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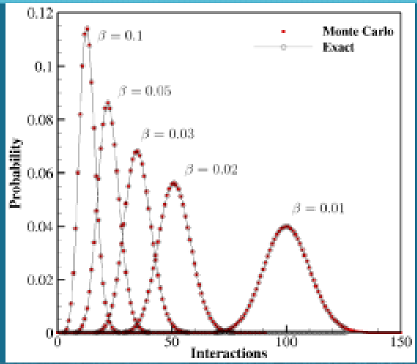
Probability Distribution Functions of the Number of Scattering Collisions in Electron Slowing Down



ITS

THE UNIVERSITY OF NEW MEXICO

Brian C. Franke (Sandia National Laboratories)

Anil K. Prinja (University of New Mexico)

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Outline

Derivation of Analytical Probability Distributions

- Exponential energy-loss model

Numerical Probability Distributions

- Exponential energy-loss model
- Physics models
- Physics-informed exponential energy-loss models

Possible approaches for analytical models that better represent the physics

Backward Master Equation for Collision-Number Probability

We begin with a backward master equation for the probability that a particle will experience n collisions in slowing down from energy E to below energy E_u.

$$P_n(E) = \widetilde{f}_s(E)\,\delta_{n,1} + \int_{E_u}^{E} dE'\, f_s(E \to E') P_{n-1}(E') \qquad 0 < E < \infty, \quad n = 0, 1, \cdots$$

with $P_n(E) = 0$ for $n \leq 0$.

The particle necessarily undergoes at least one interaction. It may undergo only one collision with the following probability:

$$\widetilde{f}_s(E) = \int_0^{E_u} dE'\, f_s(E \to E')$$

This energy-loss distribution is related to the electron-electron differential scattering cross section.

$$\Sigma_s(E \to E') = \Sigma_s(E)\, f_s(E \to E')$$

Discrete Transform to the Probability Generating Function

We use the discrete transform

$$G(z,E) = \sum_{0}^{\infty} z^n P_n(E) \qquad\qquad |z| \leq 1$$

to convert the infinite set of coupled integral equations into a single integral equation, the probability generating function:

$$G(z,E) = z\,\widetilde{f}_s(E) + z\int_{E_u}^{E} dE'\, f_s(E \to E')G(z,E)$$

Analytical solution with exponential energy-loss scattering

To obtain an analytical solution for the pdf, we model the peaked differential cross section as an exponential function

$$\Sigma_s(E \to E') = \frac{\Sigma_0}{\beta}\, e^{-\frac{(E-E')}{\beta}}, \quad 0 \le E' \le E$$

For E_u << E, the energy-loss distribution terms are given approximately, to exponentially small terms, by

$$f_s(E \to E') = \frac{1}{\beta}\, e^{\frac{-(E-E')}{\beta}}$$

$$\widetilde{f}_s(E) = e^{-\frac{(E-E_u)}{\beta}}$$

Collision number distribution is almost Poisson for an exponential energy-loss distribution.

Since *z* appears only parametrically in the probability generating function

$$G(z,E) = z\,\widetilde{f}_s(E) + z\int_{E_u}^{E} dE'\, f_s(E \to E')G(z,E)$$

this equation can be solved by converting it to a differential equation.

$$G(z,E) = z\left[\frac{1}{\widetilde{f}_s(E)}\right]^{z-1} = z\left(\frac{e^{\frac{E}{\beta}}-1}{e^{\frac{E_u}{\beta}}-1}\right)^{z-1}$$

$P_n(E)$ is recovered by expanding this solution for *G* in a series in *z* to obtain

$$P_n(E) = \frac{1}{\Gamma(n)}\left[\ln\frac{1}{\widetilde{f}_s(E)}\right]^{n-1} e^{-\ln\frac{1}{\widetilde{f}_s(E)}}, \quad n \geq 1$$

This is almost a Poisson distribution with parameter $\ln\frac{1}{\widetilde{f}_s(E)}$
but with index *n*-1 instead of *n*.

Summary of the Collision Number Distribution with Exponential Energy-Loss Scattering

Again, the solution is

$$P_n(E) = \frac{1}{\Gamma(n)} \left[\ln \frac{1}{\widetilde{f}_s(E)} \right]^{n-1} e^{-\ln \frac{1}{\widetilde{f}_s(E)}}, \quad n \geq 1$$

The mean and variance are

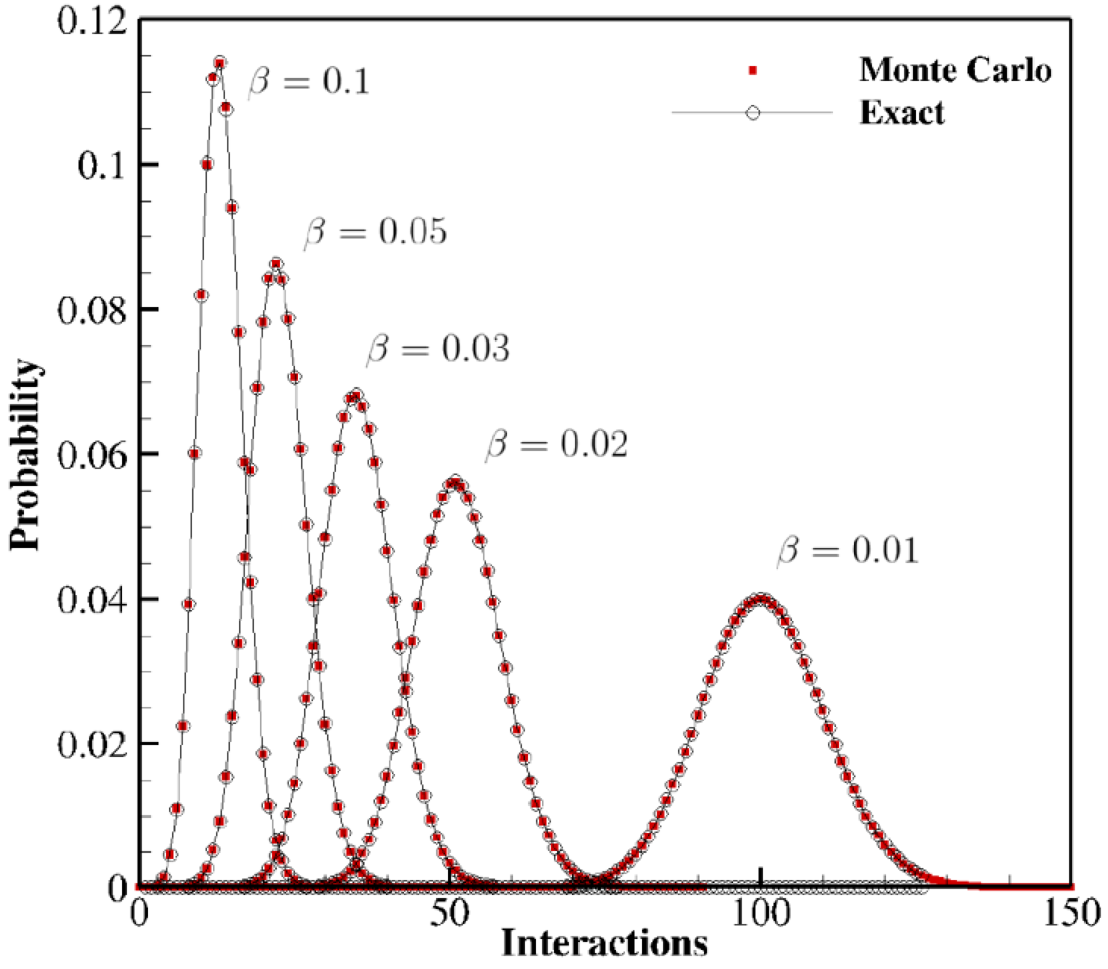
$$\overline{n}(E) = 1 + \ln \left[\frac{1}{\widetilde{f}_s(E)} \right] \qquad V(E) = \ln \left[\frac{1}{\widetilde{f}_s(E)} \right] \approx \overline{n}(E)$$

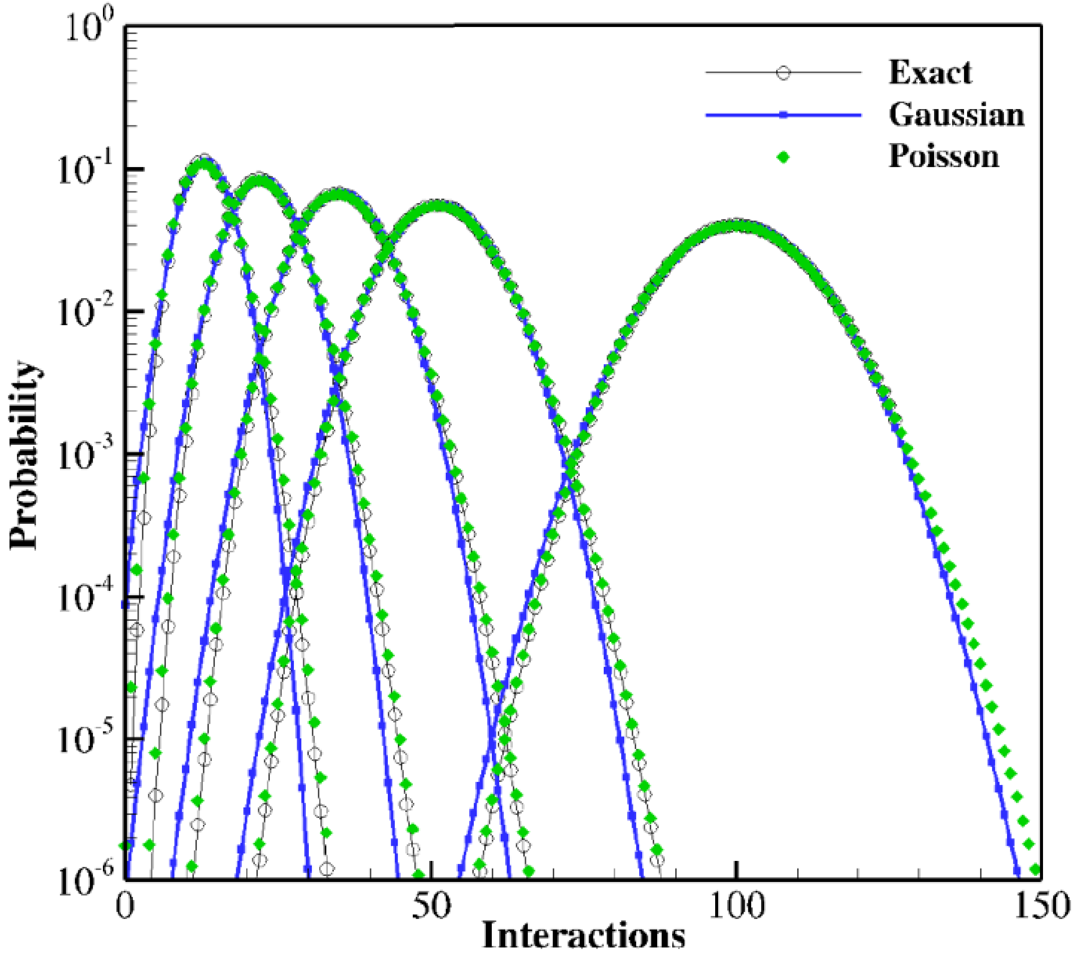
Neglecting exponentially small terms, the model parameters can be related to the stopping power, *S*, and energy-loss straggling coefficient, *T*.

$$\beta = \frac{T}{2S} \qquad f_s(E \to E') = \frac{1}{\beta} e^{\frac{-(E-E')}{\beta}}$$

$$\Sigma_0 = \frac{2S^2}{T} \qquad \widetilde{f}_s(E) = e^{-\frac{(E-E_u)}{\beta}}$$

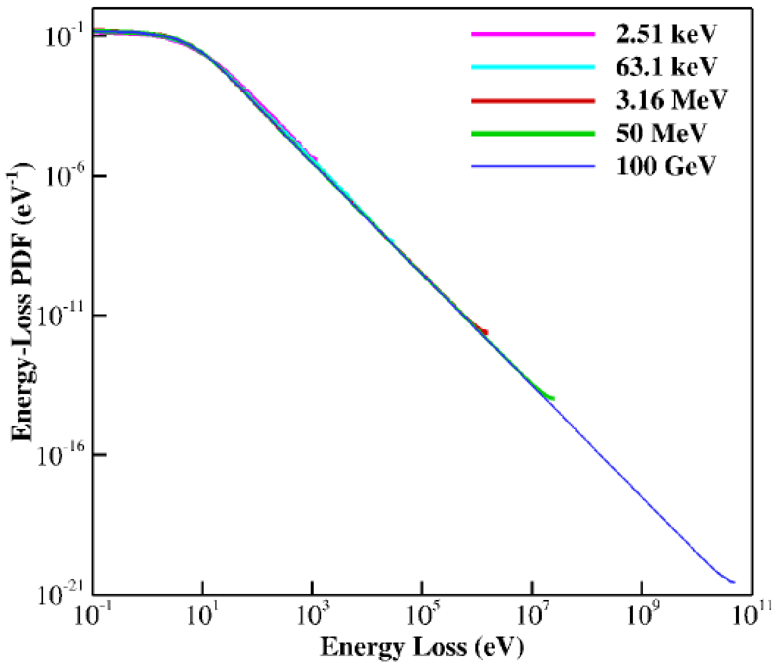
Numerical Results with Exponential Energy Loss



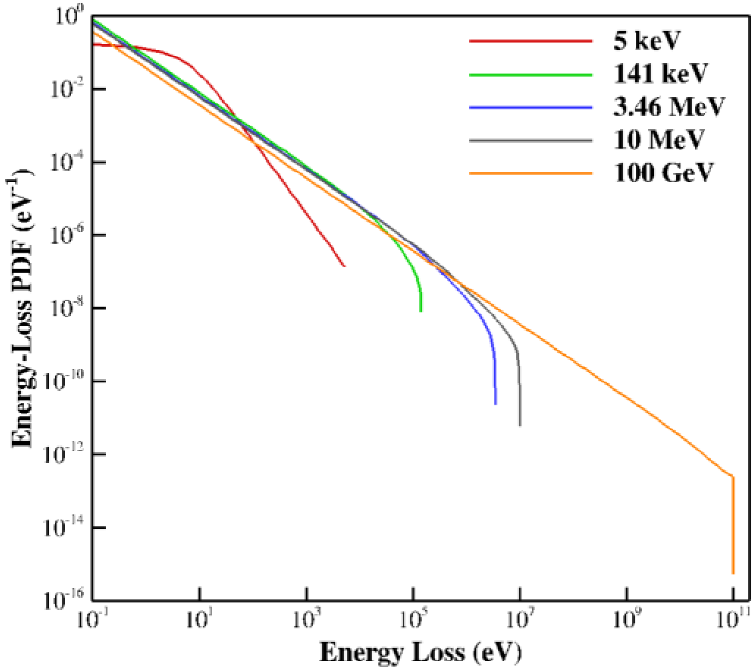


Energy-Loss Distributions

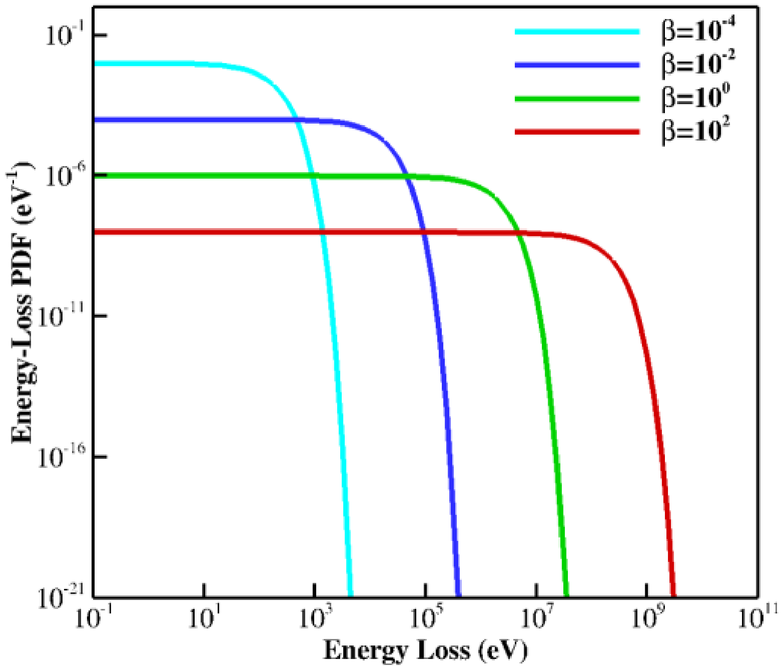
Electro-ionization Energy-Loss



Bremsstrahlung Energy-Loss

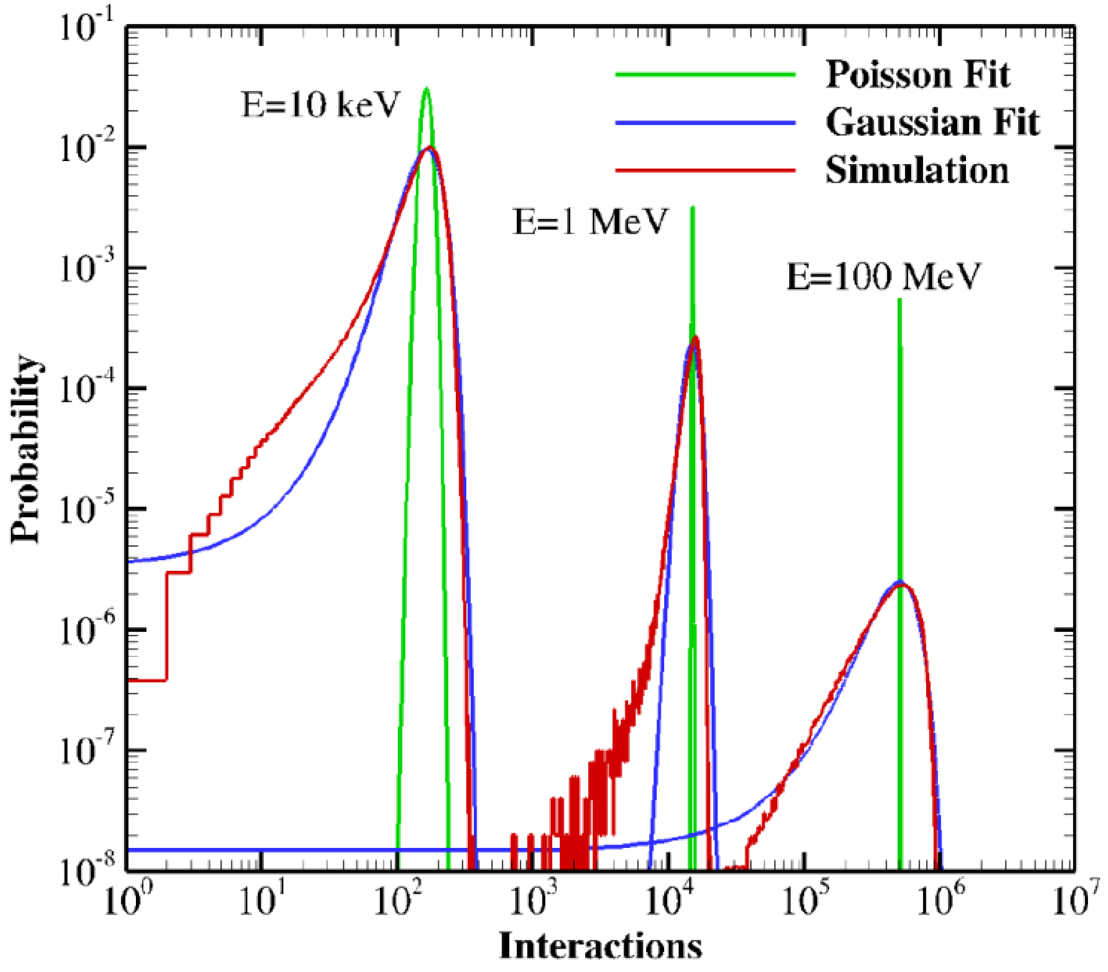


Exponential Energy-Loss

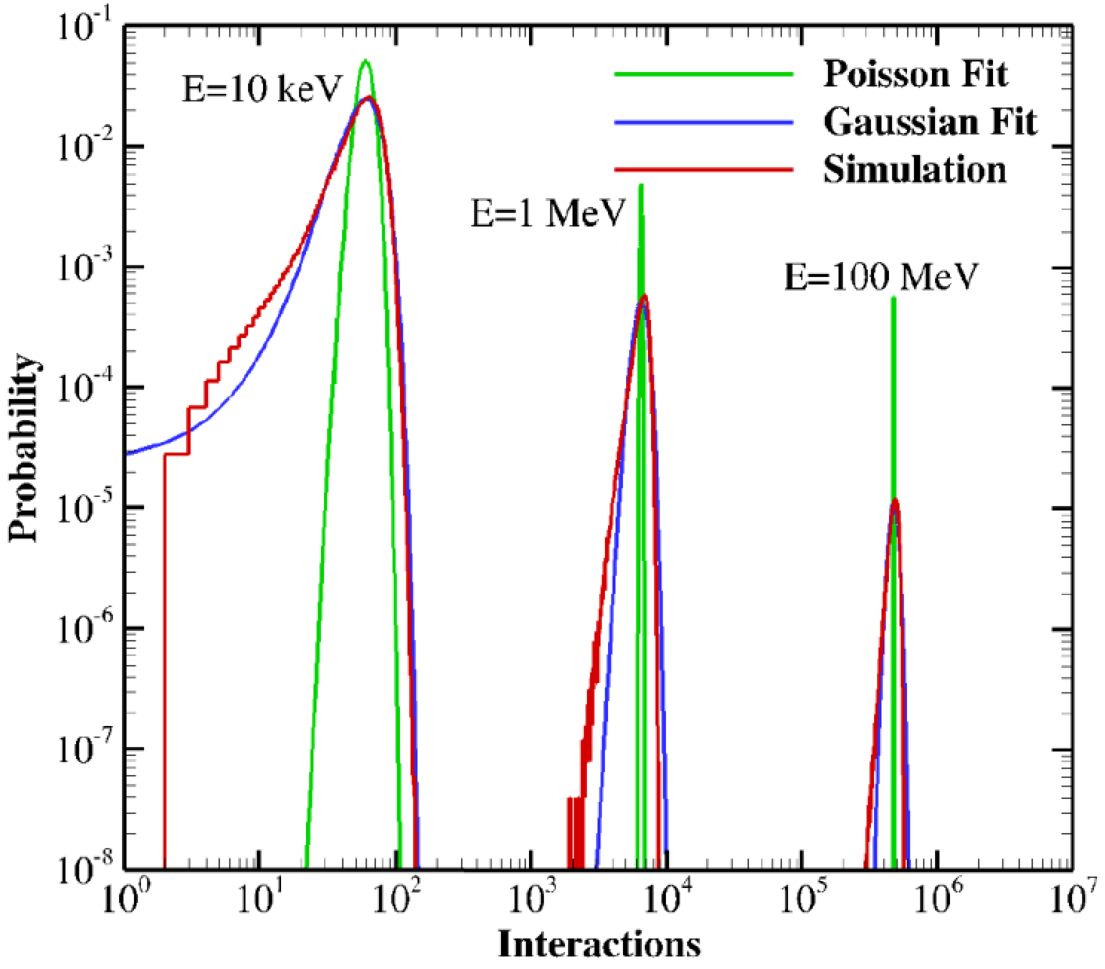


Numerical Results with EEDL Electron Energy-Loss Models

Full Electron Physics

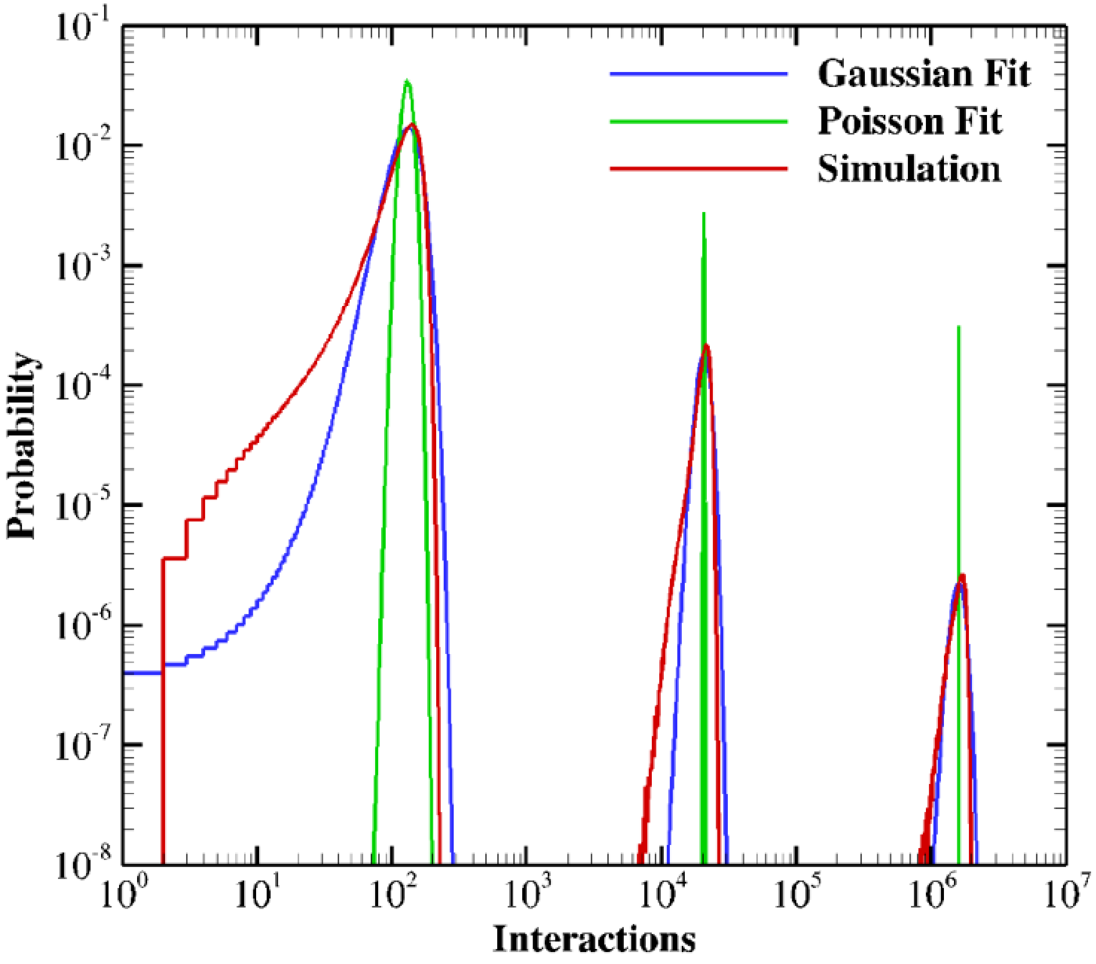


Electro-Ionization Interactions Only

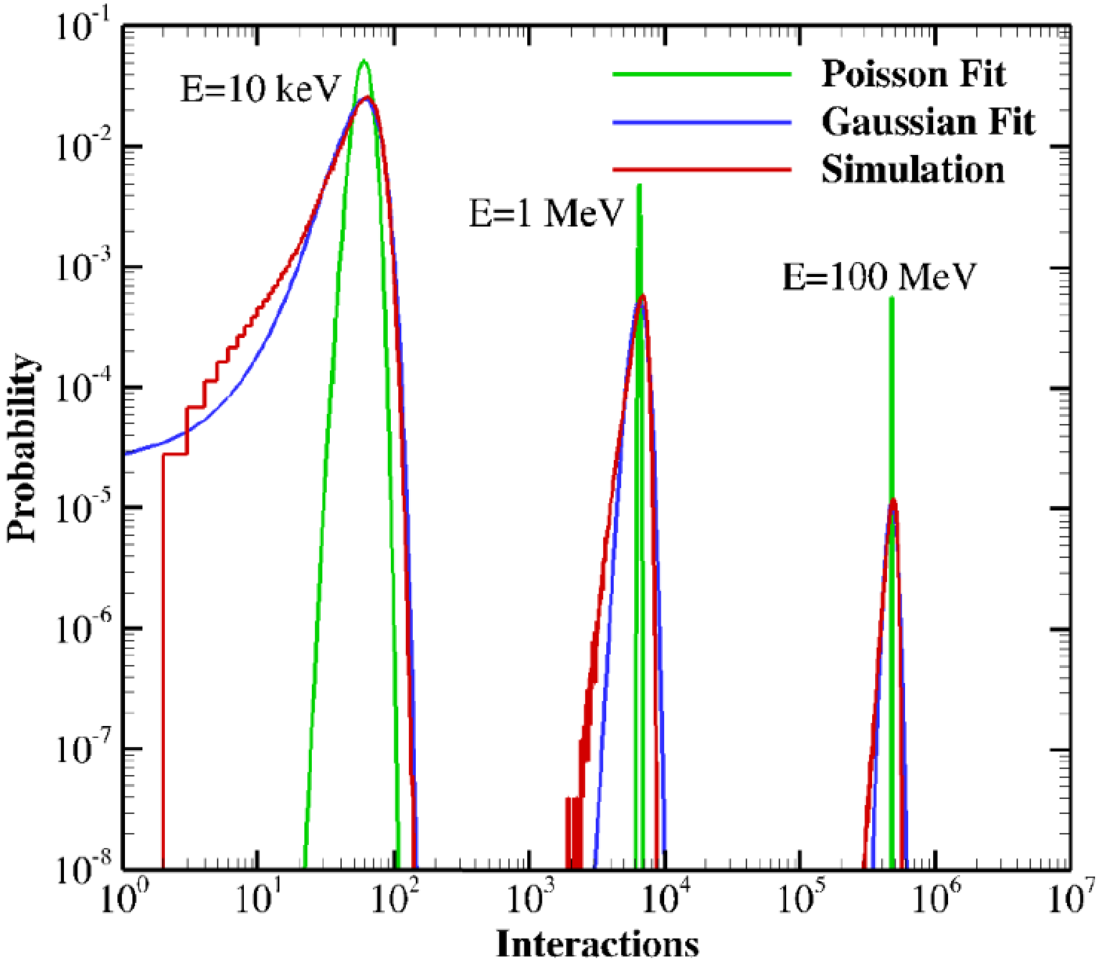


Numerical Results with EEDL Electron Energy-Loss Models

Hydrogen Background Material

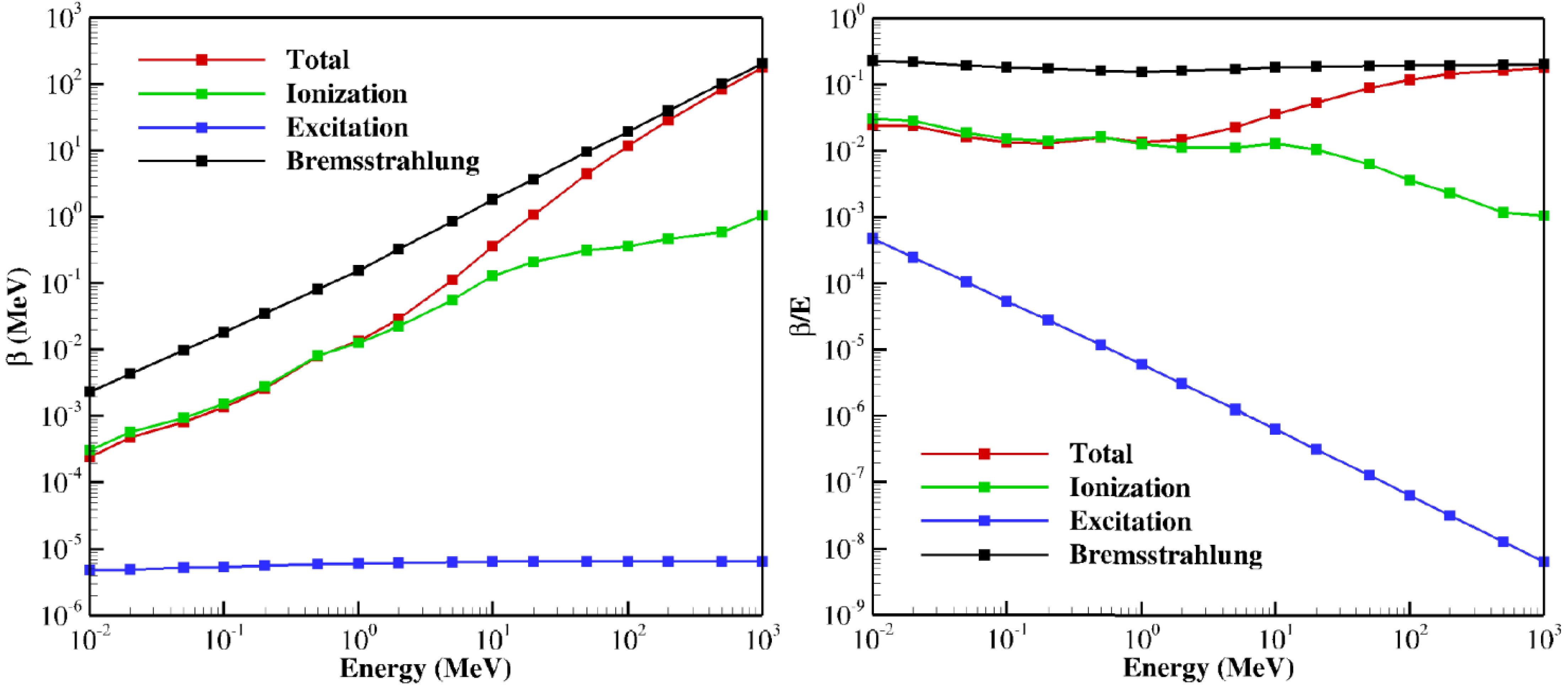


Aluminum Background Material



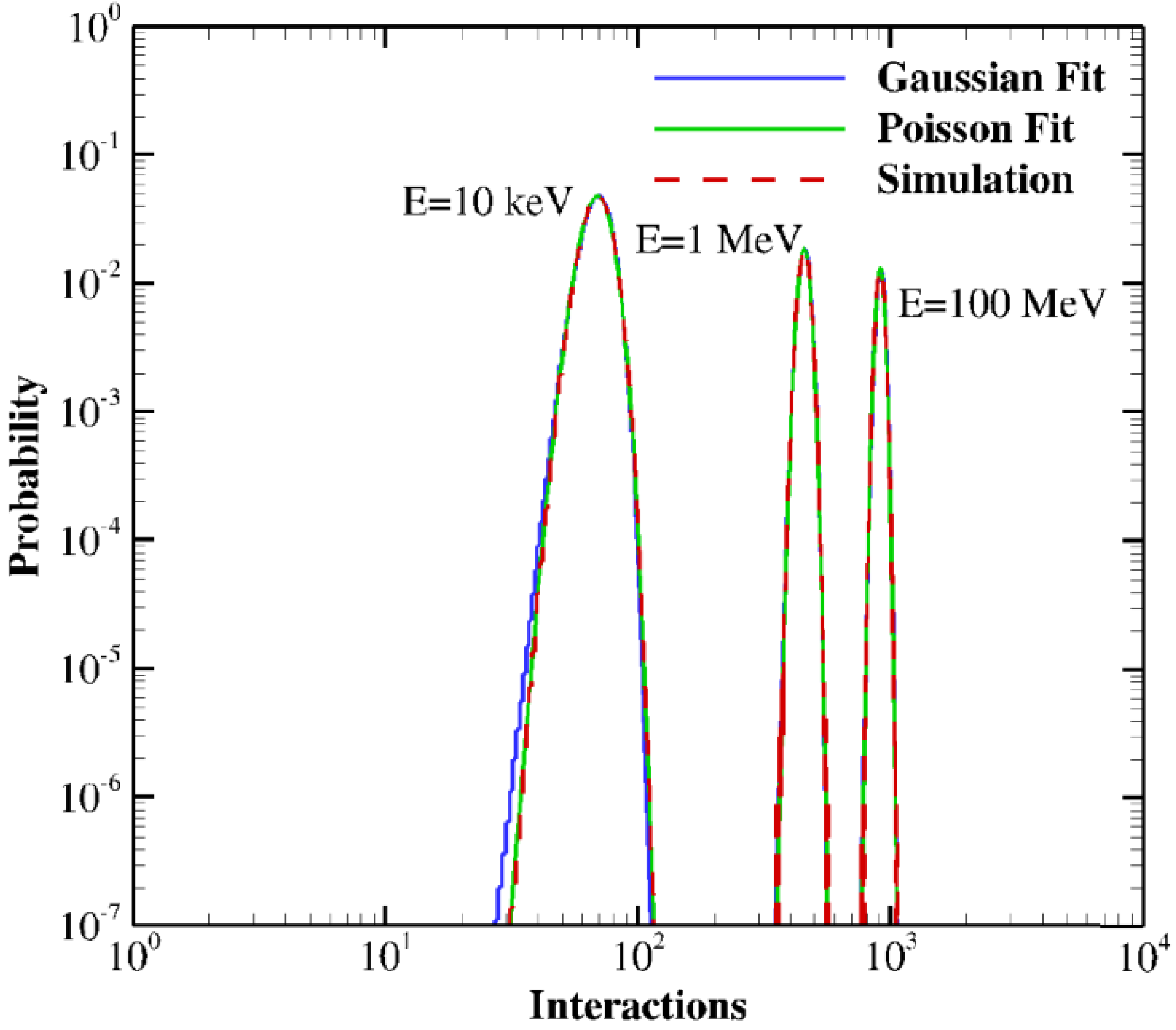
Energy-Loss Parameter from Physics Models

$$\beta = \frac{T}{2S}$$

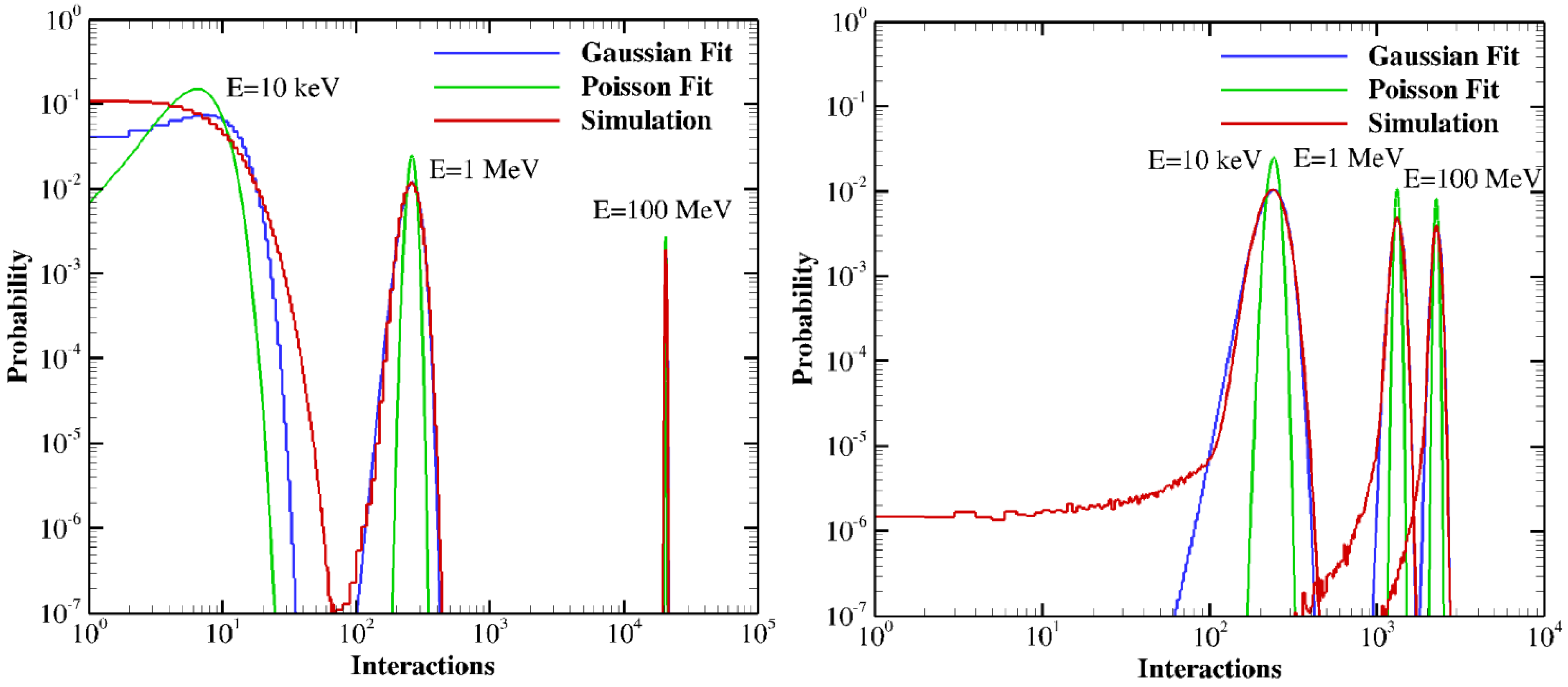


Exponential Energy Loss with Energy-Dependent Parameter

$$\beta = E/100$$

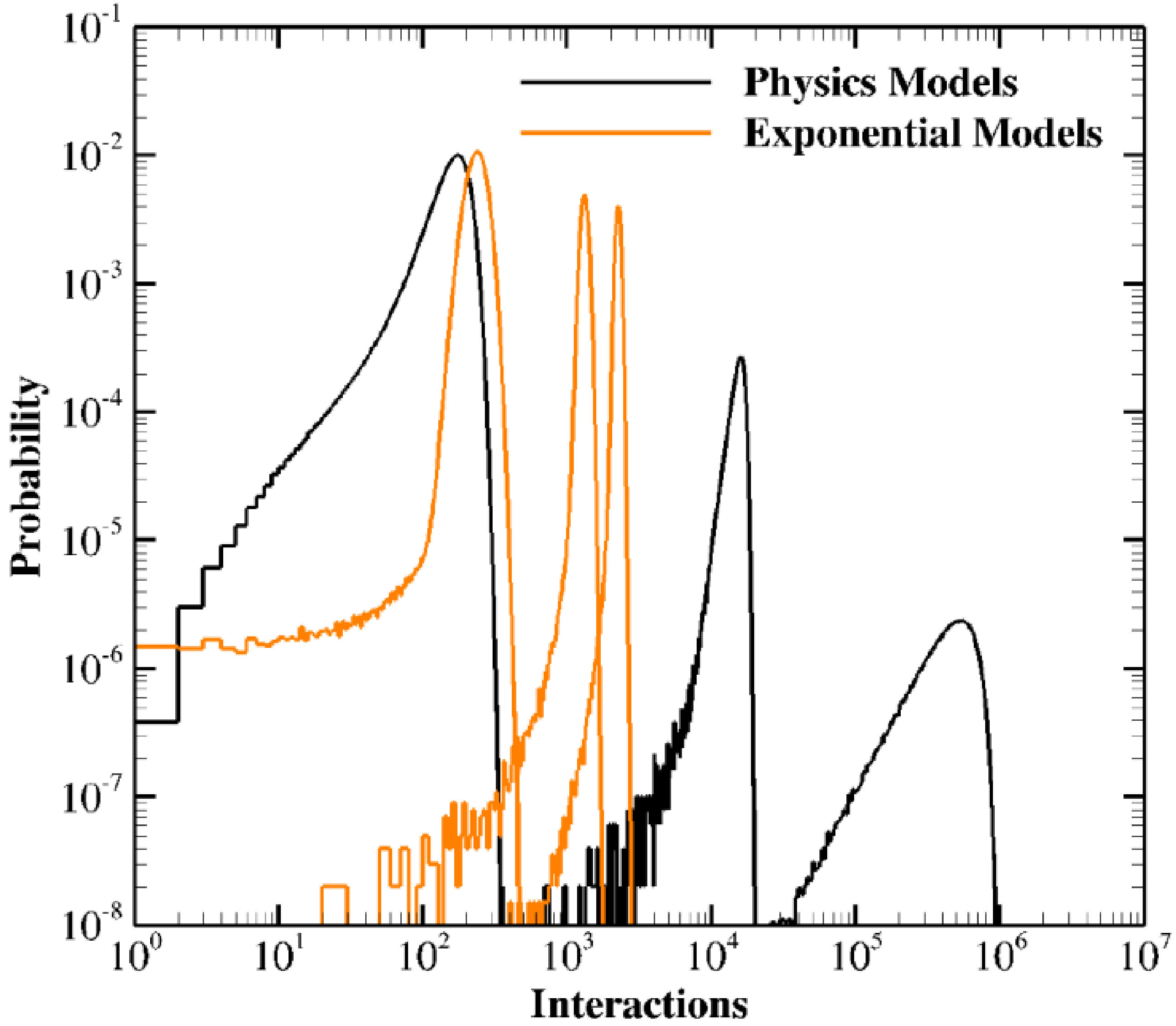


Exponential Energy Loss with Mixed Parameter Values



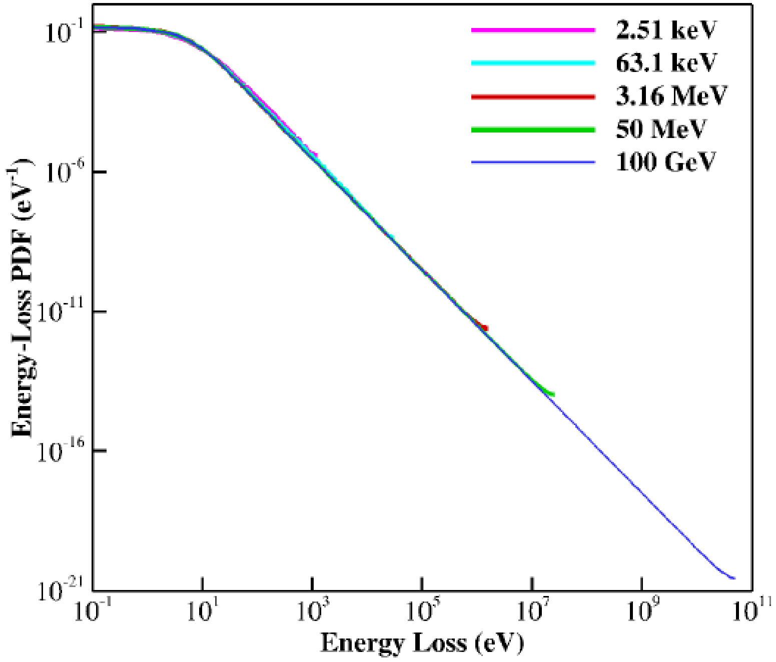
Electron ionization	$\beta = 0.01$	Electron ionization	$\beta = E/100$
Bremsstrahlung	$\beta = 0.1$	Bremsstrahlung	$\beta = E/5$
Excitation	$\beta = 0.00001$	Excitation	$\beta = E/10000$

Energy-Dependent Exponentials versus Physics Models

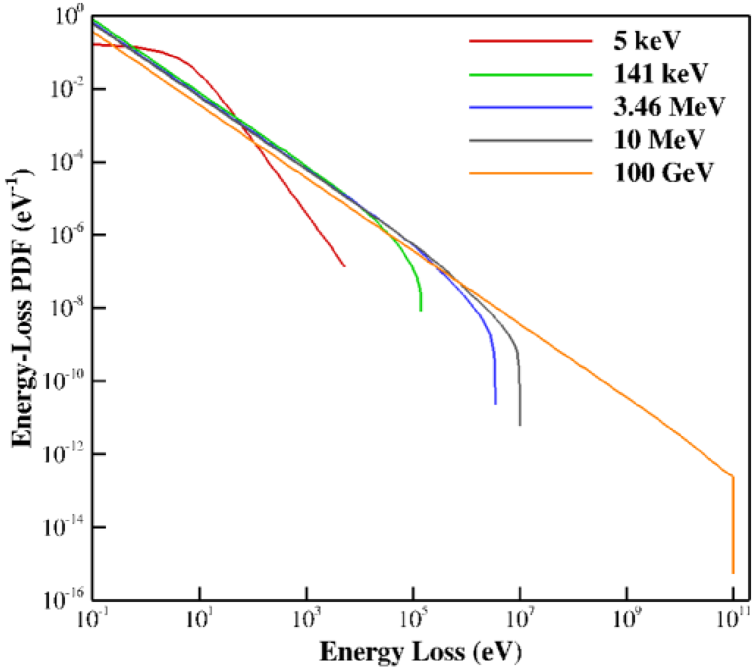


Energy-Loss Distributions

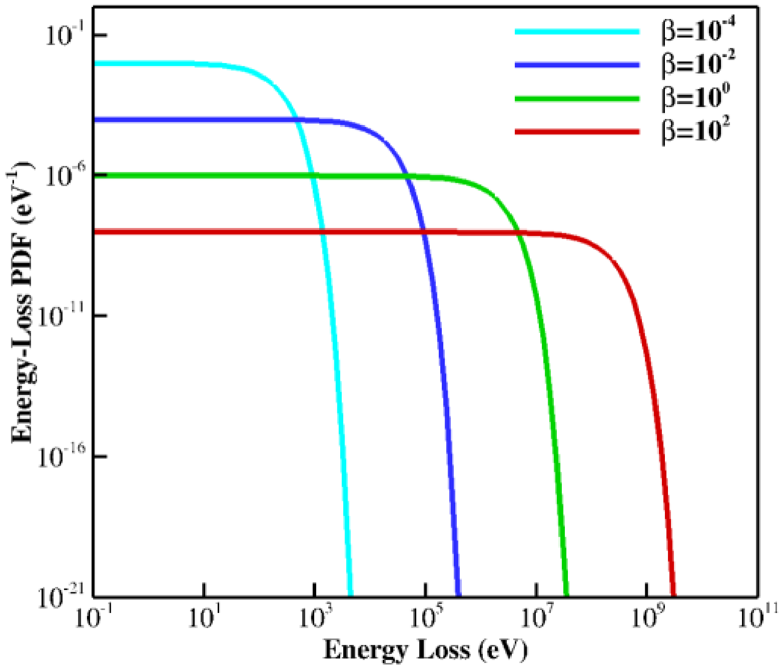
Electro-ionization Energy-Loss



Bremsstrahlung Energy-Loss



Exponential Energy-Loss

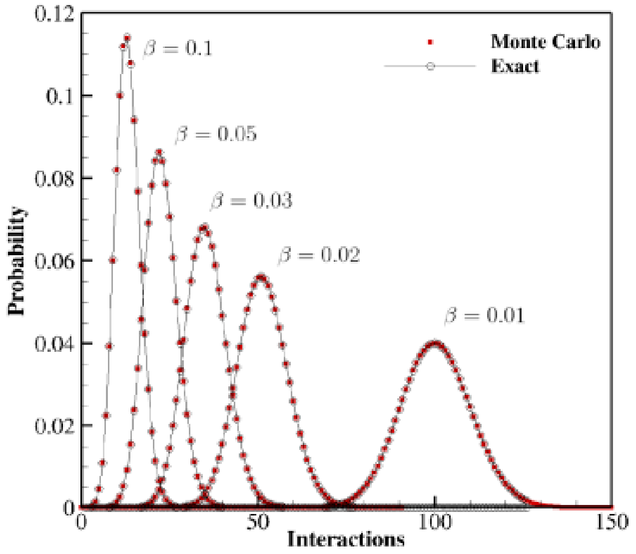


Conclusions

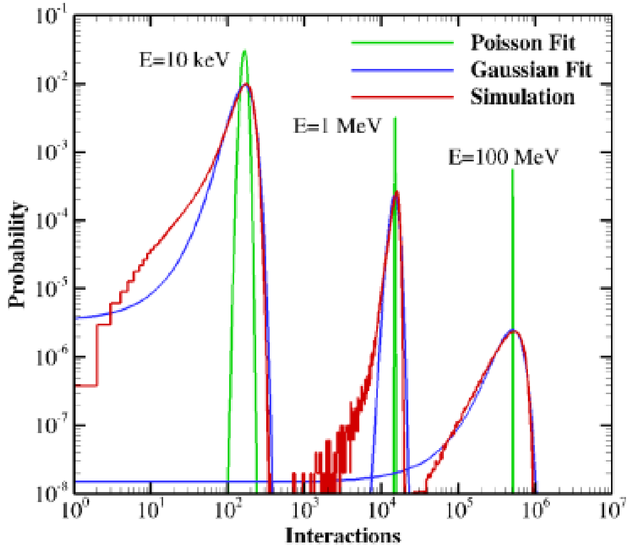
$$P_n(E) = \frac{1}{\Gamma(n)} \left[\ln \frac{1}{\widetilde{f}_s(E)} \right]^{n-1} e^{-\ln \frac{1}{\widetilde{f}_s(E)}}$$

Starting from a backward Master equation, the analytical solution for collision-number probability distribution was derived based on an exponential energy-loss kernel.

The analytical result was shown to agree with Monte Carlo simulation results.



Comparisons were made with simulation results obtained using electron scattering models.



An analytical solution with a power-law energy-loss term or multiple exponential energy-loss terms could be more representative of the true physics of electrons.