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SIMPLIFIED FORMULAS FOR EXTERNAL PRESSURE DESIGN

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ABSTRACT

Design of vessels for external pressure currently requires a chart-based solution or analytical approaches which are not necessarily intuitive. In this paper, we propose simple formulas for the external pressure evaluation of pipes and other cylindrical pressure vessels. We present a conceptual comparison between the elastic and elastic-plastic stability of structural columns and that of cylindrical vessels of long, intermediate, and short length. Their common features allow an accurate and straightforward approach for external pressure design. The approach is also extended to spherical caps, conical vessels, and formed heads.

We compare the method presented to the current acceptance criteria from various design codes, including the ASME Boiler and Pressure Vessel Code Section VIII, Code Case 2286, and EN 13445-3, as well as codes for steel and aluminum structures. In further discussion, the simplified method is compared against the results of more than 500 experiments on the buckling of cylindrical and spherical vessels published over the past two centuries.

This simple but accurate approximation is conceptually intuitive, analytically straightforward, and shows potential utility in pressure vessel design codes, as well as piping design codes such as B31 that currently reference ASME VIII for external pressure design.[1]

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NOMENCLATURE

A	ASME code factor
$\mathcal{A}$	Area
B	ASME code factor
C_c	Critical slenderness ratio
D	Outer diameter
E	Young's modulus
E_t	Tangent modulus
F_{cr}	Critical buckling force on a column
I	Second moment of area
L	Axial length
P	Pressure
P_{all}	Allowable external pressure
P_{cr}	Critical buckling pressure on a vessel
R	Outer radius
k	Buckling end condition constant
n	Buckling lobe number
r	Radius of gyration, $\equiv \sqrt{I/A}$
t	Wall thickness
α	Interpolation parameter
λ	Slenderness ratio, $\equiv kL/r$ for a structural column
ν	Poisson's ratio
σ	Stress
σ_{all}	Allowable stress
σ_{cr}	Critical buckling stress
σ_y	Yield stress

INTRODUCTION

The ASME Boiler and Pressure Vessel Code [1], in its Design-by-Rule sections (including Section VIII, Division 1; Section VIII, Division 2, Part 4; and various subsections of Section III, Division 1), provides two techniques for the external pressure design of pipes, vessels, and heads. The first option is to apply the external pressure charts in Section II-D of the Code. Those charts, developed from the works of Sturm [2], Windenburg and Trilling [3], and others in the 1930s through 50s [4], are based on classic elastic stability theory and confirmed by tests on cylinders of various material. The second method, originally presented in Code Case 2286 [5] and since incorporated into Section VIII, Division 2 of the Code, is based on work carried out by Miller and others in the 1980s and 90s [6–9]. This method is a best-fit approach, informed by elastic theory and based on extensive testing of carbon steel vessels.

This paper shows that the collapse of cylinders and spheres under external pressure is analogous to the buckling of an axially-loaded column. The column-equivalent slenderness ratio λ produces the following simple formulas for external pressure design of cylindrical vessels:

$$\lambda = \min\left(\frac{3D}{t}, \quad 2.75\left[\frac{D}{t}\right]^{0.75}\left[\frac{L}{D}\right]^{0.5}\right) \tag{1}$$

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_y}} \tag{2}$$

$$\sigma_{all} = \begin{cases} 0.2\sigma_y\left(\lambda/C_c\right)^{-2}, & \text{if} \quad \lambda/C_c \geq 0.833 \\ \left(0.888 - 0.72\lambda/C_c\right)\sigma_y, & \text{if} \quad 0.4 \leq \lambda/C_c < 0.833 \\ 0.6\sigma_y, & \text{if} \quad \lambda/C_c < 0.4 \end{cases} \tag{3}$$

$$P_{all} = \frac{2t\sigma_{all}}{D} \tag{4}$$

This method is derived in the following paragraphs. It is intuitively straightforward for the designer, closely aligns with the Code Case 2286 allowables, and compares favorably with the significant experimental work performed by others on this topic, dating back nearly 200 years.

OVERVIEW OF COLUMN BUCKLING

In 1744, Leonhard Euler developed the well-known formula expressing the critical load that will induce buckling in an elastic column [10, 11]:

$$F_{cr} = \frac{\pi^2 EI}{L^2} \tag{5}$$

The average stress in the column at the critical load defines the critical buckling stress,

$$\sigma_{cr} = \frac{\pi^2 EI}{L^2\mathcal{A}} = \frac{\pi^2 E}{\left(\frac{L}{r}\right)^2} = \frac{\pi^2 E}{\lambda^2} \tag{6}$$

where $r \equiv \sqrt{\frac{I}{\mathcal{A}}}$ is the column's radius of gyration, and $\lambda \equiv \frac{L}{r}$ is the column's slenderness ratio. The boundary conditions (e.g., fixed ends, pinned ends) at each end of the column determine the buckling mode shape. For engineering design purposes, the mode shape is often accounted for by incorporating an end-condition constant, k, into the slenderness ratio, such that $\lambda \equiv \frac{kL}{r}$ [12].

Though Euler's column formula is accurate for very long columns, the formula predicts buckling loads for shorter columns greatly in excess of those seen in experiment [13, 14]. Subsequent generations of engineers developed two parallel approaches to address this problem. Engesser [15], Considère [16], Kármán [17], Southwell [18], and Shanley [19,20] developed the approach of modifying the elastic modulus to account for elastic-plastic material behavior, culminating in Shanley's tangent modulus theory.

$$\sigma_{cr} = \frac{\pi^2 E_t}{\lambda^2}, \text{ where } E_t = f\left(\lambda\right) \text{ for a given material} \tag{7}$$

Meanwhile, Hodgkinson [21], Rankine [22], and Johnson [23] developed empirical equations that described column behavior in the plastic region, resulting in the well known Rankine-Gordon formula,

$$\frac{\sigma_{cr}}{\sigma_y} = \frac{1}{1 + \frac{\sigma_y\lambda^2}{\pi^2 E}} = \frac{1}{1 + 2(\lambda/C_c)^2} \tag{8}$$

as well as the J.B. Johnson parabolic formula that for years formed the basis of the AISC allowable stress design criteria [12, 24], and that also sees widespread use in mechanical design [25].

$$\frac{\sigma_{cr}}{\sigma_y} = \begin{cases} 1 - \frac{\sigma_y\lambda^2}{4\pi^2 E} = 1 - 0.5(\lambda/C_c)^2, & \text{if} \quad \lambda/C_c < 1 \\ \frac{\pi^2 E}{\sigma_y\lambda^2} = 0.5(\lambda/C_c)^{-2}, & \text{if} \quad \lambda/C_c \geq 1 \end{cases} \tag{9}$$

Osgood [26] unified the two approaches by showing that the Rankine-Gordon, J.B. Johnson, and similar empirical formulas were equivalent to assuming particular shapes of a material's stress-strain curve.

An examination of the foregoing equations shows the key role played by the slenderness ratio parameter λ. Effectively, the slenderness ratio provides a way to combine the various geometric properties of the structure (including length, moment of inertia, cross-sectional area, and buckling mode shape) into a single dimensionless value. This allows the calculation of the critical buckling load to be divided into two steps. First, the geometry and buckling mode shape determine the slenderness ratio. Second, the slenderness ratio and the material's stress-strain properties are compared to determine the critical stress and load. These two steps, geometric and material, are considered below for geometries and materials applicable to piping and pressure vessel design.

GEOMETRIC CONSIDERATIONS; SLENDERNESS

A Column Analogy for Vessels under External Pressure

The similarity between the buckling of axially-loaded structural columns or struts and the collapse of vessels and pipes under external pressure has long been recognized. In 1875, Unwin stated, "The metal of the flue [a boiler tube under external pressure] is in the same condition as a straight column of length πD subjected to a compression of the same intensity σ. In investigating the strength of flues, the well-known laws of resistance of long columns may be applied" [27]. In the same manner, in proposing formulas for the plastic buckling of tubes, Southwell noted, "... An infinitely long tube collapses under external pressure in a manner which has many points of resemblance to strut failure.... The two problems, in fact, are almost identical." [18]

Though Unwin and Southwell used this analogy primarily as a conceptual stepping stone in the development of their expressions for external pressure buckling, the analogy can also be applied in a more quantitative manner. The equations of elastic stability for more complex geometries and buckling modes can be arranged into the form of Eq. (6). This permits a slenderness ratio to be defined for other geometries. This slenderness ratio can then be used to apply the equations of elastic or elastic-plastic column buckling to these additional geometries. This method is used in the following sections to define slenderness ratios for a variety of shells under external pressure.

Buckling of Thin Rings under External Pressure

The first and simplest case to be examined is the buckling of a thin circular ring under uniform external pressure. Timoshenko and Gere [28] provide an expression for the critical pressure in this case:

$$P_{cr} = \frac{3EI}{R^3} = \frac{2Et^3}{D^3} \tag{10}$$

For a column, the compressive stress was simply $\sigma = F/\mathcal{A}$. For a ring, the analogous compressive stress is the hoop stress, $\sigma = F/\mathcal{A} = PD/2t$. At the critical buckling pressure,

$$\sigma_{cr} = \frac{P_{cr}D}{2t} = \frac{Et^2}{D^2} \tag{11}$$

Comparing Eq. (11) to Eq. (6) reveals the analogous slenderness ratio for a thin ring.

$$\sigma_{cr} = \frac{Et^2}{D^2} = \frac{\pi^2 E}{\lambda^2} \tag{12}$$

$$\lambda = \frac{\pi D}{t} \tag{13}$$

This expression allows for a better intuitive grasp of the effect of geometry on the external pressure capacity of a ring. As Unwin observed, a thin ring can be loosely thought of as buckling under external pressure at the same average stress as an equivalent axially-loaded column, which has a length equal to the ring's circumference and a radius of gyration equal to the ring's thickness. Just as for a column, the slenderness ratio brings together the ring's geometry and buckling mode shape into a single value.

Long Cylindrical Vessels and Pipes

The same process can be used to define the slenderness of a long cylindrical pipe or vessel. A cylindrical vessel of significant length (roughly $L/D > 10$) is the plane strain analog of the thin ring, with the effective Young's modulus reduced by a factor of $1-\nu^2$. The result is the Bryan-Bresse equation [3]:

$$P_{cr} = \frac{2Et^3}{(1-\nu^2)D^3} \tag{14}$$

Again, the critical buckling stress is the average compressive hoop stress at the critical pressure:

$$\sigma_{cr} = \frac{P_{cr}D}{2t} = \frac{Et^2}{(1-\nu^2)D^2} \tag{15}$$

$$\lambda = \pi\sqrt{1-\nu^2}\frac{D}{t} \tag{16}$$

Conveniently, when $\nu \approx 0.3$, as for many metals, this can be closely approximated as

$$\lambda = \frac{3D}{t} \tag{17}$$

For a given D/t ratio, Eq. (17) is equivalent to the vertical section of the curve in the upper portion of Figure G of the ASME Boiler and Pressure Vessel Code, Section II-D, used to determine the factor A.

Short Cylindrical Vessels

If the axial length of a cylindrical vessel or pipe is less than around ten diameters, the length plays a role, in addition to the diameter and wall thickness, in determining the buckling pressure. Richard von Mises [29–31] determined the critical buckling pressure for these vessels.

$$P_{cr} = \left(\frac{1}{3}\left[n^2 + \left(\frac{\pi D}{2L}\right)^2\right]^2 \frac{2E}{1-\nu^2}\left(\frac{t}{D}\right)^2 + \frac{2E(t/D)}{n^2(\frac{2L}{\pi D})^2+1}\right) \times \frac{1}{n^2 + \frac{1}{2}(\frac{\pi D}{2L})^2} \tag{18}$$

The Mises formula was used as the basis for the external pressure requirements of BS5500 [32], and from there incorporated into EN 13445-3 [33]. However, this formula requires knowledge of the lobe number n, and therefore iteration, charts, or tables. Such charts were developed for BS5500 based on the Mises formula; the ASME Section II-D charts are instead based on an enveloping expression independent of the lobe number, first proposed by Windenburg and Trilling at the US Experimental Model Basin [3]:

$$P_{cr} = \frac{\frac{4.5\pi}{10.5^{3/4}}\frac{E}{(1-\nu^2)^{3/4}}\left(\frac{t}{D}\right)^{5/2}}{\frac{L}{D} - \frac{\pi\sqrt{t/D}}{4[10.5(1-\nu^2)]^{1/4}}} \tag{19}$$

Following the same approach as for a long pipe, a slenderness ratio for short pipes can be defined based on the formula of Windenburg and Trilling:

$$\lambda = \sqrt{\frac{10.5^{3/4}\pi(1-\nu^2)^{3/4}}{2.25}}\left(\frac{D}{t}\right)^{3/4}\sqrt{\frac{L}{D} - \frac{\pi\sqrt{t/D}}{4[10.5(1-\nu^2)]^{1/4}}} \tag{20}$$

In the case that $\nu = 0.3$, this expression simplifies to

$$\lambda = 2.75\left(\frac{D}{t}\right)^{3/4}\sqrt{\frac{L}{D} - 0.447\sqrt{\frac{t}{D}}} \tag{21}$$

Most often, $\frac{L}{D} \gg \sqrt{\frac{t}{D}}$ and the expression can be further simplified to

$$\lambda = 2.75\left(\frac{D}{t}\right)^{3/4}\left(\frac{L}{D}\right)^{1/2} \tag{22}$$

Eq. (22) is always more conservative than Eq. (21), but the two equations differ little except for especially short or thick vessels. For example, with $\frac{L}{D} = 1.0$ and $\frac{D}{t} = 20$, the difference is 5.3%.

The enveloping Windenburg-Trilling formula, as captured in Eq. (21) or Eq. (22), traces out the diagonal curves in the lower and rightmost portions of ASME Section II-D, Figure G.

Intermediate Length Cylindrical Vessels

When the values for λ obtained with Eq. (17) and Eqs. (21) or (22) are of the same order of magnitude, it will always be conservative to take the lower value of the slenderness ratio. When less conservatism is needed, the slenderness ratio may be determined by interpolation. With λ_{long} determined by Eq. (17) and λ_{short} by Eq. (21) or Eq. (22),

$$\alpha = \min\left(\frac{D}{t}, 15\right) \tag{23}$$

$$\lambda = \left(\lambda_{long}^{-\alpha} + \lambda_{short}^{-\alpha}\right)^{-\frac{1}{\alpha}} \tag{24}$$

This duplicates with reasonable accuracy the rounded transition range of the curves in ASME Section II-D, Figure G as well as Figure 8.5.3 from EN 13445-3. Figure 1 compares the slenderness ratios as calculted in this paper to those obtained from the figures in the design codes, using the relationship $A = \frac{\pi^2}{\lambda^2}$. The excellent agreement is unsurprising, since the design code figures are based on the same sources used in this paper to define the slenderness ratio.

Spheres and Spherical Caps

The classical solution of the elastic buckling stress of a spherical shell was first determined by Zoelly [34], who found

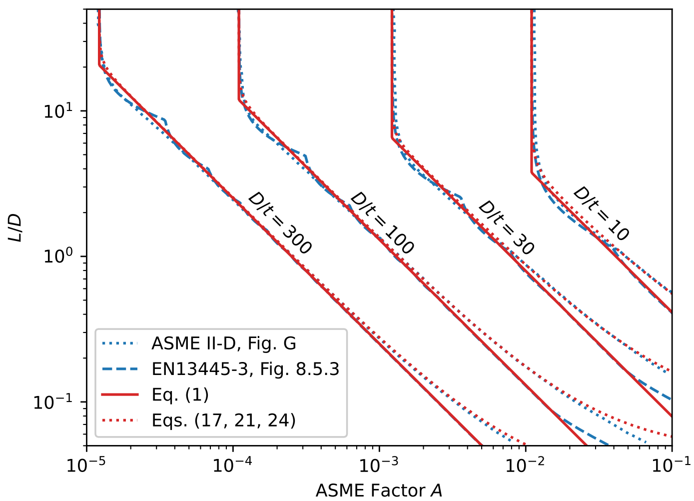

FIGURE 1. CALCULATED SLENDERNESS RATIOS COMPARED WITH BPVC AND EN 13445-3

the critical pressure to be

$$P_{cr} = \frac{2E}{\sqrt{3(1-\nu^2)}}\left(\frac{t}{R}\right)^2 \tag{25}$$

Since the biaxial stress in a pressurized spherical shell is

$$\sigma = \frac{PR}{2t} \tag{26}$$

the slenderness ratio of a sphere would be expected to be

$$\lambda = \pi\sqrt[4]{3(1-\nu^2)}\sqrt{\frac{R}{t}} \approx 4.038\sqrt{\frac{R}{t}} \tag{27}$$

However, a discrepancy between the theoretical and practical buckling strength of spherical shells has long been recognized [35], and a visual comparison with published experimental data (see Figure 4 and related discussion) seems to indicate that this value of λ should be larger by a factor of approximately $\sqrt{2}$, which gives

$$\lambda = \pi\sqrt[4]{12(1-\nu^2)}\sqrt{\frac{R}{t}} \approx 5.711\sqrt{\frac{R}{t}} \tag{28}$$

This expression for the slenderness ratio of a sphere or spherical cap is very similar to the geometric parameter introduced by Kaplan and Fung [36]; in fact, their parameter matches Eq. (28) for a hemisphere. Unlike their parameter, though, Eq. (28) depends entirely on the radius of curvature with no dependence on the height of the spherical head, even for very shallow caps or complete spheres. This approach, without including a dependence on cap height, has so far produced the best agreement with the published experimental data.

Conical Vessels and Formed Heads

The Code provides a method for analyzing conical vessel sections as equivalent cylinders, as well as methods for analyzing formed ellipsoidal and torispherical heads as equivalent spherical caps. Through the use of those existing procedures, the methods of this paper can be extended to these geometries

MATERIAL CONSIDERATIONS; DETERMINATION OF ALLOWABLE STRESS

Through the use of the slenderness λ, a comparison can be made between pressure vessel code requirements and the column buckling theories discussed above, as well as between pressure vessel and structural codes. ASME Section II-D contains a collection of charts for a variety of materials at several temperatures, which are used to determine the ASME factor B and the allowable external pressure in ASME Section VIII, Division 1. As described in Appendix 3 of Section II-D, these plots are pseudo-stress-strain curves for each material from which the factor B, correctly incorporating the Young's modulus (in the elastic range) or tangent modulus (in the plastic range), can be directly read [37,38]. In effect, these charts implement the tangent modulus column buckling theory of Engesser and Shanley for externally-pressurized vessels. The charts can be re-plotted in terms of λ by using the relationship $A = \frac{\pi^2}{\lambda^2}$.

ASME Code Case 2286 [5], since incorporated with refinements into Section VIII, Division 2 of the Code, provides explicit formulas as an alternative to the graphical method. These formulas are again equivalent to applying the tangent modulus buckling theory, in this case using a generalized stress-strain curve. The Code Case 2286 equations for cylinders under external pressure cannot formally be expressed as a function of λ as derived above; however, they can be closely approximated by

$$\sigma_{all} = \begin{cases} 0.2\sigma_y\left(\lambda/C_c\right)^{-2}, & \text{if } \lambda/C_c \geq 0.833 \\ 0.24\sigma_y\left(\lambda/C_c\right)^{-1}, & \text{if } 0.4 \leq \lambda/C_c < 0.833 \\ 0.6\sigma_y, & \text{if } \lambda/C_c < 0.4 \end{cases} \tag{29}$$

Figure 2 shows a comparison of both the external pressure charts and Code Case 2286 with the allowable stress in compression per the 1989 AISC Manual of Steel Construction [12] and the 2015 AA Aluminum Design Manual [39]. In the plastic buckling region ($\lambda/C_c \leq 1$), the Code Case 2286 allowable stresses

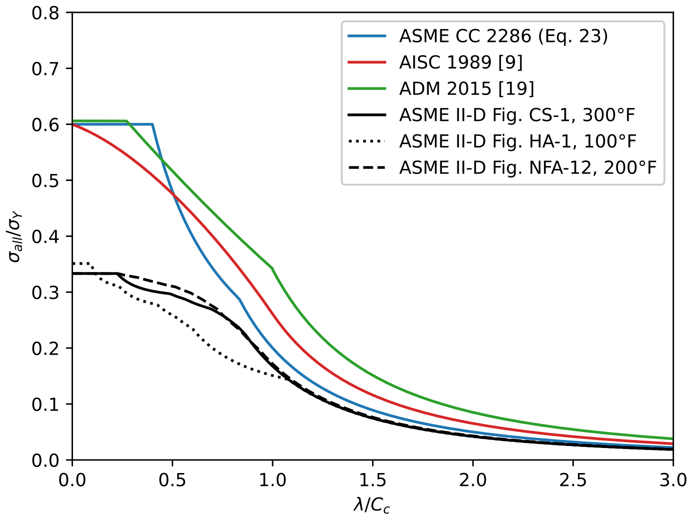

FIGURE 2. COMPARISON OF ALLOWABLE STRESSES BETWEEN CODES

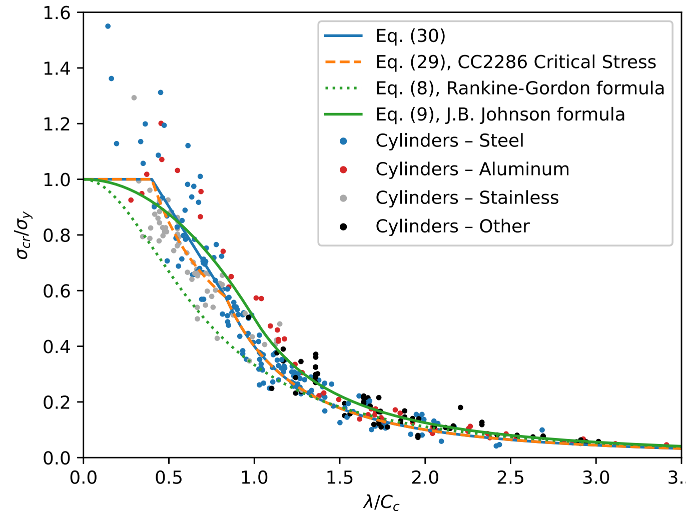

FIGURE 3. COLLAPSE PRESSURE OF EXTERNALLY-PRESSURIZED CYLINDERS – EXPERIMENTAL DATA

are quite similar to those from structural design codes and significantly higher than those using the external pressure charts. In the elastic region, the Code Case 2286 and external pressure charts give similar allowable stresses, which are somewhat lower than those allowed by the structural codes.

It is worth noting that both the 1989 AISC and 2015 ADM codes appear to be based on empirical buckling relations (J.B. Johnson in the case of AISC, and a straight-line formula for the ADM) rather than tangent modulus buckling theory, likely because these empirical formulas are more straightforward to apply while still being acceptably accurate. This suggests that applying such empirical formulas to pressure vessel design may produce simplified design formulas that nonetheless produce designs with acceptable margins of safety.

COMPARISON WITH EXPERIMENT AND DISCUSSION

The value of any engineering design approach is ultimately its ability to provide an adequate design margin against failure. To that end, a number of published experimental studies on the collapse of cylinders due to external pressure were reviewed [2,3,6,40–52], dating from the 1850s to the present. This set of data [53] covered a wide range of dimensions, pressures, slenderness ratios, and materials of construction. The experimental data is plotted in Fig. 3, along with the estimated critical stress using the method of Code Case 2286 (including the geometric factor, but with the factor of safety removed). Empirical column formulas Eq. (8) (Rankine-Gordon formula) and Eq. (9) (J.B. Johnson formula) are also shown. Using λ, these plots correspond to the column curve diagrams often shown in structural design textbooks [24–26].

In Fig. 3, it is readily apparent that the data points fall into a narrow band that is well-approximated by both the Code Case 2286 and by the Rankine-Gordon and J.B. Johnson formulas.

Though the code case method was developed specifically for carbon steel, it is clearly also applicable for other materials, if slightly conservative for some (e.g., certain copper and aluminum alloys). The approach of having different charts for each material, though sound from a theoretical standpoint, seems unnecessary since the differences between materials do not appear to greatly exceed the experimental scatter within each material type.

For brittle materials, materials at very high temperature, or other cases where a material's stress-strain curve differs greatly from that of cold carbon steel, the use of established column design formulas appropriate for the application could be considered as a simpler alternative to graphically or analytically duplicating the stress-strain curve of the material.

It is not immediately apparent that the "cusp" in the Code Case 2286 curve at $\lambda = 0.833$ is reflected in the experimental data; it is likely this cusp is an artifact of the curve fitting process used to develop the equation. Therefore the following slightly less conservative equation is proposed for the critical stress:

$$\sigma_{cr} = \begin{cases} 0.4\sigma_y\left(\lambda/C_c\right)^{-2}, & \text{if} \quad \lambda/C_c \geq 0.833 \\ \left(1.3914 - 0.9785\lambda/C_c\right)\sigma_y, & \text{if} \quad 0.4 \leq \lambda/C_c < 0.833 \\ \sigma_y, & \text{if} \quad \lambda/C_c < 0.4 \end{cases} \tag{30}$$

Applying the same variable factor of safety included in Code Case 2286 produces the following expression for allowable

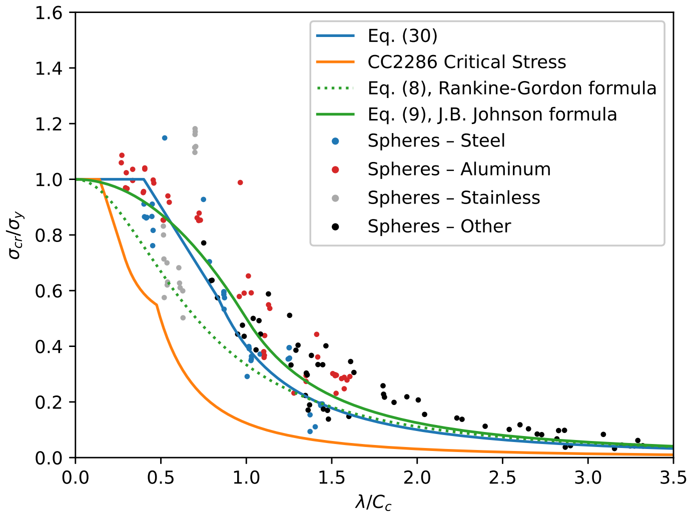

FIGURE 4. COLLAPSE PRESSURE OF EXTERNALLY-PRESSURIZED SPHERES – EXPERIMENTAL DATA

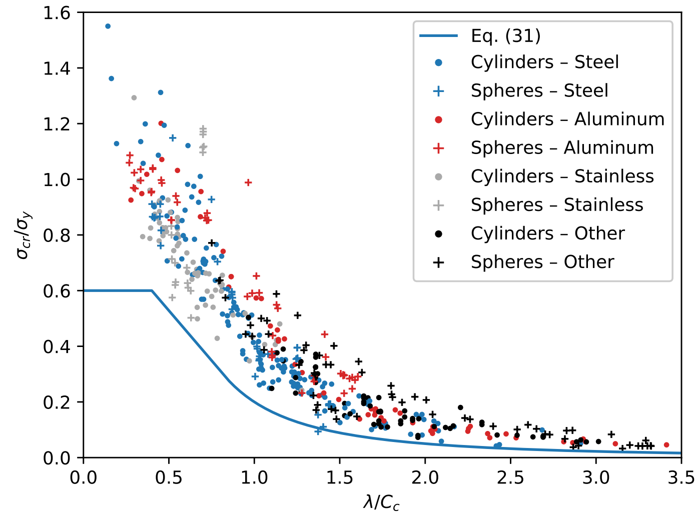

FIGURE 5. PROPOSED DESIGN CURVE

stress, as an alternative to Eq. (29).

$$\sigma_{all} = \begin{cases} 0.2\sigma_y \left(\lambda/C_c\right)^{-2}, & \text{if} \quad \lambda/C_c \geq 0.833 \\ \left(0.888 - 0.72\lambda/C_c\right)\sigma_y, & \text{if} \quad 0.4 \leq \lambda/C_c < 0.833 \\ 0.6\sigma_y, & \text{if} \quad \lambda/C_c < 0.4 \end{cases} \tag{31}$$

Similarly, from the 1950s to the present, a number of experimental studies have been published [36,54–65] covering the external pressure buckling of spheres, hemispheres, and spherical caps . These studies were reviewed [53], and the data is plotted in Figure 4 using the slenderness ratio as calculated per Eq. (28). The data for spheres and spherical caps, as for cylinders, is broadly in agreement with the J.B. Johnson formula, and considerably above the Code Case 2286 curve. However, there is a much larger variance in the data than for cylinder tests. This wider scatter has long been recognized and represents the high imperfection sensitivity of spherical shell buckling [66].

For values of λ below 0.4, the critical stresses continue to increase, though the code allowable stress is limited to $0.6\sigma_y$. Consideration should be given to whether the variable factor of safety can be reduced further in this region (for a similar discussion, see [67]).

CONCLUSIONS

The slenderness ratio method to calculate pipe and vessel buckling pressure is useful in several ways:

1. The basic design process, Eqs. (1–4), is simple enough for a back-of-the-envelope calculation, with more exact refinements such as Eq. (24) optional for additional precision. The simplified process always errs conservatively.
2. The slenderness ratio and column curve diagram are familiar to engineers and provide an intuitive understanding of the stability of the system in relation to pressure, geometry, and material properties. The well-known J.B. Johnson formula can be used to provide a good estimate of the critical pressure.
3. Strengths, curves, and design factors can be compared between geometries and among codes and standards for pressure vessel, civil/structural design, and aerospace structures.
4. The slight excess conservatism apparent in Code Case 2286 in the range $0.4 < \lambda/C_c < 0.833$ can be reduced.

The proposed design curve, Eq. (4), is shown in Figure 5. For pipes and vessels, this represents a modest evolution from the Code Case 2286 allowable stresses, with only the range $0.4 < \lambda/C_c < 0.833$ substantially affected. For spheres, the allowable stresses would be greatly increased, and this should be approached with caution. The greater scatter of the test data for spheres may warrant a larger design factor.

The slenderness ratio method compares well against the existing design-by-rule methods, which require either chart-based solutions or the use of formulas which are, by comparison, more complex and less intuitive. This method should be considered for incorporation into the Boiler and Pressure Vessel Code as an allowable method of calculating external pressure. A simplified method based on Eq. (17) may be particularly well-suited to applications in the B31 codes where long cylinders are predominant.

EXAMPLE

Using the methods of this paper, predict the collapse pressure and allowable stress for a steel pipe ($\sigma_y = 31$ ksi, $E = 28,000$ ksi) with an outside diameter of 16 in., a length of 6 in., and a wall thickness of 0.051 in. (Model #32, Windenburg & Trilling [3])

SOLUTION:

For this material,

$$C_c = \sqrt{\frac{2\pi^2 E}{\sigma_y}} = 133.5$$

This is a short pipe with an L/D ratio of 0.375. Using Eq. (22),

$$\lambda = 2.75 \left(\frac{D}{t}\right)^{0.75} \left(\frac{L}{D}\right)^{0.5} = 125.5$$

$$\lambda / C_c = 0.940$$

The collapse pressure can be predicted using the J.B. Johnson formula, Eq. (9):

$$\sigma_{cr} = \sigma_y \left(1 - 0.5\,(\lambda / C_c)^2\right) = 17.3 \text{ ksi}$$

$$P_{cr} = \frac{2\sigma_{cr} t}{D} = 110 \text{ psi}$$

Using Eq. (31), the allowable stress and allowable external pressure are:

$$\sigma_{all} = 0.2\sigma_y\,(\lambda / C_c)^{-2} = 7.02 \text{ ksi}$$

$$P_{all} = \frac{2\sigma_{all} t}{D} = 44.7 \text{ psi}$$

For comparison, the allowable pressures using paragraph UG-28 and Code Case 2286 would be 44.4 psi and 47.5 psi, respectively. This model failed experimentally at an external pressure of 107 psi.

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