SAND2021-7439C

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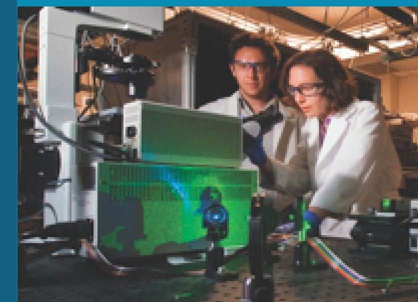
Application of the Level-2 Quantum Lasserre Hierarchy in Quantum Approximation Algorithms



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Quantum Bits Live in a Sphere

State space

Classical bit: (bit)

OR

+1 = Head

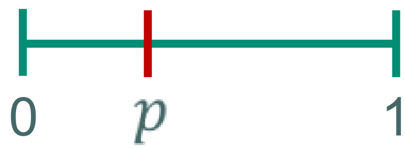
-1 = Tail

{+1, -1}

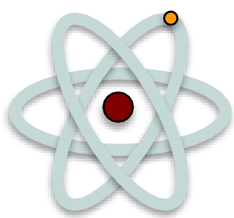
Prob. bit: (p-bit)

+1 with probability p

-1 with probability 1-p



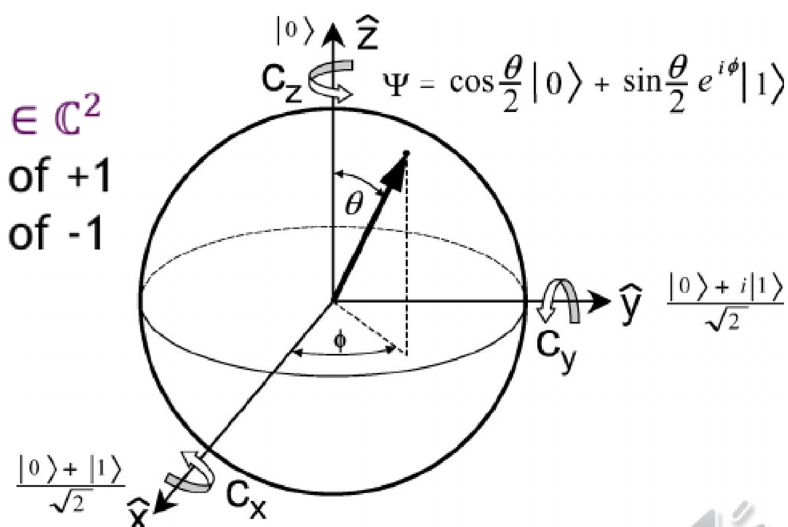
Quantum bit: (qubit)



$\alpha|+1\rangle + \beta|-1\rangle \in \mathbb{C}^2$

$|\alpha|^2$-probability of +1

$|\beta|^2$-probability of -1



How to describe a Generic distribution of qubits?

Generic quantum state:

$$\rho \in \mathbb{C}^{2^n \times 2^n},\quad \rho \text{ is Hermitian},\quad Tr(\rho) = 1,\quad \rho \succcurlyeq 0$$

$$\begin{bmatrix} \mathbb{P}[(A_1, \ldots A_n) = (+1^n)] \\ \mathbb{P}[(A_1, \ldots, A_n) = (+1^{n-1}, -1)] \\ \mathbb{P}[(A_1, \ldots, A_n) = (+1^{n-2}, +1, -1)] \\ \vdots \\ \mathbb{P}[(A_1, \ldots A_n) = (-1^n)] \end{bmatrix} \text{ vs. } \begin{bmatrix} \mathbb{P}(A_1 = +1) \\ \mathbb{P}(A_2 = +1) \\ \vdots \\ \mathbb{P}(A_n = +1) \\ \mathbb{P}(A_1 = +1, A_2 = +1) \\ \vdots \\ \mathbb{P}(A_1 = +1, \ldots, A_n = +1) \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Fact- $\{\gamma_1 \otimes \gamma_2 \ldots \otimes \gamma_n : \gamma_i \in \{I, X, Y, Z\}\}$ is an operator basis for Pauli matrices

Notation- Let σ_i for $\sigma \in \{X, Y, Z\}$ be Pauli σ on qubit i, i.e. $Z_2 = I \otimes Z \otimes I \ldots$
$Z_1 Z_3 = Z \otimes I \otimes Z \otimes I \ldots$

$$\begin{bmatrix} Tr(\rho X_1) \\ Tr(\rho Y_1) \\ Tr(\rho Z_1) \\ Tr(\rho X_2) \\ \vdots \\ Tr(\rho Z_1 \ldots Z_n) \end{bmatrix}$$

Local Statistics determined by low-order moments

$$\rho_1 = \frac{I + Tr(\rho X_1)X + Tr(\rho Y_1)Y + Tr(\rho Z_1)Z}{2}$$

Moment Matrices

$(\;_1\;_2)$

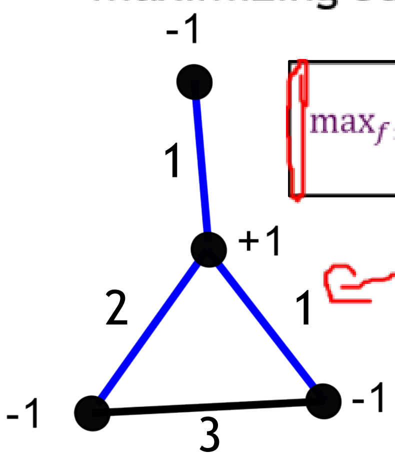
$_1 \quad _1 \quad _1 \quad _2 \; \ldots \quad _1 \quad _2 \ldots \quad Z_{n-1} Z_n \quad _1 \quad _2 \quad _3 \quad \ldots \quad _1 \ldots$

$$
\begin{array}{c}
I \\ X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ \vdots \\ Z_n \\ X_1 X_2 \\ \vdots \\ Z_{n-1} Z_n \\ X_1 X_2 X_3 \\ \vdots \\ Z_1 Z_2 \ldots Z_n
\end{array}
\begin{bmatrix}
1 & & & & & & & & & & & & \\
& 1 & & & & & & & & & & & \\
& & 1 & & & & & & & & & & \\
& & & 1 & & & & & & & & & \\
& & & & 1 & & & & & & & & \\
& & & & & \ddots & & & & & & & \\
& & & & & & 1 & & & & & & \\
& & & & & & & 1 & & & & & \\
& & & & & & & & \ddots & & & & \\
& & & & & & & & & 1 & & & \\
& & & & & & & & & & 1 & & \\
& & & & & & & & & & & \ddots & \\
& & & & & & & & & & & & 1
\end{bmatrix}
$$

- Redundant Description of Quantum State
- Must be PSD since $v^\dagger M\, v = Tr\left(S^\dagger S\, \rho\right) \geq 0$
- Equivalent to Density matrix description

“Quantumizing” Max Cut

- Assign $\{\pm 1\}$ to vertices of a graph, maximizing edges “cut”

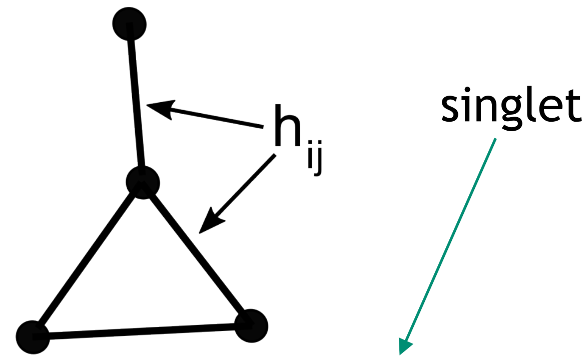


$$\max_{f:V\to\{\pm 1\}} \sum_{ij\in E} w_{ij} \frac{1 - f(i)f(j)}{2}$$

$$= 1/2(-)$$

$$\begin{matrix} x_i, x_j = +1,+1 \\ x_i, x_j = +1,-1 \\ x_i, x_j = -1,+1 \\ x_i, x_j = -1,-1 \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\max Tr(\rho \;\; \textstyle\sum_{ij\in E} g_{ij})$$
ρ diagonal



$$h \; = 1/4(- - -)$$

“How close” to the singlet on each edge?

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_{max}(\textstyle\sum_{ij\in E} h_{ij}) = \max Tr(\rho \;\; \textstyle\sum_{ij\in E} h_{ij})$$
ρ generic

Motivation

Problem

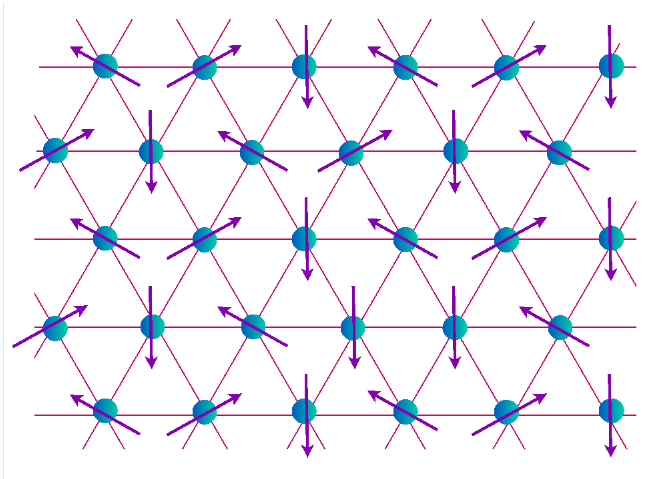
Find max-energy state of $\sum(I - X_iX_j - Y_iY_j - Z_iZ_j)$

($\equiv$ Find min-energy state of $\sum(X_iX_j + Y_iY_j + Z_iZ_j)$,

but different from approximation point of view)

Motivation

The Heisenberg model is fundamental for describing quantum magnetism, superconductivity, and charge density waves. Beyond 1 dimension, the properties of the anti-ferromagnetic Heisenberg model are notoriously difficult to analyze.



Anti-ferromagnetic Heisenberg model: roughly neighboring quantum particles aim to align in opposite directions. This kind of Hamiltonian appears, for example, as an effective Hamiltonian for so-called Mott insulators.
[Image: Sachdev, arxiv:1203.4565]

Moment Matrix Picture

In the Moment matrix picture, define C:

$$
\begin{array}{c|cccccccc}
 & \square & \square_1\square_2 & \square_1\square_3 & \cdots & Z_{n-1}Z_n & \square_1\square_2\square_3 & \cdots \\
\hline
\square & \sum_{ij}\frac{w_{ij}}{4} & 0 & -\frac{w_{12}}{8} & -\frac{w_{13}}{8} & \cdots & -\frac{w_{n-1,n}}{8} & 0 & \cdots \\
\vdots & 0 & & & & & & & \\
\square_1\square_2 & -\frac{w_{12}}{8} & & & & & & & \\
\square_1\square_3 & -\frac{w_{13}}{8} & & & & \mathbf{0} & & & \\
\vdots & \vdots & & & & & & & \\
Z_{n-1}Z_n & -\frac{w_{n-1,n}}{8} & & & & & & & \\
\square_1\square_2\square_3 & 0 & & & & & & & \\
 & \vdots & & & & & & &
\end{array}
$$

Quantum Max Cut

$\max C \cdot M$

$s.t.\ M$ is a valid moment matrix

vs.

$\max \sum_{ij\in E} Tr(\rho\, h_{ij})$

ρ generic

Different representations of the same problem!!

- QMA-hard, so we seek approximations

Approximation Algorithms and Ansatze

$$\frac{Alg(G,\{w_{ij}\})}{\lambda_{max}(H)} \geq \alpha$$

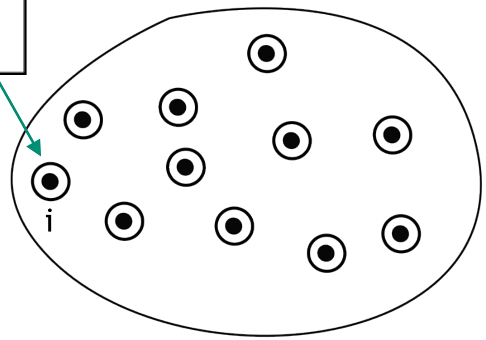
Runs in poly time in n,
provable guarantee independent of instance

- Unlike classical Max-cut not clear what kind of description is best
- Ansatz- "kind" of quantum state the algorithm outputs.

Product State Ansatz

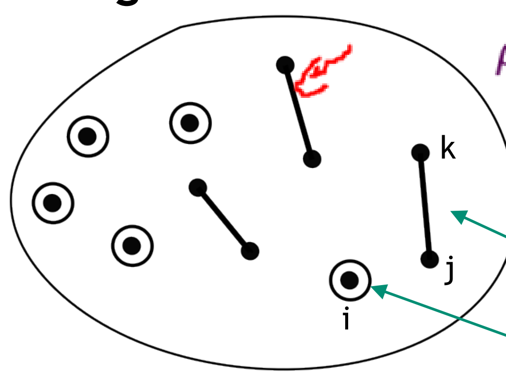
$$\rho_i = \frac{I + \alpha_i X_i + \beta_i Y_i + \gamma_i Z_i}{2}$$

$= \prod$



Singlets+Product States

$$\rho = \prod_i \rho_i \cdot \prod_{jk} \rho_{jk}$$



$$\rho_{jk} = \frac{I - X_j X_k - Y_j Y_k - Z_j Z_k}{4}$$

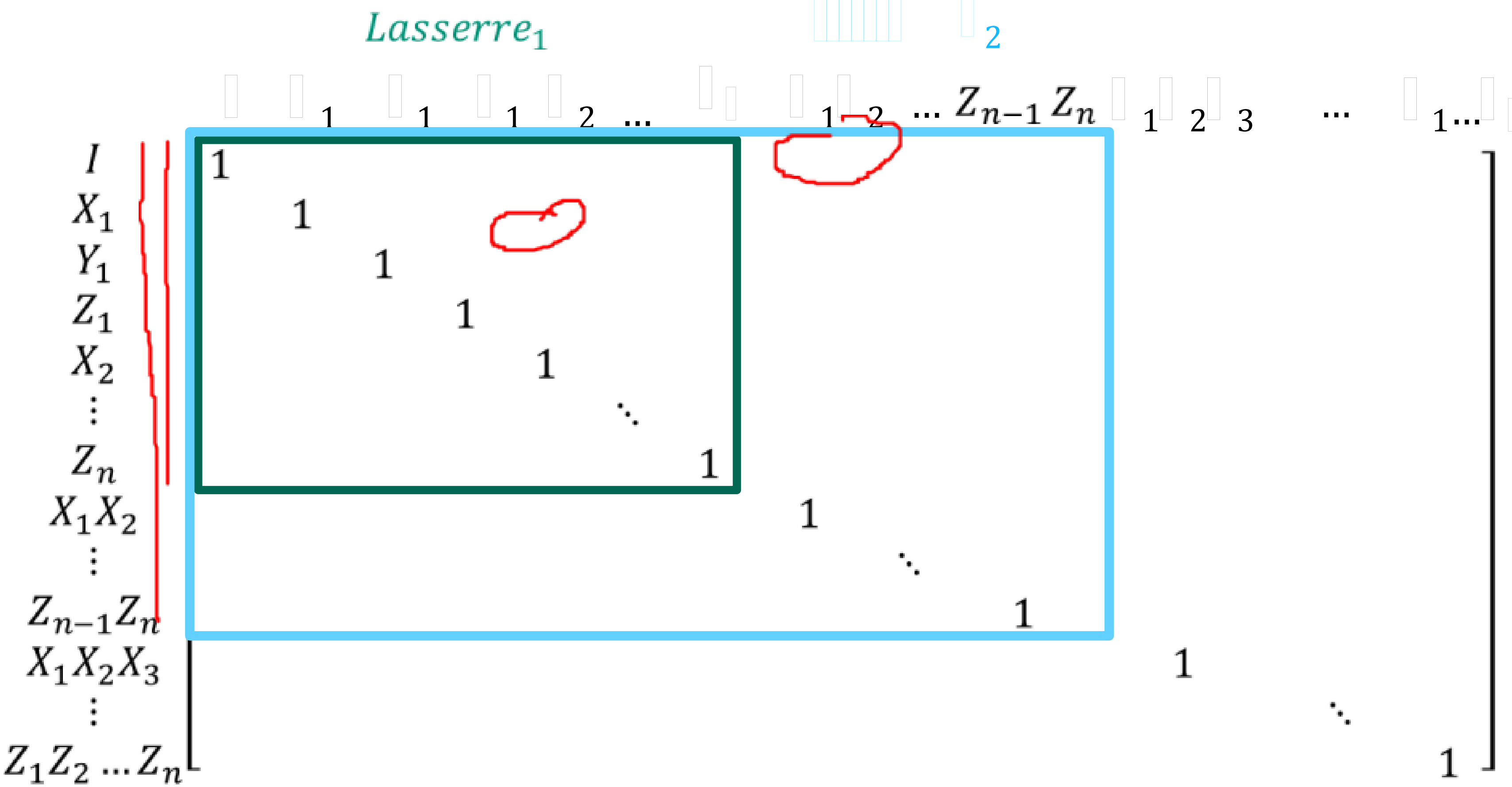
$$\rho_i = \frac{I + \alpha_i X_i + \beta_i Y_i + \gamma_i Z_i}{2}$$

Previous Work

Reference	Approximation Factor Achieved
Bansal, Bravyi, Terhal, 07’	PTAS (for planar instances)
Gharibian, Kempe, 12’	PTAS (for dense instances)
Brandao, Harrow, ‘16	PTAS (for dense instances)
Harrow, Montanaro, ‘17	Graph Dependent constant
Bravyi, Gosset, Koenig, Temme, ‘18	$\Omega(1/\log(n))$
Gharibian, Parekh, ‘19	0.498
Parekh, Thompson, ‘20	0.467
Anshu, Gosset, Morenz, ‘20	0.531
This work	0.533

- Constant factor algorithms for QMC (without additional assumptions) is an active area of research
- All except AGM20 produce product states
- Performance is limited because generic states (i.e. maximal e-vectors of QMA-complete) are highly “non-product”

Relaxing the Moment Matrix



Can optimize over a (polynomial large) submatrix

- ➢ Still guaranteed PSD
- ➢ Satisfies all equality constraints the matrix intersects with
- ➢ Relaxation because submatrix likely not embeddable

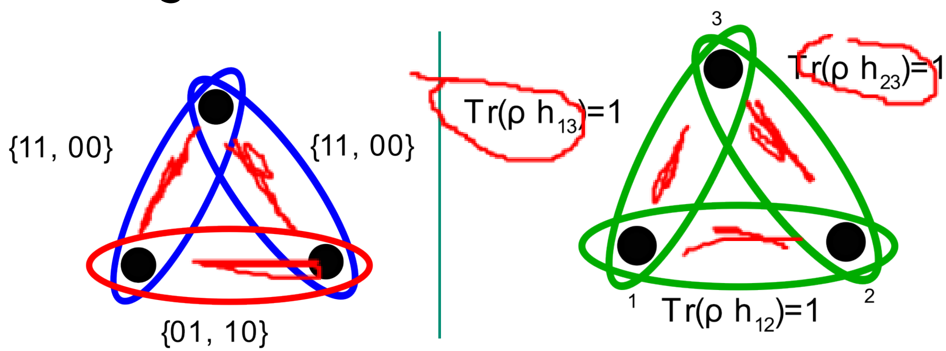
[Lasserre '01]

[Pironio, Navascués, Acìn, '10]

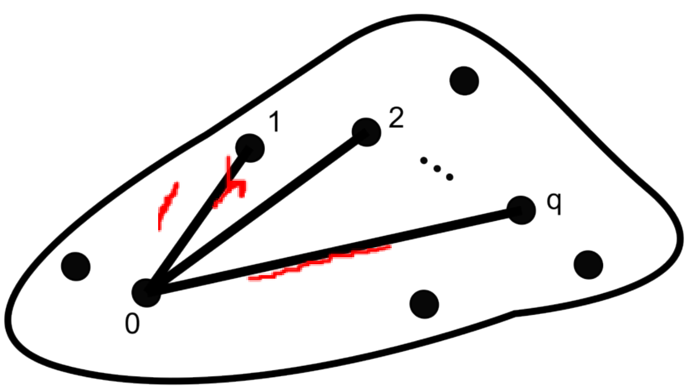
Relaxation Quality

- Higher levels are tighter relaxations on quantum states (⇔Moment matrices)
- Key observation in our analysis is that $Lasserre_2$ captures **monogamy of entanglement** inequalities
- **Entanglement-** “Quantum” correlation between subsystems.

- Monogamy of entanglement- can’t enforce inconsistent *quantum* marginals



Star Bound [Lieb, Mattis, ’62]
[Anshu, Gosset, Morenz, ‘20]



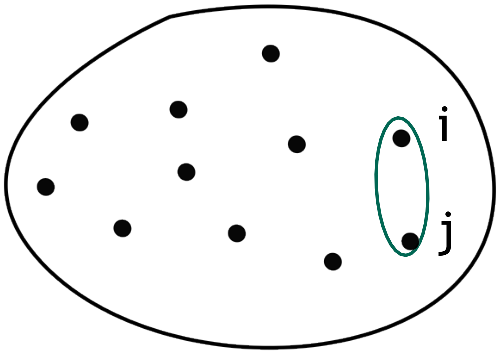
$$\sum_{i=1}^{m} Tr(\rho\, h_{0i}) \le \frac{q+1}{2}$$

- $Lasserre_1$ gets q
- $Lasserre_{1.5}$ gets $\approx 2q/3$
- $Lasserre_2$ gets $(q+1)/2$

$$Lasserre_2\left(\sum_i h_{0i}\right) \le \frac{q+1}{2}$$

Rounding Algorithm

Employing Singlets+Product state Ansatz



$$v_{ij} := \frac{-M(X_iX_j, I) - M(Y_iY_j, I) - M(Z_iZ_j, I)}{3}$$

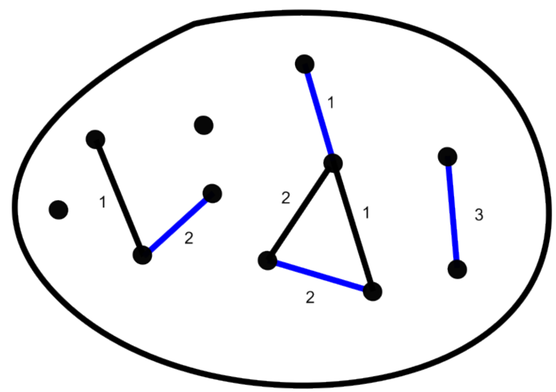
$-\frac{1}{3} \leq v_{ij} \leq 1$, if $v_{ij} \approx 1$ then $Lasserre_2$ "thinks" that edge should be a singlet.

Choose edges to be singlets with "guidance" from $Lasserre_2$.

Fix d to be some integer ≥ 1, and let $\alpha(d) = \frac{d+3}{3(d+1)}$

Algorithm

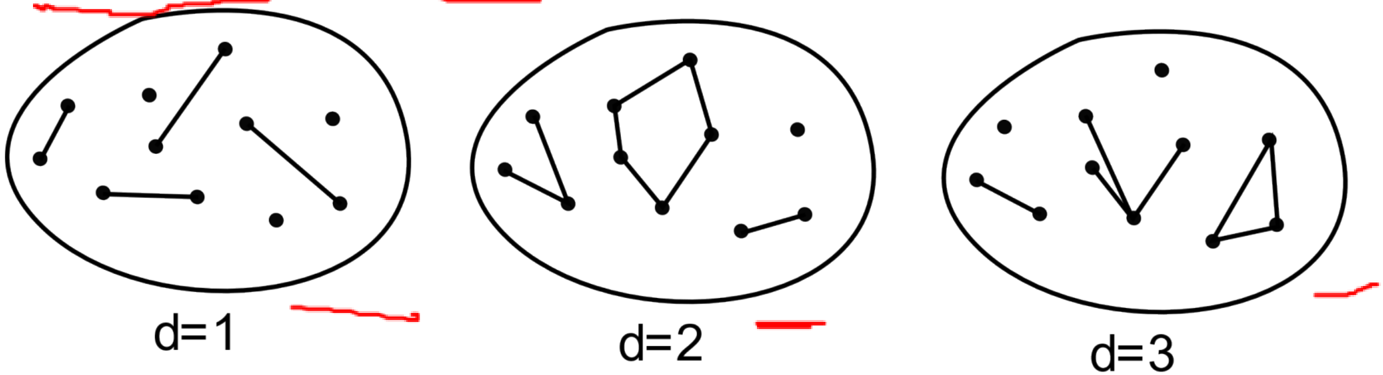
1. Solve $Lasserre_2$ to get M
2. Initialize $L = \{\}$
3. For all ij calculate v_{ij}. If $v_{ij} > \alpha(d)$ add ij to L.
4. Find Maximum matching on L.
5. Consider two states
   1. Singlets on MM, product state on rest
   2. PS rounding from [GP 19']
6. Take whichever has better objective.



Analysis

[Goemans, Williamson, '94]

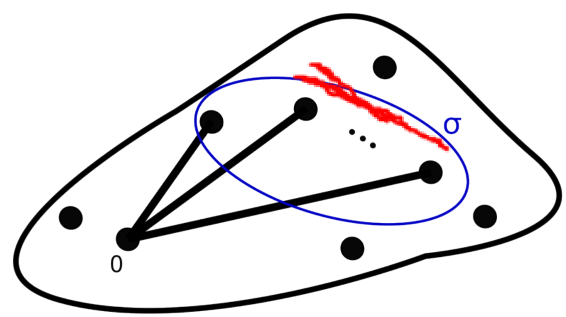
- Standard approach: try to bound objective loss from rounding
- What if most edges have large value and L has high degree?
  - Star bound implies L has low degree



- Most edges have large value ⇒ MM has good performance
- Most edges have small value ⇒ Don't need entanglement to get good obj.
- Tradeoff in d:
  - d is too small ⇒ product state rounding bad
  - d is too large ⇒ matching is bad

d=2 for our results

- Additional proof techniques
  - Symmeterization over transformations
  - "Sum of Squares" proofs



Implications

- Demonstrated that $Lasserre_2$ satisfies physically motivated constraints, possibly opening the door to additional approximation algorithms.
  - "low-order" quasi-description of a state can look "entangled"
  - Demonstrate explicit gap in "representational power" of different levels of Lasserre

- Classical approximation algorithms follow a standard "meta" algorithm,
  1. Solve SDP
  2. Use solution to round to feasible point
- Only other known algorithm which produces entangled ansatz [Anshu, Gosset, Morenz '20] does not follow this format
  - By bringing in the meta algorithm we have opened the door to using the rich background of classical techniques for combinatorial opt.

Open Questions

- No known hardness results for approximation (say under unique games)

- Likely only scratching the surface of the power of $Lasserre_2$
  - What other kinds of graphs is $Lasserre_2$ exact on?
  - Are moments subject to other monogamy inequalities?
  - Can these be used to further improve approx. factor?

- More generally, what kind of physical constraints are present in $Lasserre_k$ for $k = O(1)$?

- Singlets + product state still *locally* entangled. Can we get more entangled ansatz? i.e. tensor network states?

- Genuinely quantum Approximation algorithms? i.e. alg requires quantum computer and produces quantum state