SAND2021-12822C

Stability and convergence analysis of the harmonic balance method for a Duffing oscillator with freeplay nonlinearity

Brian Evan Saunders[1], Rui M.G. Vasconcellos[2], Robert J. Kuether[3], and Abdessattar Abdelkefi[1]

[1] Mechanical & Aerospace Engineering Department
New Mexico State University, Las Cruces, NM, USA

[2] Campus of São João da Boa Vista
São Paulo State University, São João da Boa Vista, Brazil

[3] Sandia National Laboratories, Albuquerque, NM, USA

ABSTRACT

In this work, we determine the quality of the harmonic balance method (HBM) using a single degree-of-freedom forced Duffing oscillator with freeplay. HBM results are compared to results obtained using direct time integration with an event location procedure to properly capture contact behavior and identify non-periodic motions. The comparison facilitates an evaluation of the accuracy of nonlinear, periodic responses computed with HBM, specifically by comparing super- and sub-harmonic resonances, regions of periodic and non-periodic (i.e., quasiperiodic or chaotic) responses, and discontinuity-induced bifurcations, such as grazing bifurcations. Convergence analysis of HBM determines the appropriate number of harmonics required to capture nonlinear contact behavior while satisfying the governing equations. Hill's method and Floquet theory are used to compute the stability of periodic solutions and identify the types of bifurcations in the system. Extensions to multi-degree-of-freedom oscillators will be discussed as well.

Keywords: Harmonic balance, piecewise-smooth, bifurcations, Floquet stability, convergence analysis

INTRODUCTION

The harmonic balance method (HBM) is a well-known and popular method for obtaining periodic solutions of dynamical systems, whether linear or nonlinear, with few or many degrees of freedom (DOF). Its largest advantages over time-integration numerical methods include much lower computational costs and the ability to capture unstable solution responses otherwise undetectable even in experiments. Other valuable uses of HBM include computing the nonlinear normal modes (NNMs) of a system [1,2], bifurcation tracking [3-5], and stability analysis [3,6]. In the past decade, HBM has seen increased application to problems involving contact, impact, or friction between parts. These kinds of phenomena are non-smooth and can lead to very complex behavior even in simple systems. Further, a larger number of harmonics is often required to obtain converged results than is required for smooth systems. Alcorta et al. [4], for example, used 15 modes/harmonics when analyzing a single-DOF system with both cubic and contact nonlinearities while Detroux et al. [3] analyzed a 37-DOF system and used five harmonics in their calculations. Residual error analysis can be used to determine the approximate number of harmonics required to get results accurate to a given error tolerance and to ensure that sub- and super-harmonic resonances are not being missed.

In this work, we determine the quality of the harmonic balance method for a single degree-of-freedom forced Duffing oscillator with freeplay [7]. HBM results are compared to time-integration results to facilitate an evaluation of the accuracy of nonlinear periodic responses computed with HBM. Super- and sub-harmonic resonances, regions of periodic and non-periodic (i.e., quasiperiodic or chaotic) responses, and discontinuity-induced bifurcations, such as grazing bifurcations are all analyzed to study HBM solution quality. Convergence analysis is performed to find the appropriate number of harmonics required to capture nonlinear contact behavior for various nonlinearity strengths. Hill's method and Floquet theory are used to compute the stability of periodic solutions and identify bifurcations. In the final conference presentation, extensions to multi degree-of-freedom oscillators will also be discussed.

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

SYSTEM MODELING AND NUMERICAL METHODS

The equations of motion for the Duffing-freeplay system as used by deLangre et al. [7] are given by:

$$\ddot{x} + 2\omega_n\zeta\dot{x} + \omega_n^2 x + \frac{\alpha}{m}x^3 + \frac{F_c}{m} = \frac{p}{m}cos(\omega t), F_c = \begin{cases} K_c(x + j_1), & x < -j_1 \\ 0, & -j_1 < x < j_2 \\ K_c(x - j_2), & x < j_2 \end{cases} \#(1)$$

where α denotes the cubic stiffness, K_c is the contact stiffness, and j_1,j_2 represent the freeplay boundaries. MATLAB® ode45 with *Event Location* was used to provide time-integration solutions used as reference data [8] to compare the HBM solution. The harmonic balance works by assuming the solution, nonlinear forcing terms, and external forcing term(s) can all be expressed as a Fourier series with the following general form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{nl}(\mathbf{x},\dot{\mathbf{x}}) = \mathbf{f}_{ext}(t), \qquad \mathbf{x}(t) = \frac{\mathbf{c}_0^x}{\sqrt{2}} + \sum_{k=1}^{N_h}\left[\mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t)\right] \#(2)$$

The mathematical details are omitted here but are given by Detroux et al. [3]. The result is a system of nonlinear algebraic equations. The piecewise-smooth contact force is regularized with a hyperbolic tangent approximating function [9]. This system is solved iteratively using the Newton-Raphson method. The step-size is adaptively controlled within the continuation algorithm using the strategy of Colaïtis et al. [10] and also controlled by the curvature at each solution point.

CONVERGENCE ANALYSIS

Figure 1(a) shows the frequency response curves of the system for a moderately hard contact stiffness and a small symmetric freeplay gap. The colored dots correspond to time integration results for various initial conditions (IC), one color for each IC, and the solid lines correspond to HBM results for 12, 24, and 36 harmonics along with the zero-frequency/DC component. Overall, all three HBM cases capture the majority of the behavior with good agreement including the isolated subharmonic resonance between 15-28 Hz. The agreement breaks down at the lower frequencies, though, as more harmonics are required to capture the various superharmonic resonance responses present between 0-4 Hz. There also appear to be multiple isolated responses not captured, in addition to chaotic responses which will naturally also not be captured. Figure 1(b) shows the Floquet stability of the previous HBM result curves excluding the subharmonic resonance. Dots indicate unstable regions. Nearly all the unstable regions start and end at vertical tangents which indicate saddle-node bifurcations have occurred. The relative residual error of the three HBM results in the superharmonic resonance region is inset in the upper right and indicates the error is highest in the frequency ranges where amplitude is large enough for contact to occur. Error reduces by a factor of three between each curve ($N_H = 12 \rightarrow 24 \rightarrow 36$).

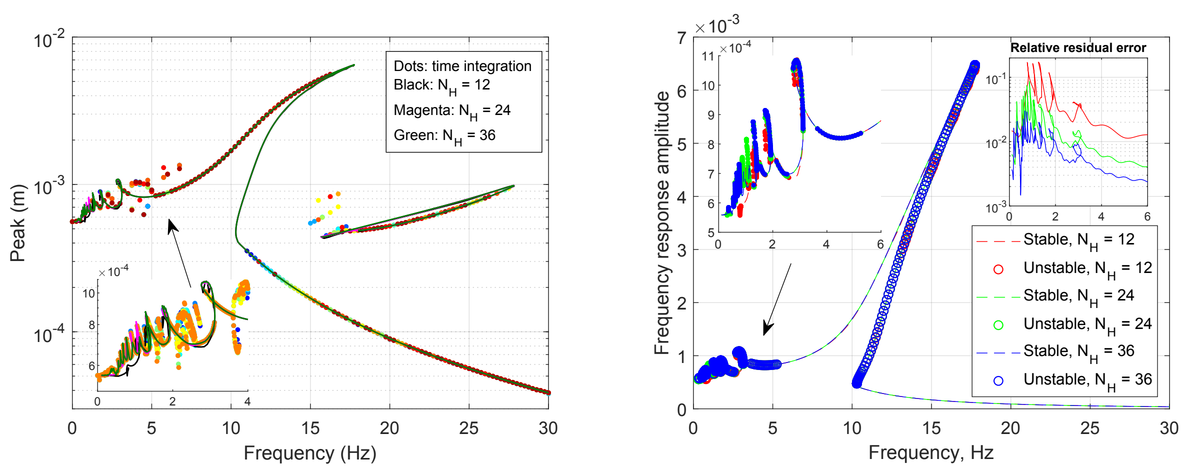

Fig. 1 (a) Frequency response curves of the system comparing HBM with different numbers of harmonics to time integration and (b) frequency response divided into stable and unstable regions with relative residual error also inset.

CONCLUSIONS

In this work, we presented convergence and stability analyses of a single degree-of-freedom forced Duffing oscillator with freeplay using the harmonic balance method. We showed that, for this combination of contact stiffness and freeplay gap size, the majority of the frequency response can be captured with 12 harmonics including the behavior of the isolated subharmonic resonance branch. However, the superharmonic resonances can require many more harmonics in order to be captured. Floquet analysis showed that the system experiences a large number of saddle-node bifurcations at low frequency. The continuation procedure was subject to some persistent difficulties near turning points; the procedure was robust against turning back on itself, but it was prone to “jumping” to a previously solved portion of a branch and continuing in reverse. A more robust step-size control or error control may mitigate this issue.

ACKNOWLEDGEMENTS

The authors B. Saunders and A. Abdelkefi gratefully acknowledge the support from Sandia National Laboratories. This study describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government. Supported by the Laboratory Directed Research and Development program at Sandia National Laboratories, a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia LLC, a wholly owned subsidiary of Honeywell International Inc. for the U.S. Department of Energy’s National Nuclear Security Administration under contract DE-NA0003525. SAND**???**. R. Vasconcellos acknowledges the financial support of the Brazilian agency CAPES (grant 88881.302889/2018-01).

REFERENCES

[1] Kerschen, G., Peeters, M., Golinval, J.C., and Vakakis, A.F., 2009, "Nonlinear normal modes, Part I: A useful framework for the structural dynamicist," Mechanical systems and signal processing, **23**(1), pp.170-194. Doi: https://doi.org/10.1016/j.ymssp.2008.04.002

[2] Peeters, M., Viguié, R., Sérandour, G., Kerschen, G., and Golinval, J.C., 2009, "Nonlinear normal modes, Part II: Toward a practical computation using numerical continuation techniques," Mechanical Systems and Signal Processing, **23**(1) pp. 195-216. Doi: https://doi.org/10.1016/j.ymssp.2008.04.003

[3] Detroux, T., Renson, L., Masset, L., Kerschen, G., 2015, "The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems," Computer Methods in Applied Mechanics and Engineering, **296**, pp. 18-38. Doi: https://doi.org/10.1016/j.cma.2015.07.017

[4] Alcorta, R., Baguet, S., Prabel, B., Piteau, P., and Jacquet-Richardet, G., 2019, “Period doubling bifurcation analysis and isolated sub-harmonic resonances in an oscillator with asymmetric clearances,” Nonlinear Dynamics, **98**, pp. 2939–2960. Doi: https://doi.org/10.1007/s11071-019-05245-6

[5] Xie, L., Baguet, S., Prabel, B., and Dufour, R., 2017, "Bifurcation tracking by Harmonic Balance Method for performance tuning of nonlinear dynamical systems," Mechanical Systems and Signal Processing, **88**(1), pp. 445-461. Doi: https://doi.org/10.1016/j.ymssp.2016.09.037

[6] Lazarus, A. and Thomas, O., 2010, "A harmonic-based method for computing the stability of periodic solutions of dynamical systems," Comptes Rendus Mécanique, **338**(9), pp. 510-517. Doi: https://doi.org/10.1016/j.crme.2010.07.020

[7] De Langre, E. and Lebreton, G., “An Experimental and Numerical Analysis of Chaotic Motion in Vibration with Impact," *ASME 8th International Conference on Pressure Vessel Technology*, Montreal, Quebec, Canada, 1996.

[8] Saunders, B.E., Vasconcellos, R., Kuether, R.J., and Abdelkefi, A., 2021, “Relationship between the contact force strength and numerical inaccuracies in piecewise-smooth systems," International Journal of Mechanical Sciences, in press. Doi: https://doi.org/10.1016/j.ijmecsci.2021.106729

[9] Saunders, B.E., Vasconcellos, R., Kuether, R.J., and Abdelkefi, A., “Insights on the continuous representation of piecewise-smooth nonlinear systems: limits of applicability and effectiveness," Nonlinear Dynamics, 2021. Doi: https://doi.org/10.1007/s11071-021-06436-w

[10] Colaïtis, Y. and Batailly, A., 2021, "The harmonic balance method with arc-length continuation in blade-tip/casing contact problems," Journal of Sound and Vibration, **502**, p.116070. Doi: https://doi.org/10.1016/j.jsv.2021.116070