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Stability and convergence analysis of the harmonic balance method for a Duffing oscillator with freeplay nonlinearity

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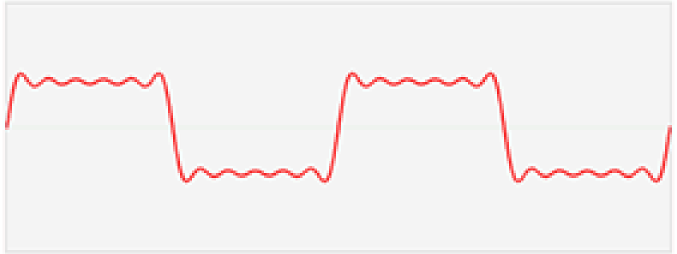
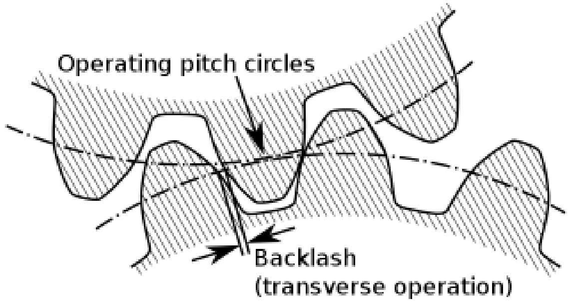
Presentation outline

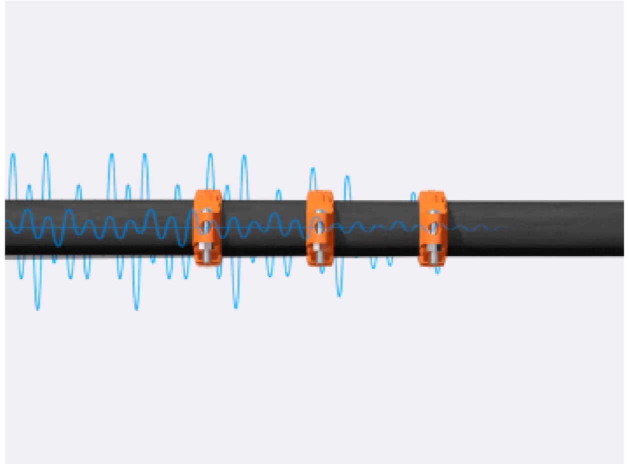
- **Introduction and motivation**
- **System modeling and numerical methods**
- **System dynamics**
- **Convergence analysis**
- **Stability analysis**
- **Conclusions**

Introduction and motivation

- **The harmonic balance method (HBM) is very popular for analyzing dynamical systems**
  - Computes periodic solutions using a Galerkin/Fourier series approximation
  - Useful for linear/nonlinear, SDOF/MDOF systems, faster than time integration, computes unstable solution branches, etc.
  - Growing interest in using HBM for engineering systems with contact and/or friction
  - Contact is a non-smooth nonlinearity than can induce very complex dynamics







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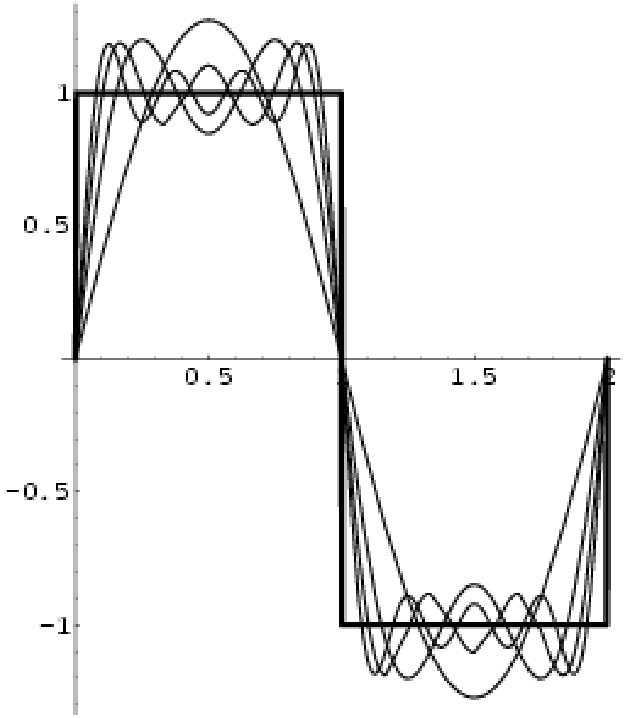
Introduction and motivation

- **Difficulties in pairing HBM with contact systems include:**
  - Significant nonlinear complexity compared to smooth systems
  - Large numbers of harmonics required due to discontinuous derivatives and Gibbs phenomenon effects
- **Questions:**
  - What are the limits of applicability of HBM for a system subject to freeplay contact?
  - How many harmonics would you need to get an accurate solution?
- **Goal of this work:**
  - Determine the limits of applicability of HBM for a nonlinear system with freeplay
  - Evaluate the accuracy of nonlinear periodic responses computed with HBM
  - Perform a stability analysis of the system using HBM



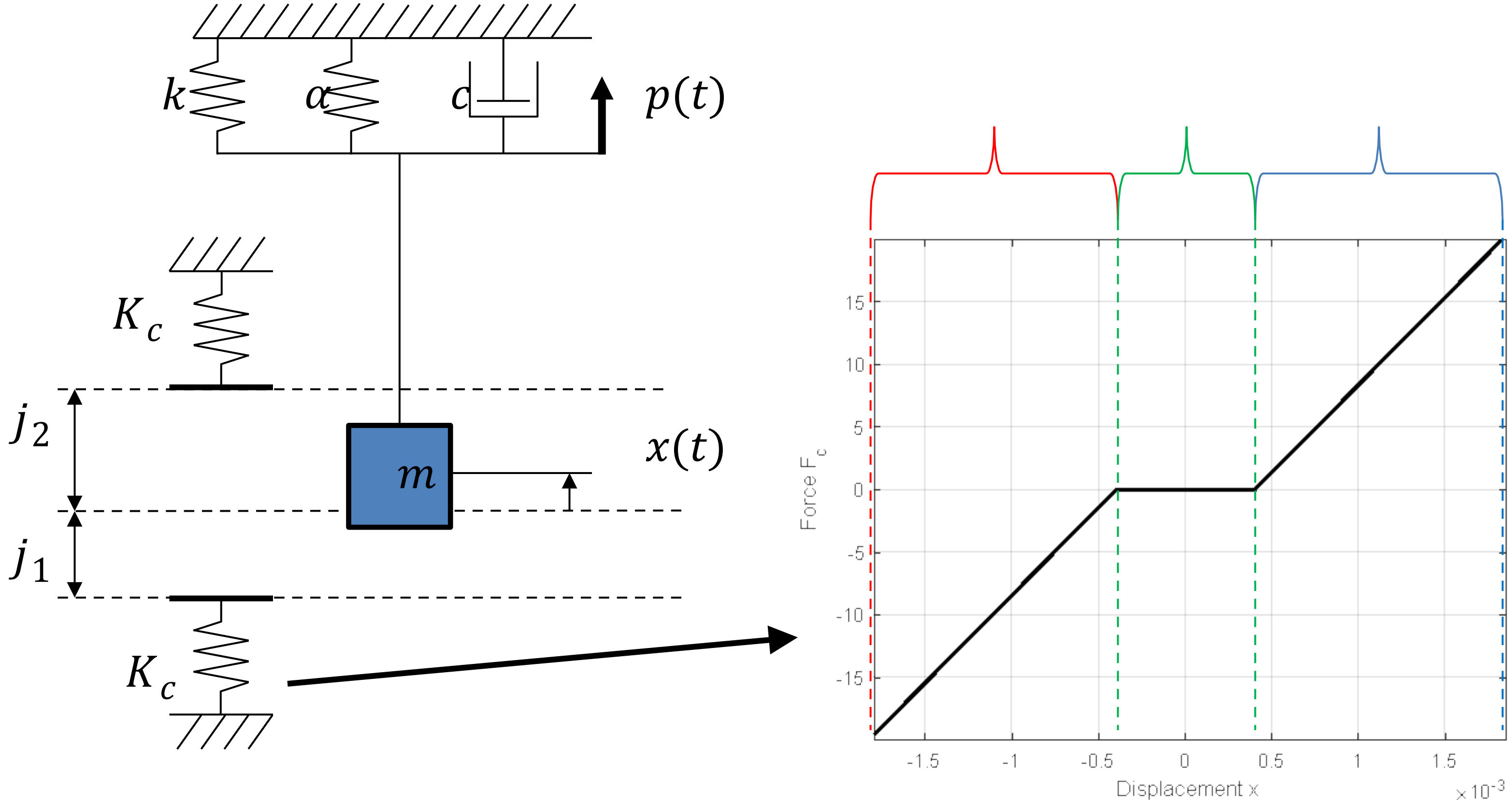
Example of Gibbs phenomenon

Weisstein, E, 1999 "Gibbs Phenomenon," https://archive.lib.msu.edu/crcmath/math/math/g/g164.htm

System modeling and numerical methods

❑ Forced Duffing oscillator with freeplay

$$\ddot{x} + 2\omega_n \zeta \dot{x} + \omega_n^2 x + \frac{\alpha}{m} x^3 + \frac{F_c(x)}{m} = \frac{p}{m} cos(\omega t),$$



$$F_c = \begin{cases} K_c(x + j_1), & x < -j_1 \\ 0, & -j_1 \leq x \leq j_2 \\ K_c(x - j_2), & x > j_2 \end{cases}$$

- ➢ Parameter values taken from deLangre et al. (1996)
- ➢ Contact is modeled with piecewise-smooth penalty stiffness
- ➢ Contact damping is negligible
- ➢ Past work showed that harder contact stiffness or smaller gap sizes increase the amount of nonlinear behavior in the system

De Langre, E., Lebreton, G., 1996, "An Experimental and Numerical Analysis of Chaotic Motion in Vibration with Impact," ASME 8 th International Conference on Pressure Vessel Technology, Montreal, Quebec, Canada, July 21 - 26, 1996.

System modeling and numerical methods

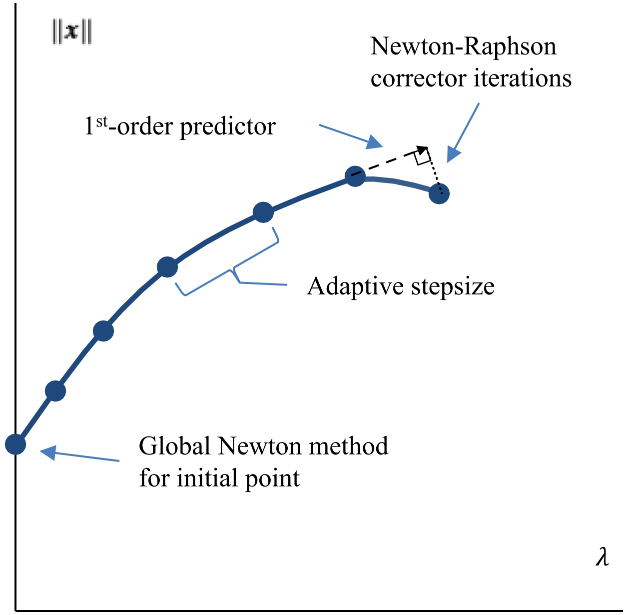
- **Result data is obtained using the harmonic balance method (HBM)**
  - The form of the solution is assumed to be a Fourier series:

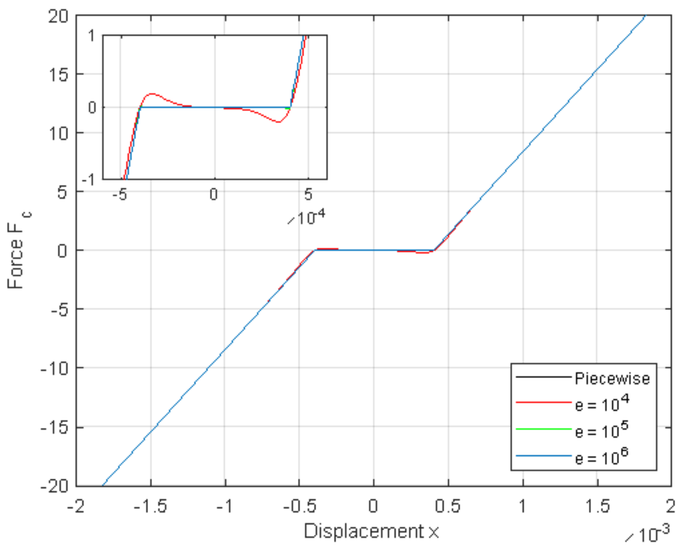
$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}_{ext}(t),$$

$$\mathbf{x}(t) = \frac{\mathbf{c}_0^x}{\sqrt{2}} + \sum_{k=1}^{N_h} [\mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t)]$$

  - This is combined with pseudo-arclength continuation to trace out solution branches
  - The freeplay force is approximated with a fully smooth (regularized) function:

$$F_c = K_c\left(\frac{1}{2}[1 - tanh(e(x + j_1))](x + j_1) + \frac{1}{2}[1 + tanh(e(x - j_2))](x - j_2)\right)$$





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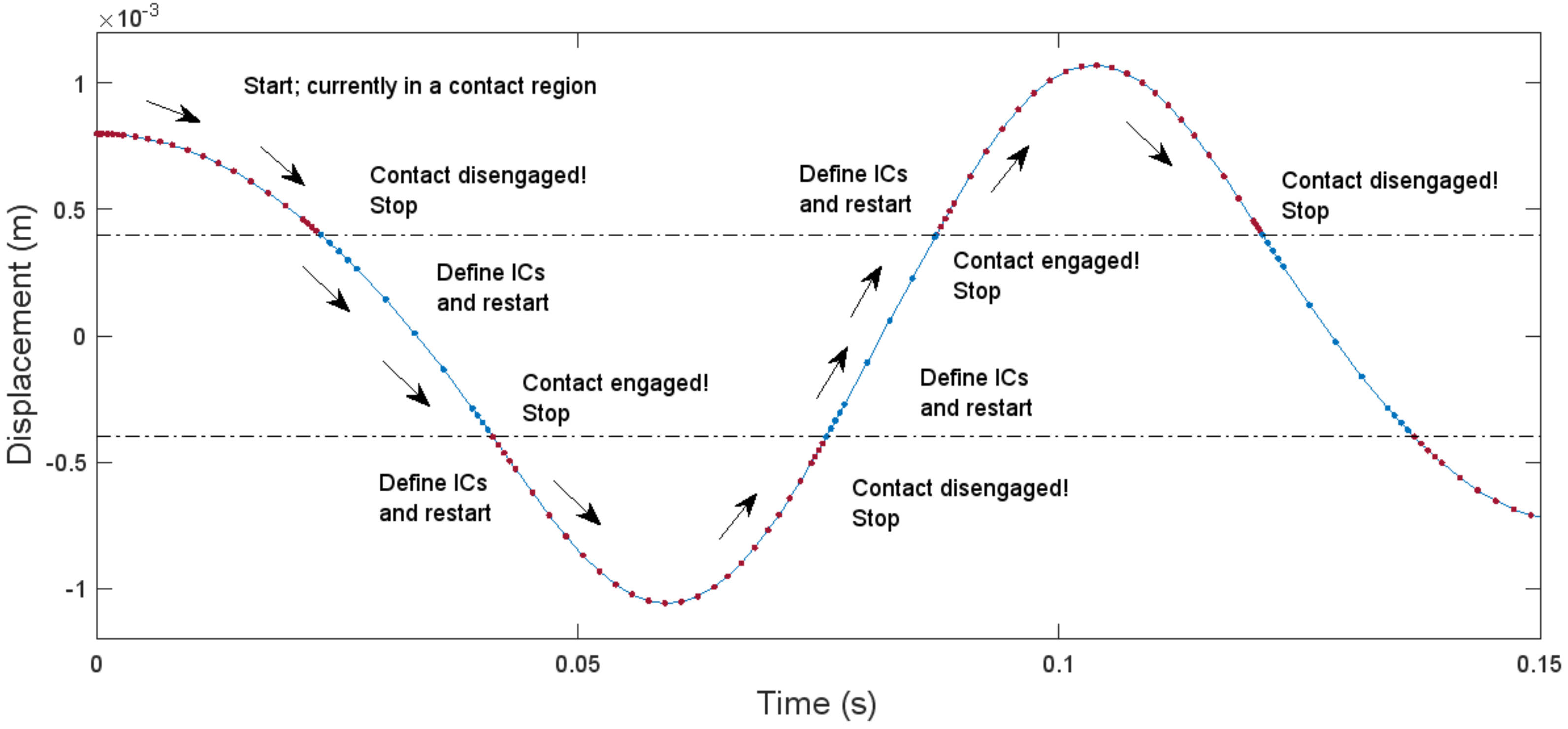
Saunders, B.E., Vasconcellos, R., Kuether, R.J., and Abdelkefi, A., 2021, "Insights on the continuous representation of piecewise-smooth nonlinear systems: limits of applicability and effectiveness," Nonlinear Dynamics. Doi: https://doi.org/10.1007/s11071-021-06436-w

System modeling and numerical methods

- **Reference data is obtained using Matlab® ode45 with *Event Location***
  - Piecewise time integration, which prevents accumulating roundoff error
  - A timestep is always forced at every instance of contact to ensure accuracy
  - Past validation has shown good results



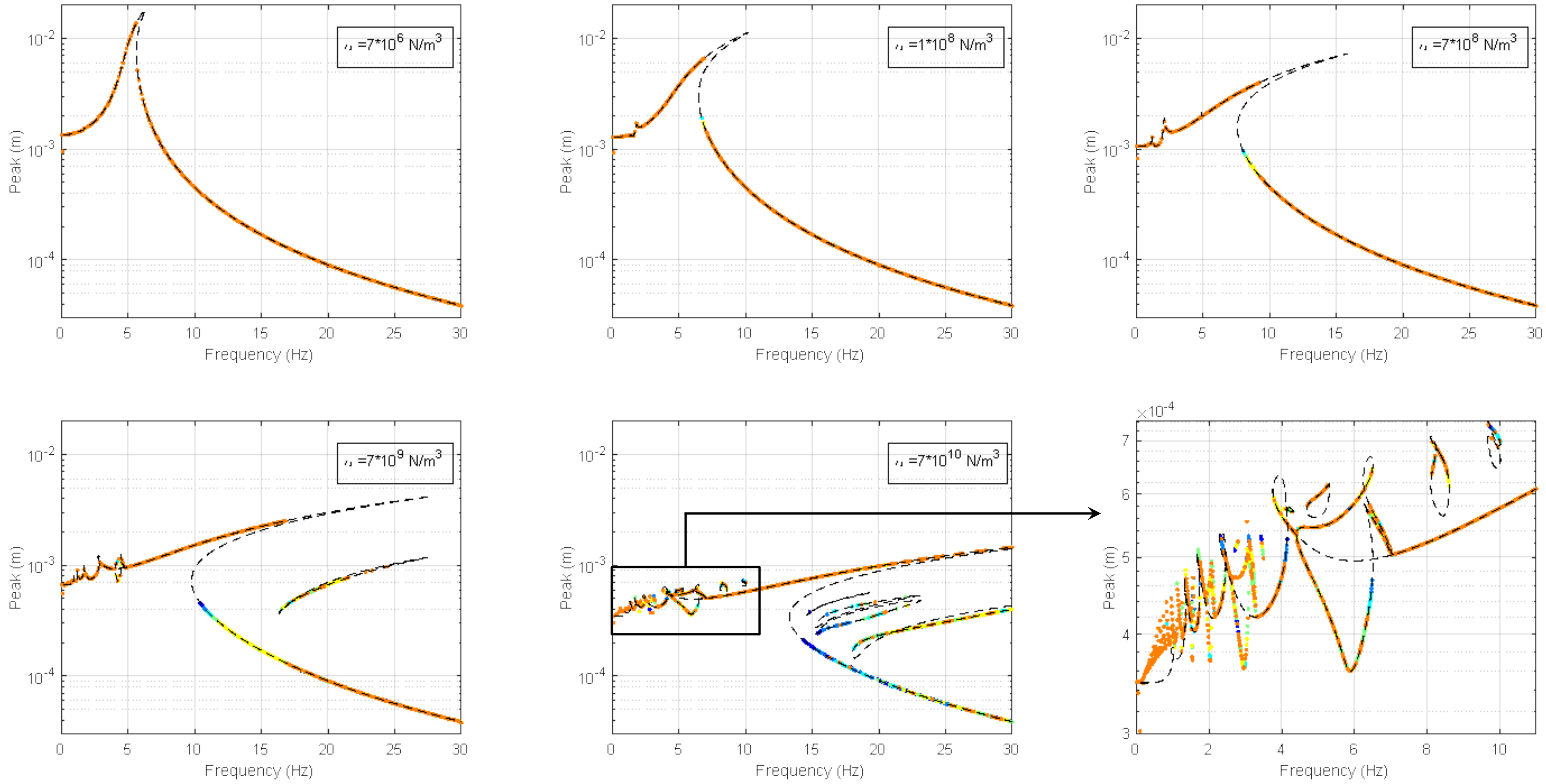
Saunders, B.E., Vasconcellos, R., Kuether, R.J., and Abdelkefi, A., 2021, "Relationship between the contact force strength and numerical inaccuracies in piecewise-smooth systems," International Journal of Mechanical Sciences, 210, 106729. Doi: https://doi.org/10.1016/j.ijmecsci.2021.106729

System dynamics

❑ Initial verification with fully smooth system:

- ➢ $K_c = 0, \alpha = 0 - 7 * 10^{10} N/m^3$ (classical Duffing)

Dots: time integration, various ICs
Dashed lines: HBM harmonics 0-12

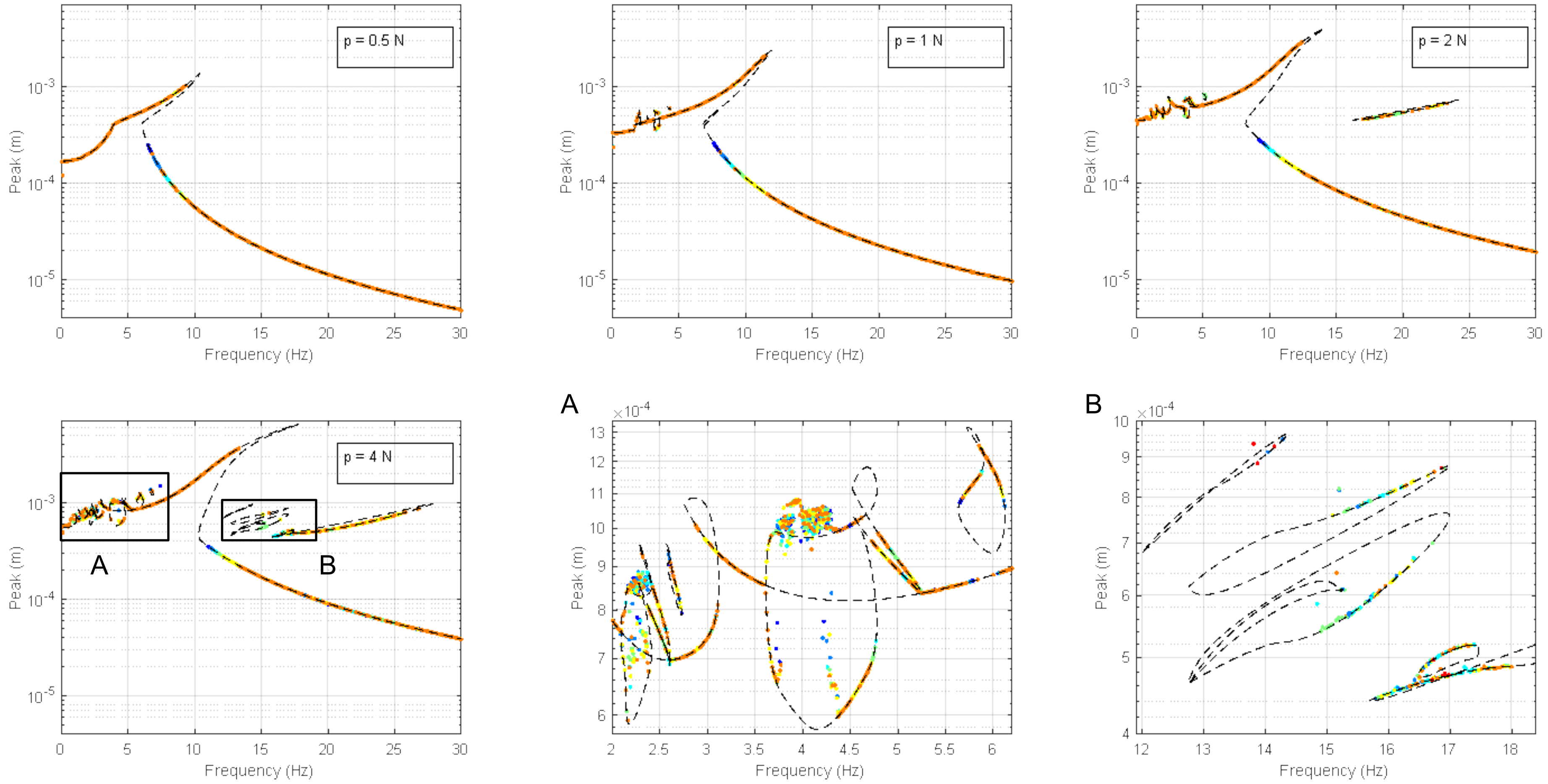


Takeaways:

- ➢ 12 harmonics gives excellent quantitative agreement for weak and strong cubic nonlinearity, e.g. beyond the limits of perturbation theory
- ➢ It is also sufficient for very strong cubic nonlinearity, as numerous super- and sub-harmonic resonance structures appear
- ➢ More harmonics are needed for the ultra-subharmonic resonances at low frequencies

System dynamics

- **Full Duffing-freeplay system, varying forcing magnitude:**
  - $\alpha = 7 * 10^8 \frac{N}{m^3}, K_c = 1.4 * 10^4 \frac{N}{m}, j_1 = j_2 = 0.4\, mm, p = 0.5 - 4N$



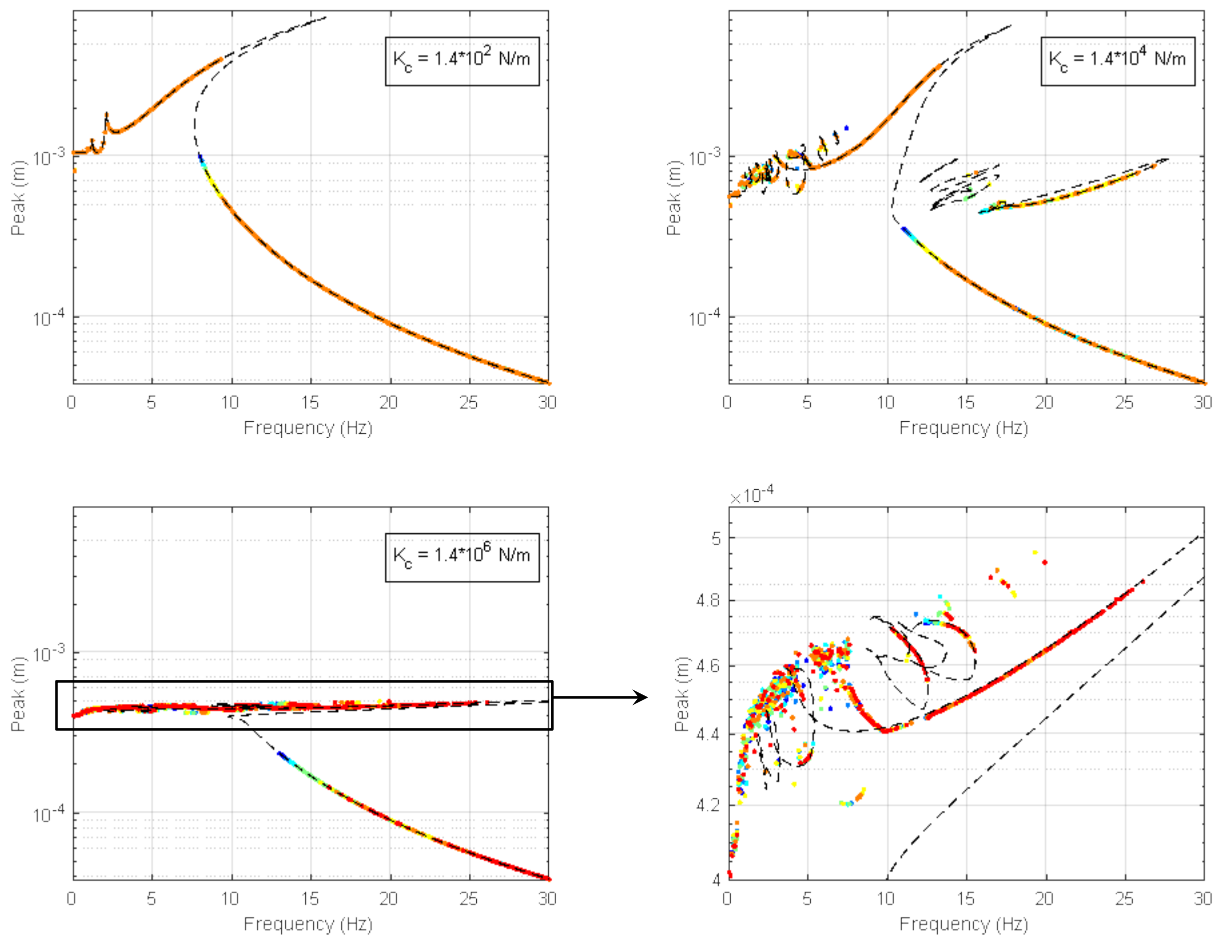
Takeaways:

- 12 harmonics still gives excellent quantitative agreement for weak and strong forcing magnitude
- HBM is able to capture the numerous resonances that occur in the system
- HBM, as expected, fails to capture some chaotic regions at higher forcing

System dynamics

❑ Full Duffing-freeplay system, varying contact stiffness:

➢ $\alpha = 7 * 10^8 \frac{N}{m^3}, p = 4N, K_c = 1.4 * 10^2 - 10^6 \frac{N}{m}$



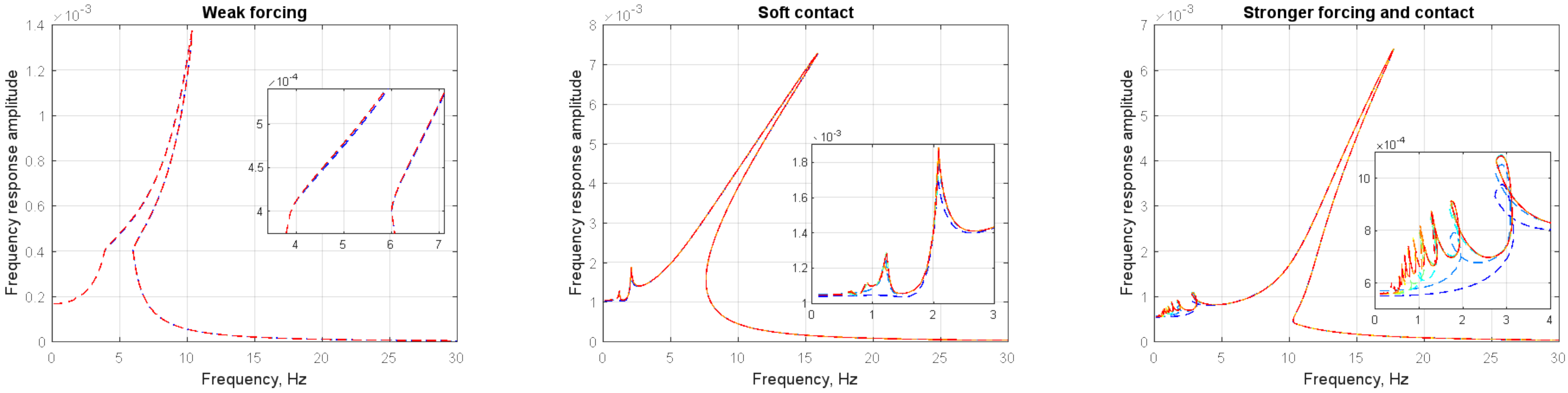
Takeaways:

- ➢ 12 harmonics gives excellent quantitative agreement for soft and medium contact stiffnesses
- ➢ For hard contact, 12 harmonics gives good agreement for much of the main response branch
- ➢ This breaks down below 10 Hz, as this region transitions to chaos
- ➢ On the order of 72 harmonics are now required to adequately capture superharmonic and ultra-subharmonic resonance solutions

Convergence analysis

- **Main resonance branch for three nonlinearity strengths is plotted with multiple harmonics**
- **Visually, the only real differences are at low frequency**
  - Stronger contact leads to many superharmonic resonances that require more harmonics to capture

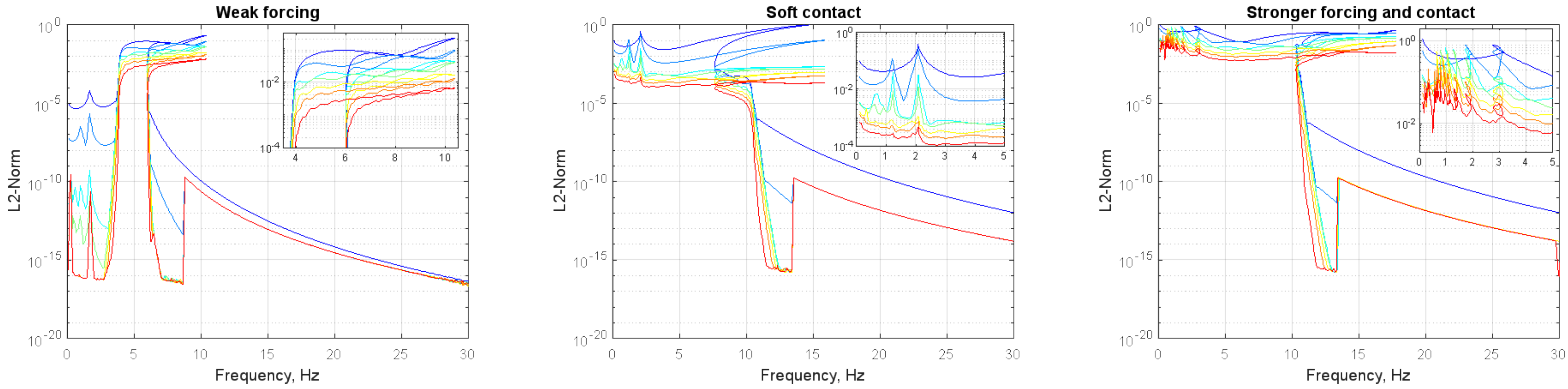
Legend:
0:3
0:6
0:9
0:12
0:18
0:24
0:36



Convergence analysis

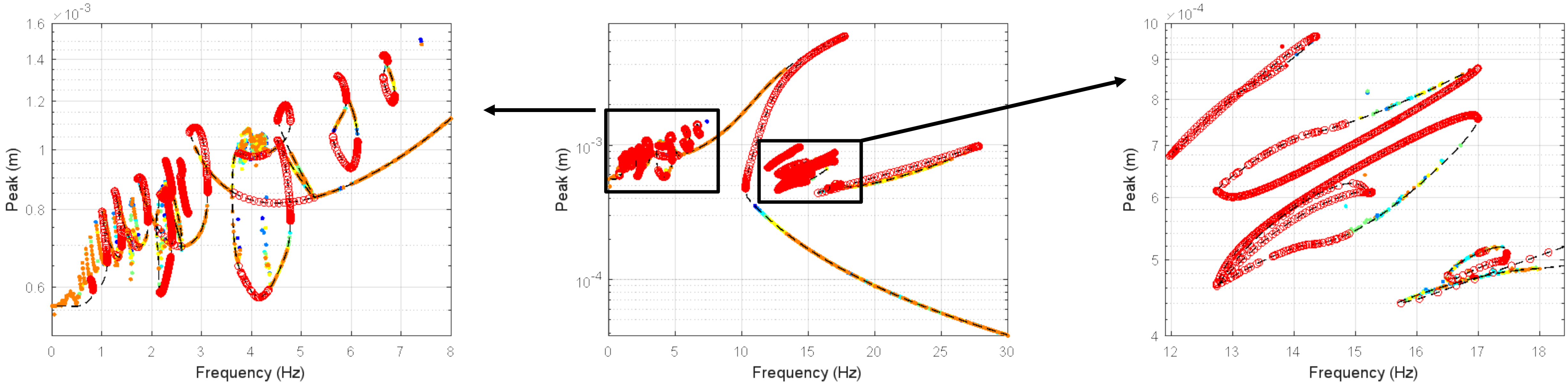
- **L_2-norm of the residual is plotted to show accuracy vs. # of harmonics**
  - Error is highest near the primary resonance peak, where amplitude is highest
  - Error is also high near superharmonic resonances
  - Using more harmonics produces an asymptotic error curve

Legend:
0:3
0:6
0:9
0:12
0:18
0:24
0:36



Stability analysis

- **Floquet stability is calculated using Hill's method**
  - There are numerous turning points/saddle-node bifurcations on all solution branches
  - Several branch points/pitchfork bifurcations occur at low frequency
  - Some occur on the isolated subharmonic resonances as well
  - The response is shown to be unstable where chaotic responses occur



Conclusions

- Harmonic balance method (HBM) was applied to an SDOF nonlinear oscillator system with freeplay
- The accuracy of nonlinear periodic responses computed with HBM was evaluated
- HBM is able to capture many types of nonlinear behavior with relatively few harmonics
  - 12 harmonics was sufficient to capture nearly all behavior except for cases with hard contact
  - Superharmonic resonances and isolated subharmonic resonances
  - Isolated ultra-subharmonic resonances tended to require more harmonics (~3 times as many)
  - Chaotic responses cannot be captured, but this is expected
  - Hard contact required ~6 times as many harmonics
- Stability analysis showed that numerous saddle-node and pitchfork bifurcations occur, particularly at low frequency
  - Although HBM cannot resolve chaotic responses, it can detect unstable regions where it may occur

Acknowledgements

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Thank you for your attention!
Any questions?

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