SAND2021-12151PE

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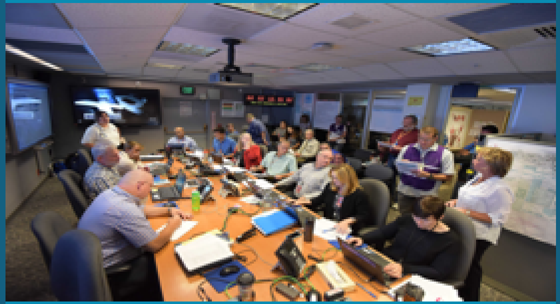
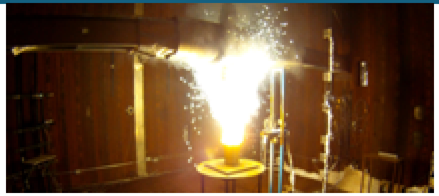
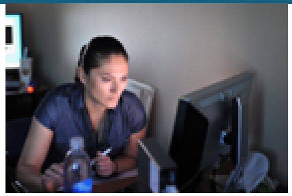
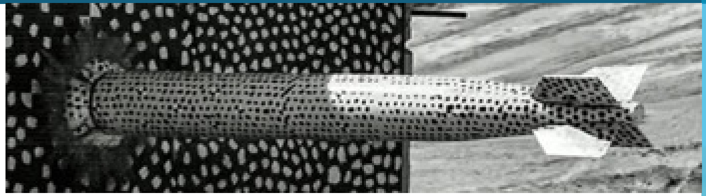
Algebraic Multigrid for mixed finite element discretizations of the Stokes Equations









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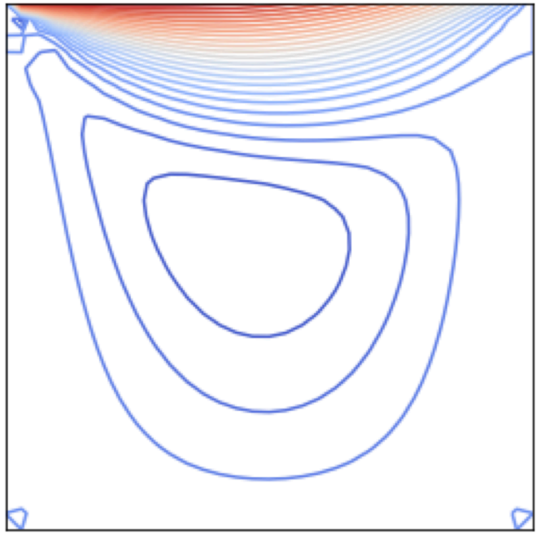
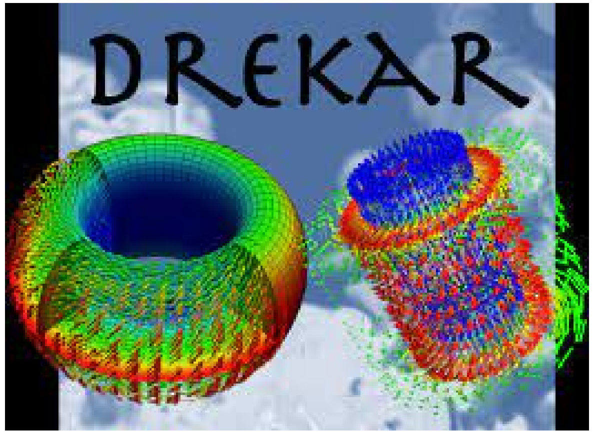
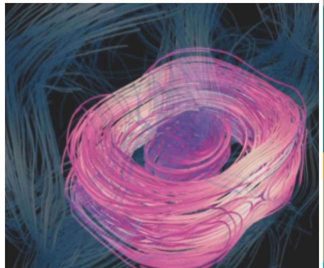
Outline

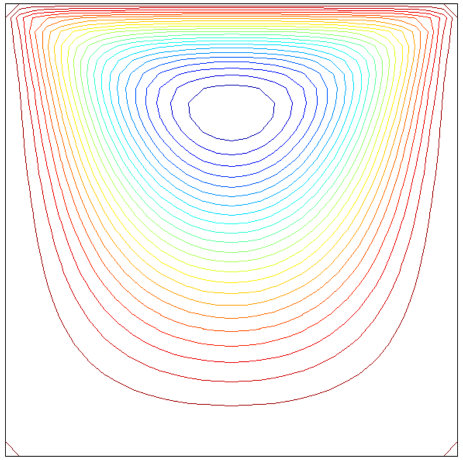
- Stokes Equations & mixed finite elements
  - Inf-sup stability
- AMG: scalar PDEs $\longrightarrow$ PDE systems
- AMG challenges for mixed FE discretizations
- Defect correction & auxiliary preconditioning methods
- A geometric MG preconditioner for $\mathbb{Q}_2/\mathbb{Q}_1$ Stokes
  - $\mathbb{Q}_1 \, iso \, \mathbb{Q}_2/\mathbb{Q}_1$ discretization
- AMG for $\mathbb{Q}_2/\mathbb{Q}_1$ QStokes
  - aggregation & inf-sup stability?
  - guarantees?
- Conclusions







Stokes Equations



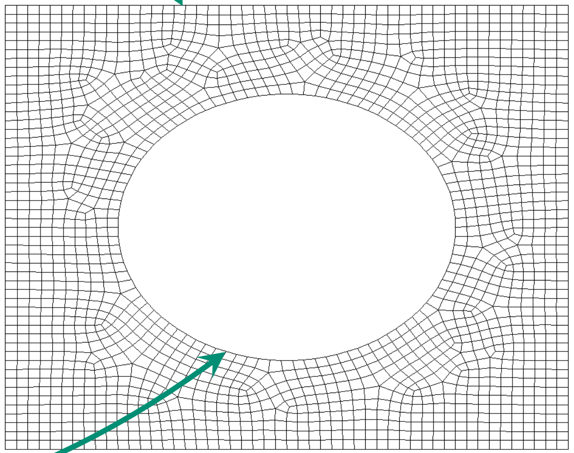
$$-\nabla^2 \vec{u} + \nabla p = \vec{f}$$
$$-\nabla \cdot \vec{u} = 0$$

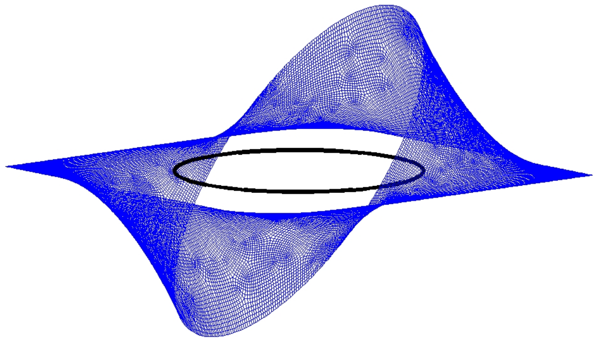
Simplification of Navier-Stokes when advective forces are negligible (or viscosities are large)

<u>Dirichlet BCs</u>

velocities = 0
on exterior boundary

tangent velocity of 1
on circle

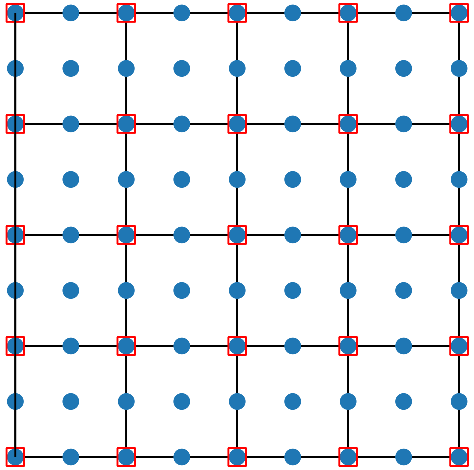




computed horizontal velocity

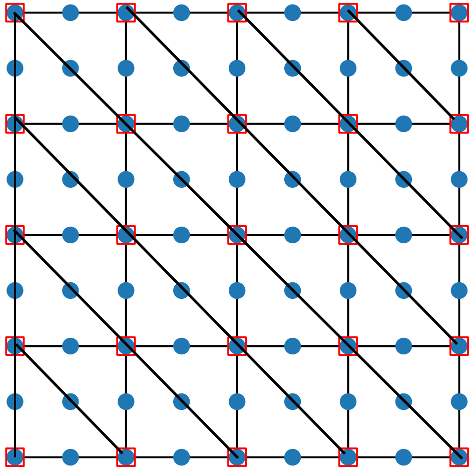
Stokes Equations & mixed finite elements

$$
\begin{aligned}
-\nabla^2 \vec{u} + \nabla p &= \vec{f} \\
-\nabla \cdot \vec{u} &= 0
\end{aligned}
\quad \xrightarrow[\text{using Taylor-Hood elements}]{\text{Stable Mixed FE Assembly}} \quad
Ax = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \vec{u} \\ p \end{bmatrix} = \begin{bmatrix} \vec{f} \\ 0 \end{bmatrix}
$$



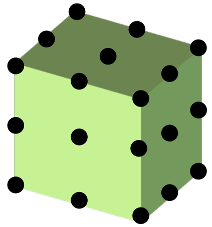
$\mathbb{Q}_2/\mathbb{Q}_1$

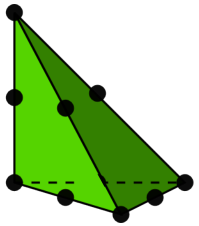
Bases
- Biquadratic for velocity
- Bilinear for pressure



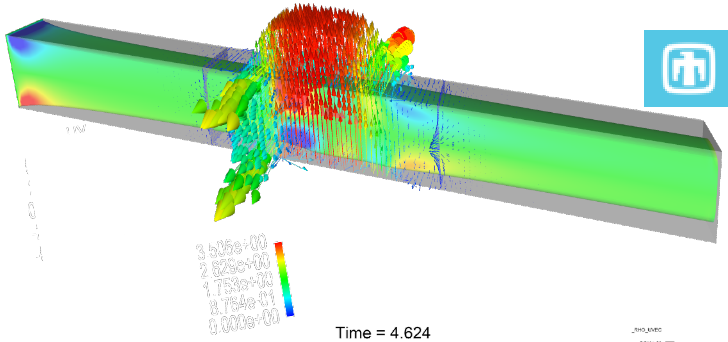
$\mathbb{P}_2/\mathbb{P}_1$

Bases
- quadratic for velocity
- linear for pressure

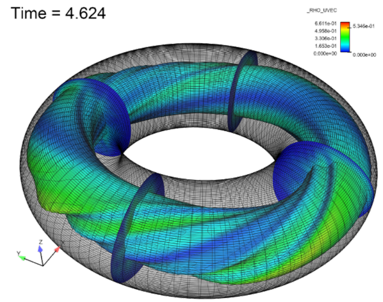




Sandia interest: MHD (magnetohydrodynamics)



- Fluids & electro-magnetics (Navier-Stokes & Maxwell)



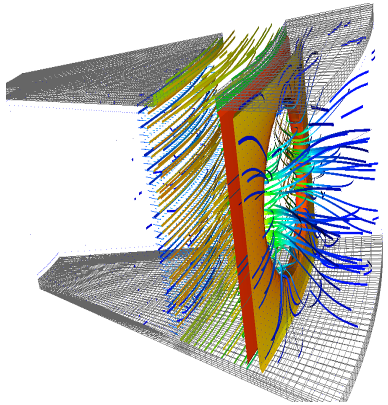
e.g.,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \nabla\cdot\nu\,\nabla\boldsymbol{u} + \nabla p + \nabla\cdot\left(-\frac{1}{\mu_0}\boldsymbol{B}\otimes\boldsymbol{B} + \frac{1}{2\mu_0}\|\,\boldsymbol{B}\,\|^2\boldsymbol{I}\right) = 0$$

$$\nabla\cdot\boldsymbol{u} = 0$$

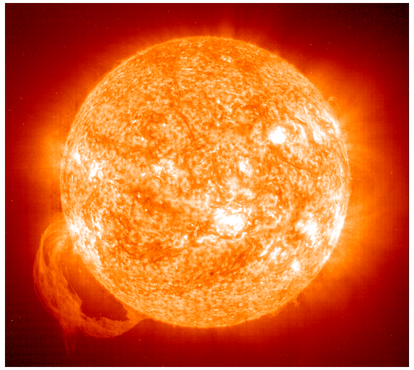
$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \nabla\times\frac{\eta}{\mu_0}\nabla\times\boldsymbol{B} + \nabla r = 0$$

$$\nabla\cdot\boldsymbol{B} = 0$$



$\boldsymbol{u}$: velocity, p: pressure,
μ : viscosity,
$\boldsymbol{B}$: magnetic induction,
η : resistivity,
μ_0: magnetic permeability of free space

$$\begin{bmatrix} J_u & B_u^T & Z & \\ B_u & S_p & & \\ Y & & J_I & B_b^T \\ & & B_b & S_r \end{bmatrix}\begin{bmatrix}\delta u\\ \delta p\\ \delta b\\ \delta r\end{bmatrix} = rhs$$



Sandia often uses standard equal order FEs, but is also interested in mixed FE.

Advantages of mixed finite elements

- when greater accuracy needed for some components
- preserve a property of discrete system (e.g., edge elements to represent curl operators)

- satisfy inf-sup or LBB (Ladyzhenskaya–Babuška–Brezzi) condition & avoid stability issues arising in some PDE systems

  $\Rightarrow$ unique solution to saddle point system that depends continuously on input without artifacts such as spurious oscillations

Unstable Example

$u_{xx} = f$ with periodic BCs

reformulated as 1st order system

$u_x = v$ &. $v_x = f$

& discretize with linear nodal FE

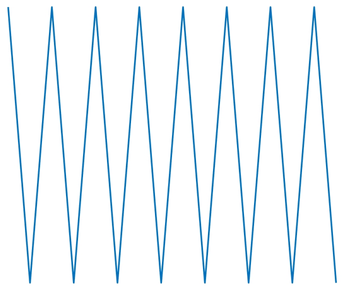
$$\begin{pmatrix} I & -B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} v_h \\ u_h \end{pmatrix} = \begin{pmatrix} 0 \\ f_h \end{pmatrix}$$

Schur complement is BB^T

stencil B is -1 0 1

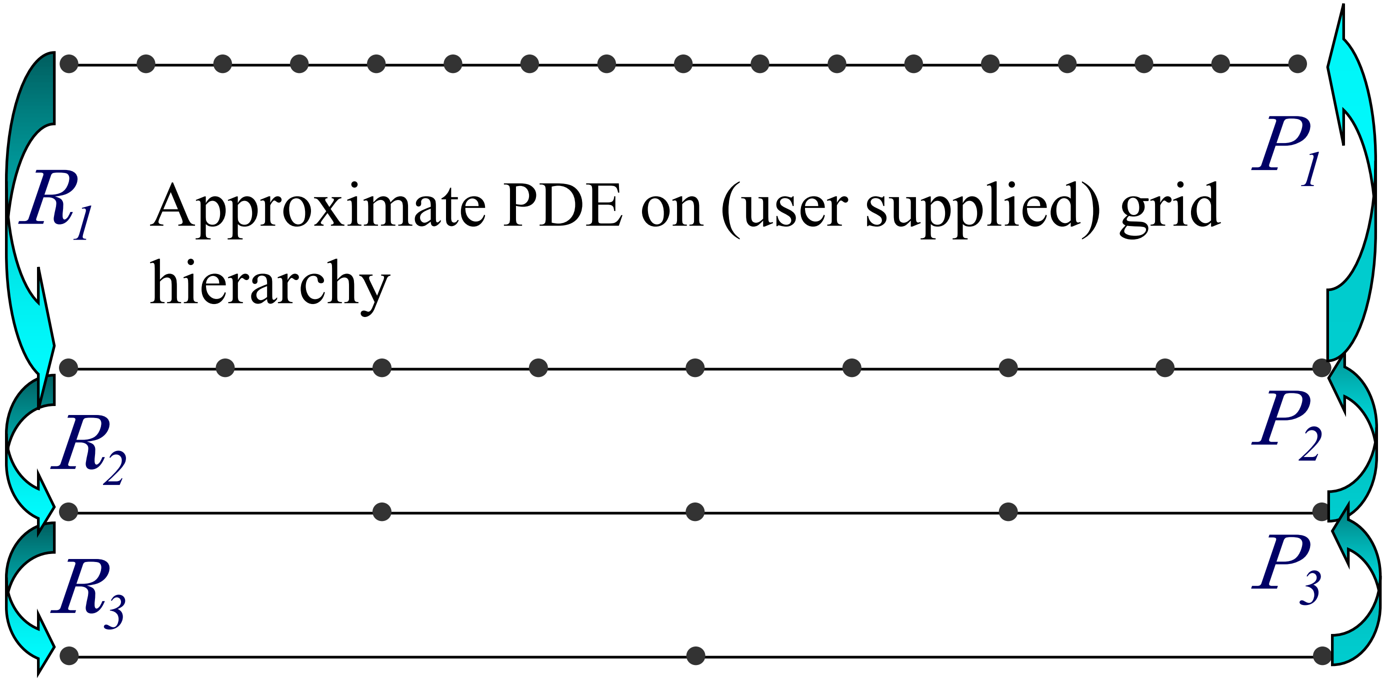
stencil BB^T is -1 0 2 0 -1

$BB^T q = 0$

where $q =$ 

$\Rightarrow$ unstable

Geometric Multigrid



Solve

$A_0\ u_0 = f_0$

$A_1\ u_1 = f_1$

$A_2\ u_2 = f_2$

$A_3\ u_3 = f_3$

Develop <u>relaxation methods</u> (approximate solve on a level)

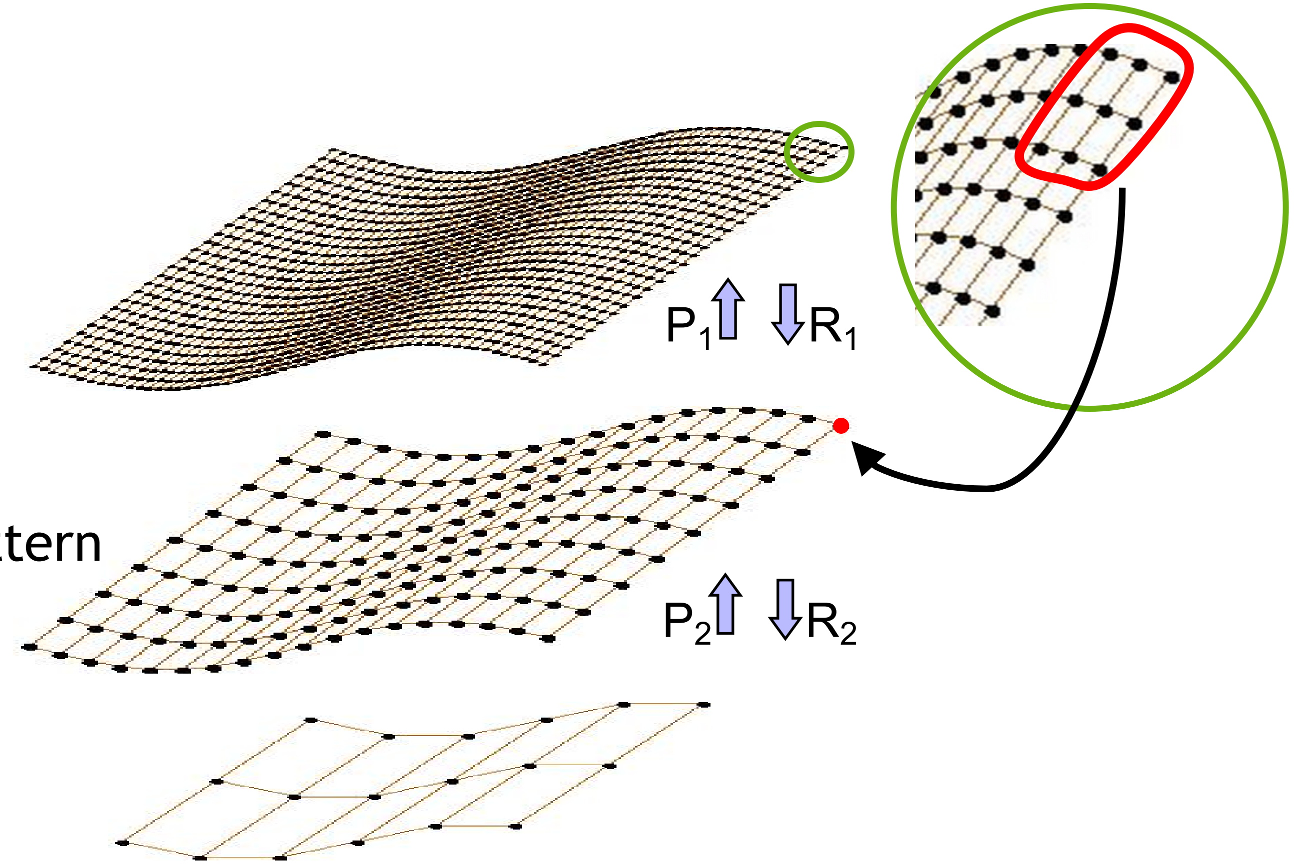
Jacobi, Gauss-Seidel, CG, etc.

Develop grid transfers (e.g. linear interpolation)

Use coarse A_k's to accelerate convergence for A_0

Algebraic Multigrid (AMG)

Solve $A_0\, u_0 = f_0$

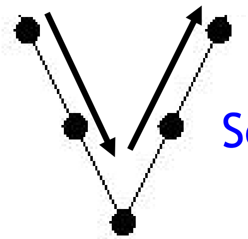


- Construct Graph & Coarsen
- Determine P_i & R_i sparsity pattern
- Determine P_i & R_i's coefs
- Project: $A_i = R_i\, A_{i-1}\, P_i$

Relax $A_0 u_0 = f_0$. Set $f_1 = R_1 r_0$

Relax $A_1 u_1 = f_1$. Set $f_2 = R_2 r_1$

Solve $A_2 u_2 = f_2$ directly.



Set $u_0 = u_0 + P_1 u_1$. Relax $A_0 u_0 = f_0$

Set $u_1 = u_1 + P_2 u_2$. Relax $A_1 u_1 = f_1$

Algebraic MG behavior

- High & low frequencies not available algebraically.

  These notions are replaced with $\| . \|_{A_k}$ or $\| . \|_{A_k^2}$

$$\| e_k \|_{A_k} \text{ or } \| e_k \|_{A_k^2} \text{ small} \Rightarrow \text{low frequency}$$

$$\| e_k \|_{A_k} \text{ or } \| e_k \|_{A_k^2} \text{ large} \Rightarrow \text{high frequency}$$

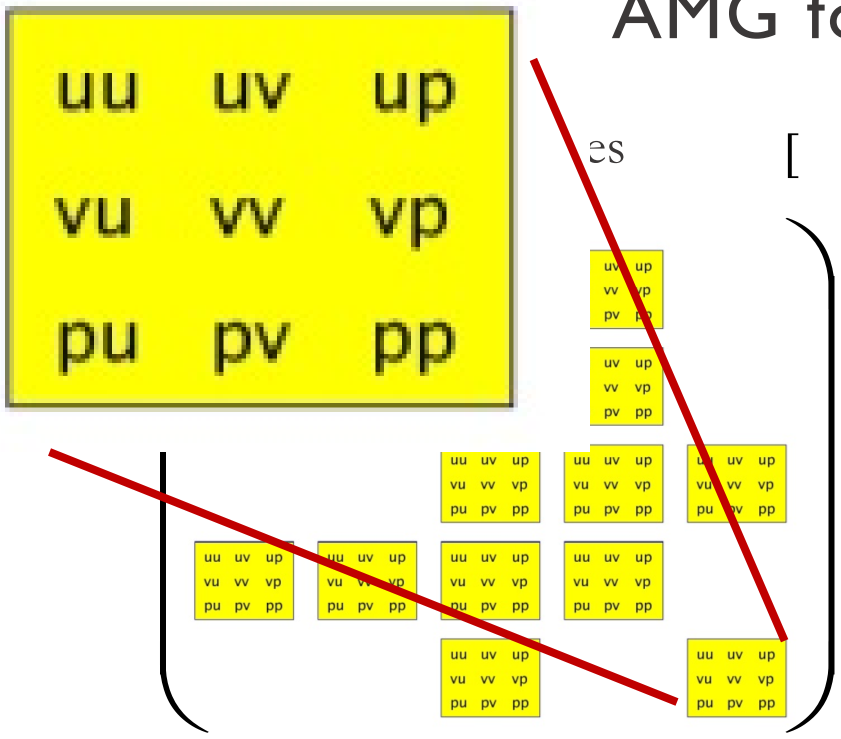
<u>Properties of AMG methods</u>

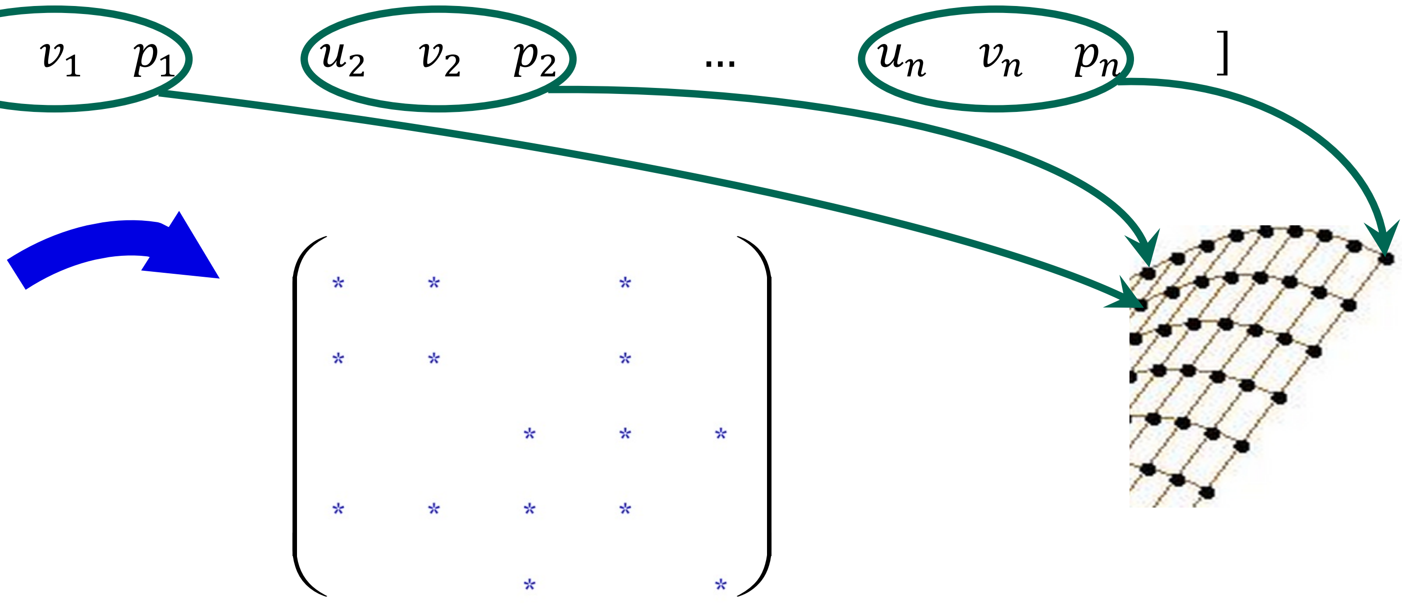
- relaxation smooths errors with high energy ($\| e_k \|_{A_k}$ large).
- P_k must accurately interpolate low energy errors (small $\| e_k \|_{A_k}$).
- P_k must interpolate errors not damped by smoothing.

Note:

$$\| v \|_A = \sqrt{v^T A v}$$

AMG for “co-located” PDE systems





- ➢ Most AMG software is setup in this block way & most PDE systems employing AMG are solved in this block way
  - ➢ Transform point matrix graph to block matrix graph
  - ➢ Coarsen block matrix graph
  - ➢ Determine grid transfer sparsity pattern based on block matrix graph
  - ➢ Grid transfer coefficients based on point matrix information

AMG challenges specific to mixed finite elements

- Cannot apply "co-located" approach to mixed FE system

- Inf-sup stability on coarse level A_i's ?

- Most AMG methods don't *like* *0*'s on matrix diagonal

  e.g., • smoothed aggregation AMG performs a $[\,diag(A_i)\,]^{-1} A_i$ step, which is not defined when $diag(A_i)$ has zeros

  • classical AMG theory based on M-matrices, which we are far from

- AMG methods *tend* to generate interpolation *resembling* linear basis functions

- AMG methods *tend* to more aggressively coarsen dense rows

  $\Rightarrow$ velocities coarsen more rapidly than pressure

Auxiliary matrix & defect correction

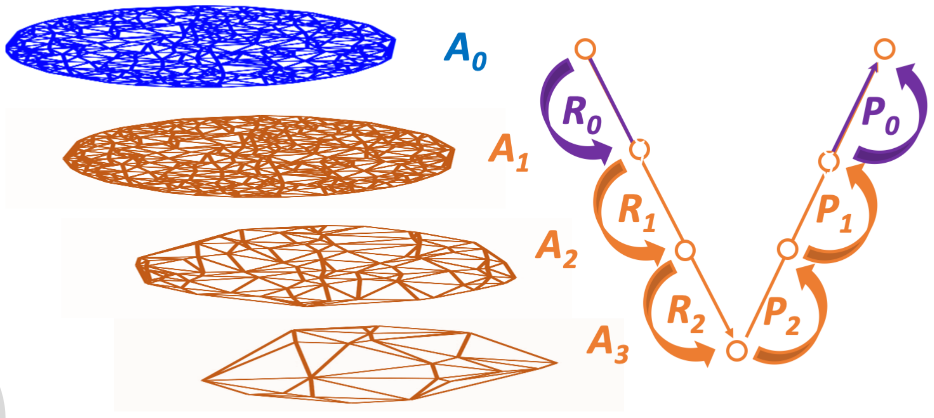
Since AMG prefers matrices based on low order discretizations, consider an auxiliary matrix approach …

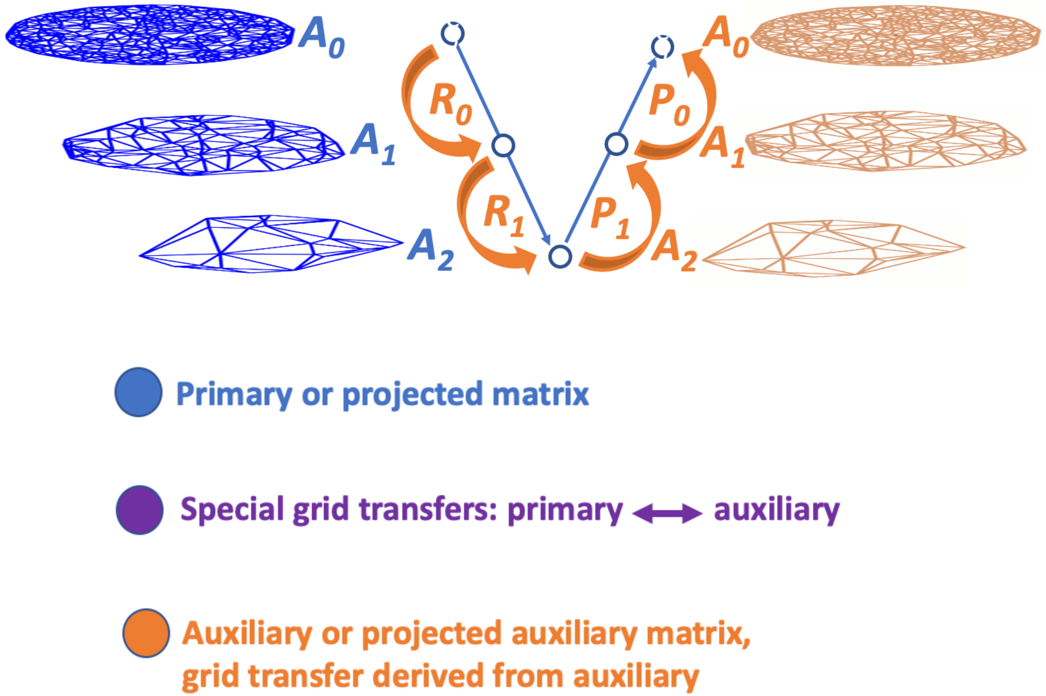
Auxiliary Methods

1) Devise preconditioner for A_0 by applying AMG to auxiliary operator A_1

2) Devise preconditioner for A_0 by generating AMG components (e.g., R_i or P_i) via algorithms applied to auxiliary operator A_0

Premise: Auxiliary operator is more amenable to AMG ideas than A_0





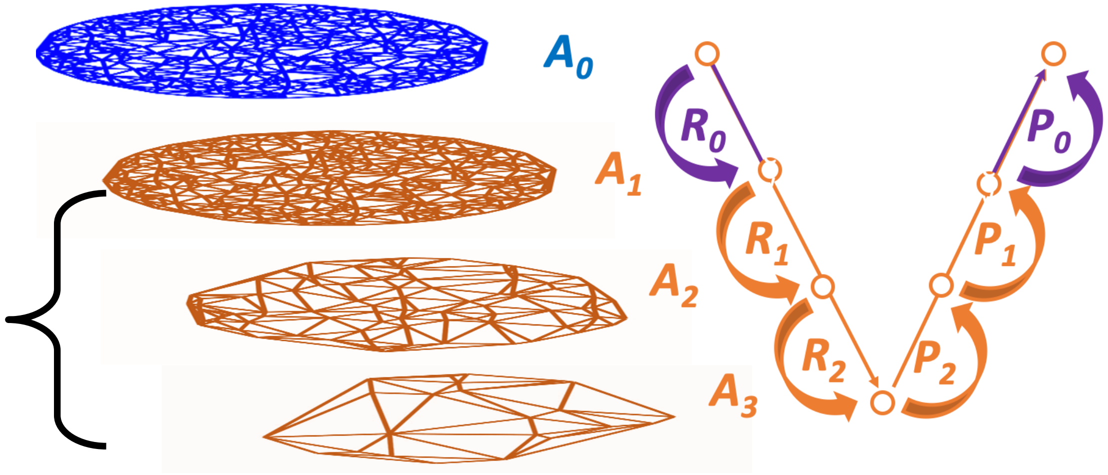
Not a new idea, e.g. high order discretizations (via defect correction methods), shifted Laplacians for Helmholtz & H-curl/H-div solvers

[Xu, The Auxiliary space method …, Computing 56 (1996)]

Precondition via low-order FE discretization

discretize via $\mathbb{Q}_2/\mathbb{Q}_1$

discretize via $\mathbb{Q}_1/\mathbb{Q}_1$

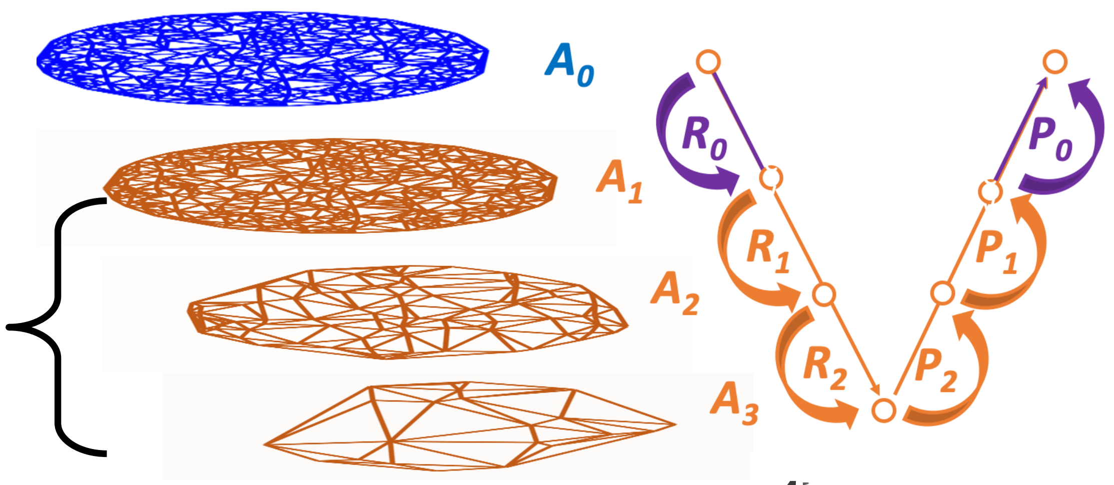


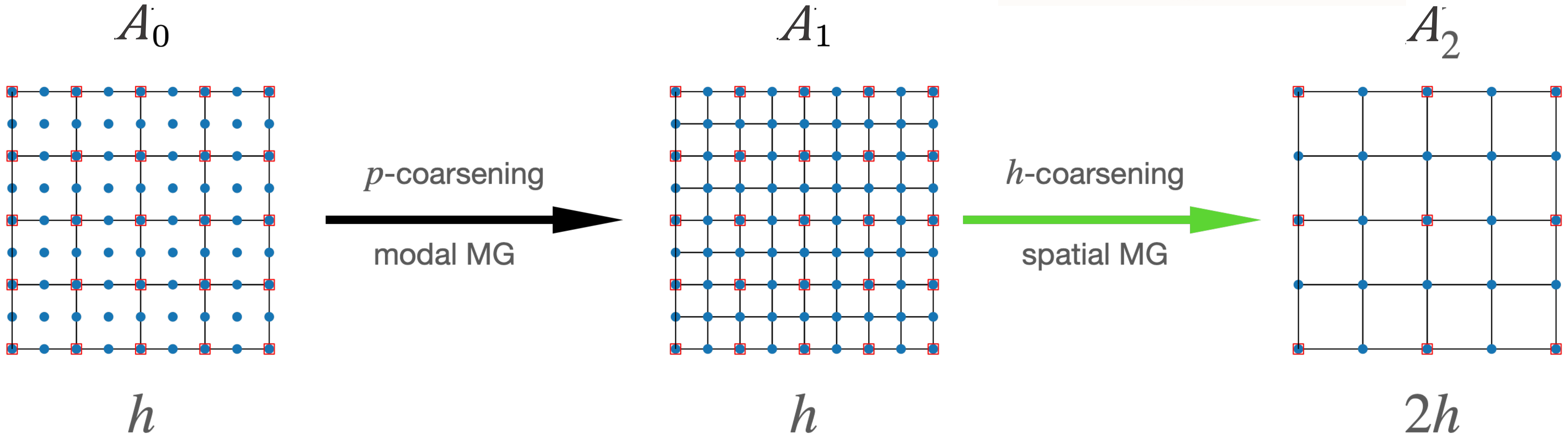
1^{st} Step: investigate idea via GMG

discretize via $\mathbb{Q}_2/\mathbb{Q}_1$

discretize via $\mathbb{Q}_1 iso\, \mathbb{Q}_2/\mathbb{Q}_1$

GMG hierarchy





$R_0 = I\,,\ P_0 = I,\ A_1 = \mathbb{Q}_1 iso\, \mathbb{Q}_2/\mathbb{Q}_1$ Stokes discretization $\neq\ R_0 A_0 P_0$

P_k based on linear interpolation for $k > 0,\ \ R_k = (P_k)^T\ \ \&\ \ A_{k+1} = R_k A_k P_k$ for $k \geq 0$

$\mathbb{Q}_2/\mathbb{Q}_1$ preconditioned via $\mathbb{Q}_1 iso\, \mathbb{Q}_2/\mathbb{Q}_1$ + GMG

Need MG relaxation procedures

- Inexact Braess-Sarazin
- Additive Vanka

Special saddle point incompressible flow relaxation procedures that employ damping parameters

$\Rightarrow$ Fourier analysis to obtain damping parameters

Inexact Braess-Sarazin Relaxation

$$A\,\delta x = \begin{bmatrix} G & \\ B & -S \end{bmatrix} \begin{bmatrix} I & G^{-1}B^T \\ & I \end{bmatrix} \begin{bmatrix} \delta_{\vec{u}} \\ \delta_p \end{bmatrix} = \begin{bmatrix} r_{\vec{u}} \\ r_p \end{bmatrix}$$

Solve

$$S\delta_p = BD^{-1}r_{\vec{u}} - \alpha \cdot r_p$$ Solved via weighted Jacobi iteration

$$\delta_{\vec{u}} = \frac{1}{\alpha} D^{-1}(r_{\vec{u}} - B^T\delta_p)$$ Solved via direct computation

introduces Jacobi damping factor, β

where

$$D = \mathrm{diag}(G)$$
$$S = BD^{-1}B^T$$

correction damped via ω $(i.e.,\ x \leftarrow x + \omega\,\delta x)$

Additive Vanka Relaxation

$$A\,\delta x = \begin{bmatrix} G & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \delta_{\vec{u}} \\ \delta_p \end{bmatrix} = \begin{bmatrix} r_{\vec{u}} \\ r_p \end{bmatrix}$$

$$\delta x = \sum_{i=1}^{N_p} (V_i^T W_i A_i^{-1} V_i) r,$$

where

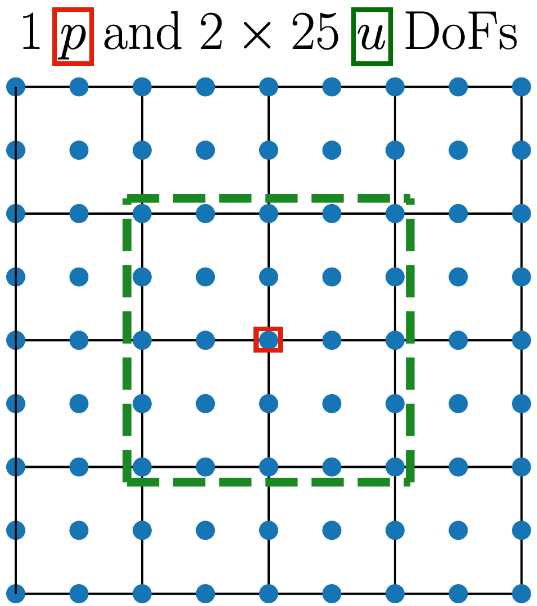
$$A_i = V_i A V_i^T$$

$$V_i = \{\text{Global DoF to } i^{th} \text{ patch restriction}\}$$

$$W_i = \{\text{diagonal weighting matrix}\}$$

$w_i = 1/$(# patches containing DoF i)

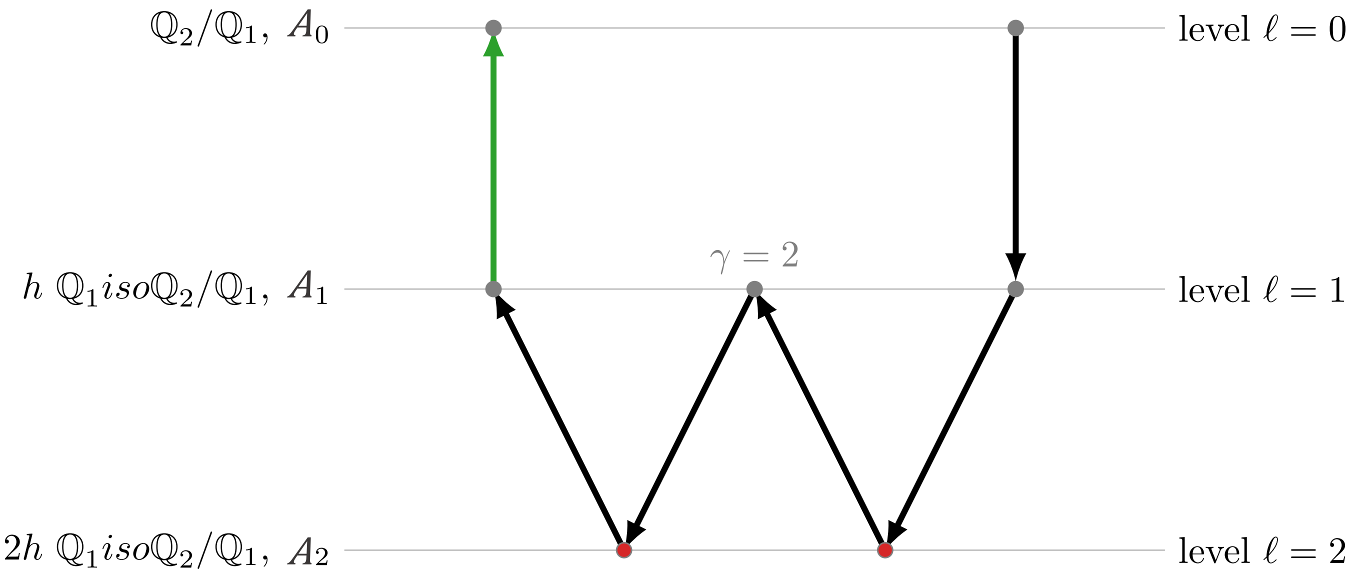
correction damped via ω ($i.e.,\ x \leftarrow x + \omega\,\delta x$)



MG parameters

$$E = \left(I - \omega_0 M_0{}^{-1} A_0\right)^{\nu_2} \left(I - \tau (I - Z_1^{\gamma}) A_1^{-1} A_0\right) \left(I - \omega_0 M_0{}^{-1} A_0\right)^{\nu_1}$$
$$Z_1 = (I - \omega_1 M_1{}^{-1} A_1)(I - P A_2^{-1} R A_1)(I - \omega_1 M_1{}^{-1} A_1)$$

- Pre/Post relaxation (ν_1, ν_2)
- h-GMG cycles: γ
- Solution updates: τ, ω_0 and ω_1
- Relaxation for $\mathbb{Q}_2/\mathbb{Q}_1$ and $\mathbb{Q}_1 \boldsymbol{iso} \mathbb{Q}_2/\mathbb{Q}_1$
  - Braess-Sarazin: α, β

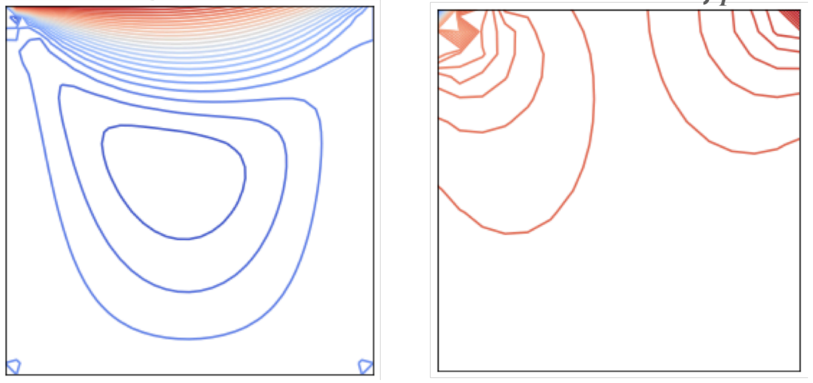


Use Local Fourier Analysis to find optimal parameters for 2-grid convergence

$\mathbb{Q}_2/\mathbb{Q}_1$ preconditioned via $\mathbb{Q}_1 \, iso \, \mathbb{Q}_2/\mathbb{Q}_1$ + multilevel GMG

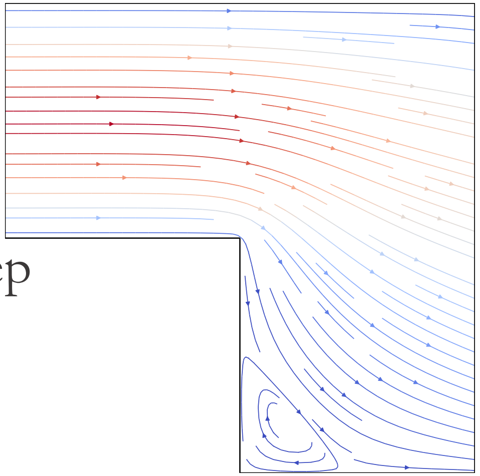
Lid Driven Cavity

Velocity field, $\vec{u}$ Pressure field, p



	(ν_1, ν_2, γ)	ρ_{LFA}	Cycle	Measured ρ^{P}	ρ^{D}	GMG m^{P}	m	w/ FGMRES m^{P}	m
BS	$(1,0,1)$		V	.17	.20	13	14	12	12
		.18	W	.17	.17	13	13	12	12
	$(1,1,1)$		V	.25	.55	13	33	9	11
		.09	W	.08	.08	9	10	8	9
V	$(1,0,2)$		V	.19	.20	14	14	10	11
		.21	W	.19	.20	14	14	10	10
	$(1,1,1)$		V	.99	.53	100+	30	24	17
		.29	W	.33	.30	18	17	15	15

Backward Facing Step



velocity field stream-plot

		FGMRES iterations for Method, (ν_1, ν_2, γ)			
		BS, $(1,1,1)$		V, $(1,0,2)$	
DoFs	GMG levels	V-cycle	W-cycle	V-cycle	W-cycle
515	2	10	10	11	11
1891	3	10	10	11	11
7235	4	11	10	11	11
28 291	5	12	10	11	11
111 875	6	13	9	12	11
444 931	7	14	9	12	11

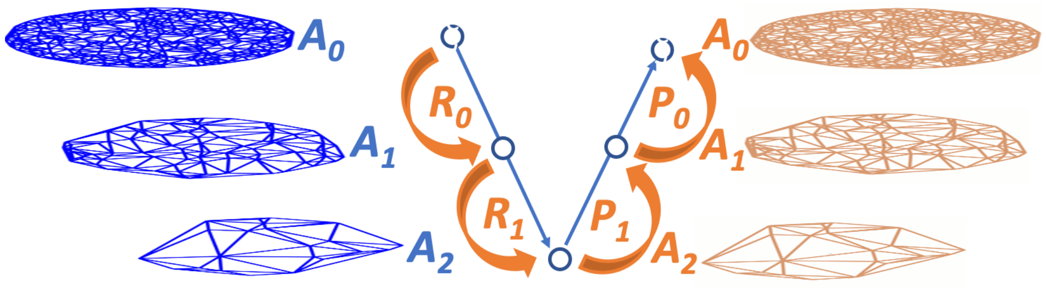
Toward AMG for $\mathbb{Q}_1 iso\, \mathbb{Q}_2/\mathbb{Q}_1$ preconditioners to $\mathbb{Q}_2/\mathbb{Q}_1$

- ➢ Cannot apply "co-located" approach to mixed FE system
- ➢ Inf-sup stability on coarse level A_i's ?
- ➢ Most AMG methods don't *like* 0's on matrix diagonal
- ✓ ~~AMG methods *tend* to generate interpolation *resembling* linear basis functions~~
- ✓ ~~Velocities coarsen more rapidly than pressure~~
- ➢ Fourier analysis not directly applicable to AMG

wrap smoothers within GMRES to choose damping parameters (at least good for Vanka). Could use Chebyshev smoothing.

AMG for $\mathbb{Q}_1 \, iso \, \mathbb{Q}_2/\mathbb{Q}_1$ preconditioner to $\mathbb{Q}_2/\mathbb{Q}_1$

- ~~Cannot apply "co-located" approach to mixed FE system~~
- Inf-sup stability on coarse level A_i's ?
- ~~Most AMG methods don't *like* *0*'s on matrix diagonal~~



Recall 2nd auxiliary approach …

Devise preconditioner for A_0 by generating AMG component (e.g., R_i or P_i) via algorithms applied to aux. op. A_i

For **only** the purpose of generating grid transfer operators apply AMG to

$$\begin{pmatrix} \bar{G} & \\ & W \end{pmatrix}$$

where

$\bar{G}$ is (1,1) block of A_1 ($\mathbb{Q}_1$ discretization of velocities on refined mesh)

$W = B\,B^T$ or $W = A_p$ (pressure Laplacian)

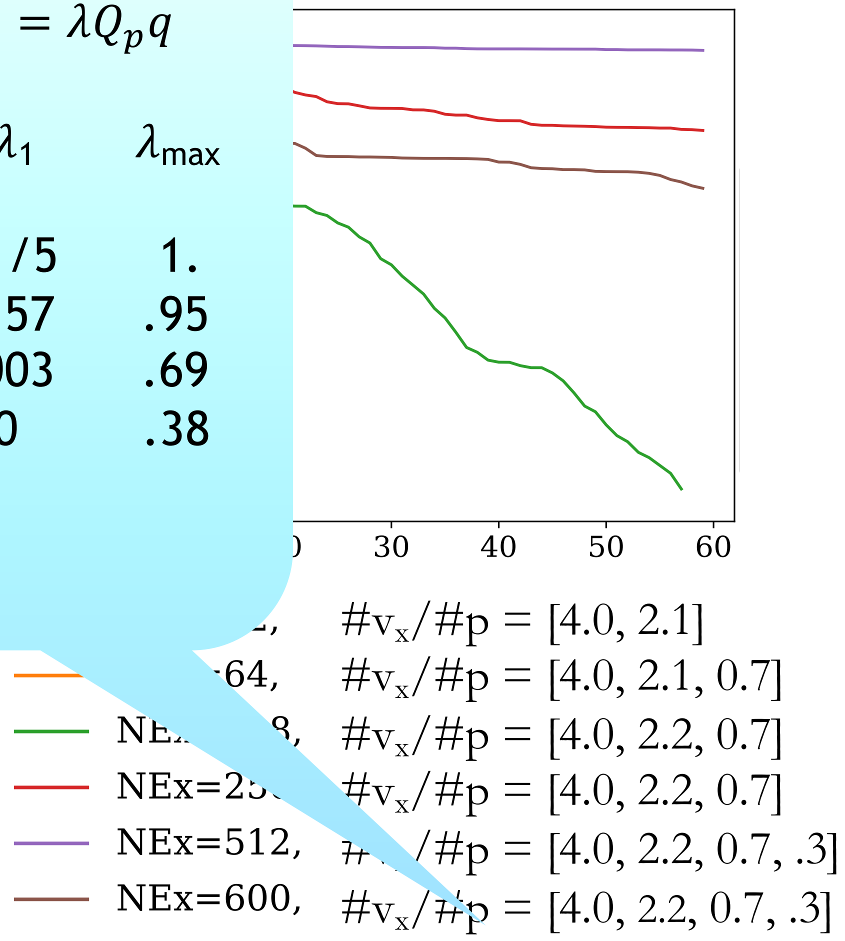
GMRES + AMG applied to lid driven cavity Stokes on Quad mesh

preliminary

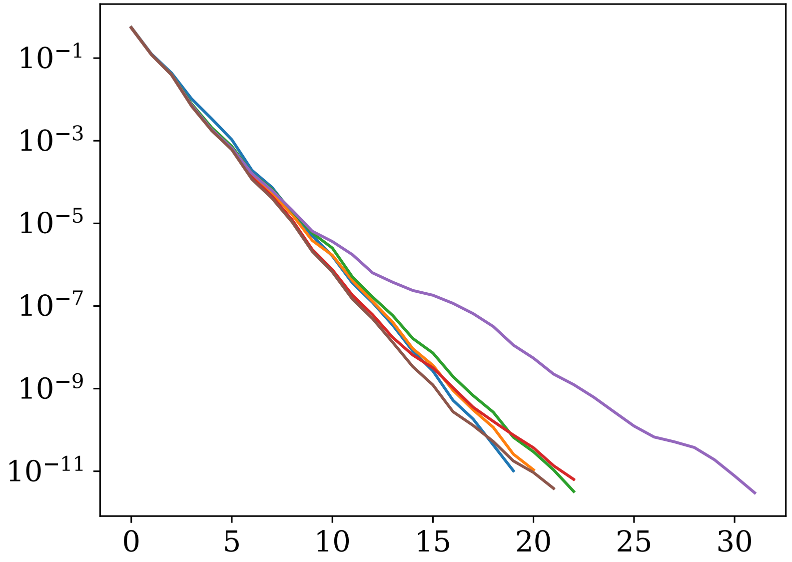
$$BG^{-1}B^T q = \lambda Q_p q$$

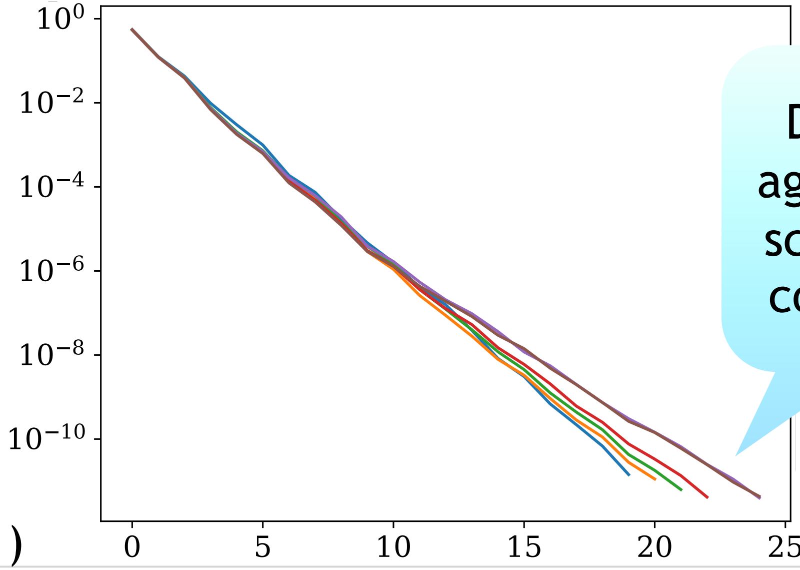
level	λ_1	$\lambda_{\max}$
0	≈1/5	1.
1	.157	.95
2	.003	.69
3	≈ 0	.38

$\mathbb{Q}_2/\mathbb{Q}_1$



$\mathbb{Q}_1 iso\, \mathbb{Q}_2/\mathbb{Q}_1$





Different aggregation scheme for coarsening

<u>Algo. Details</u>

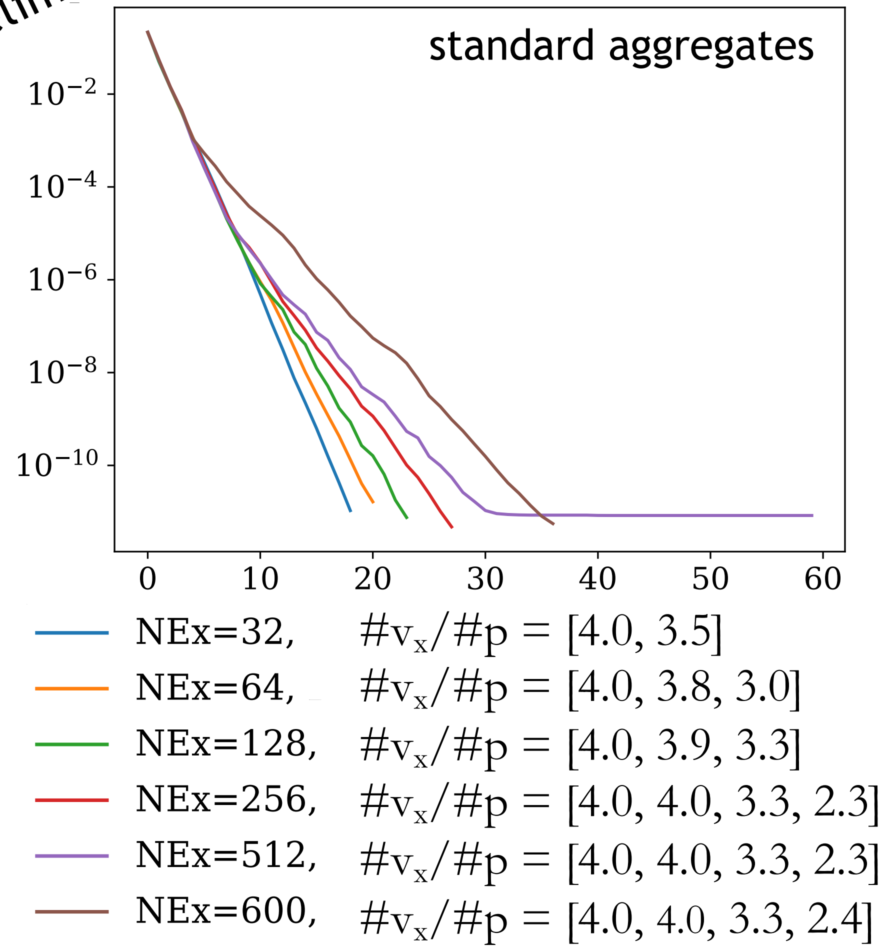
pressure: aggregation uses A_p, interp coefficients via Emin-AMG($B\,B^T$)

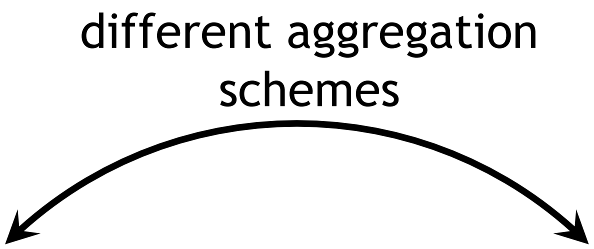
velocity: aggregation uses $\bar{G}$, interp coefficients via SA-AMG ($\bar{G}$)

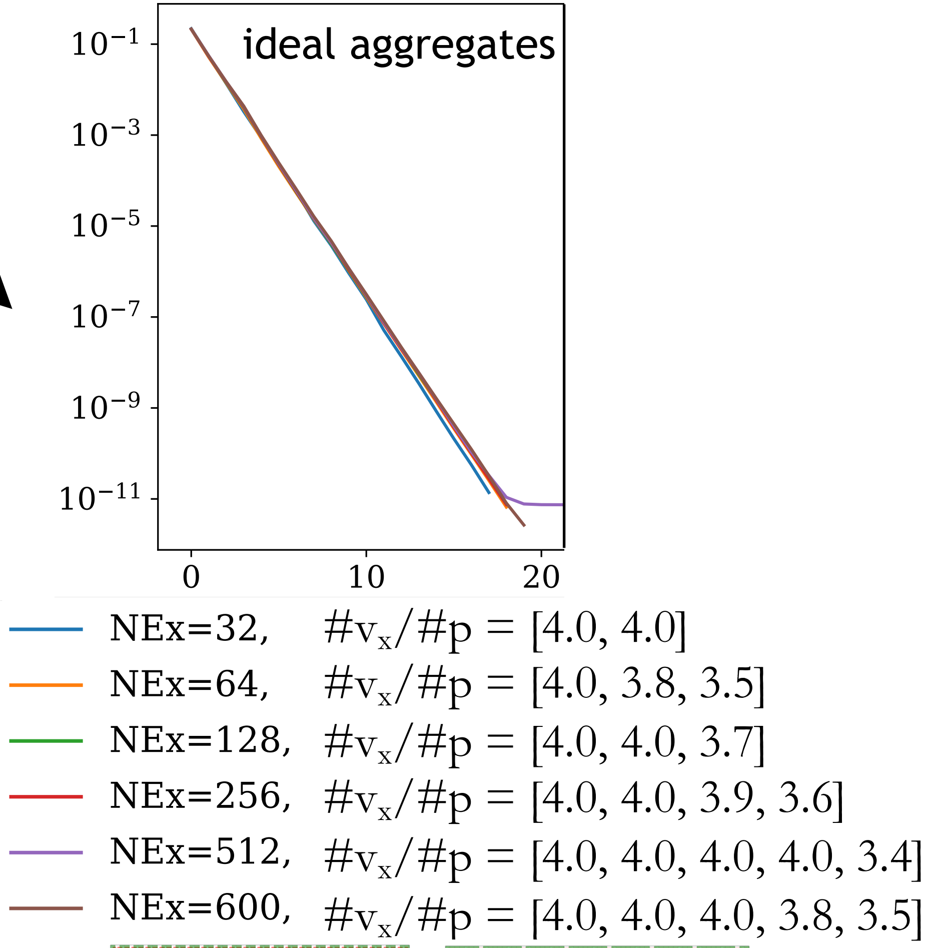
V(2,2) cycle, 2 its of GMRES(Vanka with $\omega = .77$ $(left)$ or $.35$ $(right)$)

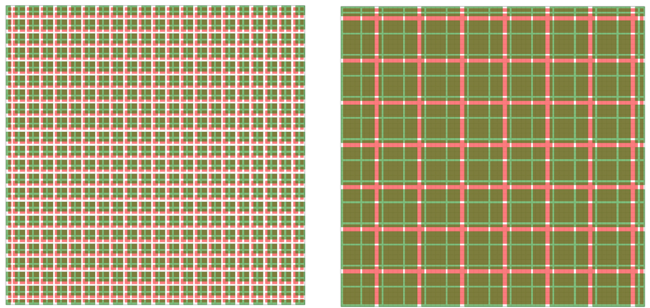
FGMRES(AMG($\mathbb{P}_1\, iso\, \mathbb{P}_2/\mathbb{P}_1$)): Triangular Element

preliminary









<u>Algo. Details</u>

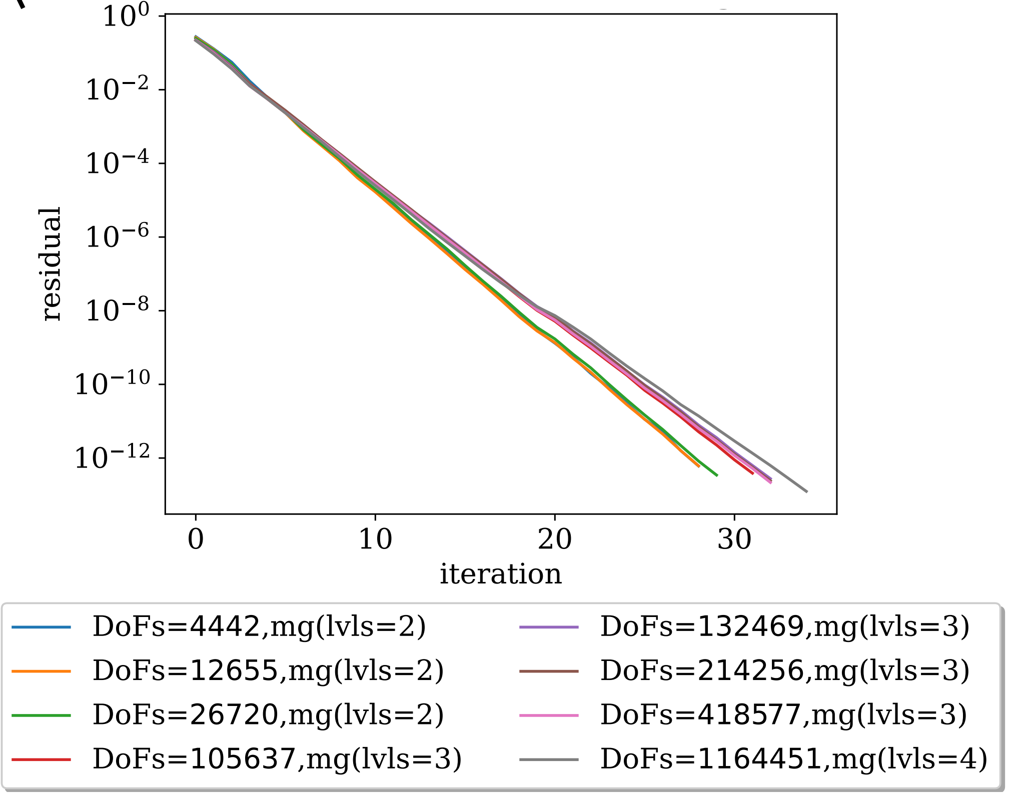
pressure: aggregation uses A_p, interp coefficients via Emin-AMG($B\,B^T$)

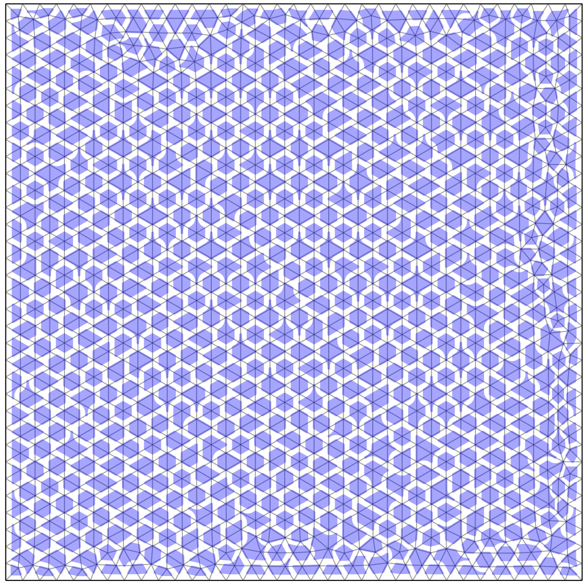
velocity: aggregation uses $\bar{G}$, interp coefficients via SA-AMG ($\bar{G}$)

V(2,2) cycle, Vanka relaxation

FGMRES(AMG($\mathbb{P}_1$ *iso* $\mathbb{P}_2$/$\mathbb{P}_1$)): best unstructured aggregation results

some cherry picking involved





sample aggregates

Algo. Details

pressure: aggregation uses A_p, interp coefficients via SA-AMG($B\,B^T$), evolution strength measure (tuned)

velocity: aggregation uses $\bar{G}$, interp coefficients via Emin-AMG ($\bar{G}$), sym. evolution strength measure (tuned)

V(2,2) cycle, Vanka with $\omega = .29$

Conclusion

- Direct application of AMG to $\mathbb{Q}_2/\mathbb{Q}_1$ is problematic
  - Loss of mesh independent convergence
- Use of $\mathbb{Q}_1 \mathit{iso}\, \mathbb{Q}_2/\mathbb{Q}_1$ system to precondition $\mathbb{Q}_2/\mathbb{Q}_1$ formulation facilitates AMG use
  - Not perfect, but much closer to mesh independence
- Some issues remain with applying AMG to $\mathbb{Q}_1 \mathit{iso}\, \mathbb{Q}_2/\mathbb{Q}_1$ system
- Careful examination of aggregation algorithms may help improve scalability
  - Should aggregation of pressure dofs & velocity dofs be correlated?
- Inf-sup guarantees would be nice on coarse grids …