-30), (B-20) and that from $\mathcal{L}_H$ we have $M_p^2 = \langle\theta^2\rangle/6$, then

$$M_p^2 = \frac{1}{6}\langle\theta^2\rangle = \frac{1}{6}\,\frac{\langle\phi^2\rangle}{\xi^2}\Big(1 + \frac{\xi^2\,\xi_h^2}{6\lambda}\Big) \quad \text{(B-31)}$$

which is similar to eq.(32) in the text. Using (B-20), (B-21), (B-30) we also find that $\Lambda = \langle\phi^2\rangle/4$. From eq.(B-31) for $\xi_h/\lambda \to 0$ which decouples the Higgs, then $\Lambda = (3/2)\,\xi^2\,M_p^2$ as in eq.(18). Eqs.(B-30), (B-31) also apply for the case discussed in the text, see eq.(38) and Section B.3. Finally, from $\phi^2 = -\tilde R$ then on the ground state $R = -4\Lambda$ (assuming $\langle\omega_\mu\omega^\mu\rangle = 0$).

B.3 SM in integrable geometry (local Weyl symmetry case)

This case discussed in the text in Section 3.1 can be recovered directly from the SMW (Section B.2): the action and the equations of motion can be found from those for the SMW by setting $F_{\mu\nu} = 0$. For convenience we present these equations below. First, the quadratic term in $\mathcal{L}_H$ (B-12) is linearised with the aid of ϕ, leading to eq.(B-15) with $F_{\mu\nu} = 0$. We do not include in the discussion here the $C^2_{\mu\nu\rho\sigma}$ term. From eq.(B-15) $\mathcal{L}_H$ becomes

$$\mathcal{L}_H = \sqrt{g}\Big\{ -\frac{1}{12}\theta^2 R - \frac{\alpha}{4}\omega^\mu \nabla_\mu K + \frac{\alpha^2}{8} K\,\omega_\mu\omega^\mu + |D_\mu H|^2 - \frac{1}{4}F^2_{Y\,\mu\nu} - V\Big\} \tag{B-32}$$

with notation as in (B-16) to (B-18). This gives that

$$\omega_\mu = \frac{1}{\alpha}\nabla_\mu \ln K, \qquad K = \frac{\phi^2}{\xi^2} + 2\,(\xi_h + 1)\,H^\dagger H. \tag{B-33}$$

All previous equations of SMW remain valid if one uses this value of ω_μ. The action becomes

$$\mathcal{L}_H = \sqrt{g}\Big\{ -\frac{1}{12}\theta^2 R - \frac{g^{\mu\nu}}{8}\,\frac{1}{K}\nabla_\mu K\,\nabla_\nu K + |D_\mu H|^2 - \frac{1}{4}F^2_{Y\,\mu\nu} - V\Big\} \tag{B-34}$$

In the unitary gauge

$$\mathcal{L}_H = \sqrt{g}\,\Big\{ -\frac{1}{12}\theta^2 R - \frac{g^{\mu\nu}}{8}\,\frac{1}{K}\nabla_\mu K\,\nabla_\nu K + \frac{1}{2}g^{\mu\nu}\partial_\mu h\,\partial_\nu h + \frac{1}{2}\,h^2 g^{\mu\nu}\mathcal{E}_{\mu\nu} - \frac{1}{4}F^2_{Y\,\mu\nu} - V\Big\} \tag{B-35}$$

with $K = \phi^2/\xi^2 + (\xi_h + 1)\,h^2$. This can also be expressed in terms of final Higgs (σ)

$$\mathcal{L}_H = \sqrt{g}\Big\{ -\frac{1}{12}\theta^2 R - \frac{1}{2}\,(\partial_\mu\theta)^2 + \frac{1}{2}\,\theta^2\,\Big(\partial_\mu\frac{\sigma}{\theta}\Big)^2 + \frac{1}{2}\,h^2 g^{\mu\nu}\mathcal{E}_{\mu\nu} - \frac{1}{4}F^2_{Y\,\mu\nu} - V\Big\} \tag{B-36}$$

which recovers eq.(30) in the text. Above we used $h = \theta\sinh(\sigma/\theta)$, so $K = \theta^2\cosh^2\sigma/\theta$.

The equation of the metric from (B-35) is a particular case of (B-24):

$$\begin{aligned}
\frac{1}{\sqrt{g}}\frac{\delta\mathcal{L}_H}{\delta g^{\mu\nu}} &= \frac{1}{12}\Big\{ -\theta^2\Big[R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,R\Big] - \Big[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\Big]\theta^2\Big\} \\
&- \frac{1}{8}\frac{1}{K}\,\Big[\nabla_\mu K\nabla_\nu K - \frac{1}{2}\,g_{\mu\nu}(\nabla_\alpha K)^2\Big] + \frac{1}{2}\,\partial_\mu h\partial_\nu h - \frac{1}{4}\,g_{\mu\nu}(\partial_\alpha h)^2 \\
&+ \frac{h^2}{2}\,\Big[\mathcal{E}_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\,\mathcal{E}_{\alpha\beta}\,g^{\alpha\beta}\Big] + \frac{g_{\mu\nu}}{2}\,V - \frac{1}{2}\,\Big[g^{\alpha\beta}\,F^Y_{\mu\alpha}F^Y_{\nu\beta} - \frac{g_{\mu\nu}}{4}\,F^Y_{\alpha\beta}\,F^{Y\,\alpha\beta}\Big] = 0.
\end{aligned} \tag{B-37}$$

The last bracket is the stress energy tensor of the hypercharge gauge field. For a full SM action the above equation should also contain a similar contribution for the SU(2) and SU(3) gauge bosons and a contribution $(1/2)T^\psi_{\mu\nu}$ which accounts for the stress energy tensor of SM fermions (not included here). Taking the trace

$$\frac{1}{12}\,\theta^2 R - \frac{1}{4}\Box\theta^2 + \frac{1}{8\,K}\,(\nabla_\mu K)^2 - \frac{1}{2}\,(\partial_\alpha h)^2 - \frac{1}{2}\,h^2\mathcal{E}_{\alpha\beta}g^{\alpha\beta} + 2V = 0. \tag{B-38}$$

while the equation of motion of ϕ gives

$$R-\frac{3}{2}\left(\nabla_\mu \ln K\right)^2-3\nabla_\mu\nabla^\mu \ln K=-\phi^2, \tag{B-39}$$

In the lhs of this equation we recognise the expression of $\tilde{R}$ for Weyl integrable geometry, see eq.(8) in the text with ω_μ of (B-33); hence $\tilde{R}=-\phi^2$, which we already know from linearising the quadratic term with the aid of ϕ.

The equation of motion for the Higgs h is simply a linear combination of the last two equations. The current J^μ is in this case trivial, as seen from using ω_μ of (B-33) in J_μ of (B-28), as also discussed in the text, after eq.(47). Regarding the values of M_P and Λ, the same discussion as in (B-30) and (B-31) applies, and again $R=-4\Lambda$ on the ground state.

Acknowledgement: The work of D.G. was supported by a grant of the Romanian Ministry of Education and Research, CNCS-UEFISCDI, project number PN-III-P4-ID-PCE-2020-2255 and partially by PN 23 21 01 01/2023.

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