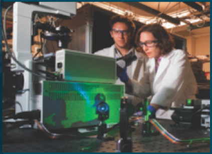
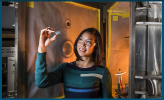
Sandia National Laboratories

Conservative and Entropy-Stable Nonconformal Interfaces with Lower Accuracy Quadrature





PRESENTED BY

Jared Crean and Travis Fisher

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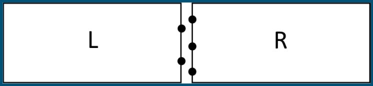
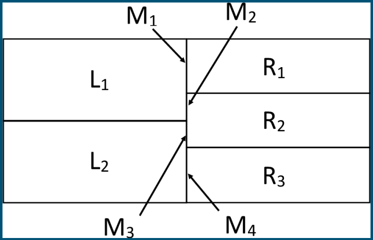
Nonconformal Interfaces



(a) A *p*-nonconformal



(b) *hp*-nonconformal

SBP Operator: Stability

The requirements for a **stable** SBP discretization are:

1. Differentiation operators D_x, D_y that satisfy, at minimum, $D_x\mathbf{1} = \mathbf{0}$ and $D_y\mathbf{1} = \mathbf{0}$
2. Symmetric boundary operators B_x, B_y that satisfy, at minimum, $\mathbf{1}^T B_x \mathbf{1} = \mathbf{0}$ and $\mathbf{1}^T B_y \mathbf{1} = \mathbf{0}$.
3. A diagonal mass matrix P that forms a quadrature rule of degree $2p - 1$, at least
4. Weak form integration matrices $Q_x = PD_x$ and $Q_y = PD_y$.
5. Skew-symmetric splitting $Q_x = S_x + \frac{1}{2}B_x$ and $Q_y = S_y + \frac{1}{2}B_y$ where S_x and S_y are skew-symmetric matrices.

Nonconformal Discretization

Define interpolation from the face to the M grid integration points P_{LM} and P_{RM}. The discretization is then:

$$P\frac{du_L}{dt} + (S_x \circ F_x(u_L, u_L))\mathbf{1} + \left(S_y \circ F_y(u_L, u_L)\right)\mathbf{1} = -\frac{1}{2}\sum_x \left(\left(R_L^T P_{LM}^T \underbrace{B_M N_{x,M}}_{\text{use M integration rule}} P_{RM} R_R \right) \circ F_x(u_L, u_R) \right)\mathbf{1} + W.F.T, \quad (1)$$

The face operator has similar structure to simplex SBP.

Mapped SBP Construction

Must satisfy SBP properties with M quadrature rule.
Define the metrics:

$$\Lambda_{\xi_i,x_j} \approx \text{diag}\left(\left[\left(J\frac{\partial \xi_i}{\partial x_j}\right)_1 \cdots \left(J\frac{\partial \xi_i}{\partial x_j}\right)_n\right]\right), \tag{2}$$

and construct SBP operator:

$$\mathsf{S}_{x_i} = \frac{1}{2}\left(\Lambda_{\xi_1,x_i}\mathsf{Q}_{\xi_1} + \Lambda_{\xi_2,x_i}\mathsf{Q}_{\xi_2} - \mathsf{Q}_{\xi_1}^T\Lambda_{\xi_1,x_i} - \mathsf{Q}_{\xi_2,x_i}^T\Lambda_{xi_2,x_i}\right) \tag{3}$$

$$\mathsf{B}_{x_i} = \sum_{\gamma}\mathsf{R}_L^T\mathsf{P}_{LM}^T\mathsf{B}_M\mathsf{N}_{x_i,M}\mathsf{P}_{LM}\mathsf{R}_L \tag{4}$$

$$\mathsf{Q}_{x_i} = \mathsf{S}_{x_i} + \frac{1}{2}\sum_{\gamma}\mathsf{B}_{x_i} \tag{5}$$

$$\mathsf{D}_{x_i} = (\mathsf{P}|J|)^{-1}\mathsf{Q}_{x_i}. \tag{6}$$

Metric Projection

Must satisfy constant exactess:

$$\mathrm{D}_{x_i}\mathbf{1} = \mathbf{0} \quad \Rightarrow \quad \frac{1}{2}\left(\sum_j \left(-\mathrm{Q}_{\xi_j}^T \Lambda_{\xi_j, x_i}\mathbf{1}\right) + \sum_\gamma \mathrm{B}_{x_i}\mathbf{1}\right) = \mathbf{0} \tag{7}$$

Solve the (linear) optimization problem

$$\min_{\mathrm{m}} \quad \left(\mathrm{m} - \mathrm{m}_{\mathrm{targ}}\right)^T \left(\mathrm{m} - \mathrm{m}_{\mathrm{targ}}\right) \tag{8}$$

$$\text{subject to} \quad \sum_j \left(-\mathrm{Q}_{\xi_j}^T \Lambda_{\xi_j, x_i}\mathbf{1}\right) + \sum_\gamma \mathrm{B}_{x_i}\mathbf{1} = \mathbf{0} \tag{9}$$

where

$$\mathrm{m}_{\mathrm{targ}} = \begin{bmatrix} \Lambda_{\xi_1, x_i}\mathbf{1} \\ \Lambda_{\xi_2, x_i}\mathbf{1} \end{bmatrix} \tag{10}$$

Accuracy

For straight-sided elements:

$$Q_{x_i} = \sum_j \left(A \frac{\partial \xi_j}{\partial x_i} Q_{\xi_j} \right) - \frac{1}{2} \sum_\gamma \left(R_L^T B_L R_L^T n_{x_i} \right) + \frac{1}{2} \sum_\gamma \left(R_L^T P_{LM}^T B_M P_{LM} R_L n_{x_i} \right)$$

Q_{x_i} will be accurate for polynomials if the final two terms cancel.

- Holds if B_L and B_M are exact for $2p$ polynomials
- Otherwise, lose 1 order of accuracy (p rather than $p+1$)

Linear Advection

Test the linear advection equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + S = 0 \quad (11)$$

Manufactured solution

$$u(x,t) = \exp(x + y + t) \quad \text{for} \quad x, y \in \begin{bmatrix} 0 & 1 \end{bmatrix} \quad (12)$$

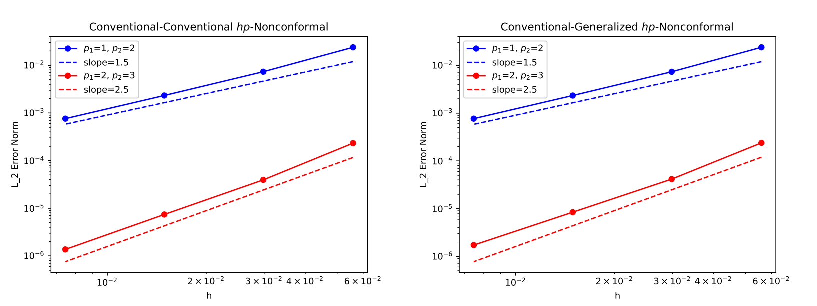
with curved grid

$$x_2 = x_1 + \alpha \sin(\frac{x_1 - c_x}{\beta_x}) \quad (13)$$

$$y_2 = y_1 + \alpha \sin(\frac{y_1 - c_y}{\beta_y}), \quad (14)$$

Linear Advection: *hp*-Nonconformal Convergence

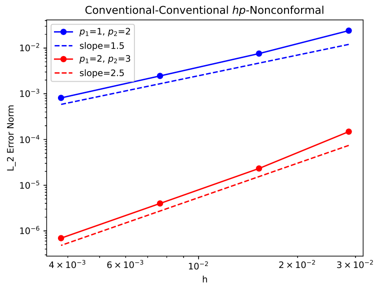
Straight-sided elements:



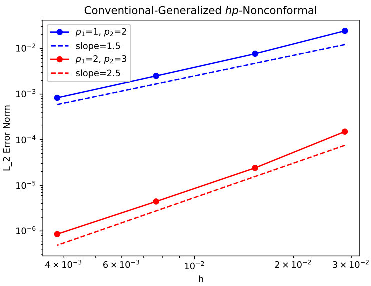
(a) Conventional elements for all blocks (b) Generalized elements in middle block, Conventional elsewhere

Linear Advection: *hp*-Nonconformal Convergence

Curved elements:



(a) Conventional elements for all blocks



(b) Generalized elements in middle block, Conventional elsewhere

Euler Equations

$$\frac{\partial u}{\partial t} + \frac{\partial f_i(u)}{\partial x_i} = 0, \tag{15}$$

where

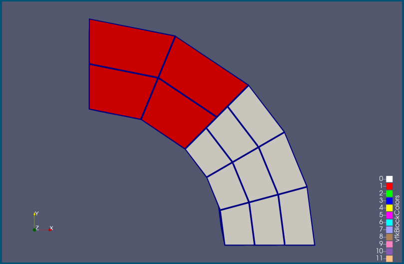
$$\mathrm{u} = [\rho, \rho u, \rho v, E] \tag{16}$$

Test problems

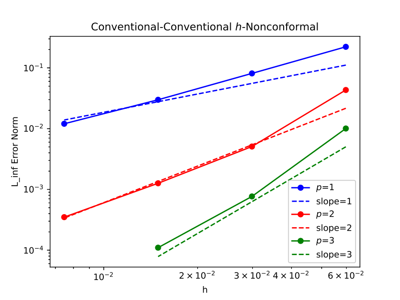
- Steady isentropic vortex: convergence rate
- Unsteady isentropic vortex (periodic domain): entropy stability

Entropy stable dissipation is used for all problems.

Steady Isentropic Vortex

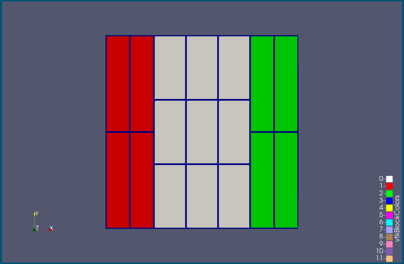


(a) First mesh

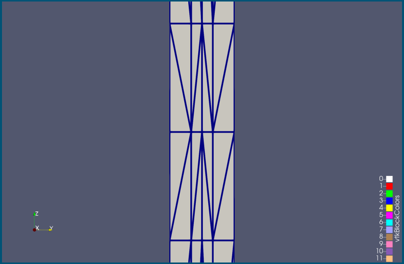


(b) Convergence results

Unsteady Isentropic Vortex

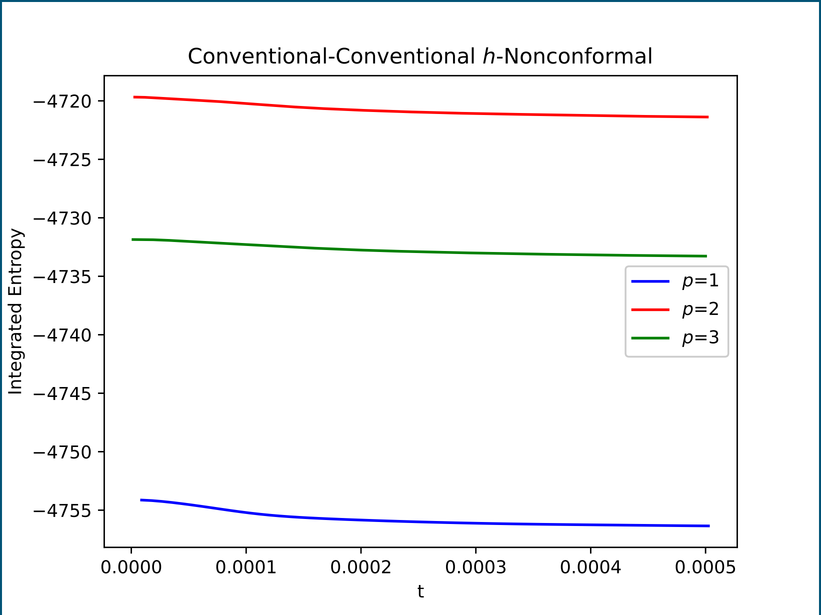


(a) Mesh



(b) *M* Mesh

Unsteady Isentropic Vortex



Conclusions

We have presented a new *hp*-nonconformal discretization is:

- Conservative
- Entropy stable
- High-order accurate
- Applicable to SBP operators with $2p - 1$ quadratures