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Capacitive/Inductive Corrections for Numerical Implementation of Thin-Slot Transmission Line Models and Other Useful Formulas

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Capacitive/Inductive Corrections for Numerical Implementation of Thin-Slot Transmission Line Models and Other Useful Formulas

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ABSTRACT

Capactiance/inductance corrections for grid induced errors for a thin slot models are given for both one and four point testing on a rectangular grid for surface currents surrounding the slot. In addition a formula for translating from one equivalent radius to another is given for the thin-slot transmission line model. Additional formulas useful for this slot modeling are also given.

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I. Transmission Line Equations for the Slot Filamentary Magnetic Current in an Infinite Ground Plane

For a rectangular slot in an infinite ground plane separating two half spaces Fig. 1 (slot length ℓ is much greater than the slot depth d and width w (or cross sectional dimensions in general)) we can write the integro-differential equation for the filamentary slot equivalent magnetic current $I_m(z)$ as [1]

$$H_z^{>}(a,z) + \frac{1}{4}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = -H_z^{inc}(a,z) \tag{1}$$

$$H_z^{>}(a,z) = \frac{i}{\omega\mu_0}\left(\frac{d^2}{dz^2} + k^2\right)\int_{-h}^{h} \frac{e^{ikR_a}}{4\pi R_a} I_m\left(z'\right) dz' \tag{2}$$

where $I_m \equiv I_m^- = -2V$, V is voltage difference across the slot between the left and right conductors (Fig. 1., [1]), the slot is along the z-axis, $R_a = \sqrt{a^2 + (z-z')^2}$ where a is the equivalent radius of the slot with no gasket and no losses present. As in [1] $e^{-i\omega t}$ time dependence is assumed and the incident wave impinges on the slot from the $y < -\frac{d}{2}$ side. For a rectangular slot of depth d and width w in an infinite, perfectly conducting ground plane the equivalent radius a is given by eqs. 21-24 and 39 [3]. As $\frac{d}{w} \to 0$, $a \to \frac{w}{4}$ and a useful uniformly valid approximation to $a \approx \frac{w}{4}e^{-\frac{\pi d}{2w}}$. The local admittance per unit length Y_{ρ_0} and the total series impedance Z_{ρ_0} [1] depend on the radius ρ_0 , which splits the problem into a local transmission line region and a global region, the quantities ΔY_L and ΔY_C

$$\Delta Y_L = \frac{1}{Z_{\rho_0}} + \frac{1}{i\omega L_{\rho_0}^{ext^0}} \tag{3}$$

$$\Delta Y_C = Y_{\rho_0} + i\omega C_{\rho_0}^0 = G - i\omega C_{\rho_0} + i\omega C_{\rho_0}^0 \tag{4}$$

$$L_{\rho_0}^{ext^0} = \frac{\pi\mu_0}{2\ln(\rho_0/a)} \tag{5}$$

$$C_{\rho_0}^0 = \frac{2}{\pi}\epsilon_0 \ln(\rho_0/a) \tag{6}$$

are independent of ρ_0. The transmission line parameters $Y_{\rho_0} = G - i\omega C_{\rho_0}$, $Z_{\rho_0} = \frac{-i\omega\Phi}{I\rho_0}$, ΔY_L, ΔY_C are obtained from the static two dimensional problem of an infinitely long slot in an ground plane of infinite extent [1] . These parameters depend on the depth, width, and conductivity of the slot walls. A gasket with permeability μ_0, conductivity σ_g, and relative permittivity ε_g may be added to the slot interior and effects Y_{ρ_0}. For the case that wall losses are not small and $d >> w$ we may use the parallel splitting eq. 40 [1]

$$\frac{1}{Z_{\rho_0}} = \frac{1}{-i\omega L_{\rho_0}^{ext}} + Y^{int} \tag{7}$$

eq. 41 [1]

$$\Delta Y_L = \frac{1}{-i\omega L_{\rho_0}^{ext}} + Y^{int} + \frac{1}{i\omega L_{\rho_0}^{ext^0}} \tag{8}$$

and eq. 76 [1] which is valid for no gasket or a gasket with permeability μ_0 (restrictions are given in [1])

$$Y^{int} \approx \frac{2Z_s/d}{i\omega\mu_0 \frac{w}{d}\left(\frac{2Z_s}{d} - i\omega\mu_0\frac{w}{d}\right)} \tag{9}$$

If the slot has a gasket with permeability other than μ_0 one may find a work-around in Section 4.2 of [2]. For the remainder of this paper we assume the permeability of the gasket is μ_0 .

$$\Delta Y_L = \frac{1}{-i\omega L_{\rho_0}^{extr}} + \frac{1}{-i\omega L_{\rho_0}^{intr}} + Y^{int} + \frac{1}{i\omega L_{\rho_0}^{extr^0}} + \frac{1}{i\omega L_{\rho_0}^{intr0}}$$

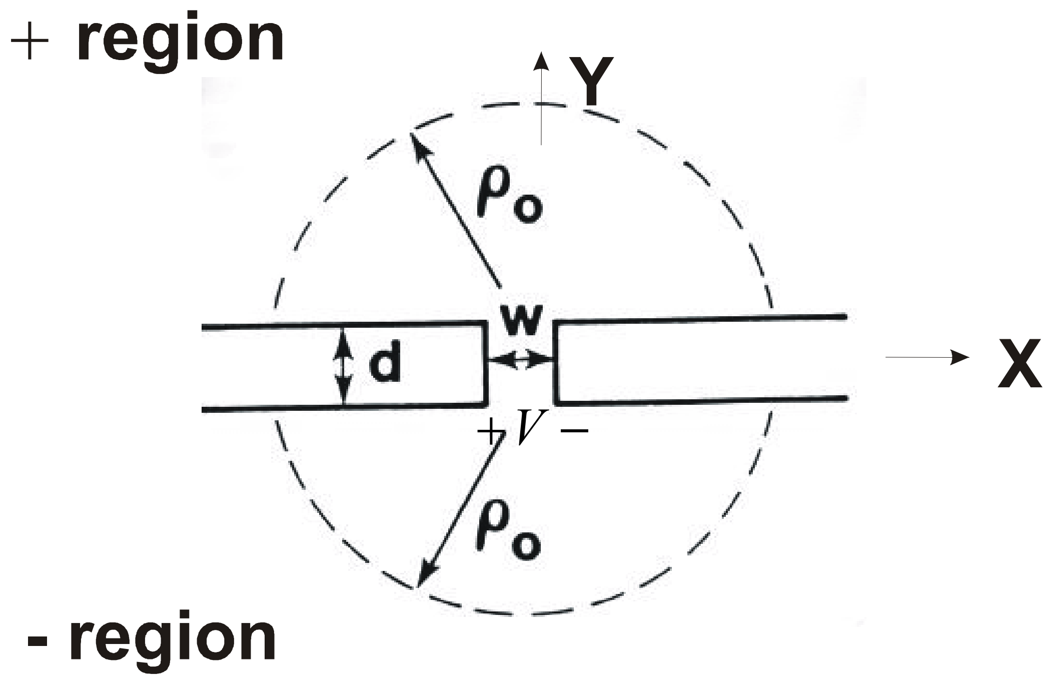

Figure 1: A slot of width w and depth d in an infinite ground plane.

$$\approx Y^{int} \tag{10}$$

The losses interior to the slot dominate and the parts of C_{ρ_0} and $C^0_{\rho_0}$exterior to the slot cancel

$$\Delta Y_C = G - i\omega C_{\rho_0} + i\omega C^0_{\rho_0}$$
$$\approx -i\omega\varepsilon_0 \left(\varepsilon_g + \frac{\sigma_g}{i\omega\varepsilon_0}\right)\frac{d}{w} + i\omega\varepsilon_0 \frac{d}{w} \tag{11}$$

II. Transformation of Radius in Thin-Slot Transmission Line Model

Equivalently, the integro-differential equations for the slot equivalent magnetic current at ρ_0 eqs.$(25-26)$ [1] are

$$H_z^{>}(\rho_0, z) + \frac{1}{4}\left(\frac{1}{Z_{\rho_0}}\frac{d^2}{dz^2}I_m - Y_{\rho_0}I_m\right) = -H_z^{inc} \tag{12}$$

$$H_z^{>}(\rho_0, z) = \frac{i}{\omega\mu_o}(\frac{d^2}{dz^2} + k^2)\int_{-h}^{h}\frac{e^{ikR_{\rho_0}}}{4\pi R_{\rho_0}}I_m(z')dz' \tag{13}$$

$$\Delta Y_L = \frac{1}{Z_{\rho_0}} \tag{14}$$

$$\Delta Y_C = Y_{\rho_0} = G - i\omega C_{\rho_0} \tag{15}$$

where $I_m \equiv I_m^- = -2V$ due to the assumption $(\ell, \lambda >> \rho_0 >> d)$.

Eqs. (12-15) may be translated from ρ_0 to a by adding $\frac{1}{i\omega L_{\rho_0}^{ext0}}$ to ΔY_L and $i\omega C_{\rho_0}^{0}$ to ΔY_C. This is seen by using the approximation [6]

$$\int_{-h}^{h}\frac{e^{ikR_a}}{R_a}I_m(z')dz' \approx \Omega_e I_m(z) \tag{16}$$

where $\Omega_e = \Omega + C_e$ is Hallen's antenna fatness parameter, $\Omega = 2\ln(2h/a)$ and C_e may be approximated as $C_e = 2(\ln 2 - 7/3)$ and $2h$ is the length of the slot.

$$\int_{-h}^{h}\frac{e^{ikR_{\rho_0}}}{R_{\rho_0}}I_m(z')dz' \approx (2\ln(2h/\rho_0) + C_e)\, I_m(z)$$

$$= (2\ln(2h/a) + C_e)\, I_m(z) - 2\ln(\rho_0/a)\, I_m(z)$$

$$\approx \int_{-h}^{h} \frac{e^{ikR_a}}{R_a} I_m(z')dz' - 2\ln(\rho_0/a)\, I_m(z) \tag{17}$$

Equation 17 may be used to approximate 12 as

$$\frac{i}{4\pi\omega\mu_o}(\frac{d^2}{dz^2} + k^2)\left[\int_{-h}^{h} \frac{e^{ikR_a}}{R_a} I_m(z')dz' - 2\ln(a/\rho_0)\, I_m(z)\right]$$

$$+\frac{1}{4}\left(\frac{1}{Z_{\rho_0}}\frac{d^2}{dz^2} I_m - Y_{\rho_0} I_m\right) = -H_z^{inc} \tag{18}$$

$$\frac{i}{4\pi\omega\mu_o}(\frac{d^2}{dz^2} + k^2)\left[\int_{-h}^{h} \frac{e^{ikR_a}}{R_a} I_m(z')dz'\right]$$

$$+\frac{1}{4}\left[\left(\frac{1}{Z_{\rho_0}} + \frac{1}{i\omega}\frac{2}{\pi\mu_o}\ln(\rho_0/a)\right)\frac{d^2}{dz^2} I_m - \left(Y_{\rho_0} + \frac{i\omega 2\varepsilon_0}{\pi}\ln(\rho_0/a)\right) I_m\right] = -H_z^{inc} \tag{19}$$

Using eqs. (5-6) eqs. (12-15) becomes eqs.(1-6).

To mitigate gridding difficulties with extremely small equivalent radii, a transformation to the alternate equivalent radius $a_0 = ae^{\frac{\pi d}{2w}}$ may be carried out when modeling a rectangular slot.

$$H_z^{>}(a_0, z) + \frac{1}{4}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = -H_z^{inc} \tag{20}$$

$$H_z^{>}(a_0, z) = \frac{i}{\omega\mu_0}\left(\frac{d^2}{dz^2} + k^2\right)\int_{-h}^{h} \frac{e^{ikR_{a_0}}}{4\pi R_{a_0}} I_m(z')\, dz' \tag{21}$$

$$\Delta Y_L \approx Y^{int} - \frac{1}{i\omega L_0} \tag{22}$$

$$\approx \frac{1}{-i\omega}\frac{1}{\mu_0}\frac{d}{w} + Y^{int}$$

$$\Delta Y_C \approx -i\omega\varepsilon_0 \left(\varepsilon_g + \frac{\sigma_g}{i\omega\varepsilon_0}\right)\frac{d}{w} \tag{23}$$

where $L_0 = \mu_o \frac{w}{d}$, and $C_0 = \varepsilon_0 \frac{d}{w}$, the parallel plate inductance and capacitance of interior rectangular slot region.

III. Transmission Line Equations for the Slot Filamentary Magnetic Current when no Ground Plane is Present

Multiplying eq. 20 by 2 yields

$$2H_z^{>}(a_0, z) + \frac{1}{2}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = -2H_z^{inc}(a_0, z) \tag{24}$$

Since $\rho_0 << \lambda$, $H_z^{inc}(a, z) \approx H_z^{inc}(a_0, z)$.

If the structure surrounding the slot is not an infinite ground plane, as a slot in a finite rectangular box with no ground planes, one may need compute unknown currents on the surface as well as the slot equivalent magnetic current. In this case the incident field $H_z^{inc}(a_0, z)$ and the magnetic slots currents I_m are not imaged so eq. 24 may be re-written as

$$H_z^{inc}\left(a_0^-, z\right) + H_z^{scatt}\left(J^-, a_0^-, z\right) + \frac{1}{2}H_z^{>}\left(a_0^-, z\right) - H_z^{scatt}\left(J^+, a_0^+, z\right) - \frac{1}{2}H_z^{>}\left(a_0^+, z\right)$$

$$+\frac{1}{2}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = 0 \tag{25}$$

$$H_z^{>}\left(a_0^{\pm}, z\right) = \pm \frac{-i}{\omega\mu_0}\left(\frac{d^2}{dz^2} + k^2\right)\int_{-h}^{h} \frac{e^{ikR_{a_0}}}{4\pi R_{a_0}} I_m\left(z'\right) dz' \tag{26}$$

where $H_z^{scatt}\left(J^{\pm}, a_0^{\pm}, z\right)$ is the magnetic field scattered by the surface current $\mathbf{J}^{\pm}\left(a_0^{\pm}, z\right)$ at the slot location and $\frac{1}{2}H_z^{>}\left(a_0^{\pm}, z\right)$ is the magnetic field scattered by $\mp\frac{I_m}{2}$ (with no ground plane image present). Note that

$$\mathbf{H}_{\tan} = -\,\widehat{n} \times \mathbf{J} \tag{27}$$

where $\mathbf{H}_{\tan}$ is the tangential component of the total $\mathbf{H}$ field and the unit normal $\widehat{n}$ points away from the metal.

$$\mathbf{H}_{\tan}\left(a_0^{-}, z\right) = H_z^{inc}\left(a_0^{-}, z\right) + H_z^{scatt}\left(J^{-}, a_0^{-}, z\right) + \frac{1}{2}H_z^{>}\left(a_0^{-}, z\right) = -\,\widehat{n^{-}} \times \mathbf{J}^{-}\left(a_0^{-}, z\right) \tag{28}$$

$$\mathbf{H}_{\tan}\left(a_0^{+}, z\right) = H_z^{scatt}\left(J^{+}, a_0^{+}, z\right) + \frac{1}{2}H_z^{>}\left(a_0^{+}, z\right) = -\,\widehat{n^{+}} \times \mathbf{J}^{+}\left(a_0^{+}, z\right) \tag{29}$$

and eq 25 becomes

$$-\,\widehat{n^{-}} \times \mathbf{J}^{-}\left(a_0^{-}, z\right) + \,\widehat{n^{+}} \times \mathbf{J}^{+}\left(a_0^{+}, z\right)$$

$$+\frac{1}{2}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = 0 \tag{30}$$

$$\Delta Y_L \approx \frac{1}{-i\omega}\frac{1}{\mu_0}\frac{d}{w} + Y^{int} \tag{31}$$

$$\Delta Y_C \approx -i\omega\varepsilon_0 \left(\varepsilon_g + \frac{\sigma_g}{i\omega\varepsilon_0}\right)\frac{d}{w} \tag{32}$$

A capacitive correction from a 2-d electro-static problem that follows may be added to eq. 30 to account for resolution errors in the charge associated with $J^{\pm}$

$$-\ \widehat{n^-} \times \mathbf{J}^- \left(a_0^-, z\right) + \ \widehat{n^+} \times \mathbf{J}^+ \left(a_0^+, z\right)$$

$$-\frac{1}{2}\Delta Y_P \left(\frac{1}{k^2}\frac{d^2}{dz^2} + 1\right) I_m + \frac{1}{2}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = 0 \tag{33}$$

$$\Delta Y_P = -i\omega \left(C_{\rho_0}^0 \left(a_0\right) - C_{\rho_0}^P\right) \tag{34}$$

where $C_{\rho_0}^0 \left(a_0\right)$ is defined by eq. 6 with a replaced by $a_0 = ae^{\frac{\pi d}{2w}}$ and $C_{\rho_0}^P$ is numerically computed capacitance on both the $\rho_0^{\pm}$ sides between the two portions of the ground planes (with potential difference V) out to radius ρ_0. In the calculation of $C_{\rho_0}^P$ there is no attempt to resolve a_0 so it has been set to 0. The unit term in parenthesis times ΔY_P is is a capacitive correction while the $\frac{1}{2k^2}\Delta Y_p$ is an inductive correction which the may be obtained from a dual 2-d magneto-static problem or simply by relating the inductance term to the capacitance term by the velocity of light. If the slot depth is small, a correction for losses exterior to the slot may be added to this inductive term.

IV. Capacitance Correction for One Point Test and Piecewise Constant Rectangular Basis Functions

The derivation of the difference capacitance for a one point test at the center of a rectangular patch is as follows. We take two half planes to be thin and at potential $\phi\left(x\right) = (V/2)\text{sgn}(x)$.

The surface charge (which is odd in x) is expanded as

$$\sigma = \sum_{n=1}^{\infty} \sigma_n \left\{ \Pi_n \left(x/\Delta \right) - \Pi_n \left(-x/\Delta \right) \right\} \tag{35}$$

where $\Pi_n (t) = 1$ for $n - 1 < t < n$ and vanishes otherwise. It is noted that σ_n includes charges from both sides (in y) of the thin ground plane. If only one side is gridded the final ΔY_P correction is divided by 2. The basis function width is taken to be the uniform value Δ. The patch capacitance per unit length is thus

$$C_{\rho_0}^{P} = \frac{\Delta}{V} \sum_{n=1}^{N} \sigma_n \tag{36}$$

here $N\Delta = \rho_0$. The potential is thus

$$\phi (x, 0) = -\frac{1}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \sigma_n \int_{(n-1)\Delta}^{n\Delta} \left[\ln \left| x - x' \right| - \ln \left| x + x' \right| \right] dx'$$

$$= -\frac{\Delta}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \sigma_n \int_{n-1}^{n} \left[\ln \left| t - t' \right| - \ln \left| t + t' \right| \right] dt' \tag{37}$$

Using

$$\int \ln \left| t' \pm t \right| dt' = \left(t' \pm t \right) \left[\ln \left| t' \pm t \right| - 1 \right] \tag{38}$$

and setting to the location of the quadrature point $t = m - 1/2$, $m \geq 1$ gives

$$-\frac{\Delta}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \sigma_n \left[(n - m + 1/2) \left\{ \ln |n - m + 1/2| - 1 \right\} - (n + m - 1/2) \left\{ \ln |n + m - 1/2| - 1 \right\} \right.$$

$$\left. - (n - 1/2 - m) \left\{ \ln |n - 1/2 - m| - 1 \right\} + (n - 3/2 + m) \left\{ \ln |n - 3/2 + m| - 1 \right\} \right] = V/2 \tag{39}$$

or

$$-\sum_{n=1}^{\infty} S_n \left[(n-m+1/2)\left\{\ln|n-m+1/2|-1\right\} - (n+m-1/2)\left\{\ln|n+m-1/2|-1\right\} \right.$$

$$\left. -(n-1/2-m)\left\{\ln|n-1/2-m|-1\right\} + (n-3/2+m)\left\{\ln|n-3/2+m|-1\right\}\right] = \pi^2/2 \tag{40}$$

where

$$S_n = \frac{\pi\Delta\sigma_n}{2\epsilon_0 V} \sim \frac{1}{n-1/2},\ n \to \infty \tag{41}$$

$$C_{\rho_0}^{P} = \frac{2}{\pi}\epsilon_0 \sum_{n=1}^{N} S_n \tag{42}$$

The capacitance per unit length

$$C_{\rho_0}^{ng} = \frac{2}{\pi}\epsilon_0 \int_{\Delta}^{N\Delta} \frac{dx}{x} = \frac{2}{\pi}\epsilon_0 \ln(N) \tag{43}$$

The average charge density on the nth interval which goes with this capacitance is

$$\sigma_n^{ng} = \frac{2}{\pi\Delta}\epsilon_0 V \int_{(n-1)\Delta}^{n\Delta} \frac{dx}{x} = \frac{2}{\pi\Delta}\epsilon_0 V \ln\left(\frac{n}{n-1}\right) \tag{44}$$

or

$$C_{\rho_0}^{ng} = \frac{\Delta}{V}\sum_{n=2}^{N} \sigma_n^{ng} = \frac{2}{\pi}\epsilon_0 \sum_{n=2}^{N} S_n^{ng} \tag{45}$$

$$S_n^{ng} = \ln\left(\frac{n}{n-1}\right) \tag{46}$$

Subtracting S_n^{ng} from S_n and adding the resulting term to the right hand side of eq. 40 as well gives

$$-\sum_{n=1}^{\infty} \Delta S_n \left[(n-m+1/2)\left\{\ln|n-m+1/2|-1\right\}-(n+m-1/2)\left\{\ln|n+m-1/2|-1\right\}\right.$$

$$\left.-(n-1/2-m)\left\{\ln|n-1/2-m|-1\right\}+(n-3/2+m)\left\{\ln|n-3/2+m|-1\right\}\right]=\pi^2/2$$

$$+\sum_{n=2}^{\infty} \ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2)\left\{\ln|n-m+1/2|-1\right\}-(n+m-1/2)\left\{\ln|n+m-1/2|-1\right\}\right.$$

$$\left.-(n-1/2-m)\left\{\ln|n-1/2-m|-1\right\}+(n-3/2+m)\left\{\ln|n-3/2+m|-1\right\}\right] \qquad (47)$$

where $\Delta S_n = S_n - S_n^{ng}$, for $n \geq 2$ and $\Delta S_1 = S_1$. If we now truncate the left hand side at $n = N$ we find

$$-\sum_{n=1}^{N} \Delta S_n \left[(n-m+1/2)\left\{\ln|n-m+1/2|-1\right\}-(n+m-1/2)\left\{\ln|n+m-1/2|-1\right\}\right.$$

$$\left.-(n-1/2-m)\left\{\ln|n-1/2-m|-1\right\}+(n-3/2+m)\left\{\ln|n-3/2+m|-1\right\}\right]=\pi^2/2$$

$$+\sum_{n=2}^{\infty} \ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2)\left\{\ln|n-m+1/2|-1\right\}-(n+m-1/2)\left\{\ln|n+m-1/2|-1\right\}\right.$$

$$\left.-(n-1/2-m)\left\{\ln|n-1/2-m|-1\right\}+(n-3/2+m)\left\{\ln|n-3/2+m|-1\right\}\right] \qquad (48)$$

To accurately sum the right hand side, the remainder will be applied for large $n \geq n'$. Thus letting

$$R_{n'} = \sum_{n=n'}^{\infty} \ln\left(\frac{n}{n-1}\right) \left[(n-m+1/2)\left\{\ln|n-m+1/2| - 1\right\} - (n+m-1/2)\left\{\ln|n+m-1/2| - 1\right\} \right.$$

$$\left. - (n-1/2-m)\left\{\ln|n-1/2-m| - 1\right\} + (n-3/2+m)\left\{\ln|n-3/2+m| - 1\right\} \right] \qquad (49)$$

To approximate $R_{n'}$ the Taylor series approximations eqs. (50-59) will be used.

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + O\left(x^4\right) \qquad (50)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O\left(x^4\right) \qquad (51)$$

$$\ln\left(\frac{n}{n-1}\right) = -\ln\left(1-\frac{1}{n}\right) \sim \left\{\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{4n^4}\right\} \qquad (52)$$

$$\ln(n-m+1/2) - 1 \sim \ln n - 1 - \frac{m-1/2}{n} - \frac{(m-1/2)^2}{2n^2} - \frac{(m-1/2)^3}{3n^3} \qquad (53)$$

$$\ln(n+m-1/2) - 1 \sim \ln n - 1 + \frac{m-1/2}{n} - \frac{(m-1/2)^2}{2n^2} + \frac{(m-1/2)^3}{3n^3} \qquad (54)$$

$$\ln\left(1+\frac{m-1/2}{n}\right) \sim \frac{m-1/2}{n} - \frac{(m-1/2)^2}{2n^2} + \frac{(m-1/2)^3}{3n^3} \qquad (55)$$

$$\ln(n-1/2-m) - 1 \sim \ln n - 1 - \frac{m+1/2}{n} - \frac{(m+1/2)^2}{2n^2} - \frac{(m+1/2)^3}{3n^3} \qquad (56)$$

$$\ln\left(1-\frac{m+1/2}{n}\right) \sim -\frac{m+1/2}{n} - \frac{(m+1/2)^2}{2n^2} - \frac{(m+1/2)^3}{3n^3} \qquad (57)$$

$$\ln(n-3/2+m) - 1 \sim \ln n - 1 + \frac{m-3/2}{n} - \frac{(m-3/2)^2}{2n^2} + \frac{(m-3/2)^3}{3n^3} \qquad (58)$$

$$\ln\left(1+\frac{m-3/2}{n}\right) \sim \frac{m-3/2}{n} - \frac{(m-3/2)^2}{2n^2} + \frac{(m-3/2)^3}{3n^3} \qquad (59)$$

$$R_{n'} = \sum_{n=n'}^{\infty} \ln\left(\frac{n}{n-1}\right) \left[(n-m+1/2)\left\{\ln|n-m+1/2|-1\right\} - (n+m-1/2)\left\{\ln|n+m-1/2|-1\right\}\right.$$

$$\left. - (n-1/2-m)\left\{\ln|n-m-1/2|-1\right\} + (n-3/2+m)\left\{\ln|n+m-3/2|-1\right\}\right] \tag{60}$$

$$R_{n'} \sim \sum_{n=n'}^{\infty} \left[1+\frac{1}{2n}+\frac{1}{3n^2}+\frac{1}{4n^3}\right] \left[\left(1-\frac{m-1/2}{n}\right)\left\{\ln n - 1 - \frac{m-1/2}{n} - \frac{(m-1/2)^2}{2n^2} - \frac{(m-1/2)^3}{3n^3}\right\}\right.$$

$$-\left(1+\frac{m-1/2}{n}\right)\left\{\ln n - 1 + \frac{m-1/2}{n} - \frac{(m-1/2)^2}{2n^2} + \frac{(m-1/2)^3}{3n^3}\right\}$$

$$-\left(1-\frac{m+1/2}{n}\right)\left\{\ln n - 1 - \frac{m+1/2}{n} - \frac{(m+1/2)^2}{2n^2} - \frac{(m+1/2)^3}{3n^3}\right\}$$

$$\left.+\left(1+\frac{m-3/2}{n}\right)\left\{\ln n - 1 + \frac{m-3/2}{n} - \frac{(m-3/2)^2}{2n^2} + \frac{(m-3/2)^3}{3n^3}\right\}\right] \tag{61}$$

Using the approximations 62-65

$$\left[1+\frac{1}{2n}+\frac{1}{3n^2}+\frac{1}{4n^3}\right]\left(1-\frac{m-1/2}{n}\right) \sim 1-\frac{m-1}{n}-\frac{m-7/6}{2n^2}-\frac{m-5/4}{3n^3} \tag{62}$$

$$\left[1+\frac{1}{2n}+\frac{1}{3n^2}+\frac{1}{4n^3}\right]\left(1+\frac{m-1/2}{n}\right) \sim 1+\frac{m}{n}+\frac{m+1/6}{2n^2}+\frac{m+\frac{1}{4}}{3n^3} \tag{63}$$

$$\left[1+\frac{1}{2n}+\frac{1}{3n^2}+\frac{1}{4n^3}\right]\left(1-\frac{m+1/2}{n}\right) \sim 1-\frac{m}{n}-\frac{m-1/6}{2n^2}-\frac{m-\frac{1}{4}}{3n^3} \tag{64}$$

$$\left[1+\frac{1}{2n}+\frac{1}{3n^2}+\frac{1}{4n^3}\right]\left(1+\frac{m-3/2}{n}\right) \sim 1+\frac{m-1}{n}+\frac{m-5/6}{2n^2}+\frac{m-3/4}{3n^3} \tag{65}$$

eq. 61 becomes

$$R_{n'} \sim \sum_{n=n'}^{\infty}\left[\left(1-\frac{m-1}{n}-\frac{m-7/6}{2n^2}-\frac{m-5/4}{3n^3}\right)\left\{\ln n-1-\frac{m-1/2}{n}-\frac{(m-1/2)^2}{2n^2}-\frac{(m-1/2)^3}{3n^3}\right\}\right.$$

$$-\left(1+\frac{m}{n}+\frac{m+1/6}{2n^2}+\frac{m+\frac{1}{4}}{3n^3}\right)\left\{\ln n-1+\frac{m-1/2}{n}-\frac{(m-1/2)^2}{2n^2}+\frac{(m-1/2)^3}{3n^3}\right\}$$

$$-\left(1-\frac{m}{n}-\frac{m-1/6}{2n^2}-\frac{m-\frac{1}{4}}{3n^3}\right)\left\{\ln n-1-\frac{m+1/2}{n}-\frac{(m+1/2)^2}{2n^2}-\frac{(m+1/2)^3}{3n^3}\right\}$$

$$\left.+\left(1+\frac{m-1}{n}+\frac{m-5/6}{2n^2}+\frac{m-3/4}{3n^3}\right)\left\{\ln n-1+\frac{m-3/2}{n}-\frac{(m-3/2)^2}{2n^2}+\frac{(m-3/2)^3}{3n^3}\right\}\right]$$

$$=\sum_{n=n'}^{\infty}\left[(1-2m)\frac{1}{n^2}+(1-2m)\frac{1}{n^3}\right] \tag{66}$$

$$R_{n'} \sim (1-2m)\sum_{n=n'}^{\infty}\frac{1}{n^2}+(1-2m)\sum_{n=n'}^{\infty}\frac{1}{n^3}=$$

$$(1-2m)\left[\frac{\pi^2}{6}-\sum_{n=1}^{n=n'-1}\frac{1}{n^2}\right]+(1-2m)\left[1.2020569032-\sum_{n=1}^{n'-1}\frac{1}{n^3}\right] \tag{67}$$

$$-\sum_{n=1}^{N}\Delta S_n\left[(n-m+1/2)\{\ln|n-m+1/2|-1\}-(n+m-1/2)\{\ln|n+m-1/2|-1\}\right.$$

$$\left.-(n-1/2-m)\{\ln|n-1/2-m|-1\}+(n-3/2+m)\{\ln|n-3/2+m|-1\}\right]=\pi^2/2$$

$$+\sum_{n=2}^{n'-1}\ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2)\{\ln|n-m+1/2|-1\}-(n+m-1/2)\{\ln|n+m-1/2|-1\}\right.$$

$$-(n-1/2-m)\{\ln|n-1/2-m|-1\}+(n-3/2+m)\{\ln|n-3/2+m|-1\}]+R_{n'} \quad (68)$$

Solving eq. 68 numerically for the difference coefficients with $N = 100$ and $n' = 2000$, gives $\Delta S_1 \approx 2.44637$, $\Delta S_2 \approx -0.12842$, $\Delta S_3 \approx -0.01168$, $\Delta S_4 \approx -0.00591$, $\Delta S_5 \approx -0.00269$, $\Delta S_6 \approx -0.00149$, $\Delta S_7 \approx -0.00090$, $\Delta S_8 \approx -0.00059$, $\Delta S_9 \approx -0.00041$, $\Delta S_{10} \approx -0.00029$.

$$\sum_{n=1}^{\infty} \Delta S_n \approx 2.2927 \quad (69)$$

$$\Delta C_{\rho_0}^P = C_{\rho_0}^P - C_{\rho_0}^{ng} \approx \frac{2}{\pi}\epsilon_0 \sum_{n=1}^{\infty} \Delta S_n \quad (70)$$

Using eqs. 6 and 43, we can write

$$C_{\rho_0}^0 - C_{\rho_0}^{ng} = \frac{2}{\pi}\epsilon_0 \ln\left(\Delta/a_0\right) \quad (71)$$

or

$$C_{\rho_0}^0 - C_{\rho_0}^P = C_{\rho_0}^0 - C_{\rho_0}^{ng} - \left(C_{\rho_0}^P - C_{\rho_0}^{ng}\right) \approx \frac{2}{\pi}\epsilon_0 \left(\ln\left(\Delta/a_0\right) - 2.2927\right) \approx \frac{2}{\pi}\epsilon_0 \ln\left(\frac{\Delta/9.902}{a_0}\right) \quad (72)$$

It is emphasized that eq. 72 is the correction to the thin-slot transmission line model out to radius ρ_0 [1] when the surfaces on both slides of the slot are gridded using the same rectangular basis functions and one test point at the center of the basis function.

V. Capacitance Correction for Four Point Testing with a Uniform Grid of Rectangular Basis Functions

Equations 35-38 remain valid for using a four point test on a uniform grid of rectangular basis functions. For this analysis we consider the variation along the length of the slot constant and only consider variations perpendicular to the length of the slot. Using eq. 38 and setting t to its value at the quadrature point $t = m - 1/2 + d$, where [5]

$$d = \pm \frac{1}{2\sqrt{3}} \tag{73}$$

and $m \geq 1$ yields

$$\sum_{n=1}^{\infty} \sigma_n \int_{n-1}^{n} \left[\ln \left| t - t' \right| - \ln \left| t + t' \right| \right] dt' =$$

$$\sum_{n=1}^{\infty} \sigma_n \left[(n - m + 1/2 - d) \left\{ \ln |n - m + 1/2 - d| - 1 \right\} - (n + m - 1/2 + d) \left\{ \ln |n + m - 1/2 + d| - 1 \right\} \right.$$

$$\left. - (n - m - 1/2 - d) \left\{ \ln |n - m - 1/2 - d| - 1 \right\} + (n + m - 3/2 + d) \left\{ \ln |n + m - 3/2 + d| - 1 \right\} \right] \tag{74}$$

Note that in the Galerkin scheme the results from the two different d values have equal weights so their results as well as the right hand sides ($V/2$ and $V/2$) may be added to obtain

$$\phi \left(t\Delta, 0 \right) =$$

$$-\frac{\Delta}{2\pi\epsilon_0} \sum_{n=1}^{\infty} \sigma_n \left[(n - m + 1/2 - d) \left\{ \ln |n - m + 1/2 - d| - 1 \right\} - (n + m - 1/2 + d) \left\{ \ln |n + m - 1/2 + d| - 1 \right\} \right.$$

$$- (n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\} + (n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}]$$

$$-\frac{\Delta}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\sigma_n\left[(n-m+1/2+d)\left\{\ln|n-m+1/2+d|-1\right\} - (n+m-1/2-d)\left\{\ln|n+m-1/2-d|-1\right\}\right.$$

$$- (n-m-1/2+d)\left\{\ln|n-m-1/2+d|-1\right\} + (n+m-3/2-d)\left\{\ln|n+m-3/2-d|-1\right\}]$$

$$= V \tag{75}$$

where d now denotes the positive value of d in eq. 73

$$-\frac{1}{2}\sum_{n=1}^{\infty}S_n\left[(n-m+1/2-d)\left\{\ln|n-m+1/2-d|-1\right\} - (n+m-1/2+d)\left\{\ln|n+m-1/2+d|-1\right\}\right.$$

$$- (n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\} + (n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}]$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}S_n\left[(n-m+1/2+d)\left\{\ln|n-m+1/2+d|-1\right\} - (n+m-1/2-d)\left\{\ln|n+m-1/2-d|-1\right\}\right.$$

$$- (n-m-1/2+d)\left\{\ln|n-m-1/2+d|-1\right\} + (n+m-3/2-d)\left\{\ln|n+m-3/2-d|-1\right\}]$$

$$= \pi^2/2 \tag{76}$$

where S_n and $C_{\rho 0}^{P}$ are defined by eqs. 41 and 42. The capacitance per unit length $C_{\rho 0}^{ng}$ is given by eqs. 43 and 45. The average charge density which goes with this capacitance is σ_n^{ng} (eq. 44). Thus subtracting $S_n^{ng}$from S_n and adding resulting term to the right hand side of matrix equation eq. 76 gives

$$-\frac{1}{2}\sum_{n=1}^{\infty}\Delta S_n\left[(n-m+1/2-d)\left\{\ln|n-m+1/2-d|-1\right\} - (n+m-1/2+d)\left\{\ln|n+m-1/2+d|-1\right\}\right.$$

$$-(n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\}+(n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}]$$

$$-\frac{1}{2}\sum_{n=1}^{\infty}\Delta S_n\left[(n-m+1/2+d)\left\{\ln|n-m+1/2+d|-1\right\}-(n+m-1/2-d)\left\{\ln|n+m-1/2-d|-1\right\}\right.$$

$$\left.-(n-m-1/2+d)\left\{\ln|n-m-1/2+d|-1\right\}+(n+m-3/2-d)\left\{\ln|n+m-3/2-d|-1\right\}\right]$$

$$=\pi^2/2+\frac{1}{2}\sum_{n=2}^{\infty}\ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2-d)\left\{\ln|n-m+1/2-d|-1\right\}\right.$$

$$-(n+m-1/2+d)\left\{\ln|n+m-1/2+d|-1\right\}$$

$$\left.-(n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\}+(n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}\right]$$

$$+\frac{1}{2}\sum_{n=2}^{\infty}\ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2+d)\left\{\ln|n-m+1/2+d|-1\right\}\right.$$

$$-(n+m-1/2-d)\left\{\ln|n+m-1/2-d|-1\right\}$$

$$\left.-(n-m-1/2+d)\left\{\ln|n-m-1/2+d|-1\right\}+(n+m-3/2-d)\left\{\ln|n+m-3/2-d|-1\right\}\right] \tag{77}$$

where $\Delta S_n = S_n - S_n^{ng}$, for $n \geq 2$ and $\Delta S_1 = S_1$. If we now truncate the left hand side at $n = N$ we find

$$-\frac{1}{2}\sum_{n=1}^{N}\Delta S_n\left[(n-m+1/2-d)\left\{\ln|n-m+1/2-d|-1\right\}-(n+m-1/2+d)\left\{\ln|n+m-1/2+d|-1\right\}\right.$$

$$-(n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\}+(n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}]$$

$$-\frac{1}{2}\sum_{n=1}^{N}\Delta S_n\left[(n-m+1/2+d)\left\{\ln|n-m+1/2+d|-1\right\}-(n+m-1/2-d)\left\{\ln|n+m-1/2-d|-1\right\}\right.$$

$$\left.-(n-m-1/2+d)\left\{\ln|n-m-1/2+d|-1\right\}+(n+m-3/2-d)\left\{\ln|n+m-3/2-d|-1\right\}\right]$$

$$=\pi^2/2+\frac{1}{2}\sum_{n=2}^{\infty}\ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2-d)\left\{\ln|n-m+1/2-d|-1\right\}\right.$$
$$-(n+m-1/2+d)\left\{\ln|n+m-1/2+d|-1\right\}$$

$$\left.-(n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\}+(n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}\right]$$

$$+\frac{1}{2}\sum_{n=2}^{\infty}\ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2+d)\left\{\ln|n-m+1/2+d|-1\right\}\right.$$
$$-(n+m-1/2-d)\left\{\ln|n+m-1/2-d|-1\right\}$$

$$\left.-(n-m-1/2+d)\left\{\ln|n-m-1/2+d|-1\right\}+(n+m-3/2-d)\left\{\ln|n+m-3/2-d|-1\right\}\right] \tag{78}$$

To accurately sum the right hand side, the remainder will be applied for large $n \geq n'$. Thus letting

$$R_{n'}(m+d)=+\frac{1}{2}\sum_{n=n'}^{\infty}\ln\left(\frac{n}{n-1}\right)\left[(n-m+1/2-d)\left\{\ln|n-m+1/2-d|-1\right\}\right.$$
$$+-(n+m-1/2+d)\left\{\ln|n+m-1/2+d|-1\right\}$$
$$\left.-(n-m-1/2-d)\left\{\ln|n-m-1/2-d|-1\right\}+(n+m-3/2+d)\left\{\ln|n+m-3/2+d|-1\right\}\right] \tag{79}$$

$$R_{n'}(m)=$$

$$+\frac{1}{2}\sum_{n=n'}^{\infty}\ln\left(\frac{n}{n-1}\right)[(n-m+1/2)\{\ln|n-m+1/2|-1\}-(n+m-1/2)\{\ln|n+m-1/2|-1\}$$

$$-(n-m-1/2)\{\ln|n-m-1/2|-1\}+(n+m-3/2)\{\ln|n+m-3/2|-1\}] \qquad (80)$$

which is the same as eq. 60. Thus $R_{n'}(m+d)$ is approximated by inserting $(m+d)$ into eq. 67.

$$-\frac{1}{2}\sum_{n=1}^{N}\Delta S_n\,[(n-m+1/2-d)\{\ln|n-m+1/2-d|-1\}-(n+m-1/2+d)\{\ln|n+m-1/2+d|-1\}$$

$$-(n-m-1/2-d)\{\ln|n-m-1/2-d|-1\}+(n+m-3/2+d)\{\ln|n+m-3/2+d|-1\}]$$

$$-\frac{1}{2}\sum_{n=1}^{N}\Delta S_n\,[(n-m+1/2+d)\{\ln|n-m+1/2+d|-1\}-(n+m-1/2-d)\{\ln|n+m-1/2-d|-1\}$$

$$-(n-m-1/2+d)\{\ln|n-m-1/2+d|-1\}+(n+m-3/2-d)\{\ln|n+m-3/2-d|-1\}]$$

$$=\pi^2/2+\frac{1}{2}\sum_{n=2}^{n'-1}\ln\left(\frac{n}{n-1}\right)[(n-m+1/2-d)\{\ln|n-m+1/2-d|-1\}$$

$$(-(n+m-1/2+d)\{\ln|n+m-1/2+d|-1\}$$

$$-(n-m-1/2-d)\{\ln|n-m-1/2-d|-1\}+(n+m-3/2+d)\{\ln|n+m-3/2+d|-1\}]$$

$$+\frac{1}{2}\sum_{n=2}^{n'-1}\ln\left(\frac{n}{n-1}\right)[(n-m+1/2+d)\{\ln|n-m+1/2+d|-1\}$$

$$-(n+m-1/2-d)\{\ln|n+m-1/2-d|-1\}-(n+m-1/2-d)\{\ln|n+m-1/2-d|-1\}$$

$$-(n-m-1/2+d)\{\ln|n-m-1/2+d|-1\}+(n+m-3/2-d)\{\ln|n+m-3/2-d|-1\}]$$

$$+R_{n'}(m+d)+R_{n'}(m-d) \qquad (81)$$

where $d = \frac{1}{2\sqrt{3}}$.

Solving eq. 81 numerically for the difference coefficients with $N = 100$ and $n' = 2000$, gives $\Delta S_1 \approx 0.29264073E + 01$, $\Delta S_2 \approx -0.29883585$, $\Delta S_3 \approx 0.94811390E - 02$, $\Delta S_4 \approx -0.11380113E - 01$, $\Delta S_5 \approx -0.30238274E - 02$, $\Delta S_6 \approx -0.20373822E - 02$, $\Delta S_7 \approx -0.11872146E - 02$, $\Delta S_8 \approx -0.78313399E - 03$, $\Delta S_9 \approx -0.53814088E - 03$, $\Delta S_{10} \approx -0.38625512E - 03$ where

$$\sum_{n=1}^{\infty} \Delta S_n \approx 2.6160479 \tag{82}$$

and following the same procedure for the one point quadrature gives

$$C^0_{\rho_0} - C^P_{\rho_0} = C^0_{\rho_0} - C^{ng}_{\rho_0} - \left(C^P_{\rho_0} - C^{ng}_{\rho_0}\right) \approx \frac{2}{\pi}\epsilon_0 \left(\ln\left(\Delta/a_0\right) - 2.6160479\right) \approx \frac{2}{\pi}\epsilon_0 \ln\left(\frac{\Delta/13.681546}{a_0}\right) \tag{83}$$

It is emphasized that eq. 83 is the correction to the thin-slot transmission line model out to radius ρ_0 [1] when the surfaces on both slides of the slot are gridded using the same rectangular basis functions and four point test.

VI. A Cavity Backed Aperture above a Slot in an Infinite Ground Plane

If the structure surrounding the slot is an infinite ground plane for $y = -\frac{d}{2}$, and cavity backed slot for $y \geq \frac{d}{2}$, one may need compute unknown currents on the surface as well as the slot equivalent magnetic current. In this case eqs. 33 and 34 may be re-written to account for the presence of the infinite ground plane

$$2H_z^{inc}(0,z) + H_z^{>}\left(a_0^-, z\right) + \widehat{n^+} \times \mathbf{J}^+\left(a_0^+, z\right)$$

$$-\frac{1}{2}\Delta Y_P \left(\frac{1}{k^2}\frac{d^2}{dz^2} + 1\right) I_m + \frac{1}{2}\left(\Delta Y_L \frac{d^2}{dz^2} I_m - \Delta Y_C I_m\right) = 0 \tag{84}$$

$$H_z^{>}\left(a_0^-, z\right) = \frac{i}{\omega\mu_0}\left(\frac{d^2}{dz^2} + k^2\right)\int_{-h}^{h}\frac{e^{ikR_{a_0}}}{4\pi R_{a_0}} I_m\left(z'\right) dz' \tag{85}$$

$$\Delta Y_P = -i\omega\left(C_{\rho_0}^0 - C_{\rho_0}^P\right) = \left\{\begin{array}{c} \frac{1}{\pi}\epsilon_0 \ln\left(\frac{\Delta/9.902}{a_0}\right) \text{ for 1 point testing on rectangles} \\ \frac{1}{\pi}\epsilon_0 \ln\left(\frac{\Delta/13.681546}{a_0}\right) \text{ for 4 point testing on rectangles} \end{array}\right\} \tag{86}$$

$$\Delta Y_L \approx \frac{1}{-i\omega}\frac{1}{\mu_0}\frac{d}{w} + Y^{int} \tag{87}$$

$$Y^{int} = \frac{2Z_s/d}{i\omega\mu_0\frac{w}{d}\left(\frac{2Z_s}{d} - i\omega\mu_0\frac{w}{d}\right)} \tag{88}$$

$$\Delta Y_C \approx -i\omega\varepsilon_0\left(\varepsilon_g + \frac{\sigma_g}{i\omega\varepsilon_0}\right)\frac{d}{w} \tag{89}$$

where the capacitive correction only includes the charge on the $y = \frac{d}{2}$ side since it is the only side that is gridded.

VII. Conclusion

The grid corrections ΔY_P are useful when $\frac{d}{w}$ is moderate in size but requires rectangular gridding in the vicinity of the slot, as $\frac{d}{w}$becomes large the capacitance of the slot interior dominates and this correction has little effect and is no longer needed.

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