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Differential geometric approaches to momentum-based formulations for fluids

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ABSTRACT

This SAND report documents CIS Late Start LDRD Project 22-0311, "Differential geometric approaches to momentum-based formulations for fluids". The project primarily developed geometric mechanics formulations for momentum-based descriptions of nonrelativistic fluids, utilizing a differential geometry/exterior calculus treatment of momentum and a space+time splitting. Specifically, the full suite of geometric mechanics formulations (variational/Lagrangian, Lie-Poisson Hamiltonian and Curl-Form Hamiltonian) were developed in terms of exterior calculus using vector-bundle valued differential forms. This was done for a fairly general version of semi-direct product theory sufficient to cover a wide range of both neutral and charged fluid models, including compressible Euler, magnetohydrodynamics and Euler-Maxwell.

As a secondary goal, this project also explored the connection between geometric mechanics formulations and the more traditional Godunov form (a hyperbolic system of conservation laws). Unfortunately, this stage did not produce anything particularly interesting, due to unforeseen technical difficulties.

There are two publications related to this work currently in preparation, and this work will be presented at SIAM CSE 23, at which the PI is organizing a mini-symposium on geometric mechanics formulations and structure-preserving discretizations for fluids. The logical next step is to utilize the exterior calculus based understanding of momentum coupled with geometric mechanics formulations to develop (novel) structure-preserving discretizations of momentum. This is the main subject of a successful FY23 CIS LDRD "Structure-preserving discretizations for momentum-based formulations of fluids".

CONTENTS

LIST OF FIGURES

1. INTRODUCTION AND EXECUTIVE SUMMARY OF RESULTS

This SAND report documents CIS Late Start LDRD Project 22-0311, "Differential geometric approaches to momentum-based formulations for fluids". This project primarily developed geometric mechanics formulations for momentum-based descriptions of nonrelativistic fluids, utilizing a differential geometry/exterior calculus treatment of momentum and a space+time splitting.

Specifically, momentum (and fluid velocity) are treated as (vector) bundle-valued differential forms (BVDFs) [10, 9, 14, 8, 24]. BVDFs are an extension of the more standard scalar-valued differential forms (SVDFs) [23, 3] to differential forms that take values in a vector bundle. Classic examples of a vector bundle are the tangent bundle (vectors) and cotangent bundle (covectors) on a manifold. In this treatment, momentum is a covector-valued twisted n-form, while velocity is a vector-valued straight 0-form. These two objects are dual to each other under the appropriate pairing, through a generalized Hodge star for BVDFs.

This treatment of momentum was then used to derive geometric mechanics formulations for non-relativistic fluids: variational (Euler-Poincaré) [13, 11] and the associated Lie-Poisson Hamiltonian and Curl-Form Hamiltonian formulations [18, 21, 11] for semi-direct product theory (a special case of matched pair dynamics) [6, 7]. Semi-direct product theory is sufficient to treat a wide range of both neutral and charged fluid models, such as compressible Euler, magnetohydrodynamics and Euler-Maxwell.

The BVDF-based treatment of momentum extends the usual vector or tensor calculus based formulations, and provides an approach that works seamlessly for general manifolds and arbitrary (Eulerian) coordinate systems. Additionally, it will serve as the basis for structure-preserving discretizations of momentum-based descriptions of fluids, just as SVDFs form the basis for structure-preserving discretization for velocity-based descriptions of fluids.

As a secondary goal, this project also explored the connection between geometric mechanics formulations and the more traditional Godunov form (a hyperbolic system of conservation laws) [4, 16, 19, 20]. Unfortunately, this stage did not produce anything particularly interesting, due to unforeseen technical difficulties which are detailed more in Section 6.

More details on the research and work done for this project can be found in Sections 2, 3, 4 and 5, on the results in Section 6, and on anticipated outcomes and impact (including additional future work) in Section 7. Finally, conclusions are found in Section 8, a review of exterior calculus in Appendix A, and a copy of the ending project review presentation in Appendix B (including a descriptive quad-chart).

2. GEOMETRIC MECHANICS FORMULATIONS BASED ON $\mathbf{u}_{\mathrm{T}}^{0}$

Using the material from Appendix A, we are now ready to derive geometric mechanics formulations based on the vector-valued 0-form $\mathbf{u}_{\mathrm{T}}^{0}$ that describes the fluid velocity. This is

mainly a review of existing material from [13], written in the language of bundle-valued differential forms as in [10].

The fluid is assumed to be fully described by the velocity 0-form $\mathbf{u}^0_{\mathrm{T}}$ and an arbitrary set of *advected quantities* $a \in \{\Lambda^k, \tilde{\Lambda}^k\}$ and $\mathbf{b} \in \{\Lambda^k(T), \Lambda^k(T^\star), \tilde{\Lambda}^k(T), \tilde{\Lambda}^k(T^\star)\}$. Typically these are quantities such as mass density, entropy density, electromagnetic fields/potentials, etc. The advected quantities are assumed to obey the *transport equations*

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} a = 0 \tag{1}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \mathbf{b} = 0 \tag{2}$$

In other words, they *Lie dragged* or *Lie transported*.

2.1. Variational formulation

We will consider the equations of motion generated by a generic Lagrangian $\mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b}] = \int_\Omega \tilde{l}^3(\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b})$ that is a functional of velocity $\mathbf{u}^0_{\mathrm{T}} \in \Lambda^0(T)$ and an arbitrary number of advected SVDF's $a \in \{\Lambda^k, \tilde{\Lambda}^k\}$ and BVDF's $\mathbf{b} \in \{\Lambda^k(T), \Lambda^k(T^\star), \tilde{\Lambda}^k(T), \tilde{\Lambda}^k(T^\star)\}$, where $\tilde{l}^3(\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b})$ is the *Lagrangian density* twisted n-form. By assumption, if the domain Ω has a boundary $\partial\Omega$ then material boundary conditions hold: $\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mu}^n = \tilde{u}^{n-1} = 0$ on $\partial\Omega$. The *action* associated with the Lagrangian $\mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b}]$ is

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b}] \tag{3}$$

The equations of motion arise from applying an appropriate *variational principle* to (3). Specifically, we use an *Euler-Poincaré* variational principle and take *variations* of the action (3)

$$\delta\mathcal{S} = \delta \int_{t_1}^{t_2} \mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b}] = 0 \tag{4}$$

subject to the constraints

$$\delta \mathbf{u}^0_{\mathrm{T}} = \partial_t \zeta^0_{\mathrm{T}} + [\mathbf{u}^0_{\mathrm{T}}, \zeta^0_{\mathrm{T}}] \tag{5}$$

$$\delta a = -\mathrm{L}_{\zeta^0_{\mathrm{T}}} a \tag{6}$$

$$\delta \mathbf{b} = -\mathrm{L}_{\zeta^0_{\mathrm{T}}} \mathbf{b} \tag{7}$$

for arbitrary variations $\zeta^0_{\mathrm{T}} \in \Lambda^0(T)$ that satisfy $\zeta^0_{\mathrm{T}}(t_1) = \zeta^0_{\mathrm{T}}(t_2) = 0$ and $\mathrm{i}_{\zeta^0_{\mathrm{T}}} \tilde{\mu}^n = \tilde{\zeta}^{n-1} = 0$ on $\partial\Omega$. These constraints (known as *Lin constraints*) enforce the transport equations (1).

Remark 2.1. *This variational principle is a special case of the more general theory of matched pair dynamics [6, 7], specifically a version of semi-direct product theory where the advected quantities do not have any internal dynamics. In this case the semi-direct product is* $\mathfrak{X}(\Omega) \times \Lambda$*, where* $\mathfrak{X}(\Omega) = \Lambda^0(T)$ *is the space of vector fields and* Λ *represents the advected differential*

forms. This will be sufficient to treat most of examples in this report, but not models such as electrohydrodynamics where the advected electrodynamic fields directly interact with each other. The more sophisticated case of advected quantities with internal dynamics is discussed in Section 5.

Introducing the momentum $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} = \frac{\delta \mathcal{L}}{\delta \mathbf{u}^0_{\mathrm{T}}} \in \tilde{\Lambda}^n(T^\star)$, the variations of (3) give

$$\begin{aligned}
\int_{t_1}^{t_2} \left[\left\langle \left\langle \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, \delta \mathbf{u}^0_{\mathrm{T}} \right\rangle \right\rangle_\chi + \sum \left\langle \left\langle \frac{\delta \mathcal{L}}{\delta a}, \delta a \right\rangle \right\rangle + \sum \left\langle \left\langle \frac{\delta \mathcal{L}}{\delta \mathbf{b}}, \delta \mathbf{b} \right\rangle \right\rangle_\chi \right] &= 0 \\
\int_{t_1}^{t_2} \left[\left\langle \left\langle \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, \partial_t \zeta^0_{\mathrm{T}} + [\mathbf{u}^0_{\mathrm{T}}, \zeta^0_{\mathrm{T}}] \right\rangle \right\rangle_\chi + \sum \left\langle \left\langle \frac{\delta \mathcal{L}}{\delta a}, -\mathrm{L}_{\zeta^0_{\mathrm{T}}} a \right\rangle \right\rangle + \sum \left\langle \left\langle \frac{\delta \mathcal{L}}{\delta \mathbf{b}}, -\mathrm{L}_{\zeta^0_{\mathrm{T}}} \mathbf{b} \right\rangle \right\rangle_\chi \right] &= 0 \\
\int_{t_1}^{t_2} \left[-\left\langle \left\langle \partial_t \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, \zeta^0_{\mathrm{T}} \right\rangle \right\rangle_\chi - \left\langle \left\langle \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, \zeta^0_{\mathrm{T}} \right\rangle \right\rangle_\chi + \sum \left\langle \left\langle \frac{\delta \mathcal{L}}{\delta a} \diamond a, \zeta^0_{\mathrm{T}} \right\rangle \right\rangle + \sum \left\langle \left\langle \frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond \mathbf{b}, \zeta^0_{\mathrm{T}} \right\rangle \right\rangle_\chi \right] &= 0
\end{aligned}$$

Since ζ^0_{T} is arbitrary, this gives the *Euler-Poincaré equations*

$$\frac{\partial}{\partial t} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} - \sum \frac{\delta \mathcal{L}}{\delta a} \diamond a - \sum \frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond \mathbf{b} = 0 \tag{8}$$

A choice of advected quantities a and $\mathbf{b}$ and an expression for the Lagrangian $\mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b}]$ will close the dynamics and give explicit forms for $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}$, $\frac{\delta \mathcal{L}}{\delta a}$ and $\frac{\delta \mathcal{L}}{\delta \mathbf{b}}$. Some examples from various fields of fluid dynamics are given in Section 4.

2.2. Lie-Poisson Hamiltonian formulation

Associated with the variational formulation and Euler-Poincaré equations is a Lie-Poisson Hamiltonian formulation that comes from a Legendre transform from $\mathbf{u}^0_{\mathrm{T}}$ to $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}$

$$\mathsf{H}[\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, a, \mathbf{b}] = \left\langle \left\langle \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, \mathbf{u}^0_{\mathrm{T}} \right\rangle \right\rangle_\chi - \mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, a, \mathbf{b}] = \int \tilde{h}^3 = \int \mathbf{u}^0_{\mathrm{T}} \mathbin{\lrcorner} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} - \tilde{l}^3 \tag{9}$$

where $\tilde{h}^3$ is the *Hamiltonian density* twisted n-form. The functional derivatives of H are

$$\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}} = \mathbf{u}^0_{\mathrm{T}} \qquad \frac{\delta \mathsf{H}}{\delta a} = -\frac{\delta \mathcal{L}}{\delta a} \qquad \frac{\delta \mathsf{H}}{\delta \mathbf{b}} = -\frac{\delta \mathcal{L}}{\delta \mathbf{b}} \tag{10}$$

Thus we can write the Euler-Poincaré (8) and transport (1) equations as

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \sum \frac{\delta \mathsf{H}}{\delta a} \diamond a + \sum \frac{\delta \mathsf{H}}{\delta \mathbf{b}} \diamond \mathbf{b} &= 0 && (11) \\
\frac{\partial a}{\partial t} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} a &= 0 && (12) \\
\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} \mathbf{b} &= 0 && (13) \\
& && (14)
\end{aligned}$$

Remark 2.2. *The process above works only when the Lagrangian is non-degenerate i.e. the Legendre transform between $\mathcal{L}$ and $\mathcal{H}$ is invertible. Certain choices of Lagrangians (such as the Lagrangian for the hydrostatic primitive equations from geophysical fluid dynamics) violate this assumption i.e. they are degenerate and require a more careful treatment.*

The associated Lie-Poisson Hamiltonian formulation can be written in *symplectic form* as

$$\frac{\partial}{\partial t}\begin{bmatrix} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} \\ a \\ \mathbf{b} \end{bmatrix} = -\begin{bmatrix} \mathrm{L}_{\square}\,\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} & \square\diamond a & \square\diamond \mathbf{b} \\ \mathrm{L}_{\square}\, a & 0 & 0 \\ \mathrm{L}_{\square}\,\mathbf{b} & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}} \\ \frac{\delta \mathsf{H}}{\delta a} \\ \frac{\delta \mathsf{H}}{\delta \mathbf{b}} \end{bmatrix} \tag{15}$$

Note that the elements of the first column are all Lie derivatives of the predicted quantities with respect to vector field $\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}$, while the corresponding terms in the momentum equation are just the *diamond* ($\diamond$) operators of the advected quantities and their associated functional derivatives. The anti-symmetry of $\mathbb{J}$ rests on two things: that the diamond $\diamond$ operators are the (negative) adjoints of the Lie derivative with respect to the vector field, and that the $\mathrm{L}_{\square}\,\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}$ operator is self-(negative)adjoint with respect to the advecting vector field. The latter fact is easily established

$$\left\langle\left\langle \mathbf{a}^0_T, \mathrm{L}_{\mathbf{b}^0_T}\,\mathbf{c}^n_{T^\star}\right\rangle\right\rangle_\chi = -\left\langle\left\langle \mathbf{c}^n_{T^\star}, [\mathbf{b}^0_T, \mathbf{a}^0_T]\right\rangle\right\rangle_\chi = \left\langle\left\langle \mathbf{c}^n_{T^\star}, [\mathbf{a}^0_T, \mathbf{b}^0_T]\right\rangle\right\rangle_\chi = -\left\langle\left\langle \mathbf{b}^0_T, \mathrm{L}_{\mathbf{a}^0_T}\,\mathbf{c}^n_{T^\star}\right\rangle\right\rangle_\chi \tag{16}$$

where we have used the antisymmetry of the Lie bracket $[\mathbf{a}^0_T, \mathbf{b}^0_T] = -[\mathbf{b}^0_T, \mathbf{a}^0_T]$.

The Poisson brackets corresponding to $\mathbb{J}$ are

$$\{\mathsf{A},\mathsf{B}\} = \left\langle\left\langle \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, [\frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}]\right\rangle\right\rangle_\chi - \sum\left\langle\left\langle \frac{\delta \mathsf{A}}{\delta a}, \mathrm{L}_{\frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\, a\right\rangle\right\rangle - \sum\left\langle\left\langle \frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \frac{\delta \mathsf{B}}{\delta a}\diamond a\right\rangle\right\rangle_\chi \tag{17}$$

where we have used Lie derivatives, Lie brackets and diamond operators to make clear the universal semi-direct product structure underlying the brackets. It is very important to note that this is only one of many possible Poisson brackets. In particular, for each a and $\mathbf{b}$, there are three equivalent forms:

$$\{\mathsf{A},\mathsf{B}\}_{a,mixed} = -\left\langle\left\langle \frac{\delta \mathsf{A}}{\delta a}, \mathrm{L}_{\frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\, a\right\rangle\right\rangle - \left\langle\left\langle \frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \frac{\delta \mathsf{B}}{\delta a}\diamond a\right\rangle\right\rangle_\chi \tag{18}$$

$$\{\mathsf{A},\mathsf{B}\}_{a,\mathrm{L}} = -\left\langle\left\langle \frac{\delta \mathsf{A}}{\delta a}, \mathrm{L}_{\frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\, a\right\rangle\right\rangle + \left\langle\left\langle \frac{\delta \mathsf{B}}{\delta a}, \mathrm{L}_{\frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\, a\right\rangle\right\rangle \tag{19}$$

$$\{\mathsf{A},\mathsf{B}\}_{a,\diamond} = \left\langle\left\langle \frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \frac{\delta \mathsf{A}}{\delta a}\diamond a\right\rangle\right\rangle_\chi - \left\langle\left\langle \frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \frac{\delta \mathsf{B}}{\delta a}\diamond a\right\rangle\right\rangle_\chi \tag{20}$$

related through the definition of the $\diamond$ operator (with similar results for $\mathbf{b}$). Additionally, the Lie bracket term can be written as

$$\{\mathsf{A},\mathsf{B}\}_{LB} = \left\langle\left\langle \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}, [\frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}]\right\rangle\right\rangle_\chi = \left\langle\left\langle \frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \mathrm{L}_{\frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\, \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}\right\rangle\right\rangle_\chi = -\left\langle\left\langle \frac{\delta \mathsf{A}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}, \mathrm{L}_{\frac{\delta \mathsf{B}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\, \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}\right\rangle\right\rangle_\chi \tag{21}$$

Under the assumption of material boundaries, all of these brackets are truly Poisson brackets- they satisfy the Jacobi identity. However, the introduction of more general boundaries (such as inflow/outflow) can lead to violation of the Jacobi identity for certain choices of bracket. The mixed form $\{\mathsf{A},\mathsf{B}\}_{a,mixed}$ (along with the 3rd type of LB bracket $\{\mathsf{A},\mathsf{B}\}_{LB}$) is connected with symplectic form, and are what is used in (17).

3. GEOMETRIC MECHANICS FORMULATIONS BASED ON v^1

In addition to formulations based on $\mathbf{u}^0_{\mathrm{T}}$, approaches based on a related straight 1-form are commonly used. Start by defining the momentum straight 1-form m^1 as

$$\mathrm{m}^1 = (\tilde{\star}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat = \mathbb{T}\,\mathbf{m}^0_{\mathrm{T}^\star} \tag{22}$$

where $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} = \mathbf{m}^0_{\mathrm{T}^\star} \dot{\wedge} \tilde{\mu}^n$. Note that $\mathbf{m}^0_{\mathrm{T}^\star} = \sharp_1 \tilde{\star}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}$.

Then assume the existence of a total mass density twisted n-form $\tilde{D}^n$ that is a linear combination of advected twisted n-forms $\tilde{a}^n$ (i.e. densities):

$$\tilde{D}^n = \sum c_i \tilde{a}^n_i \tag{23}$$

with associated straight 0-form $D^0 = \tilde{\star}\tilde{D}^n$.

Using m^1 and D^0, the velocity straight 1-form v^1 is

$$\mathrm{v}^1 = \frac{1}{D^0} \wedge (\tilde{\star}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat = \frac{1}{D^0} \wedge \mathrm{m}^1 \tag{24}$$

3.1. Kelvin Noether Form

The evolution equation for m^1 is given by $\tilde{\star}(\frac{\partial \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}{\partial t})^\flat$, which is

$$\frac{\partial \mathrm{m}^1}{\partial t} + (\tilde{\star}\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat - \sum \left(\tilde{\star}[\frac{\delta \mathcal{L}}{\delta a} \diamond a] \right)^\flat - \sum \wedge \left(\tilde{\star}[\frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond \mathbf{b}] \right)^\flat = 0 \tag{25}$$

Using $\diamond_{\mathrm{kn}}$ we can rewrite (25) as

$$\frac{\partial \mathrm{m}^1}{\partial t} + (\tilde{\star}\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat - \sum \frac{\delta \mathcal{L}}{\delta a} \diamond_{\mathrm{kn}} a - \sum \frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond_{\mathrm{kn}} \mathbf{b} = 0 \tag{26}$$

Now note that $\frac{\partial \mathrm{m}^1}{\partial t} = D^0 \wedge \frac{\partial \mathrm{v}^1}{\partial t} + \frac{\partial D^0}{\partial t} \wedge \mathrm{v}^1$ and that $\frac{\partial D^0}{\partial t} + \tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{D}^n = 0$ and therefore we have

$$D^0 \wedge \frac{\partial \mathrm{v}^1}{\partial t} - \mathrm{v}^1 \wedge \tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{D}^n + (\tilde{\star}\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat - \sum \frac{\delta \mathcal{L}}{\delta a} \diamond_{\mathrm{kn}} a - \sum \frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond_{\mathrm{kn}} \mathbf{b} = 0 \tag{27}$$

We also have

$$\mathrm{v}^1 \wedge \tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{D}^n = \mathrm{v}^1 \wedge \tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} D^0 \wedge \tilde{\mu}^n = D^0 \wedge \mathrm{v}^1 \wedge \tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mu}^n = m^1 \wedge \tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} \tilde{\mu}^n \tag{28}$$

Now consider the $(\tilde{\star}\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat$ term. Start by noting that $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}=\mathbf{m}^0_{\mathrm{T}^\star}\wedge\tilde{\mu}^n$ where $\mathbf{m}^0_{\mathrm{T}^\star}=\mathbb{I}\,\mathrm{m}^1$. This gives

$$\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}=\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\mathbf{m}^0_{T^\star}\wedge\tilde{\mu}^n=\mathbf{m}^0_{T^\star}\wedge\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mu}^n+\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\mathbf{m}^0_{T^\star}\wedge\tilde{\mu}^n \tag{29}$$

Since $\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mu}^n=\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mu}^n=\mathrm{d}\,\tilde{u}^{n-1}$ we have

$$(\tilde{\star}\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star})^\flat=\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}m^1+m^1\wedge\tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mu}^n=D^0\wedge\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}v^1+m^1\wedge\tilde{\star}\,\mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mu}^n \tag{30}$$

This yields finally the *Kelvin-Noether form* of the Euler-Poincaré equations

$$D^0\wedge\left(\frac{\partial}{\partial t}\mathrm{v}^1+\mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\mathrm{v}^1\right)-\sum\frac{\delta\mathcal{L}}{\delta a}\diamond_{\mathrm{kn}}a-\sum\frac{\delta\mathcal{L}}{\delta\mathbf{b}}\diamond_{\mathrm{kn}}\mathbf{b}=0 \tag{31}$$

3.2. Curl-Form Hamiltonian Formulation

Connected to the Kelvin-Noether equations (31) is a *curl-form Hamiltonian formulation*[1] that predicts v^1 instead of $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}$. This amounts to a change of variables in the Lie-Poisson formulation. Let $\mathcal{H}[\mathrm{v}^1,a,\mathbf{b}]=\mathsf{H}[\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star},a,\mathbf{b}]$ and use the chain rule for functional derivatives along with (24) to get

$$\frac{\delta\mathcal{H}}{\delta\mathrm{v}^1}=\tilde{\star}\left[\frac{1}{D^0}\wedge(\frac{\delta\mathsf{H}}{\delta\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}})^\flat\right]=\frac{1}{D^0}\wedge\tilde{\star}(\frac{\delta\mathsf{H}}{\delta\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}})^\flat=\mathrm{i}_{\frac{\delta\mathsf{H}}{\delta\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\tilde{D}^n \tag{32}$$

$$\frac{\delta\mathcal{H}}{\delta\tilde{a}^n}=\frac{\delta\mathsf{H}}{\delta\tilde{a}^n}+\frac{\partial\tilde{D}^n}{\partial\tilde{a}^n}\wedge\mathrm{i}_{\frac{\delta\mathsf{H}}{\delta\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\mathrm{v}^1\qquad\frac{\delta\mathcal{H}}{\delta a}=\frac{\delta\mathsf{H}}{\delta a}\qquad\frac{\delta\mathcal{H}}{\delta\mathbf{b}}=\frac{\delta\mathsf{H}}{\delta\mathbf{b}} \tag{33}$$

Using these functional derivatives, it is an exercise in exterior calculus to obtain the new Poisson brackets

$$\{\mathcal{A},\mathcal{B}\}=-\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta\mathrm{v}^1},\mathrm{i}_{\left[\tilde{\star}\frac{\delta\mathcal{B}}{\delta\mathrm{v}^1})\right]^\sharp}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0}\right\rangle\right\rangle-\sum\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta a},\mathrm{L}_{\left[\tilde{\star}\frac{\delta\mathcal{B}}{\delta\mathrm{v}^1})\right]^\sharp}\frac{a}{D^0}\right\rangle\right\rangle-\sum\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta\mathrm{v}^1},\frac{1}{D^0}\wedge\left[\frac{\delta\mathcal{B}}{\delta a}\diamond_{\mathrm{kn}}a\right]\right\rangle\right\rangle_\chi \tag{34}$$

As before, there are in fact multiple equivalent Poisson brackets when there are no boundaries or material boundary conditions are assumed:

$$\{\mathcal{A},\mathcal{B}\}_{a,mixed}=-\sum\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta a},\mathrm{L}_{\left[\tilde{\star}\frac{\delta\mathcal{B}}{\delta\mathrm{v}^1})\right]^\sharp}\frac{a}{D^0}\right\rangle\right\rangle-\sum\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta\mathrm{v}^1},\frac{1}{D^0}\wedge\left[\frac{\delta\mathcal{B}}{\delta a}\diamond_{\mathrm{kn}}a\right]\right\rangle\right\rangle_\chi \tag{35}$$

$$\{\mathcal{A},\mathcal{B}\}_{a,\mathrm{L}}=-\sum\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta a},\mathrm{L}_{\left[\tilde{\star}\frac{\delta\mathcal{B}}{\delta\mathrm{v}^1})\right]^\sharp}\frac{a}{D^0}\right\rangle\right\rangle-\sum\left\langle\left\langle\frac{\delta\mathcal{B}}{\delta a},\mathrm{L}_{\left[\tilde{\star}\frac{\delta\mathcal{A}}{\delta\mathrm{v}^1})\right]^\sharp}\frac{a}{D^0}\right\rangle\right\rangle \tag{36}$$

$$\{\mathcal{A},\mathcal{B}\}_{a,\diamond}=\sum\left\langle\left\langle\frac{\delta\mathcal{B}}{\delta\mathrm{v}^1},\frac{1}{D^0}\wedge\left[\frac{\delta\mathcal{A}}{\delta a}\diamond_{\mathrm{kn}}a\right]\right\rangle\right\rangle_\chi-\sum\left\langle\left\langle\frac{\delta\mathcal{A}}{\delta\mathrm{v}^1},\frac{1}{D^0}\wedge\left[\frac{\delta\mathcal{B}}{\delta a}\diamond_{\mathrm{kn}}a\right]\right\rangle\right\rangle_\chi \tag{37}$$

[1] Also known as *vector-invariant* or *Carter-Licnerowicz*.

and

$$\{\mathcal{A},\mathcal{B}\}_{\mathrm{v}^1} = -\left\langle\left\langle \frac{\delta\mathcal{A}}{\delta \mathrm{v}^1}, \mathrm{i}_{\left[\tilde{\star}\frac{\delta\mathcal{B}}{\delta \mathrm{v}^1}\right)^\sharp} \frac{\mathrm{d}\,\mathrm{v}^1}{D^0}\right\rangle\right\rangle = -\left\langle\left\langle \tilde{\star}\frac{\delta\mathcal{A}}{\delta \mathrm{v}^1}, \tilde{\star}\frac{\delta\mathcal{B}}{\delta \mathrm{v}^1}\wedge\tilde{\star}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0}\right\rangle\right\rangle = -\left\langle\left\langle \tilde{\star}\frac{\delta\mathcal{A}}{\delta \mathrm{v}^1}\wedge\tilde{\star}\frac{\delta\mathcal{B}}{\delta \mathrm{v}^1}, \tilde{\star}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0}\right\rangle\right\rangle \tag{38}$$

These brackets give the equations of motion

$$\begin{aligned}
\frac{\partial\,\mathrm{v}^1}{\partial t} + \mathrm{i}_{\mathbf{U}^0_{\mathrm{T}}}\mathrm{d}\,\mathrm{v}^1 + \sum\frac{1}{D^0}\wedge[\frac{\delta\mathcal{H}}{\delta a}\diamond_{\mathrm{kn}} a] - \sum\frac{1}{D^0}\wedge[\frac{\delta\mathcal{H}}{\delta \mathbf{b}}\diamond_{\mathrm{kn}}\mathbf{b}] &= 0 &(39)\\
\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{U}^0_{\mathrm{T}}} a &= 0 &(40)\\
\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{U}^0_{\mathrm{T}}}\mathbf{b} &= 0 &(41)
\end{aligned}$$

for advective velocity $\mathbf{U}^0_{\mathrm{T}} = \left[\tilde{\star}(\frac{1}{D^0}\wedge\frac{\delta\mathcal{H}}{\delta \mathrm{v}^1})\right]^\sharp = \frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}$.

Alternatively, for mass flux $\mathbf{F}^0_{\mathrm{T}} = \left[\tilde{\star}\frac{\delta\mathcal{H}}{\delta \mathrm{v}^1}\right)\Big]^\sharp$ equivalent equations:

$$\begin{aligned}
\frac{\partial\,\mathrm{v}^1}{\partial t} + \mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0} + \sum\frac{1}{D^0}\wedge[\frac{\delta\mathcal{H}}{\delta a}\diamond_{\mathrm{kn}} a] - \sum\frac{1}{D^0}\wedge[\frac{\delta\mathcal{H}}{\delta \mathbf{b}}\diamond_{\mathrm{kn}}\mathbf{b}] &= 0 &(42)\\
\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{F}^0_{\mathrm{T}}}\frac{a}{D^0} &= 0 &(43)\\
\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{F}^0_{\mathrm{T}}}\frac{\mathbf{b}}{D^0} &= 0 &(44)
\end{aligned}$$

are obtained.

When $n=2$, the velocity advection term $\mathrm{i}_{\mathbf{U}^0_{\mathrm{T}}}\mathrm{d}\,\mathrm{v}^1 = \mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}}\mathrm{d}\frac{\mathrm{v}^1}{D^0}$ can be simplified:

$$\mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0} = \tilde{\star}(\tilde{\star}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0}\wedge F^1) = \tilde{\star}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0}\wedge\tilde{F}^{n-1} = \tilde{q}^0\wedge\tilde{F}^{n-1} \tag{45}$$

where $\tilde{F}^{n-1} = \frac{\delta\mathcal{H}}{\delta \mathrm{v}^1}$ and $\tilde{D}^n\wedge\tilde{q}^0 = \mathrm{d}\,\mathrm{v}^1$.

When $n=1$, the velocity advection term $\mathrm{i}_{\mathbf{U}^0_{\mathrm{T}}}\mathrm{d}\,\mathrm{v}^1 = \mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}}\frac{\mathrm{d}\,\mathrm{v}^1}{D^0}$ is absent since $\mathrm{d}\,\mathrm{v}^1 = 0$.

4. SPECIFIC EXAMPLES

To make the rather dense and abstract presentation of the previous sections more concrete, in this section we will develop examples of several well-known fluid models by introducing the appropriate Lagrangians that describe them.

For most examples, we will need kinetic energy straight 0-form

$$K^0 = \frac{1}{2}\mathbf{g}(\mathbf{u}^0_{\mathrm{T}},\mathbf{u}^0_{\mathrm{T}}) = \frac{1}{2}\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}u^1 \tag{46}$$

where $u^1 = \mathbf{u}_{\mathrm{T}}^{0\flat}$ There is a related kinetic energy twisted n-form

$$\tilde{K}^n = \tilde{\star} K^0 = \frac{1}{2} u^1 \wedge \tilde{u}^{n-1} \tag{47}$$

where $\tilde{u}^{n-1} = \mathrm{i}_{\mathbf{u}_{\mathrm{T}}^0} \tilde{\mu}^n = \tilde{\star} u^1$.

For examples with rotation, it is useful to introduce the rotational velocity vector-valued 0-form $\mathbf{R}_{\mathrm{T}}^0$ along with the associated rotational velocity straight 1-form $R^1 = \mathbf{R}_{\mathrm{T}}^{0\flat}$ and twisted $n-1$-form $\tilde{R}^{n-1} = \mathrm{i}_{\mathbf{R}_{\mathrm{T}}^0} \tilde{\mu}^n = \tilde{\star} R^1$.

4.1. Advected Densities Model

The first model we will consider is a fluid with a single velocity and an arbitrary number of advected twisted n-forms $\tilde{D}_k^n$ (i.e. densities). In this case the various general formulations can be written as follows:

Euler-Poincaré Equations

$$\frac{\partial}{\partial t} \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}} + \mathrm{L}_{\mathbf{u}_{\mathrm{T}}^0} \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}} - \sum_k \mathbb{I}(D_k^0 \wedge \mathrm{d} \frac{\delta \mathcal{L}}{\delta \tilde{D}_k^n}) \barwedge \tilde{\mu}^n = 0 \tag{48}$$

$$\frac{\partial \tilde{\rho}^n}{\partial t} + \mathrm{d}(D_k^0 \wedge \tilde{u}^{n-1}) = 0 \tag{49}$$

Kelvin-Noether Equations

$$D^0 \wedge \left(\frac{\partial}{\partial t} \mathrm{v}^1 + \mathrm{L}_{\mathbf{u}_{\mathrm{T}}^0} \mathrm{v}^1 \right) - \sum_k D_k^0 \wedge \mathrm{d} \frac{\delta \mathcal{L}}{\delta \tilde{D}_k^n} = 0 \tag{50}$$

$$\frac{\partial \tilde{D}_k^n}{\partial t} + \mathrm{d}(D_k^0 \wedge \tilde{u}^{n-1}) = 0 \tag{51}$$

Lie-Poisson Equations

$$\frac{\partial}{\partial t} \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}} + \mathrm{L}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}}} \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}} + \sum_k \mathbb{I}(D_k^0 \wedge \mathrm{d} \frac{\delta \mathcal{L}}{\delta \tilde{D}_k^n}) \barwedge \tilde{\mu}^n = 0 \tag{52}$$

$$\frac{\partial \tilde{D}_k^n}{\partial t} + \mathrm{d}(D_k^0 \wedge \tilde{u}^{n-1}) = 0 \tag{53}$$

recalling $\tilde{u}^{n-1} = \mathrm{i}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}}} \tilde{\mu}^n$.

Curl-Form Equations

$$\frac{\partial \mathrm{v}^1}{\partial t} + \mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}} \frac{\mathrm{d}\,\mathrm{v}^1}{D^0} + \sum_k \frac{D^0_k}{D^0} \wedge \mathrm{d} \frac{\delta \mathcal{L}}{\delta \tilde{D}^n_k} = 0 \tag{54}$$

$$\frac{\partial \tilde{D}^n_k}{\partial t} + \mathrm{d}(\frac{D^0_k}{D^0} \wedge \tilde{F}^{n-1}) = 0 \tag{55}$$

recalling $\mathbf{F}^0_{\mathrm{T}} = \left[\tilde{\star} \frac{\delta \mathcal{H}}{\delta \mathrm{v}^1})\right]^\sharp$ and $\tilde{F}^{n-1} = \frac{\delta \mathcal{H}}{\delta \mathrm{v}^1}$.

By making specific choices for the densities and the associated Lagrangian, many different fluid models can be obtained. Here we will focus on three: the rotating shallow water equations, the compressible Euler equations and the multicomponent compressible Euler equations.

In the models discussed below, one of the densities (the buoyancy density for thermal shallow water and the entropy density for compressible Euler and multicomponent compressible Euler) could be replaced by the related specific straight 0-form. For example, using the specific entropy straight 0-form η^0 instead of $\tilde{S}^n$, with the two related by $\tilde{S}^n = \tilde{\rho}^n \wedge \eta^0$. The details of this are left as an exercise for the interested reader.

4.1.1. Rotating Thermal Shallow Water Equations

For the rotating thermal shallow water equations there are two advected densities: the height twisted n-form $\tilde{h}^n$ and the buoyancy density twisted n-form $\tilde{S}^n$. In this model $\tilde{D}^n = \tilde{h}^n$. Additionally, $n = 2$, although we will retain the general n notation below.

Lagrangian The Lagrangian is

$$\mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, \tilde{h}^n, \tilde{S}^n] = \left\langle \tilde{h}^n, K^n \right\rangle + \left\langle h^0, \mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} R^1 \right\rangle - \frac{1}{2} \left\langle \tilde{S}^n, \tilde{h}^n \right\rangle - \left\langle \tilde{S}^n, \tilde{b}^n \right\rangle \tag{56}$$

where $\tilde{b}^n$ is the bottom topography twisted n-form. As usual, this is just kinetic energy minus potential energy, along with a term to account for rotation.

The functional derivatives of this Lagrangian are

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}^0_{\mathrm{T}}} = \flat_1(\mathbf{u}^0_{\mathrm{T}} + \mathbf{R}^0_{\mathrm{T}}) \barwedge \tilde{h}^n = \flat_1(h^0 \barwedge [\mathbf{u}^0_{\mathrm{T}} + \mathbf{R}^0_{\mathrm{T}}]) \barwedge \tilde{\mu}^n \qquad \frac{\delta \mathcal{L}}{\delta \tilde{h}^n} = K^0 + R^0 - \frac{1}{2} S^0 \qquad \frac{\delta \mathcal{L}}{\delta \tilde{S}^n} = -\frac{1}{2} h^0 - b^0 \tag{57}$$

where $R^0 = \mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}} R^1$.

Note that the momentum $\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} = \frac{\delta \mathcal{L}}{\delta \mathbf{u}^0_{\mathrm{T}}}$ can be written in several equivalent ways:

$$\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} = \tilde{\star}(\mathrm{m}^1)^\flat = \mathbb{I}\,\mathrm{m}^1 \barwedge \tilde{\mu}^n = \mathbf{m}^0_{\mathrm{T}^\star} \barwedge \tilde{\mu}^n \tag{58}$$

where $\mathbf{m}^0_{\mathrm{T}^\star} = \mathbb{I}(h^0 \wedge [u^1 + R^1])$ and $\mathrm{m}^1 = h^0 \wedge [u^1 + R^1]$.

This gives

$$\mathrm{v}^1 = \frac{1}{h^0} \wedge \mathrm{m}^1 = u^1 + R^1 \tag{59}$$

Since $n = 2$ for the TSWE model, in the v^1 formulations the velocity advection term $\mathrm{i}_{\mathbf{F}_{\mathrm{T}}^0} \frac{\mathrm{d}\,\mathrm{v}^1}{D^0}$ simplifies to

$$\tilde{q}^0 \wedge \tilde{F}^{n-1} \tag{60}$$

where $\tilde{F}^{n-1} = \frac{\delta \mathcal{H}}{\delta \mathrm{v}^1} = h^0 \wedge \tilde{u}^{n-1}$ is the twisted mass flux $n-1$-form, and $\tilde{q}^0$ is the twisted potential vorticity 0-form defined through $\tilde{h}^n \wedge \tilde{q}^0 = \mathrm{d}\,\mathrm{v}^1$.

Lie-Poisson Hamiltonian The Hamiltonian is

$$\mathsf{H}[\tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}, \tilde{h}^n, \tilde{S}^n] = \left\langle \tilde{h}^n, K^n \right\rangle + \frac{1}{2} \left\langle \tilde{S}^n, \tilde{h}^n \right\rangle + \left\langle \tilde{S}^n, \tilde{b}^n \right\rangle \tag{61}$$

with associated functional derivatives

$$\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}} = \mathbf{u}_{\mathrm{T}}^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{h}^n} = -K^0 + \frac{1}{2} S^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{S}^n} = \frac{1}{2} h^0 + b^0 \tag{62}$$

with $\mathbf{u}_{\mathrm{T}}^0 = \frac{1}{h^0} \mathbin{\dot{\wedge}} (\tilde{\star} \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}})$.

Curl-Form Hamiltonian The Hamiltonian is

$$\mathcal{H}[\mathrm{v}^1, \tilde{h}^n, \tilde{S}^n] = \left\langle \tilde{h}^n, K^n \right\rangle + \frac{1}{2} \left\langle \tilde{S}^n, \tilde{h}^n \right\rangle + \left\langle \tilde{S}^n, \tilde{b}^n \right\rangle \tag{63}$$

with associated functional derivatives

$$\frac{\delta \mathcal{H}}{\delta \mathrm{v}^1} = h^0 \wedge \tilde{u}^{n-1} \qquad \frac{\delta \mathcal{H}}{\delta \tilde{h}^n} = K^0 + \frac{1}{2} S^0 \qquad \frac{\delta \mathcal{H}}{\delta \tilde{S}^n} = \frac{1}{2} h^0 + b^0 \tag{64}$$

recalling $u^1 = \mathrm{v}^1 - R^1$.

4.1.2. Compressible Euler

For the compressible Euler equations, there are also two advected densities: the fluid density twisted n-form $\tilde{\rho}^n$ and entropy density twisted n-form $\tilde{S}^n$. Instead of entropy density, it would be possible use the density of a general entropic variable, see [5] for details.

Lagrangian The Lagrangian is

$$\mathcal{L}[\mathbf{u}_{\mathrm{T}}^0, \tilde{\rho}^n, \tilde{S}^n] = \langle \tilde{\rho}^n, K^n \rangle + \left\langle \rho^0, \mathrm{i}_{\mathbf{u}_{\mathrm{T}}^0} R^1 \right\rangle - \langle \rho^0, u^0(\alpha^0, \eta^0) \rangle - \langle \rho^0, \phi^0 \rangle \tag{65}$$

for an arbitrary specific internal energy $u^0 = u^0(\alpha^0, \eta^0)$ and ϕ^0 geopotential straight 0-form, where $\alpha^0 = \frac{1}{\rho^0}$ is the specific volume straight 0-form and $\eta^0 = \frac{S^0}{\rho^0}$ is the specific entropy straight 0-form.

The functional derivatives are

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}_{\mathrm{T}}^0} = \flat_1(\mathbf{u}_{\mathrm{T}}^0 + \mathbf{R}_{\mathrm{T}}^0) \wedge \tilde{\rho}^n = \flat_1(\rho^0 \wedge [\mathbf{u}_{\mathrm{T}}^0 + \mathbf{R}_{\mathrm{T}}^0]) \wedge \tilde{\mu}^n \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\rho}^n} = K^0 + R^0 - \phi^0 - g^0 \qquad \frac{\delta \mathcal{L}}{\delta \tilde{S}^n} = -\frac{\partial u^0}{\partial \eta^0} = -T^0 \tag{66}$$

with temperature straight 0-form $T^0 = \frac{\partial u^0}{\partial \eta^0}$ and pressure straight 0-form $p^0 = -\frac{\partial u^0}{\partial \alpha^0}$, where $g^0 = u^0 + p^0\alpha^0 - \eta^0 T^0$ is the specific Gibbs free energy straight 0-form.

Lie-Poisson Hamiltonian The Hamiltonian is

$$\mathsf{H}[\tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}, \tilde{\rho}^n, \tilde{S}^n] = \langle \tilde{\rho}^n, K^n \rangle + \langle \rho^0, u^0(\alpha^0, \eta^0) \rangle + \langle \rho^0, \phi^0 \rangle \tag{67}$$

with associated functional derivatives

$$\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}} = \mathbf{u}_{\mathrm{T}}^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{\rho}^n} = -K^0 + \phi^0 + g^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{S}^n} = T^0 \tag{68}$$

Curl-Form Hamiltonian The Hamiltonian is

$$\mathcal{H}[\mathrm{v}^1, \tilde{\rho}^n, \tilde{S}^n] = \langle \tilde{\rho}^n, K^n \rangle + \langle \rho^0, u^0(\alpha^0, \eta^0) \rangle + \langle \rho^0, \phi^0 \rangle \tag{69}$$

with associated functional derivatives

$$\frac{\delta \mathcal{H}}{\delta \mathrm{v}^1} = \rho^0 \wedge \tilde{u}^{n-1} \qquad \frac{\delta \mathcal{H}}{\delta \tilde{\rho}^n} = K^0 + \phi^0 + g^0 \qquad \frac{\delta \mathcal{H}}{\delta \tilde{S}^n} = T^0 \tag{70}$$

4.1.3. Multicomponent Compressible Euler

For the multicomponent compressible Euler equations with m components, there are $m+1$ advected densities: the entropy density twisted n-form $\tilde{S}^n$ along with m choices from the $m+1$ fluid density twisted n-forms: the total density $\tilde{\rho}^n$ and the m component densities $\tilde{\rho}_i^n$ ($i = 1, \ldots, m$). As before, instead of entropy density, it would be possible use the density of a general entropic variable, see [5] for details.

To cover all cases, pick one arbitrary component density and denote it with d. Then the two possible sets of fluid densities are ($\tilde{\rho}^n$,$\tilde{\rho}_s^n$) and ($\tilde{\rho}_d^n$,$\tilde{\rho}_s^n$) = ($\tilde{\rho}_i^n$) where $s = 1, \ldots, m-1$. If total density $\tilde{\rho}^n$ is chosen, then $\tilde{\rho}_d^n = \tilde{\rho}^n - \sum_s \tilde{\rho}_s^n$ in the Lagrangian below. If component d density is chosen, then $\tilde{\rho}^n = \sum_i \tilde{\rho}_i^n$ in the Lagrangian below.

Lagrangian The Lagrangian is

$$\mathcal{L} = \langle \tilde{\rho}^n, K^n \rangle + \left\langle \rho^0, \mathrm{i}_{\mathbf{u}_{\mathrm{T}}^0} R^1 \right\rangle - \left\langle \rho^0, u^0(\alpha^0, \eta^0, q_i^0) \right\rangle - \left\langle \rho^0, \phi^0 \right\rangle \tag{71}$$

for an arbitrary specific internal energy $u^0 = u^0(\alpha^0, \eta^0, q_i^0)$ where $q_i^0 = \frac{\rho_i^0}{\rho^0}$ is the specific concentration straight 0-form for component i.

The functional derivatives for the $\tilde{\rho}^n$ case are

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}_{\mathrm{T}}^0} = \flat_1(\mathbf{u}_{\mathrm{T}}^0 + \mathbf{R}_{\mathrm{T}}^0) \wedge \tilde{\rho}^n = \flat_1(\rho^0 \wedge [\mathbf{u}_{\mathrm{T}}^0 + \mathbf{R}_{\mathrm{T}}^0]) \wedge \tilde{\mu}^n \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\rho}^n} = K^0 + R^0 - \phi^0 - \mu_d^0 \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\rho}_s^n} = -(\mu_s^0 - \mu_d^0) \tag{72}$$

where $\mu_i^0 = \frac{\partial u^0}{\partial q_i^0}$ is related to the chemical potential for component i.

The functional derivatives for the $\tilde{\rho}_d^n$ case are

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}_{\mathrm{T}}^0} = \flat_1(\mathbf{u}_{\mathrm{T}}^0 + \mathbf{R}_{\mathrm{T}}^0) \wedge \tilde{\rho}^n = \flat_1(\rho^0 \wedge [\mathbf{u}_{\mathrm{T}}^0 + \mathbf{R}_{\mathrm{T}}^0]) \wedge \tilde{\mu}^n \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\rho}_i^n} = K^0 + R^0 - \phi^0 - \mu_i^0 \qquad \frac{\delta \mathcal{L}}{\delta \tilde{S}^n} = -T^0 \tag{73}$$

Lie-Poisson Hamiltonian The Hamiltonian is

$$\mathsf{H} = \langle \tilde{\rho}^n, K^n \rangle + \left\langle \rho^0, u^0(\alpha^0, \eta^0, q_i^0) \right\rangle + \left\langle \rho^0, \phi^0 \right\rangle \tag{74}$$

The functional derivatives for the $\tilde{\rho}^n$ case are

$$\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}} = \mathbf{u}_{\mathrm{T}}^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{\rho}^n} = -K^0 + \phi^0 + \mu_d^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{\rho}_s^n} = \mu_s^0 - \mu_d^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{S}^n} = T^0 \tag{75}$$

The functional derivatives for the $\tilde{\rho}_d^n$ case are

$$\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}_{\mathrm{T}^\star}^{\mathrm{n}}} = \mathbf{u}_{\mathrm{T}}^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{\rho}_i^n} = -K^0 + \phi^0 + \mu_i^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{S}^n} = T^0 \tag{76}$$

Curl-Form Hamiltonian The Hamiltonian is

$$\mathcal{H} = \langle \tilde{\rho}^n, K^n \rangle + \left\langle \rho^0, u^0(\alpha^0, \eta^0, q_i^0) \right\rangle + \left\langle \rho^0, \phi^0 \right\rangle \tag{77}$$

The functional derivatives for the $\tilde{\rho}^n$ case are

$$\frac{\delta \mathcal{H}}{\delta \mathrm{v}^1} = \rho^0 \wedge \tilde{u}^{n-1} \qquad \frac{\delta \mathcal{H}}{\delta \tilde{\rho}^n} = K^0 + \phi^0 + \mu_d^0 \qquad \frac{\delta \mathcal{H}}{\delta \tilde{\rho}_s^n} = \mu_s^0 - \mu_d^0 \qquad \frac{\delta \mathcal{H}}{\delta \tilde{S}^n} = T^0 \tag{78}$$

The functional derivatives for the $\tilde{\rho}_d^n$ case are

$$\frac{\delta \mathcal{H}}{\delta \mathrm{v}^1} = \rho^0 \wedge \tilde{u}^{n-1} \qquad \frac{\delta \mathcal{H}}{\delta \tilde{\rho}_i^n} = K^0 + \phi^0 + \mu_i^0 \qquad \frac{\delta \mathcal{H}}{\delta \tilde{S}^n} = T^0 \tag{79}$$

4.2. Non-Relativistic Magnetohydrodynamics/Electrohydrodynamics

Moving beyond purely advected densities, the magnetohydrodynamics (MHD) and electrohydrodynamics (EHD) models involve advected 2-forms: specifically the magnetic field straight 2-form B^2 and the electric field twisted 2-form $\tilde{D}^2$. A key assumption of both MHD and EHD is that these quantities are purely advected, and have no internal dynamics. In this case the various general formulations can be written as follows (just showing B^2 variant, the $\tilde{D}^2$ variant is clear from this):

Euler-Poincaré Equations

$$\frac{\partial}{\partial t}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} - \sum_k \mathbb{I}(D^0_k \wedge \mathrm{d}\frac{\delta\mathcal{L}}{\delta\tilde{D}^n_k}) \dot{\wedge} \tilde{\mu}^n - \mathbb{I}(\frac{\delta\mathcal{L}}{\delta B^2} \wedge \tilde{\star}\mathrm{d}B^2 + \mathrm{i}_{\mathbf{b}^0_T}\mathrm{d}\frac{\delta\mathcal{L}}{\delta B^2}) \dot{\wedge} \tilde{\mu}^n = 0 \quad (80)$$

$$\frac{\partial \tilde{\rho}^n}{\partial t} + \mathrm{d}(D^0_k \wedge \tilde{u}^{n-1}) = 0 \quad (81)$$

$$\frac{\partial B^2}{\partial t} + \mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}B^2 + \mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}\mathrm{d}B^2 = 0 \quad (82)$$

where $\mathbf{b}^0_T = (\tilde{\star}B^2)^\sharp$.

Kelvin-Noether Equations

$$D^0 \wedge \left(\frac{\partial}{\partial t}\mathrm{v}^1 + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\mathrm{v}^1\right) - \sum_k D^0_k \wedge \mathrm{d}\frac{\delta\mathcal{L}}{\delta\tilde{D}^n_k} + \frac{\delta\mathcal{L}}{\delta B^2} \wedge \tilde{\star}\mathrm{d}B^2 + \mathrm{i}_{\mathbf{b}^0_T}\mathrm{d}\frac{\delta\mathcal{L}}{\delta B^2} = 0 \quad (83)$$

$$\frac{\partial \tilde{D}^n_k}{\partial t} + \mathrm{d}(D^0_k \wedge \tilde{u}^{n-1}) = 0 \quad (84)$$

$$\frac{\partial B^2}{\partial t} + \mathrm{d}\,\mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}B^2 + \mathrm{i}_{\mathbf{u}^0_{\mathrm{T}}}\mathrm{d}B^2 = 0 \quad (85)$$

Lie-Poisson Equations

$$\frac{\partial}{\partial t}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \mathrm{L}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \sum_k \mathbb{I}(D^0_k \wedge \mathrm{d}\frac{\delta\mathcal{L}}{\delta\tilde{D}^n_k}) \dot{\wedge} \tilde{\mu}^n + \mathbb{I}(\frac{\delta\mathcal{L}}{\delta B^2} \wedge \tilde{\star}\mathrm{d}B^2 + \mathrm{i}_{\mathbf{b}^0_T}\mathrm{d}\frac{\delta\mathcal{L}}{\delta B^2}) \dot{\wedge} \tilde{\mu}^n = 0 \quad (86)$$

$$\frac{\partial \tilde{D}^n_k}{\partial t} + \mathrm{d}(D^0_k \wedge \tilde{u}^{n-1}) = 0 \quad (87)$$

$$\frac{\partial B^2}{\partial t} + \mathrm{d}\,\mathrm{i}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}B^2 + \mathrm{i}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\mathrm{d}B^2 = 0 \quad (88)$$

recalling $\tilde{u}^{n-1} = \mathrm{i}_{\frac{\delta \mathrm{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}}\tilde{\mu}^n$.

Curl-Form Equations

$$\frac{\partial \mathrm{v}^1}{\partial t} + \mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}} \frac{\mathrm{d}\,\mathrm{v}^1}{D^0} + \sum_k \frac{D^0_k}{D^0} \wedge \mathrm{d} \frac{\delta \mathcal{L}}{\delta \tilde{D}^n_k} + \frac{1}{D^0} \wedge \frac{\delta \mathcal{L}}{\delta B^2} \wedge \tilde{\star}\, \mathrm{d} B^2 + \frac{1}{D^0} \wedge \mathrm{i}_{\mathbf{b}^0_T} \mathrm{d} \frac{\delta \mathcal{L}}{\delta B^2} = 0 \tag{89}$$

$$\frac{\partial \tilde{D}^n_k}{\partial t} + \mathrm{d}\left(\frac{D^0_k}{D^0} \wedge \tilde{F}^{n-1}\right) = 0 \tag{90}$$

$$\frac{\partial B^2}{\partial t} + \mathrm{d}\,\mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}} \frac{B^2}{D^0} + \mathrm{i}_{\mathbf{F}^0_{\mathrm{T}}} \mathrm{d} \frac{B^2}{D^0} = 0 \tag{91}$$

recalling $\mathbf{F}^0_{\mathrm{T}} = \left[\tilde{\star}\left(\frac{\delta \mathcal{H}}{\delta \mathrm{v}^1}\right)\right]^\sharp$ and $\tilde{F}^{n-1} = \frac{\delta \mathcal{H}}{\delta \mathrm{v}^{\mathrm{T}}}$.

In what follows we will consider only a single component for the fluid, the extension to multiple components is left as a (straightforward) exercise.

4.2.1. MHD

MHD shares the same advected quantities as the compressible Euler equations, with the addition of the magnetic field straight 2-form B^2. Additionally, it assumes the *involution constraint* $\mathrm{d}B^2$, which simplifies some of the expressions above. Specifically, the $\mathrm{i}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}}} \mathrm{d}B^2$ term in the B^2 equation and the $\frac{\delta \mathcal{L}}{\delta B^2} \wedge \tilde{\star}\, \mathrm{d}B^2$ term in the momentum or velocity equations both drop out.

Lagrangian The Lagrangian is

$$\mathcal{L}[\mathbf{u}^0_{\mathrm{T}}, \tilde{\rho}^n, \tilde{S}^n, B^2] = \langle \tilde{\rho}^n, K^n \rangle - \left\langle \rho^0, u^0(\alpha^0, \eta^0) \right\rangle - \frac{1}{2} \left\langle B^2, B^2 \right\rangle \tag{92}$$

This is essentially the compressible Euler Lagrangian without rotation or geopotential plus an additional term $\frac{1}{2}\left\langle B^2, B^2 \right\rangle$ to account for magnetic energy. The functional derivatives are

$$\frac{\delta \mathcal{L}}{\delta \mathbf{u}^0_{\mathrm{T}}} = \flat_1(\mathbf{u}^0_{\mathrm{T}} + \mathbf{R}^0_{\mathrm{T}}) \mathbin{\dot{\wedge}} \tilde{\rho}^n \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\rho}^n} = K^0 + R^0 - g^0 \qquad \frac{\delta \mathcal{L}}{\delta \tilde{S}^n} = -T^0 \qquad \frac{\delta \mathcal{L}}{\delta B^2} = -\tilde{\star} B^2 \tag{93}$$

Lie-Poisson Hamiltonian The Hamiltonian is

$$\mathsf{H}[\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}, \tilde{\rho}^n, \tilde{S}^n, B^2] = \langle \tilde{\rho}^n, K^n \rangle + \left\langle \rho^0, u^0(\alpha^0, \eta^0) \right\rangle + \frac{1}{2} \left\langle B^2, B^2 \right\rangle \tag{94}$$

with associated functional derivatives

$$\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}\star}} = \mathbf{u}^0_{\mathrm{T}} \qquad \frac{\delta \mathsf{H}}{\delta \tilde{\rho}^n} = -K^0 + g^0 \qquad \frac{\delta \mathsf{H}}{\delta \tilde{S}^n} = T^0 \qquad \frac{\delta \mathsf{H}}{\delta B^2} = \tilde{\star} B^2 \tag{95}$$

Curl-Form Hamiltonian The Hamiltonian is

$$\mathcal{H}[\mathrm{v}^1,\tilde{\rho}^n,\tilde{S}^n,B^2] = \langle\tilde{\rho}^n,K^n\rangle + \left\langle\rho^0,u^0(\alpha^0,\eta^0)\right\rangle + \frac{1}{2}\left\langle B^2,B^2\right\rangle \tag{96}$$

with associated functional derivatives

$$\frac{\delta\mathcal{H}}{\delta \mathrm{v}^1} = \rho^0\wedge\tilde{u}^{n-1} \qquad \frac{\delta\mathcal{H}}{\delta\tilde{\rho}^n} = K^0+g^0 \qquad \frac{\delta\mathcal{H}}{\delta\tilde{S}^n} = T^0 \qquad \frac{\delta\mathcal{H}}{\delta B^2} = \tilde{\star}B^2 \tag{97}$$

4.2.2. EHD

EHD looks exactly like MHD, except with electric field twisted 2-form $\tilde{D}^2$ instead of B^2. The additional energy term is now $\frac{1}{2}\left\langle\tilde{D}^2,\tilde{D}^2\right\rangle$, while the functional derivatives are $\frac{\delta\mathcal{L}}{\delta\tilde{D}^2} = -\tilde{\star}\tilde{D}^2$ and $\frac{\delta\mathcal{H}}{\delta\tilde{D}^2} = \frac{\delta\mathsf{H}}{\delta\tilde{D}^2} = \tilde{\star}\tilde{D}^2$. Additionally, EHD assumes the *involution constraint* $\mathrm{d}\tilde{D}^2 = \tilde{Q}^3$ for the charge density twisted n-form $\tilde{Q}^3 = q\tilde{\rho}^3$ with q the (constant) charge per unit mass.

5. ADVECTED QUANTITIES WITH INTERNAL DYNAMICS

In order to treat fluid models that have advected quantities with internal dynamics (mainly charged fluid models), a more sophisticated type of semi-direct product theory is required. Specifically, we need $\mathfrak{X}(\Omega)\times\Lambda\times T^\star$, recalling that $T^\star$ is the cotangent bundle. This generates a formulation in terms of (1-form) potentials, which can be transformed into one based on the (2-form) fields B^2 and $\tilde{D}^2$. In electrodynamics the 1-forms are the potentials, and the 2-forms are the fields. The details of this are quite complicated, and can be found in [6, 7, 19]. Redoing these calculations in the language of bundle-valued differential forms is then just an exercise in exterior calculus.

The resulting Euler-Poincaré and transport equations are:

$$\frac{\partial}{\partial t}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} - \sum\frac{\delta\mathcal{L}}{\delta a}\diamond a - \sum\frac{\delta\mathcal{L}}{\delta\mathbf{b}}\diamond\mathbf{b} - \frac{\delta\mathcal{L}}{\delta B^2}\diamond B^2 - \frac{\delta\mathcal{L}}{\delta\tilde{D}^2}\diamond\tilde{D}^2 = 0 \tag{98}$$

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}a = 0 \tag{99}$$

$$\frac{\partial\mathbf{b}}{\partial t} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\mathbf{b} = 0 \tag{100}$$

$$\frac{\partial B^2}{\partial t} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}B^2 - \mathrm{d}\frac{\delta\mathcal{L}}{\delta\tilde{D}^2} = 0 \tag{101}$$

$$\frac{\partial\tilde{D}^2}{\partial t} + \mathrm{L}_{\mathbf{u}^0_{\mathrm{T}}}\tilde{D}^2 + \mathrm{d}\frac{\delta\mathcal{L}}{\delta B^2} = 0 \tag{102}$$

The corresponding Lie-Poisson Hamiltonian equations are:

$$\frac{\partial}{\partial t}\tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star} + \sum \frac{\delta \mathsf{H}}{\delta a} \diamond a + \sum \frac{\delta \mathsf{H}}{\delta \mathbf{b}} \diamond \mathbf{b} + \frac{\delta \mathsf{H}}{\delta B^2} \diamond B^2 + \frac{\delta \mathsf{H}}{\delta \tilde{D}^2} \diamond \tilde{D}^2 = 0 \tag{103}$$

$$\frac{\partial a}{\partial t} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} a = 0 \tag{104}$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} \mathbf{b} = 0 \tag{105}$$

$$\frac{\partial B^2}{\partial t} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} B^2 + \mathrm{d}\frac{\delta \mathsf{H}}{\delta \tilde{D}^2} = 0 \tag{106}$$

$$\frac{\partial \tilde{D}^2}{\partial t} + \mathrm{L}_{\frac{\delta \mathsf{H}}{\delta \tilde{\mathbf{m}}^{\mathrm{n}}_{\mathrm{T}^\star}}} \tilde{D}^2 - \mathrm{d}\frac{\delta \mathsf{H}}{\delta B^2} = 0 \tag{107}$$

An interesting feature of these equations to note is the existence of involution constraints $\mathrm{d}B^2 = 0$ and $\mathrm{d}\tilde{D}^2 = \tilde{Q}^3 = \sum q_k \tilde{a}^n_k$ (where q_k is the constant charge per unit mass associated with density $\tilde{a}^n_k$) that, if satisfied at some initial time, are satisfied at all final times. Additionally, the Lie derivatives (and associated $\diamond$ terms in the momentum equation) will generate non-conservative terms associated with these constraints, that make proving hyperbolicity quite tricky.

As before, the system must be closed by specifying a Lagrangian and a set of advected quantities. This is straightforward, although lengthy, and can treat quite complicated systems such as Euler-Maxwell and general electrohydrodynamics with nonlinear constutitive relationships. See [20, 12, 25] for some examples of this written in terms of vector calculus.

6. RESULTS AND DISCUSSION

There were two main goals for this project:

1. Develop geometric mechanics formulations for momentum-based descriptions of nonrelativistic fluids, utilizing a differential geometry/exterior calculus treatment of momentum and a space+time splitting.
2. Explore the connections between geometric mechanics formulations and the more conventional view of momentum-based descriptions as a system of hyperbolic conservation laws (Godunov form).

6.1. Goal 1

Geometric mechanics formulations for velocity-based description of fluids can be written using in terms of exterior calculus using SVDFs, which makes application on general manifolds and/or arbitrary coordinate systems simple, and provides a good basis for structure-preserving discretization. Unfortunately, existing geometric mechanics formulations for momentum-based

descriptions of fluids (for example, [13]) are largely formulated in terms of vector or tensor calculus instead of exterior calculus. One exception is the work in [10], which treats momentum as a covector-valued twisted n-form and provides a fully exterior calculus based approach. However, the latter is specific to a few equation sets and provides only the variational/Lagrangian formulation.

In this project, we have addressed these deficiencies and provided the full suite of geometric mechanics formulations (variational/Lagrangian, Lie-Poisson Hamiltonian and Curl-Form Hamiltonian) written in terms of exterior calculus and treating momentum-based descriptions. This was done for a fairly general version of semi-direct product theory sufficient to cover a wide range of both neutral and charged fluid models, including compressible Euler, magnetohydrodynamics and Euler-Maxwell. An important part of this work was an explicit articulation of the proper way to represent the transformation between $\tilde{\mathbf{m}}^{n}_{T^\star}$ and v^1 in terms of exterior calculus, which connects Lie-Poisson and Curl-Form formulations.

6.2. Goal 2

The traditional view of momentum-based descriptions of fluids is through Godunov form: a hyperbolic system of conservation laws [4, 16, 19, 20]. A natural question to ask then is: what is the connection between Godunov form and geometric mechanics formulations? Exploring this connection was the second goal of this project. Unfortunately. this was unsuccessful due to technical difficulties. Specifically:

- Attempting to generalize the approach in [19] (termed Riemannian, since it is based on forming a metric tensor from the functional derivatives of the Hamiltonian) for 1D Hamiltonian systems of hydrodynamic form fails when attempting to go beyond 1D. Specifically, the resulting metric becomes degenerate.

- The theory of hyperbolic equations on general manifolds is poorly developed, with only limited work done [1, 2, 15, 17]. This is especially true when considering approaches based on intrinsic quantities and operators such as differential forms, exterior derivatives, etc.

Despite these difficulties, several promising directions for future research were identified:

- A strong connection between convexity of the Hamiltonian and hyperbolic has been observed. This might point the way towards a more general theory.

- We are close to developing a general conservation-type form written in terms of exterior calculus based on the covariant exterior derivative. This might provide a path towards writing the equations as a quasi-linear system of first order PDEs with an associated extra conservation law (at least in $\mathbb{R}^3$ where coordinate expressions are greatly simplified), for which several methods are known to prove hyperbolicity.

More information about the results can be found in Appendix B.

7. ANTICIPATED OUTCOMES AND IMPACTS

There are two publications related to this work currently in preparation:

- Geometric mechanics formulations for momentum-based descriptions of fluids with advected quantities: covers the case of simple semi-direct product theory for an arbitrary set of advected SVDFs and BVDFs.

- Geometric mechanics formulations for momentum-based descriptions of charged fluids with electrodynamics: extends the first paper to handle more complicated semi-direct product theory where (some of) the advected quantities have internal interactions as well (such as electrohydrodynamics).

Additionally, this work will be presented at an upcoming relevant domain-specific conference: SIAM CSE 23, at which the PI is organizing a mini-symposium on geometric mechanics formulations and structure-preserving discretizations for fluids.

The logical next step is to utilize the exterior calculus based understanding of momentum coupled with geometric mechanics formulations to develop (novel) structure-preserving discretizations of momentum. This is the main subject of a successful FY23 CIS LDRD "Structure-preserving discretizations for momentum-based formulations of fluids".

Beyond utilizing the new framework for discretization, another interesting area of follow-up work is the extension to new physical models. Specifically,

1. A version of this work with a fully spacetime treatment (without a space+time splitting) for both relativistic and non-relativistic fluids. This should include related structure-preserving discretizations.

2. A study of differential geometry based geometric mechanics formulations for general relativistic and non-relativistic kinetic/particle models, including both particle-in-cell (PIC) and fluid/particle hybrid models. This should include related structure-preserving discretizations.

3. A study of differential geometry based geometric mechanics formulations for general relativistic and non-relativistic continuum mechanics in ALE coordinates, including both solid mechanics and fluid/solid interaction. This should include related structure-preserving discretizations.

Combined, this follow-up work would provide a comprehensive treatment of differential geometry based geometric mechanics formulations for relativistic and non-relativistic particle and continuum mechanics models, and associated structure-preserving discretizations.

More information about the anticipated outcomes and impacts can be found in Appendix B.

8. CONCLUSIONS

In this CIS Late Start LDRD project, geometric mechanics formulations (variational/Lagrangian, Lie-Poisson Hamiltonian and Curl-Form Hamiltonian) for momentum-based descriptions of nonrelativistic fluids based on exterior calculus were developed, in a split space+time setting. The key insight underlying these new formulation was the representation of fluid velocity as a vector-valued straight 0-form and momentum as a covector-valued twisted n-form, both examples of BVDFs. These formulations were developed using a fairly general version of semi-direct product theory (a special case of matched pair dynamics), that was sufficient to treat a wide range both neutral and charged fluid models, including compressible Euler, magnetohydrodynamics and Euler-Maxwell. By using exterior calculus, the approach works seamlessly for general manifolds and arbitrary (Eulerian) coordinate systems. Additionally, it provides a natural basis for the development of structure-preserving discretization schemes: the new exterior calculus based geometric mechanics formulations will feature heavily in an FY23 CIS LDRD project "Structure-preserving discretizations for momentum-based formulations of fluids".

Additionally, an attempt was made to connect geometric mechanics formulations and Godunov form, but this was unsuccessful at progressing past the current state of the art due to technical difficulties. However, several promising directions were identified for future work in this area.

Beyond the project working on discretization of BVDFs, other key followup work is extending this approach to new physical models: spacetime treatments of relavisitic and nonrelativistic fluids, spacetime and space+time relativistic and nonrelativistic kinetic models, and solid mechanics. Taken together, these developments would provide a complete set of geometric mechanics formulations for relativistic and non-relativistic particle and continuum mechanics models based on exterior calculus, along with associated structure-preserving discretizations.

Finally, two publications covering this work are currently being written, and presentation of these results at an upcoming relevant domain-specific conference is planned.

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APPENDIX A. BRIEF REVIEW OF DIFFERENTIAL GEOMETRY AND EXTERIOR CALCULUS

To start, let us consider a fixed (stationary) domain $\Omega \subset \mathcal{M}$ with boundary $\partial\Omega$ where $\mathcal{M}$ is an arbitrary oriented Riemannian manifold with *metric* $\mathbf{g}$. Associated with the metric we have a unique *Leiv-Civita connection* X, which induces a *covariant derivative* ∇_{X}. There is also a volume form $\tilde{\mu}^n$ associated with $\mathbf{g}$. Let E denote an arbitrary *vector bundle* on Ω, and $E^\star$ it's dual. Of particular interest are the *tangent bundle* $T = T_{\mathbf{x}}\mathcal{M}$, the *cotangent bundle* $T^\star = T^*_{\mathbf{x}}\mathcal{M}$ (note T and $T^\star$ are dual) and the real bundle $\mathbb{R}$ (which is its own dual). A review of these concepts can be found in any standard textbook on differential geometry, such as [8, 22].

The main object of interest in this report are *(vector) bundle-valued differential forms* (BVDFs)[2] [10, 8, 14, 24, 9]. We will work with *oriented* differential forms in this report, where *twisted* forms change sign under a change of orientation, while *straight* forms do not. If the vector bundle is $\mathbb{R}$, then we obtain as an important special case the scalar-valued differential forms (SVDFs) [23, 3].

To fix notation let $\Lambda^k(E)$ denote the space of straight E-valued k-forms and $\tilde{\Lambda}^k(E)$ the space of twisted E-valued k-forms, with elements of these spaces denoted by

$$\mathbf{a}^k_E \in \Lambda^k(E) \qquad \tilde{\mathbf{a}}^k_E \in \tilde{\Lambda}^k(E) \tag{108}$$

Similarly, let $\Lambda^k(E^\star)$ denote the space of straight $E^\star$-valued k-forms and $\tilde{\Lambda}^k(E^\star)$ the space of twisted $E^\star$-valued k-forms, with elements of these spaces denoted by

$$\mathbf{a}^k_{E^\star} \in \Lambda^k(E^\star) \qquad \tilde{\mathbf{a}}^k_{E^\star} \in \tilde{\Lambda}^k(E^\star) \tag{109}$$

In the special case of $E = \mathbb{R}$, we have the straight scalar-valued k-forms Λ^k and the twisted scalar-valued k-forms $\tilde{\Lambda}^k$, with elements denoted by

$$a^k \in \Lambda^k \qquad \tilde{a}^k \in \tilde{\Lambda}^k \tag{110}$$

It will be useful to introduce special notation for the case $E = T$ and $E^\star = T^\star$. Let $\Lambda^k(T)$ denote the space of straight vector-valued k-forms, $\Lambda^k(T^\star)$ the space of straight covector-valued k-forms, $\tilde{\Lambda}^k(T)$ the space of twisted vector-valued k-forms, and $\tilde{\Lambda}^k(T^\star)$ the space of twisted covector-valued k-forms. Elements of these spaces are denoted by

$$\mathbf{a}^k_T \in \Lambda^k(T) \qquad \mathbf{a}^k_{T^\star} \in \Lambda^k(T^\star) \qquad \tilde{\mathbf{a}}^k_T \in \tilde{\Lambda}^k(T) \qquad \tilde{\mathbf{a}}^k_{T^\star} \in \tilde{\Lambda}^k(T^\star) \tag{111}$$

A.1. Operators

There are several useful operators that can be defined for BVDFs, many of which reduce to the familiar operators on SVDFs from exterior calculus when $E = \mathbb{R}$. In this section we do not give explicit formulas for these operators in terms of a chart basis, these can be found in [8, 9] or other standard textbooks on differential geometry.

[2] In this work we need only tangent (i.e. vector), cotangent (i.e. covector) and real-valued (i.e. SVDFs) versions of BVDFs.

A.1.1. Covariant Wedge Product

The first operator is the *covariant wedge product* $\dot{\wedge}$:

$$\begin{aligned}
\dot{\wedge} &: \Lambda^k(E_1), \Lambda^l(E_2) \to \Lambda^{k+l}(E_1 \otimes E_2) && (112)\\
\dot{\wedge} &: \Lambda^k(E_1), \tilde{\Lambda}^l(E_2) \to \tilde{\Lambda}^{k+l}(E_1 \otimes E_2) && (113)\\
\dot{\wedge} &: \tilde{\Lambda}^k(E_1), \Lambda^l(E_2) \to \tilde{\Lambda}^{k+l}(E_1 \otimes E_2) && (114)\\
\dot{\wedge} &: \tilde{\Lambda}^k(E_1), \tilde{\Lambda}^l(E_2) \to \Lambda^{k+l}(E_1 \otimes E_2) && (115)
\end{aligned}$$

In the case that $E_1 = E$ and $E_2 = E^\star$ (or visa versa) then $E_1 \otimes E_2 = \mathbb{R}$ and therefore $\Lambda^{k+l}(E) = \Lambda^{k+1}$ and $\tilde{\Lambda}^{k+l}(E) = \tilde{\Lambda}^{k+1}$ in the above formulas.

We have the important antisymmetry property of the covariant wedge product for $\mathbb{R}$-valued k-forms and E-valued l-forms

$$w^k \dot{\wedge} \mathbf{a}^l_E = (-1)^{kl} \mathbf{a}^l_E \dot{\wedge} w^k \tag{116}$$

with a similar definition for twisted forms.

In the case that $E_1 = E_2 = \mathbb{R}$, $\dot{\wedge}$ reduces to the familiar *wedge product* $\wedge$ from exterior calculus.

A.1.2. Covariant Hodge Star

The second operator is the *covariant Hodge star* $\tilde{\star}$:

$$\begin{aligned}
\tilde{\star} &: \Lambda^k(E) \to \Lambda^{n-k}(E^\star) && (117)\\
\tilde{\star} &: \tilde{\Lambda}^k(E) \to \tilde{\Lambda}^{n-k}(E^\star) && (118)\\
\tilde{\star} &: \Lambda^k(E^\star) \to \Lambda^{n-k}(E) && (119)\\
\tilde{\star} &: \tilde{\Lambda}^k(E^\star) \to \tilde{\Lambda}^{n-k}(E) && (120)
\end{aligned}$$

This has the important property that

$$\tilde{\star}\tilde{\star} = (-1)^{k(n-k)} \tag{121}$$

One way to define the covariant Hodge star is through

$$\mathbf{a}^k_E \dot{\wedge} \tilde{\star} \mathbf{b}^k_E = < \mathbf{a}^k_E, \mathbf{b}^k_E > \tilde{\mu}^n \tag{122}$$

where $<>$ is the pointwise L_2 inner product.

In the case when $E = E^\star = \mathbb{R}$, $\tilde{\star}$ reduces to the familiar *twisted Hodge star* $\tilde{\star}$ from exterior calculus.

A.1.3. Covariant Pairing and (L_2) Inner Product

The covariant wedge product can be used to define a *covariant pairing* $\langle\langle ,\rangle\rangle_\chi$:

$$\left\langle\left\langle \mathbf{a}^k_E, \tilde{\mathbf{b}}^{n-k}_{E^\star}\right\rangle\right\rangle_\chi = \int_\Omega \mathbf{a}^k_E \dot\wedge \tilde{\mathbf{b}}^{n-k}_{E^\star} \tag{123}$$

$$\left\langle\left\langle \tilde{\mathbf{a}}^k_E, \mathbf{b}^{n-k}_{E^\star}\right\rangle\right\rangle_\chi = \int_\Omega \tilde{\mathbf{a}}^k_E \dot\wedge \mathbf{b}^{n-k}_{E^\star} \tag{124}$$

$$\left\langle\left\langle \mathbf{a}^k_{E^\star}, \tilde{\mathbf{b}}^{n-k}_{E}\right\rangle\right\rangle_\chi = \int_\Omega \mathbf{a}^k_{E^\star} \dot\wedge \tilde{\mathbf{b}}^{n-k}_{E} \tag{125}$$

$$\left\langle\left\langle \tilde{\mathbf{a}}^k_{E^\star}, \mathbf{b}^{n-k}_{E}\right\rangle\right\rangle_\chi = \int_\Omega \tilde{\mathbf{a}}^k_{E^\star} \dot\wedge \mathbf{b}^{n-k}_{E} \tag{126}$$

In the case when $E = E^\star = \mathbb{R}$, $\langle\langle ,\rangle\rangle_\chi$ reduces to the familiar *topological pairing* $\langle\langle ,\rangle\rangle$ from exterior calculus.

The covariant wedge product and the covariant Hodge star together induce a *covariant (L_2) inner product* $\langle ,\rangle_\chi$:

$$\left\langle \mathbf{a}^k_E, \mathbf{b}^k_E\right\rangle_\chi = \int_\Omega \mathbf{a}^k_E \dot\wedge \tilde\star \mathbf{b}^k_E \tag{127}$$

$$\left\langle \tilde{\mathbf{a}}^k_E, \tilde{\mathbf{b}}^k_E\right\rangle_\chi = \int_\Omega \tilde{\mathbf{a}}^k_E \dot\wedge \tilde\star \tilde{\mathbf{b}}^k_E \tag{128}$$

$$\left\langle \mathbf{a}^k_{E^\star}, \mathbf{b}^k_{E^\star}\right\rangle_\chi = \int_\Omega \mathbf{a}^k_{E^\star} \dot\wedge \tilde\star \mathbf{b}^k_{E^\star} \tag{129}$$

$$\left\langle \tilde{\mathbf{a}}^k_{E^\star}, \tilde{\mathbf{b}}^k_{E^\star}\right\rangle_\chi = \int_\Omega \tilde{\mathbf{a}}^k_{E^\star} \dot\wedge \tilde\star \tilde{\mathbf{b}}^k_{E^\star} \tag{130}$$

The covariant pairing and the covariant inner product are related by

$$\left\langle\left\langle \mathbf{a}^k_E, \tilde{\mathbf{b}}^{n-k}_{E^\star}\right\rangle\right\rangle_\chi = \left\langle \mathbf{a}^k_E, \mathbf{b}^k_E\right\rangle_\chi \tag{131}$$

$$\left\langle\left\langle \tilde{\mathbf{a}}^k_E, \mathbf{b}^{n-k}_{E^\star}\right\rangle\right\rangle_\chi = (-1)^{k(n-k)}\left\langle \tilde{\mathbf{a}}^k_E, \tilde{\mathbf{b}}^k_E\right\rangle_\chi \tag{132}$$

$$\left\langle\left\langle \mathbf{a}^k_{E^\star}, \tilde{\mathbf{b}}^{n-k}_{E}\right\rangle\right\rangle_\chi = \left\langle \mathbf{a}^k_{E^\star}, \mathbf{b}^k_{E^\star}\right\rangle_\chi \tag{133}$$

$$\left\langle\left\langle \tilde{\mathbf{a}}^k_{E^\star}, \mathbf{b}^{n-k}_{E}\right\rangle\right\rangle_\chi = (-1)^{k(n-k)}\left\langle \tilde{\mathbf{a}}^k_{E^\star}, \tilde{\mathbf{b}}^k_{E^\star}\right\rangle_\chi \tag{134}$$

In the case when $E = E^\star = \mathbb{R}$, $\langle ,\rangle_\chi$ reduces to the familiar *(L_2) inner product* $\langle ,\rangle$ from exterior calculus.

A.1.4. Covariant Exterior Derivative

The main differential operator needed is the *covariant exterior derivative* d_{X}:

$$\begin{aligned}
\mathrm{d}_{\mathrm{X}} &: \Lambda^k(E) \to \Lambda^{k+1}(E) && (135)\\
\mathrm{d}_{\mathrm{X}} &: \tilde{\Lambda}^k(E) \to \tilde{\Lambda}^{k+1}(E) && (136)\\
\mathrm{d}_{\mathrm{X}} &: \Lambda^k(E^\star) \to \Lambda^{k+1}(E^\star) && (137)\\
\mathrm{d}_{\mathrm{X}} &: \tilde{\Lambda}^k(E^\star) \to \tilde{\Lambda}^{k+1}(E^\star) && (138)
\end{aligned}$$

The covariant exterior derivative d_{X} satisfies a Leibniz rule:

$$\mathrm{d}_{\mathrm{X}}(\mathbf{a}_E^k \mathbin{\dot\wedge} w^l) = \mathrm{d}_{\mathrm{X}}\,\mathbf{a}_E^k \mathbin{\dot\wedge} w^l + (-1)^k \mathbf{a}_E^k \mathbin{\dot\wedge} \mathrm{d}\,w^l \tag{139}$$

Unlike the exterior derivative, we do not have $\mathrm{d}_{\mathrm{X}}\,\mathrm{d}_{\mathrm{X}} = 0$ in general. Instead, $\mathrm{d}_{\mathrm{X}}\,\mathrm{d}_{\mathrm{X}}$ measures the curvature of E or $E^\star$. However, in the case of a flat bundle (such as $\mathbb{R}$) then $\mathrm{d}_{\mathrm{X}}\,\mathrm{d}_{\mathrm{X}} = 0$.

In the case that $E = T$, $E^\star = T^\star$ and $k = 0$, d_{X} reduces to the familiar *covariant derivative* ∇_{X}. In the case that $E = E^\star = \mathbb{R}$, d_{X} reduces to the familiar *exterior derivative* d.

A.1.5. Covariant Codifferential

Given the covariant inner product and the covariant exterior derivative, it is possible to define a *covariant codifferential* $\mathrm{d}_{\mathrm{X}}^\star$:

$$\begin{aligned}
\mathrm{d}_{\mathrm{X}}^\star &: \Lambda^k(E) \to \Lambda^{k-1}(E) && (140)\\
\mathrm{d}_{\mathrm{X}}^\star &: \tilde{\Lambda}^k(E) \to \tilde{\Lambda}^{k-1}(E) && (141)\\
\mathrm{d}_{\mathrm{X}}^\star &: \Lambda^k(E^\star) \to \Lambda^{k-1}(E^\star) && (142)\\
\mathrm{d}_{\mathrm{X}}^\star &: \tilde{\Lambda}^k(E^\star) \to \tilde{\Lambda}^{k-1}(E^\star) && (143)
\end{aligned}$$

This is the (formal) adjoint of d_{X} with respect to the covariant inner product:

$$\left\langle \mathrm{d}_{\mathrm{X}}\,\mathbf{a}_E^{k-1}, \mathbf{b}_E^k \right\rangle_\chi = \left\langle \mathbf{a}_E^{k-1}, \mathrm{d}_{\mathrm{X}}^\star\,\mathbf{b}_E^k \right\rangle_\chi \tag{144}$$

The covariant codifferential can be defined in terms of the covariant Hodge star and the covariant exterior derivative as

$$\mathrm{d}_{\mathrm{X}}^\star = (-1)^{n(k-1)+1}\,\tilde{\star}\,\mathrm{d}_{\mathrm{X}}\,\tilde{\star} \tag{145}$$

Just like the covariant exterior derivative, $\mathrm{d}_{\mathrm{X}}^\star\,\mathrm{d}_{\mathrm{X}}^\star \neq 0$ unless the vector bundle is flat.

In the case that $E = E^\star = \mathbb{R}$, reduces to the familiar *codifferential* δ from exterior calculus.

A.1.6. Trace and Inclusion

When $E^\star = T^\star$, there is an interpretation of $\Lambda^k(T^\star)$ as rank $(0, 1+k)$ tensors that are antisymmetric in last k indices. Similarly, Λ^k are rank $(0,k)$ tensors that are totally antisymmetric.

This motivates the inclusion operator $\mathbb{I}$:

$$\mathbb{I} \;:\; \Lambda^k \to \Lambda^{k-1}(T^\star) \tag{146}$$
$$\mathbb{I} \;:\; \tilde{\Lambda}^k \to \tilde{\Lambda}^{k-1}(T^\star) \tag{147}$$

and the trace operator $\mathbb{T}$:

$$\mathbb{T} \;:\; \Lambda^{k-1}(T^\star) \to \Lambda^k \tag{148}$$
$$\mathbb{T} \;:\; \tilde{\Lambda}^{k-1}(T^\star) \to \tilde{\Lambda}^k \tag{149}$$

where $\mathbb{T}$ is the (formal) adjoint of $\mathbb{I}$ under the inner product

$$\left\langle \mathbb{T}\,\mathbf{a}^{k-1}_{T^\star}, b^k \right\rangle = \left\langle \mathbf{a}^{k-1}_{T^\star}, \mathbb{I}\,b^k \right\rangle_\chi \tag{150}$$
$$\left\langle \mathbb{T}\,\tilde{\mathbf{a}}^{k-1}_{T^\star}, \tilde{b}^k \right\rangle = \left\langle \tilde{\mathbf{a}}^{k-1}_{T^\star}, \mathbb{I}\,\tilde{b}^k \right\rangle_\chi \tag{151}$$

It is easy to see that $\mathbb{I}$ is essentially an identity operation, but $\mathbb{T}$ is not since there are $(0, 1+k)$ tensors that are antisymmetric in last k indices which are not $(0, k+1)$ totally antisymmetric tensors. Therefore, we have $\mathbb{T}\,\mathbb{I} = \mathrm{id}$ but $\mathbb{I}\,\mathbb{T} \neq \mathrm{id}$, except for the special case of 0-forms where $(0, 1+0) = (0,1)$.

An important relationship is that trace commutes with the covariant exterior derivative

$$\mathbb{T}\,\mathrm{d}_\mathrm{X}\,\mathbf{a}^k_{T^\star} = \mathrm{d}\,\mathbb{T}\,\mathbf{a}^k_{T^\star} \tag{152}$$

with a similar formula for $\tilde{\mathbf{a}}^k_{T^\star}$.

It is also useful to introduce the coinclusion $\mathbb{I}^\star = \mathbf{g}^{ii}\,\mathbb{I}$

$$\mathbb{I}^\star \;:\; \Lambda^k \to \Lambda^{k+1}(T^\star) \tag{153}$$
$$\mathbb{I}^\star \;:\; \tilde{\Lambda}^k \to \tilde{\Lambda}^{k+1}(T^\star) \tag{154}$$

and the cotrace $\mathbb{T}^\star$

$$\mathbb{T}^\star \;:\; \Lambda^{k+1}(T^\star) \to \Lambda^k \tag{155}$$
$$\mathbb{T}^\star \;:\; \tilde{\Lambda}^{k+1}(T^\star) \to \tilde{\Lambda}^k \tag{156}$$

where again $\mathbb{T}^\star$ is the (formal) adjoint of $\mathbb{I}^\star$ under the inner product

$$\left\langle \mathbb{T}^\star\,\mathbf{a}^{k+1}_{T^\star}, b^k \right\rangle = \left\langle \mathbf{a}^{k+1}_{T^\star}, \mathbb{I}^\star\,b^k \right\rangle_\chi \tag{157}$$
$$\left\langle \mathbb{T}^\star\,\tilde{\mathbf{a}}^{k+1}_{T^\star}, \tilde{b}^k \right\rangle = \left\langle \tilde{\mathbf{a}}^{k+1}_{T^\star}, \mathbb{I}^\star\,\tilde{b}^k \right\rangle_\chi \tag{158}$$

We have the following relationships between d_X, $\mathrm{d}_X^\star$, d and δ:

$$\mathrm{d}_X \mathbb{I} a^k = \mathbb{I}^\star \delta a^k \tag{159}$$
$$\mathbb{I} \mathrm{d} a^k = \mathrm{d}_X^\star \mathbb{I}^\star a^k \tag{160}$$

with similar formulas for $\tilde{a}^k$.

A.1.7. Covariant Flat and Sharp

To convert between T-valued k-forms and $T^\star$-valued k-forms it is useful to introduce the *covariant flat* $\flat_1$ and *covariant sharp* $\sharp_1$ operators:

$$\flat_1 : \Lambda^k(T) \to \Lambda^k(T^\star) \tag{161}$$
$$\sharp_1 : \Lambda^k(T^\star) \to \Lambda^k(T) \tag{162}$$

with similar formulas for twisted forms.

These can be related to the more familiar *flat* $\flat$ and *sharp* $\sharp$ operators

$$\flat : \Lambda^0(T) \to \Lambda^1 \tag{163}$$
$$\sharp : \Lambda^1 \to \Lambda^0(T) \tag{164}$$

through the definitions

$$\flat = \mathbb{T} \flat_1 \tag{165}$$
$$\sharp = \sharp_1 \mathbb{I} \tag{166}$$

These satisfy

$$\flat\sharp = \sharp\flat = \flat_1 \sharp_1 = \sharp_1 \flat_1 = \mathrm{id} \tag{167}$$

A.1.8. Interior Product

For scalar-valued forms, it is useful to introduce the *interior product* operator $\mathrm{i}_{\mathbf{x}_T^0}$

$$\mathrm{i}_{\mathbf{x}_T^0} : \Lambda^0(T), \Lambda^k \to \Lambda^{k-1} \tag{168}$$
$$\mathrm{i}_{\mathbf{x}_T^0} : \Lambda^0(T), \tilde{\Lambda}^k \to \tilde{\Lambda}^{k-1} \tag{169}$$

where $\mathbf{x}_T^0 \in \Lambda^0(T)$ is a vector field (vector-valued 0-form).

The interior product can be written in terms of the Hodge star and the wedge product using Hirani's formula

$$\mathrm{i}_{\mathbf{u}_\mathrm{T}^0} a^k = (-1)^{k(n-k)} \tilde{\star}(\tilde{\star} a^k \wedge (\mathbf{u}_\mathrm{T}^0)^\flat) \tag{170}$$

with a similar formula for $\tilde{a}^k$.

The interior product has a Leibniz rule

$$\mathrm{i}_{\mathbf{u}^0_T}(a^k \wedge b^l) = a^k \wedge \mathrm{i}_{\mathbf{u}^0_T} b^l + (-1)^k \mathrm{i}_{\mathbf{u}^0_T} a^k \wedge b^l \tag{171}$$

with a similar formula for $\tilde{a}^k$ and $\tilde{b}^l$.

The action of the metric on vectors can be written in terms of the interior product using

$$\mathbf{g}(\mathbf{x}^0_T, \mathbf{y}^0_T) = \mathrm{i}_{\mathbf{x}^0_T} y^1 = \mathrm{i}_{\mathbf{y}^0_T} x^1 \tag{172}$$

There is an extension of the interior product from $\Lambda^0(T)$ to $\Lambda^l(T)$ (i.e. $\mathrm{i}_{\mathbf{x}^l_T} a^k$) but this generalization is not needed in this work.

A.1.9. Lie Derivative

The final operator we need is the *Lie derivative* $\mathrm{L}_{\mathbf{x}^0_T}$:

$$\mathrm{L}_{\mathbf{x}^0_T} \quad : \quad \Lambda^0(T), \Lambda^k \to \Lambda^k \tag{173}$$

$$\mathrm{L}_{\mathbf{x}^0_T} \quad : \quad \Lambda^0(T), \tilde{\Lambda}^k \to \tilde{\Lambda}^k \tag{174}$$

$$\mathrm{L}_{\mathbf{x}^0_T} \quad : \quad \Lambda^0(T), \Lambda^k(E) \to \Lambda^k(E) \tag{175}$$

$$\mathrm{L}_{\mathbf{x}^0_T} \quad : \quad \Lambda^0(T), \tilde{\Lambda}^k(E) \to \tilde{\Lambda}^k(E) \tag{176}$$

$$\mathrm{L}_{\mathbf{x}^0_T} \quad : \quad \Lambda^0(T), \Lambda^k(E^\star) \to \Lambda^k(E^\star) \tag{177}$$

$$\mathrm{L}_{\mathbf{x}^0_T} \quad : \quad \Lambda^0(T), \tilde{\Lambda}^k(E^\star) \to \tilde{\Lambda}^k(E^\star) \tag{178}$$

where $\mathbf{x}^0_T \in \Lambda^0(T)$ is a vector field (vector-valued 0-form).

In the case of Λ^k and $\tilde{\Lambda}^k$ Cartan's magic formula holds

$$\mathrm{L}_{\mathbf{x}^0_T} a^k = \mathrm{d}\, \mathrm{i}_{\mathbf{x}^0_T} a^k + \mathrm{i}_{\mathbf{x}^0_T} \mathrm{d}\, a^k \tag{179}$$

with a similar formula for $\tilde{a}^k$.

The Lie derivative of the volume form with respect to a vector field can be related to the divergence of the vector field and the exterior derivative of the associated flux as

$$\mathrm{L}_{\mathbf{x}^0_T} \tilde{\mu}^n = \mathrm{d}\, \mathrm{i}_{\mathbf{x}^0_T} \tilde{\mu}^n = \mathrm{d}\, \tilde{x}^{n-1} = \tilde{\mu}^n \mathrm{div}\ \mathbf{x}^0_T \tag{180}$$

For $\tilde{\mathbf{a}}^n_{T^\star} = \mathbf{a}^0_{T^\star} \mathbin{\hat{\wedge}} \tilde{w}^n$ there is a relationship between the Lie derivative and the covariant exterior derivative given by

$$\mathrm{L}_{\mathbf{u}^0_T} \tilde{\mathbf{a}}^n_{T^\star} = \mathrm{L}_{\mathbf{u}^0_T} (\mathbf{a}^0_{T^\star} \mathbin{\hat{\wedge}} \tilde{w}^n) = \mathrm{d}_X [\mathbf{a}^0_{T^\star} \mathbin{\hat{\wedge}} \mathrm{i}_{\mathbf{u}^0_T} \tilde{w}^n] + \mathbb{I}(\mathrm{d}_X \mathbf{u}^0_T \mathbin{\hat{\wedge}} \mathbf{a}^0_{T^\star}) \mathbin{\hat{\wedge}} \tilde{w}^n \tag{181}$$

There is an extension of the Lie derivative from $\Lambda^0(T)$ to $\Lambda^l(T)$ (i.e. $\mathrm{L}_{\mathbf{x}^l_T} a^k$), known as the *Nijenhuis–Lie derivative*, but this generalization is not needed in this work.

A.1.10. Vector Proxies

Given a straight vector-valued 0-form $\mathbf{x}^1_T$, there are several related objects that can be defined. These are the straight 1-form x^1 (circulation), the twisted $n-1$-form $\tilde{x}^{n-1}$ (flux) and the twisted covector-valued n-form $\tilde{\mathbf{x}}^n_{T^\star}$.

$$x^1 = (\mathbf{x}^0_T)^\flat \tag{182}$$
$$\tilde{x}^{n-1} = \tilde{\star} x^1 = \mathrm{i}_{\mathbf{x}^0_T} \tilde{\mu}^n \tag{183}$$
$$\tilde{\mathbf{x}}^n_{T^\star} = \tilde{\star} \mathbf{x}^1_T = \tilde{\star} (x^1)^\sharp \tag{184}$$

The twisted covector-valued n-form $\tilde{\mathbf{x}}^n_{T^\star}$ can be decomposed into a straight covector-valued 0-form $\mathbf{x}^0_{T^\star}$ and the volume form $\tilde{\mu}^n$ as

$$\tilde{\mathbf{x}}^n_{T^\star} = \mathbf{x}^0_{T^\star} \dot{\wedge} \tilde{\mu}^n \tag{185}$$

Using the trace/inclusion operators, the following relationships hold

$$x^1 = \mathbb{T} \mathbf{x}^0_{T^\star} \leftrightarrow \mathbf{x}^0_{T^\star} = \mathbb{I} x^1 \tag{186}$$
$$\tilde{x}^{n-1} = \mathbb{T}^\star \tilde{\mathbf{x}}^n_{T^\star} \leftrightarrow \tilde{\mathbf{x}}^n_{T^\star} = \mathbb{I}^\star \tilde{x}^{n-1} \tag{187}$$

These various relationships will be key for transforming between formulations based $\mathbf{u}^0_{\mathrm{T}}$ and those based on v^1, and also developing conservation form expressions.

A.2. Dual Spaces, Lie Brackets, Diamond Operators and Functional Derivatives

A.2.1. Dual Spaces and Functional Derivatives

We identify the dual space to a space of differential forms through the covariant pairing. For example,

$$\left\langle \left\langle \mathbf{a}^k_E, \tilde{\mathbf{b}}^{n-k}_{E^\star} \right\rangle \right\rangle_\chi = \int_\Omega \mathbf{a}^k_E \dot{\wedge} \tilde{\mathbf{b}}^{n-k}_{E^\star} \tag{188}$$

and therefore the dual to $\Lambda^k(E)$ is $\tilde{\Lambda}^{n-k}(E^\star)$, with with similar formulas for other spaces.

The *functional derivative* will be defined relative to this pairing, as

$$\delta \mathcal{F} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\mathcal{F}[\mathbf{x}^k_E + \varepsilon \omega^k_E] - \mathcal{F}[\mathbf{x}^k_E] \right) =: \left\langle \left\langle \omega^k_E, \frac{\tilde{\delta} \mathcal{F}}{\delta \mathbf{x}^k_E} \right\rangle \right\rangle_\chi = \int_{\mathcal{M}} \omega^k_E \dot{\wedge} \frac{\tilde{\delta} \mathcal{F}}{\delta \mathbf{x}^k_E} \quad \forall \omega^k_E \in \Lambda^k(E) \tag{189}$$

with similar formulas for other spaces. The key point is that the functional derivative lives in the dual space with respect to the variable it is taken to, ie $\frac{\tilde{\delta} \mathcal{F}}{\delta \mathbf{x}^k_E} \in \tilde{\Lambda}^{n-k}(E^\star)$

A.2.2. Lie Brackets and Their Dual

The space of vector fields naturally has a Lie bracket, denoted with $[,]$. This is antisymmetric

$$[\mathbf{a}_T^0, \mathbf{b}_T^0] = -[\mathbf{b}_T^0, \mathbf{a}_T^0] \tag{190}$$

The dual with respect to the covariant pairing is the Lie derivative for covector-valued twisted n-forms:

$$\begin{aligned}
\left\langle\left\langle \tilde{\mathbf{m}}_{T^\star}^n, [\mathbf{a}_T^0, \mathbf{b}_T^0]\right\rangle\right\rangle_\chi &= \left\langle\left\langle \tilde{\mathbf{m}}_{T^\star}^0 \mathbin{\dot\wedge} \tilde{\mu}^n, [\mathbf{a}_T^0, \mathbf{b}_T^0]\right\rangle\right\rangle_\chi && (191)\\
&= \left\langle\left\langle \tilde{\mathbf{m}}_{T^\star}^0 \mathbin{\dot\wedge} [\mathbf{a}_T^0, \mathbf{b}_T^0], \tilde{\mu}^n\right\rangle\right\rangle && (192)\\
&= \int_\Omega \mathrm{i}_{[\mathbf{a}_T^0, \mathbf{b}_T^0]}\, m^1 \wedge \tilde{\mu}^n && (193)\\
&= \int_\Omega \left[\mathrm{L}_{\mathbf{a}_T^0}\, \mathrm{i}_{\mathbf{b}_T^0}\, m^1 - \mathrm{i}_{\mathbf{b}_T^0}\, \mathrm{L}_{\mathbf{a}_T^0}\, m^1\right] \wedge \tilde{\mu}^n && (194)\\
&= \int_\Omega \left[\mathrm{L}_{\mathbf{a}_T^0}\, \mathrm{i}_{\mathbf{b}_T^0}\, m^1 - \mathbf{b}_T^0 \mathbin{\dot\wedge} \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mathbf{m}}_{T^\star}^0\right] \wedge \tilde{\mu}^n && (195)\\
&= \int_\Omega \mathrm{L}_{\mathbf{a}_T^0}\, (\mathbf{b}_T^0 \mathbin{\dot\wedge} \tilde{\mathbf{m}}_{T^\star}^0) \wedge \tilde{\mu}^n - \mathbf{b}_T^0 \mathbin{\dot\wedge} \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mathbf{m}}_{T^\star}^n + \mathbf{b}_T^0 \mathbin{\dot\wedge} \tilde{\mathbf{m}}_{T^\star}^0 \mathbin{\dot\wedge} \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mu}^n && (196)\\
&= \int_\Omega \mathrm{L}_{\mathbf{a}_T^0}\, [(\mathbf{b}_T^0 \mathbin{\dot\wedge} \tilde{\mathbf{m}}_{T^\star}^0) \wedge \tilde{\mu}^n] - \mathbf{b}_T^0 \mathbin{\dot\wedge} \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mathbf{m}}_{T^\star}^n && (197)\\
&= \int_\Omega \mathrm{d}\, \mathrm{i}_{\mathbf{a}_T^0}\, [(\mathbf{b}_T^0 \mathbin{\dot\wedge} \tilde{\mathbf{m}}_{T^\star}^0) \wedge \tilde{\mu}^n] - \mathbf{b}_T^0 \mathbin{\dot\wedge} \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mathbf{m}}_{T^\star}^n && (198)\\
&= \int_\Omega -\mathbf{b}_T^0 \mathbin{\dot\wedge} \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mathbf{m}}_{T^\star}^n && (199)\\
&= -\left\langle\left\langle \mathbf{b}_T^0, \mathrm{L}_{\mathbf{a}_T^0}\, \tilde{\mathbf{m}}_{T^\star}^n \right\rangle\right\rangle_\chi && (200)
\end{aligned}$$

where the total differential $\mathrm{d}\, \mathrm{i}_{\mathbf{a}_T^0}\, [(\mathbf{b}_T^0 \mathbin{\dot\wedge} \tilde{\mathbf{m}}_{T^\star}^0) \wedge \tilde{\mu}^n]$ vanishes under our boundary condition assumptions.

A.2.3. Diamond and Triangle Operators

In addition to the adjoint of the Lie bracket, it will be useful to introduce adjoints for the interior product and the Lie derivative with respect to the vector field. There are known as the *triangle* $\triangle$ and *diamond* $\diamond$ operators, respectively.

Start with the $\triangle$ operator

$$\begin{aligned}
\triangle &: \Lambda^k, \tilde{\Lambda}^{n-k+1} \to \Lambda^1 && (201)\\
\triangle &: \tilde{\Lambda}^k, \tilde{\Lambda}^{n-k+1} \to \Lambda^1 && (202)
\end{aligned}$$

defined through

$$\left\langle\left\langle a^k, \mathrm{i}_{\mathbf{u}_\mathrm{T}^0}\, \tilde{b}^{n-k+1} \right\rangle\right\rangle = \left\langle\left\langle \tilde{u}^{n-1}, a^k \triangle \tilde{b}^{n-k+1} \right\rangle\right\rangle \tag{203}$$

with a similar expression for $\tilde{a}^k$. Using Hirani's formula, this can be expressed explicitly as

$$a^k \triangle \tilde{b}^{n-k+1} = \tilde{\star}(\tilde{\star} a^k \wedge \tilde{\star} \tilde{b}^{n-k+1}) \tag{204}$$

again with a similar expression for $\tilde{a}^k$.

Now consider the $\diamond$ operator

$$\diamond \;:\; \Lambda^k, \tilde{\Lambda}^{n-k} \to \tilde{\Lambda}^n(T^\star) \tag{205}$$
$$\diamond \;:\; \tilde{\Lambda}^k, \Lambda^{n-k} \to \tilde{\Lambda}^n(T^\star) \tag{206}$$

defined by

$$\left\langle\left\langle a^k, \mathrm{L}_{\mathbf{u}_\mathrm{T}^0} \tilde{b}^{n-k} \right\rangle\right\rangle = -\left\langle\left\langle \mathbf{u}_\mathrm{T}^0, a^k \diamond \tilde{b}^{n-k} \right\rangle\right\rangle_\chi \tag{207}$$

with similar expressions for $\tilde{a}^k$ and b^{n-k}.

It will be useful to derive explicit expressions for $\diamond$ in the case of SVDFs. To do this, consider first the *Kelvin-Noether diamond* operator $\diamond_{\mathrm{kn}}$

$$\diamond_{\mathrm{kn}} \;:\; \Lambda^k, \tilde{\Lambda}^{n-k} \to \Lambda^1 \tag{208}$$
$$\diamond_{\mathrm{kn}} \;:\; \tilde{\Lambda}^k, \Lambda^{n-k} \to \Lambda^1 \tag{209}$$

defined through

$$\left\langle\left\langle a^k, \mathrm{L}_{\mathbf{u}_\mathrm{T}^0} \tilde{b}^{n-k} \right\rangle\right\rangle = -\left\langle\left\langle \tilde{u}^{n-1}, a^k \diamond_{\mathrm{kn}} \tilde{b}^{n-k} \right\rangle\right\rangle \tag{210}$$

with similar expressions for $\tilde{a}^k$ and b^{n-k}.

Expanding the Lie derivative using Cartan formula and then applying the definition of the triangle operator gives

$$a^k \diamond_{\mathrm{kn}} \tilde{b}^{n-k} = \mathrm{d}\, a^k \triangle \tilde{b}^{n-k} + a^k \triangle \mathrm{d}\, \tilde{b}^{n-k} \tag{211}$$

with similar expressions for $\tilde{a}^k$ and b^{n-k}.

Explicit expressions in the case $n = 3$ are

$$a^3 \diamond_{\mathrm{kn}} b^0 = \tilde{\star} a^3 \wedge \mathrm{d}\, b^0 \tag{212}$$
$$a^2 \diamond_{\mathrm{kn}} b^1 = \tilde{\star}\, \mathrm{d}\, a^2 \wedge b^1 + \mathrm{i}_{\mathbf{a}_T^0}\, \mathrm{d}\, b^1 \tag{213}$$
$$a^1 \diamond_{\mathrm{kn}} b^2 = a^1 \wedge \tilde{\star}\, \mathrm{d}\, b^2 + \mathrm{i}_{\mathbf{b}_T^0}\, \mathrm{d}\, a^1 \tag{214}$$
$$a^0 \diamond_{\mathrm{kn}} b^3 = \mathrm{d}\, a^0 \wedge \tilde{\star} b^3 \tag{215}$$

where $\mathbf{a}_T^0 = (\star a^2)^\sharp$ and $\mathbf{b}_T^0 = (\star b^2)^\sharp$.

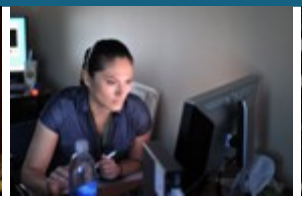
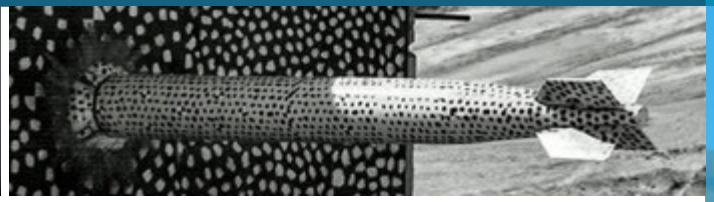
The $\diamond$ and $\diamond_{\mathrm{kn}}$ operators are related through

$$a \diamond_{\mathrm{kn}} b = (\tilde{\star}[a \diamond b])^\flat \qquad \leftrightarrow \qquad a \diamond b = \tilde{\star}(a \diamond_{\mathrm{kn}} b)^\sharp = \mathbb{I}(a \diamond_{\mathrm{kn}} b) \mathbin{\hat{\wedge}} \tilde{\mu}^n \tag{216}$$

The latter expression will be very useful in relating $\mathbf{u}_\mathrm{T}^0$ and v^1 based formulations, and also conservation form.

APPENDIX B. COPY OF ENDING PROJECT REVIEW

LDRD Ending Project Review
Differential geometric approaches to momentum-based formulations for fluids (220311)


Christopher Eldred (PI, 1442), Jim Stewart (PM)

FY22 Late Start, $95K

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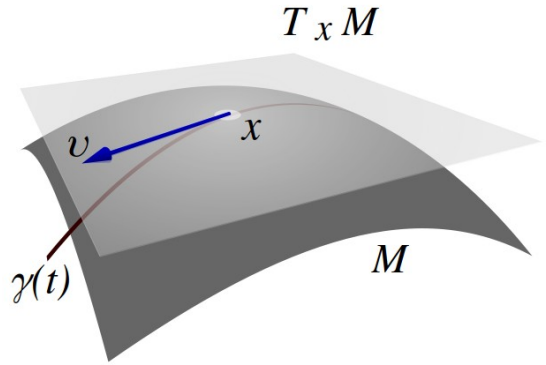
PURPOSE, GOALS AND APPROACH

Primary Goal: develop momentum-based geometric mechanics formulations for non-relativistic fluids using differential geometry (successful)

Secondary Goal: explore connection between Godunov/Hyperbolic system of conservation laws ↔ Hamiltonian mechanics (currently unsuccessful)

Why: Mimetic discretizations for momentum, ability to handle arbitrary manifolds/coordinate systems

Main Innovations: momentum as a covector-valued twisted volume form, apply to the general theory of matched pair dynamics



$$\mathcal{L}[\mathbf{u}_v^0, a, \mathbf{b}]$$

$$\tilde{\mathbf{m}}_{cv}^n = \frac{\delta \mathcal{L}}{\delta \mathbf{u}_v^0}$$

$$\mathcal{H}[\tilde{\mathbf{m}}_{cv}^n, a, \mathbf{b}]$$

Technical Details: Background/Motivation

- A **powerful approach** to formulating physical theories is:

Differential Geometry
-Exterior Calculus
-Scalar-Valued Differential Forms (SVDF)
-deRham Complex

+

Geometric Mechanics Formulations
-Lagrangian (Euler-Poincare equations)
-Hamiltonian (Lie-Poisson formulations)

$$\begin{array}{cccccc} \alpha^k & d & \wedge & \star & \mathcal{L}_{\vec{u}} \\ \delta = (-1)^{n(k-1)+1} & & & \star\, d \star & \end{array}$$

$$\delta \mathcal{S} = \delta \int_{t^1}^{t^2} \mathcal{L}[\mathbf{x}] = 0 \qquad \frac{d\mathcal{F}}{dt} = \{\mathcal{F}, \mathcal{H}\}$$

- Why? Works for **arbitrary manifolds/coordinates**, exposes commonality between physical systems, **basis for mimetic/structure-preserving discretizations**
- Associate **physical quantities** with **(oriented) geometric entities** (=differential forms)[1]
- For fluids, this approach fails when momentum is predicted instead of velocity
  - Why? Momentum is not a SVDF, it is a **(co)tangent-bundle valued differential form (BVDF)**
- Connected to the question of what $\rho\mathbf{u}$ is: **momentum density or mass flux?**

[1] E. Tonti, **Why starting from differential equations for computational physics?**

Technical Details: Vector Proxies in Exterior Calculus

Vector Proxies in Exterior Calculus

- Vector-valued 0-form (vector field) $\mathbf{u}_v^0$
- Covector-valued twisted volume form $\tilde{\mathbf{u}}_{cv}^n = \tilde{\star}\mathbf{u}_v^0$
- Scalar-valued straight 1-form (circulation) $u^1 = \left(\mathbf{u}_v^0\right)^\flat$
- Scalar-valued twisted (n-1)-form (flux) $\tilde{u}^{n-1} = \tilde{\star}u^1 = i_{\mathbf{u}_v^0} d\tilde{V}$

Connected through metric operations such as Hodge star

In $\mathbb{R}^3$ using Cartesian coordinates they look the same (“vectors”), but they are in fact distinct objects that measure different aspects of a vector

Technical Details: Representations of Velocity/Momentum

What are geometric mechanics/exterior calculus representations of fluid velocity/momentum?

- Velocity $\mathbf{u}_v^0$ → Lagrangian $\mathcal{L}[\mathbf{u}_v^0, a, \mathbf{b}]$
- Momentum Density $\mathbf{m}_{cv}^n = \dfrac{\delta \mathcal{L}}{\delta \mathbf{u}_v^0}$ → (Lie-Poisson) Hamiltonian $\mathsf{H}[\tilde{\mathbf{m}}_{cv}^n, a, \mathbf{b}]$
- Circulation $D^0 \wedge v^1 = (\tilde{\star} \mathbf{m}_{cv}^n)^\flat$ → (Curl-Form) Hamiltonian $\mathcal{H}[v^1, a, \mathbf{b}]$
- (Mass) Flux $\tilde{F}^{n-1} = \dfrac{\delta \mathcal{H}}{\delta v^1}$
- Fluid Density $\tilde{D}^n$ with $D^0 = \tilde{\star} \tilde{D}^n$

Technical Details: Semi-Direct Product Dynamics

Lagrangian $\mathcal{L}[\mathbf{u}_v^0, a, \mathbf{b}]$ with advected quantities a (SVDF) and $\mathbf{b}$ (BVDF)

Euler-Poincare Equations for $\mathbf{m}_{cv}^n = \dfrac{\delta \mathcal{L}}{\delta \mathbf{u}_v^0}$

$$\frac{\partial \tilde{\mathbf{m}}_{cv}^n}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} \tilde{\mathbf{m}}_{cv}^n - \frac{\delta \mathcal{L}}{\delta a} \diamond a - \frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond \mathbf{b} = 0 \qquad (1)$$

$$\frac{\partial a}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} a = 0 \qquad (2)$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} \mathbf{b} = 0 \qquad (3)$$

From these, can obtain Lie-Poisson Hamiltonian equations, “Conservation Form”, Curl-Form Hamiltonian equations, etc.

Treats fluids with advected quantities: compressible Euler, non-relativistic MHD/EHD, multivelocity hydrodynamics (direct product theory), Green-Naghidi, etc.

Technical Details: Matched Pair Dynamics

Lagrangian $\mathcal{L}[\mathbf{u}_v^0, a, \mathbf{b}, B^2, \tilde{D}^2]$ with electromagnetic fields B^2 and $\tilde{D}^2$

Euler-Poincare Equations

$$\frac{\partial \tilde{\mathbf{m}}_{cv}^n}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} \tilde{\mathbf{m}}_{cv}^n - \frac{\delta \mathcal{L}}{\delta a} \diamond a - \frac{\delta \mathcal{L}}{\delta \mathbf{b}} \diamond \mathbf{b} - \frac{\delta \mathcal{L}}{\delta \mathbf{B^2}} \diamond \mathbf{B^2} - \frac{\delta \mathcal{L}}{\delta \tilde{\mathbf{D}}^{\mathbf{2}}} \diamond \tilde{\mathbf{D}}^{\mathbf{2}} = 0 \quad (1)$$

$$\frac{\partial a}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} a = 0 \quad (2)$$

$$\frac{\partial \mathbf{b}}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} \mathbf{b} = 0 \quad (3)$$

$$\frac{\partial \mathbf{B^2}}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} \mathbf{B^2} - \mathrm{d} \frac{\partial \mathcal{L}}{\partial \tilde{D}^2} = 0 \quad (4)$$

$$\frac{\partial \tilde{\mathbf{D}}^{\mathbf{2}}}{\partial t} + \mathsf{L}_{\mathbf{u}_v^0} \tilde{\mathbf{D}}^{\mathbf{2}} + \mathrm{d} \frac{\partial \mathcal{L}}{\partial B^2} = 0 \quad (5)$$

From these, can obtain Lie-Poisson Hamiltonian equations, “Conservation Form”, Curl-Form Hamiltonian equations, etc. Also can work with electromagnetic potentials instead of fields

Treats fluids with matched pair dynamics: specifically, non-relativistic charged fluids (single + multifluid Euler-Poisson, Euler-Maxwell, etc.)

Technical Details: Godunov ↔ Geometric Mechanics

Key Question: can we determine when a geometric mechanics formulation leads to a hyperbolic system of conservation laws?

Existing knowledge: 1D Hamiltonian systems of hydrodynamic type → symmetric hyperbolic

So far, unsuccessful at going beyond this

Why unsuccessful?

- Extending the approach used in 1D to nD gets extremely technical, and unfortunately usually does not work
- Hyperbolic equations on manifolds still an immature subject and basically no work done (to my knowledge) using differential geometric approaches (ex. exterior derivatives and differential forms) or even tensor calculus

Some tantalizing hints for the rest of FY22 (and beyond)

- In 1D, convexity of Hamiltonian → hyperbolicity. Still true in nD? Connections with invariant domains and IDP discretizations?
- Non-hyperbolic systems with geometric mechanics formulations exist ex. multivelocity/temperature, single pressure. What makes them different from hyperbolic geometric mechanics systems?

PROJECT LEGACY

Key Result:

- General framework for momentum-based descriptions of non-relativistic fluids with matched pair dynamics based on differential geometry
- Allows treatment of very general fluid models on an arbitrary manifold in a coordinate independent way
  - compressible Euler, Green-Naghidi, MHD/EHD, Euler-Maxwell, Euler-Poisson, various multivelocity models, etc.

Impact:

- Basis for mimetic discretizations of momentum, especially charged fluid models → **FY23-25 CIS LDRD project** ($400K/year)
- Better understanding of the common geometric mechanics formulation underlying widely used continuum non-relativistic fluid models in SNL applications: EMPIRE, SPARC, Drekar, Alegra, FLEXO, etc.

PROJECT OUTPUTS, CAPABILITIES DEVELOPMENT AND CAREER DEVELOPMENT

Outputs

- 2 Papers (under preparation)
  - Geometric mechanics formulations for momentum-based descriptions of fluids with advected quantities
  - Geometric mechanics formulations for momentum-based descriptions of charged fluids with matched pair dynamics
- SAND Report (under preparation, based on papers)
- SIAM CSE 23 presentation

New Capabilities

- Improved expertise in geometric mechanics formulations/differential geometry at Sandia, for myself through doing this work and for others through discussions about this

Career Development

- PI is Early Career

TEAM BUILDING AND PARTNERSHIPS

Internal

- Put together an internal Sandia team for a successful FY23 CIS LDRD proposal
- Engaged with Sandia experts in charged fluids and hyperbolic systems of conservation laws (Travis Fisher, Eric Cyr, Allen Robinson, John Shadid, Stephen Bond, Nathan Roberts)

External

- Continued to engage with external collaborators (Francois Gay-Balmaz, Darryl Holm, Erwin Luesink, Golo Wimmer) who are experts in geometric mechanics formulations and differential geometry
- Planning on applying for FY24 University Partnership Funding (to add onto FY23 CIS LDRD) to collaborate with Ignacio Tomas (formerly 1442, now at Texas Tech)
  - Geometric mechanics formulations ↔ invariant domains + IDP methods

IA/PM PROJECT LEGACY

How did this project contribute to IA strategic goals and objectives?

- Core CIS Research area: Mathematics, Algorithms and Simulations
- Momentum-based formulations widely used in SNL applications for both neutral and charged fluids: EMPIRE, SPARC, Drekar, Alegra, FLEXO

What are the key results from this research that will be useful to other current and future projects?

- General framework for momentum-based descriptions of fluids with matched pair dynamics based on differential geometry: useful for a wide range range of neutral and charged fluid continuum models used at SNL

Technology insertion and follow-on funding for potential and realized ROI

- **Now**: FY23-25 CIS LDRD ($400K/year) on “Structure-preserving discretizations for momentum-based formulations of fluids”
- **Future**: Relativistic fluids extension i.e. differential geometry based spacetime formulations + associated discretizations? (FY24 EE LDRD or QTI and FY25-27 CIS LDRD or DOE SC/ASCR/SciDAC?)
- **Future**: Kinetic models for fluids- both differential geometry based formulations and associated discretizations? (ASCR/SciDAC FY23/24?)

Differential geometric approaches to momentum-based formulations for fluids

PI: Christopher Eldred, PM: Jim Stewart

Project goal(s)

Primary: develop momentum-based geometric mechanics formulations for non-relativistic fluids using differential geometry (successful)

Secondary: explore connection between Godunov/Hyperbolic system of conservation laws ↔ Geometric mechanics formulations (currently unsuccessful)

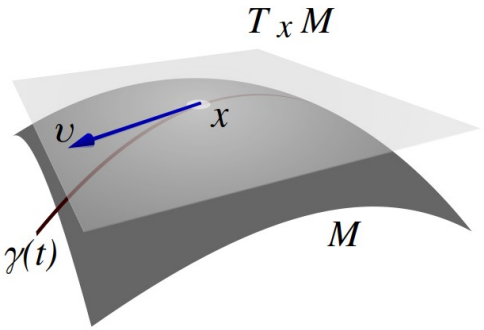
Mission Impact

Better understanding of the common geometric mechanics formulation underlying widely used continuum non-relativistic fluid models in SNL applications: EMPIRE, SPARC, Drekar, Alegra, FLEXO, etc.

Basis for future work on novel mimetic discretizations of momentum

Key Result

General framework for momentum-based descriptions of non-relativistic fluids with matched pair dynamics based on differential geometry



$$\mathcal{L}[\mathbf{u}_v^0, a, \mathbf{b}]$$

$$\tilde{\mathbf{m}}_{cv}^n = \frac{\delta \mathcal{L}}{\delta \mathbf{u}_v^0}$$

$$\mathcal{H}[\tilde{\mathbf{m}}_{cv}^n, a, \mathbf{b}]$$

Transition Plan

Pursing mimetic discretization ideas based on this approach in an FY23-25 CIS LDRD ($400K/year)

Future proposals (LDRD/ASCR/SC) to pursue similar ideas for relativistic fluids and/or kinetic models

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