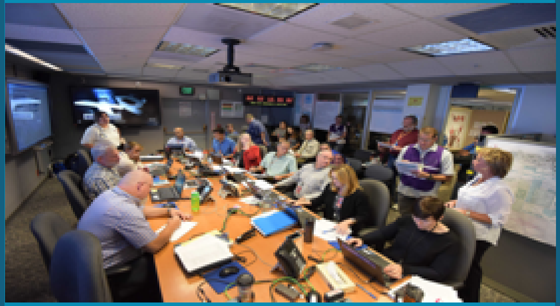
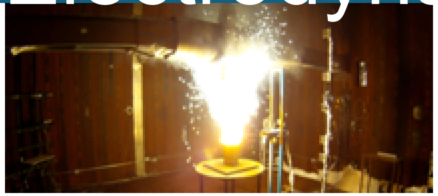
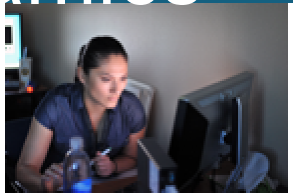
SAND2021-9644C

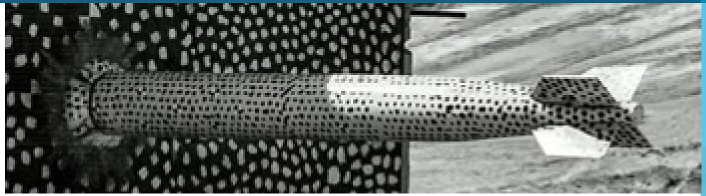
Sandia National Laboratories

Robust Relativistic Hydrodynamics to Enable Relativistic Two-Fluid Electrodynamics









PRESENTED BY

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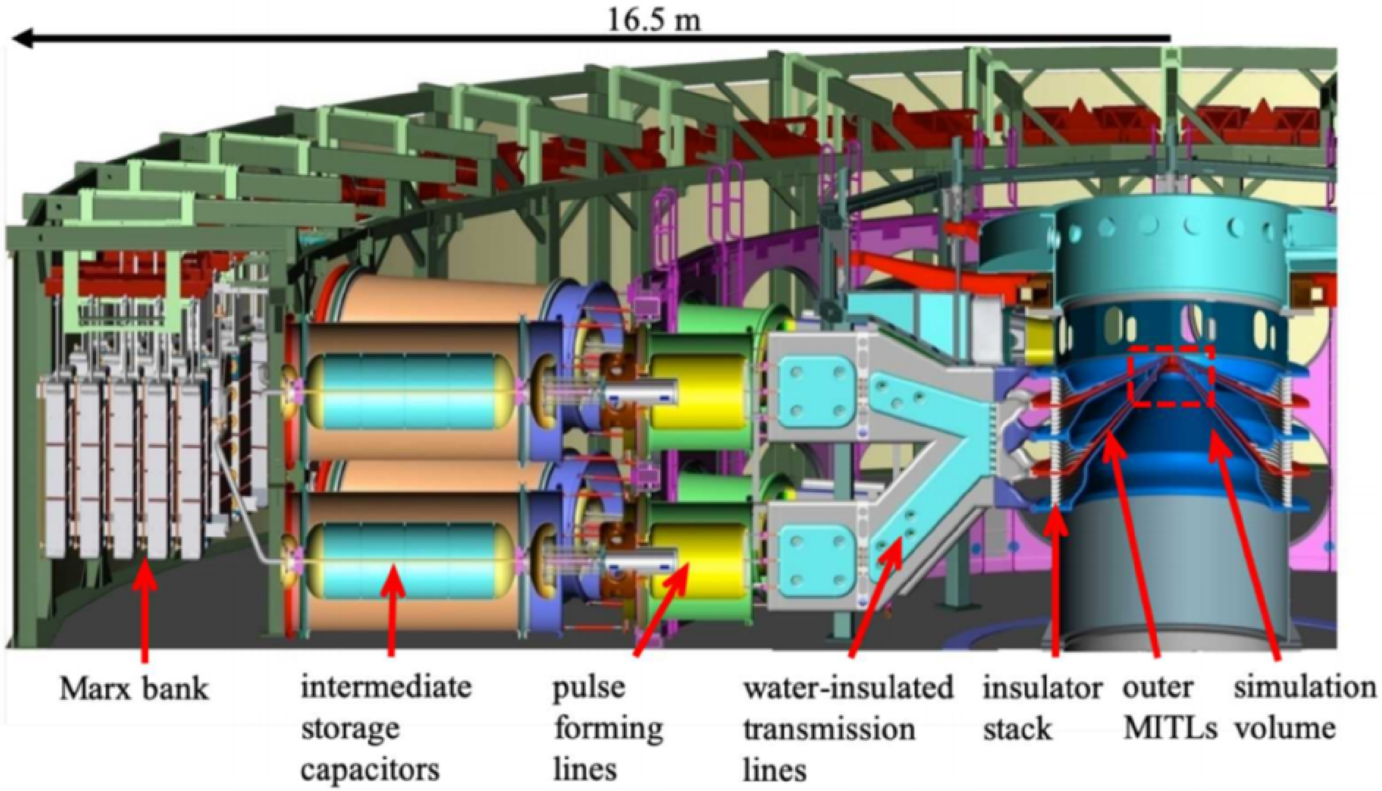
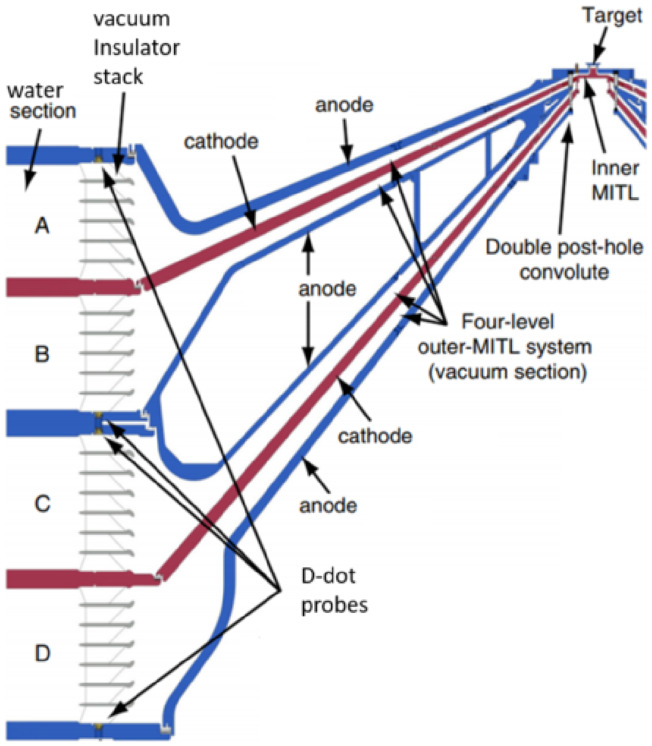
Terrestrial Application: Z Power Flow

- Z power flow
  - Conical geometry, ~2-4 MV, gap ~10mm ⇒ $\gamma \approx 5 - 9$ (Gomez et al., 2017)
  - Electron models that don't account for relativity can possess unphysical super-luminal velocities





Astrophysical Application: Magnetized Relativistic Jets

- Active Galactic Nuclei launch relativistic jets
  - $\gamma \approx 2 - 10$
- How are cosmic rays from jets accelerated?
  - Source of high energy, nonthermal ions emitted from AGN jets
  - Diffusive shock acceleration? (First order Fermi acceleration?
  - Magnetized turbulence? (Second order Fermi acceleration)
  - Magnetic Reconnection? (As suggested by PIC simulations)
- Are the jets ion-electrons or positrons-electrons?
- Multi-fluid relativistic methods are needed

Relativistic Two-Fluid Electrodynamics

Relativistic Hydrodynamics

Coupling Source Terms

$$\frac{\partial}{\partial t}(\rho_s \gamma_s) + \nabla \cdot (\rho_s \mathbf{u}_s) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{w_s}{c^2}\gamma \mathbf{u}_s\right) + \nabla \cdot \left(\frac{w_s}{c^2}\mathbf{u}_s\mathbf{u}_s + p_s \mathbf{I}\right) = \mu_s \gamma_s \rho_s \left(\mathbf{E} + \frac{\mathbf{u}_s}{\gamma_s c} \times \mathbf{B}\right) + \mathbf{R}_s$$

$$\frac{\partial}{\partial t}(w_s \gamma_s^2 - p_s) + \nabla \cdot (w_s \gamma_s \mathbf{u}_s) = \mu_s \rho_s \mathbf{u}_s \cdot \mathbf{E} + R_s^0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} - \nabla \times \mathbf{B} = -\frac{\mathbf{J}}{c}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{E} = \varrho$$

$$\nabla \cdot \mathbf{B} = 0$$

Electrodynamics

- Two (or more) charged fluids
  - Relativistic velocities and/or temperatures
  - Coupled together via Maxwell's equations
- Conservation, stability, robustness is crucial

Discontinuous Galerkin Method for Relativistic Hydrodynamics

$$\frac{\partial}{\partial t}\begin{bmatrix} D \\ \boldsymbol{M} \\ E \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho\gamma\boldsymbol{v} \\ \frac{\rho h \gamma^2}{c^2}\boldsymbol{v}\otimes\boldsymbol{v} + P\boldsymbol{I} \\ \gamma^2 \rho h \boldsymbol{v} \end{bmatrix} = \mathbf{0}$$

- Discontinuous Galerkin Method
  - Domain decomposed into cells
  - Variables approximated via polynomials over each cell

$$\mathbf{U}(\mathbf{x}) \approx \mathbf{U}^h(\mathbf{x}) = \sum_{i=1} \mathbf{U}_i \phi_i(\mathbf{x}) \quad \mathbf{x} \in \Omega_k$$

$$\phi_i(\mathbf{x}_j) = \delta_{ij}$$

$$\int_{\Omega_k} \frac{\partial \mathbf{U}^h}{\partial t}\phi(\mathbf{x})d\mathbf{x} + \oint_{\partial\Omega_k} \overline{\mathcal{F}[\mathbf{W}^h(\mathbf{U})]\cdot\mathbf{n}}\phi(\mathbf{x})ds - \int_{\Omega_k} \mathcal{F}[\mathbf{W}^h(\mathbf{U})]\cdot\nabla\phi(\mathbf{x})d\mathbf{x} = 0, \quad \forall\phi \in \{\phi_i\}$$

- Riemann Solvers to compute fluxes at surface integrals
- Gaussian quadrature used to evaluate integrals
- Difficult Pieces:
  - **Conserved to Primitive Inversion**
  - **Physicality of Conserved States**

Newtonian vs. Relativistic Conserved Variables

Newtonian Conserved Variables:

$$\boldsymbol{U} = \begin{bmatrix} D \\ \boldsymbol{M} \\ E \end{bmatrix} = \begin{bmatrix} \rho \\ \rho \boldsymbol{v} \\ \rho\left(\frac{1}{2}|\boldsymbol{v}|^2 + e\right) \end{bmatrix}$$

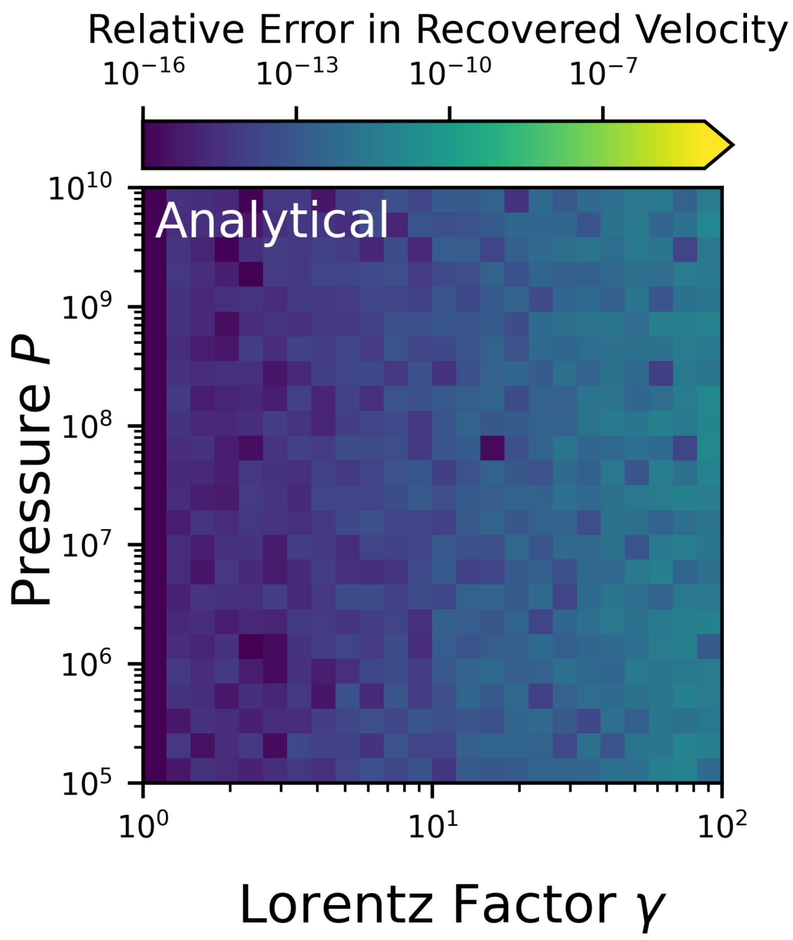
Relativistic Conserved Variables:

$$\boldsymbol{U} = \begin{bmatrix} D \\ \boldsymbol{M} \\ E \end{bmatrix} = \begin{bmatrix} \gamma\rho \\ \gamma^2 \rho h \boldsymbol{v}/c^2 \\ \gamma^2 \rho h - P \end{bmatrix}$$

$$\gamma = \frac{1}{\sqrt{1 - |\boldsymbol{v}/c|^2}} \qquad h = \frac{e + P}{\rho}$$

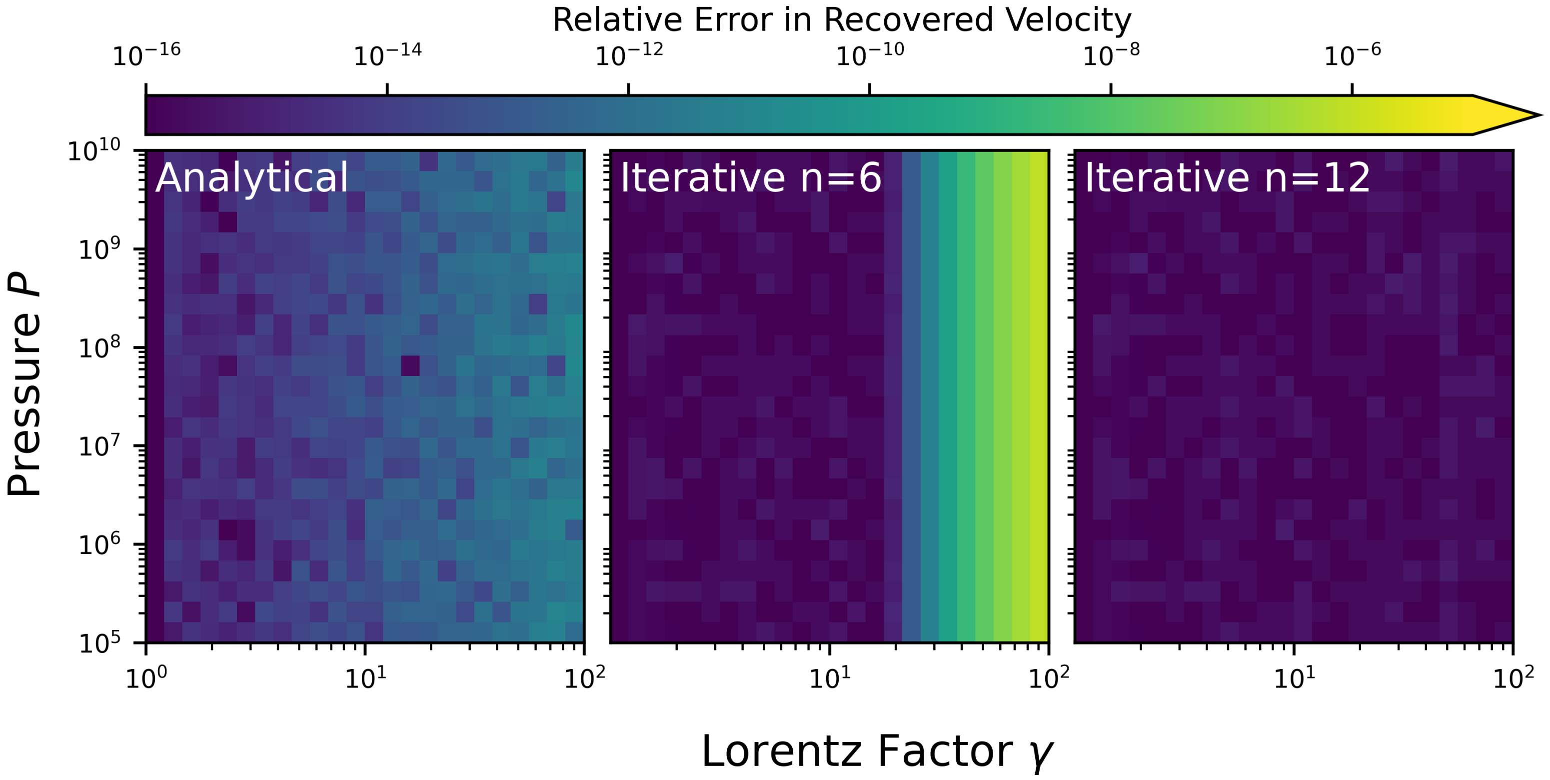
- Newtonian Hydrodynamics
  - Density, Momentum Density, Energy Density
  - Coupling between velocity and momentum is linear
- Relativistic Hydrodynamics
  - Relativistic mass density, relativistic momentum density, Energy Density (including rest mass)
  - Everything coupled though non-linear Lorentz factor
- High Lorentz factors => Velocity asymptotes to speed of light
  - Small errors in velocity lead to larger error in Lorentz factor or breaking of causality
- High Lorentz factors require robust methods
- Conserved to Primitive conversion:
  - Convert $D, \boldsymbol{M}, E \rightarrow \rho, \boldsymbol{v}, P$
  - Constrained by subluminal $\boldsymbol{v}$, positive P

Conserved to Primitive Method I: Solve the quartic analytically for v



- Analytic solver for quartic polynomials
  - Square roots, inverse trigonometry
  - Numerically unforgiving
  - Expensive to compute
- Solving for velocity
  - Solving for a small difference from speed of light
  - Machine precision can lead to superluminal velocities
  - Can fail for very relativistic temperatures
  - Small errors in velocity translate into large in Lorentz factor, other primitives

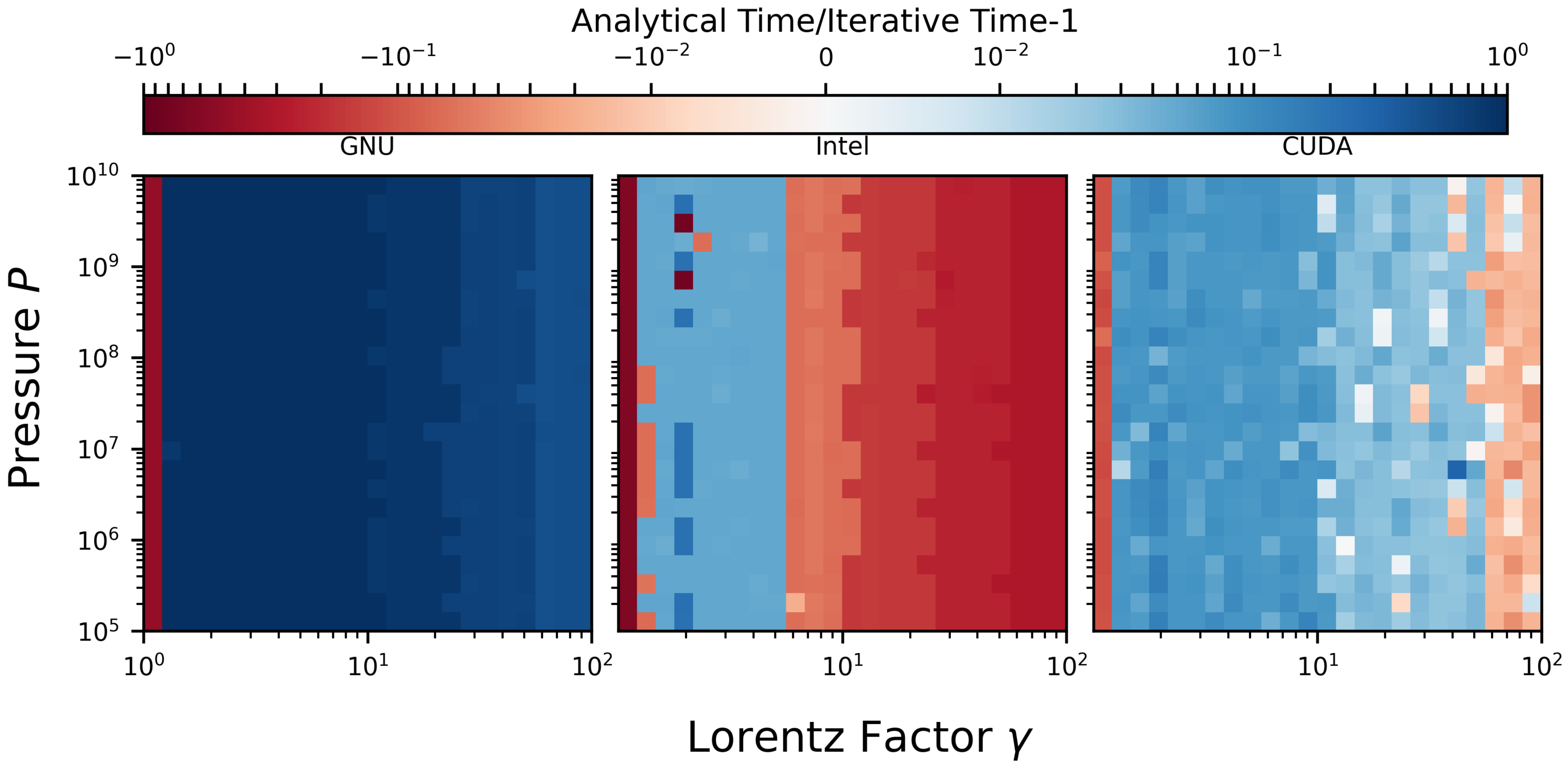
Conserved to Primitive II: Solve iteratively for w



$$v = c\frac{2w}{1+w^2}$$

- Change equations to solve in terms of "W" a velocity analogue
- Solve quartic with Newton-Raphson
  - First guess with W in [0,1] converges to physical root
  - Arbitrary accuracy
  - Robust and accurate without square roots and inverse trigonometry
- Recovers velocity to machine precession for high Lorentz factors

Iterative method can be faster than analytical method

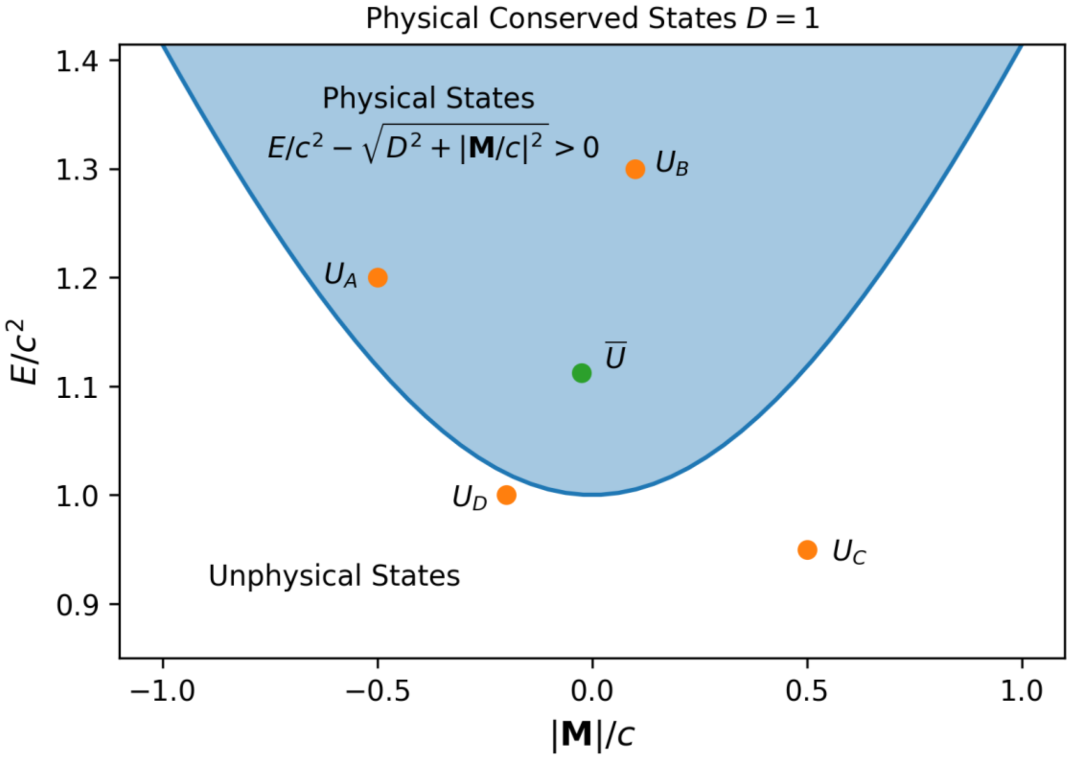


Physicality of Conserved States

- Not all conserved states are physical:
  - They don't all correspond to a primitive state
  - Can imply superluminal velocities or negative pressures
  - Conserved variables must satisfy:

$$E > 0, \qquad D > 0, \qquad E/c^2 - \sqrt{D^2 + |\boldsymbol{M}/c|^2} > 0$$

- Reconstructing conserved variables or updating conserved variables can lead to unphysical conserved states
  - Not an issue for first order or smooth flows
  - Big problem for higher order with shocks
  - Can be avoided with limiters, but by adding more diffusion



Physicality Enforcing Operator

- Conserved variables must satisfy:

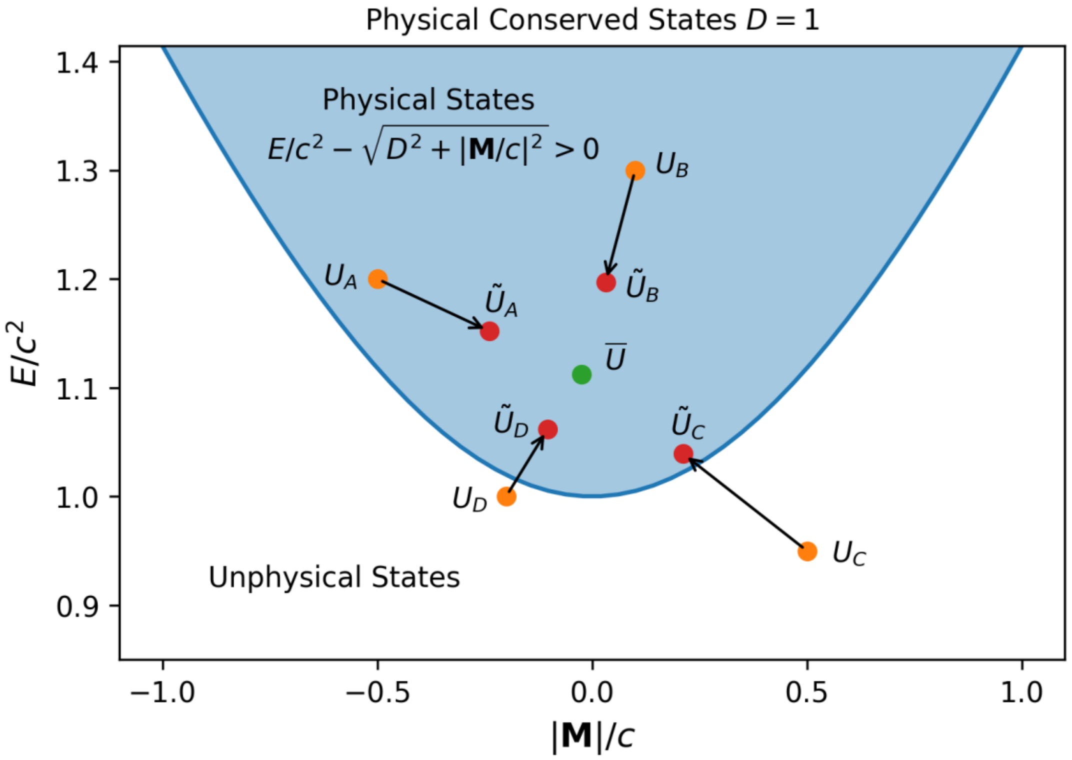
$$E > 0, \qquad D > 0, \qquad E/c^2 - \sqrt{D^2 + |\mathbf{M}/c|^2} > 0$$

- Set of physical conserved states is convex
  - If the cell volume average is physical, unphysical nodal points can be smoothed towards average

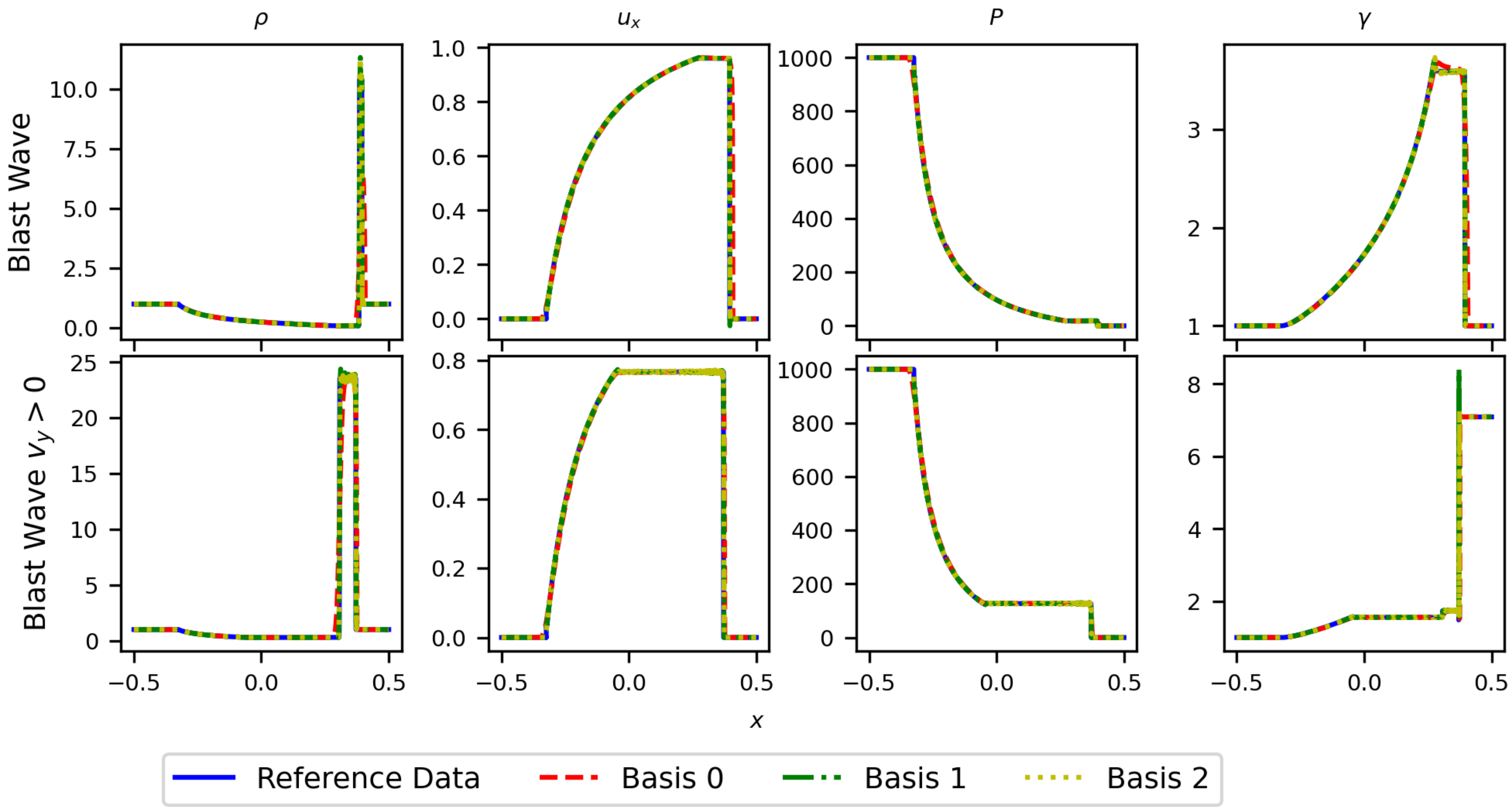
Physicality Enforcing Operator

1. Cells with unphysical nodal points are flagged
2. For each unphysical nodal point, we compute the least amount of averaging required
3. For each flagged cell, the least amount of averaging required for all points is applied

- Preserves volume average of conserved state
- Does not affect physical cells

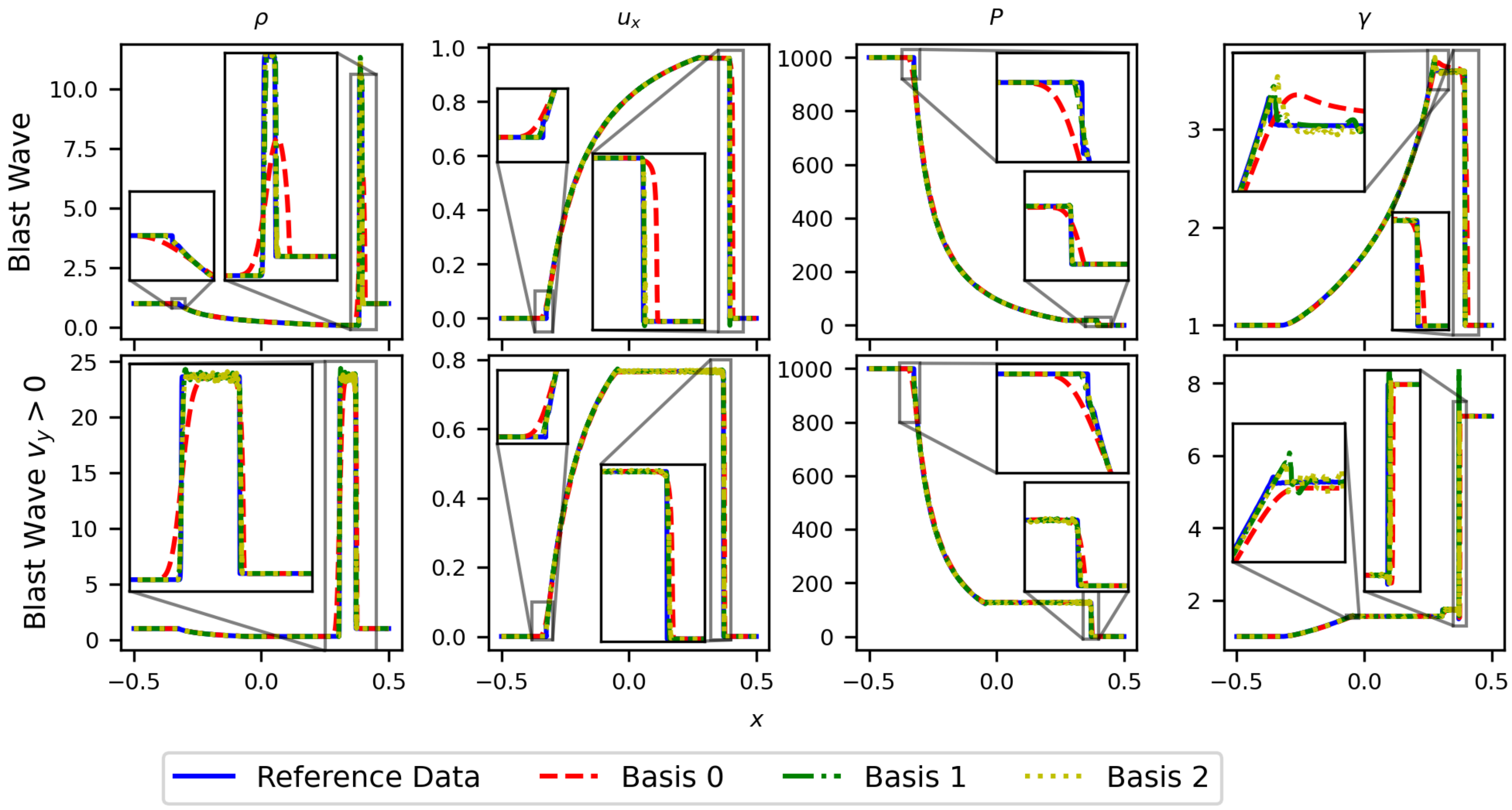


1D Shocks



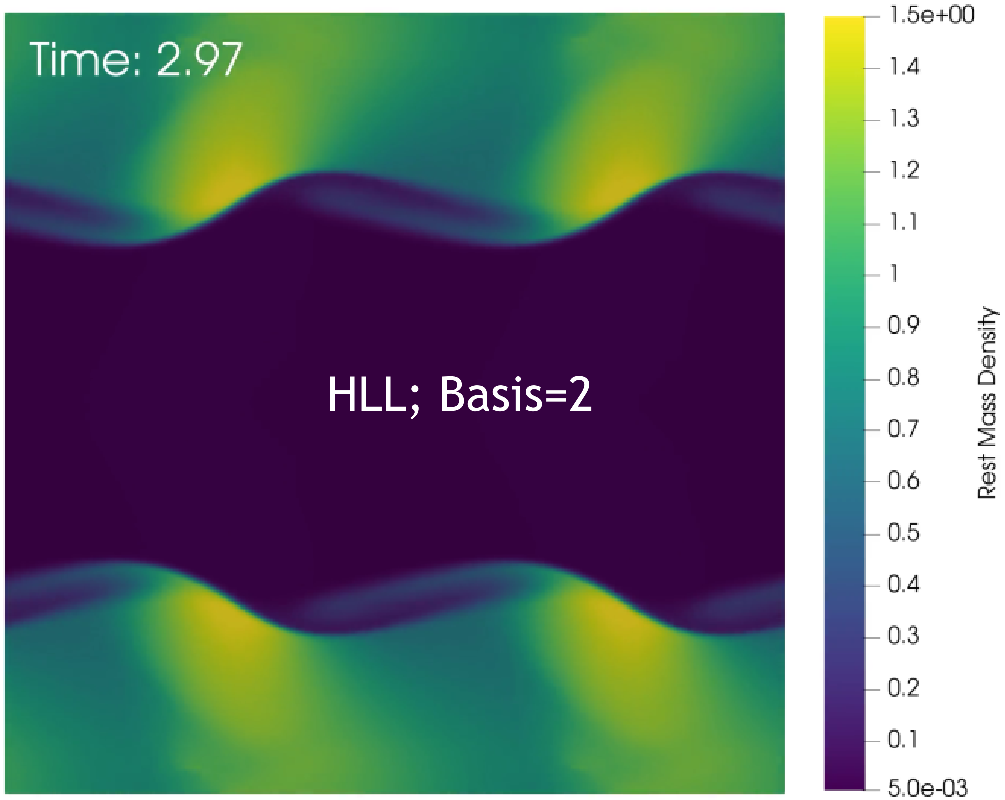
- Transverse velocity changes Lorentz factor, density, and pressure
- Conserved to Primitive solver enables high Lorentz factor
- Physicality Enforcing Operator handles low pressures

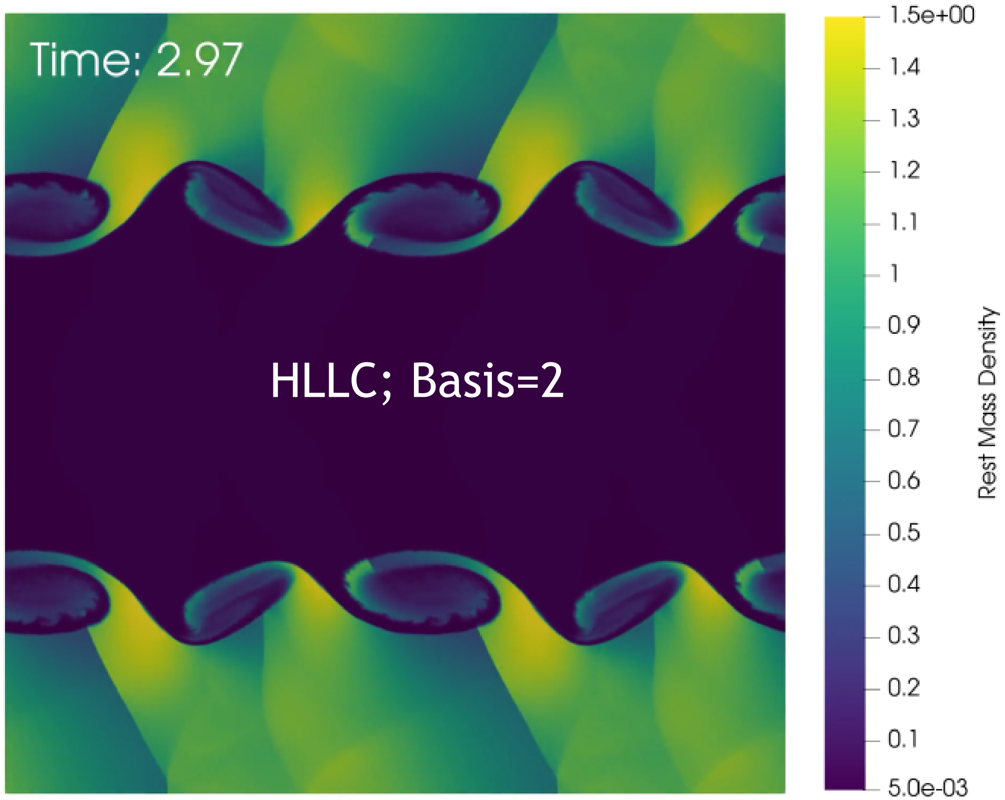
1D Shocks



- Limiters are still needed
  - Aggressive limiting can smooth out solution, but at cost of convergence
- Shocks adaptive methods could resolve issues

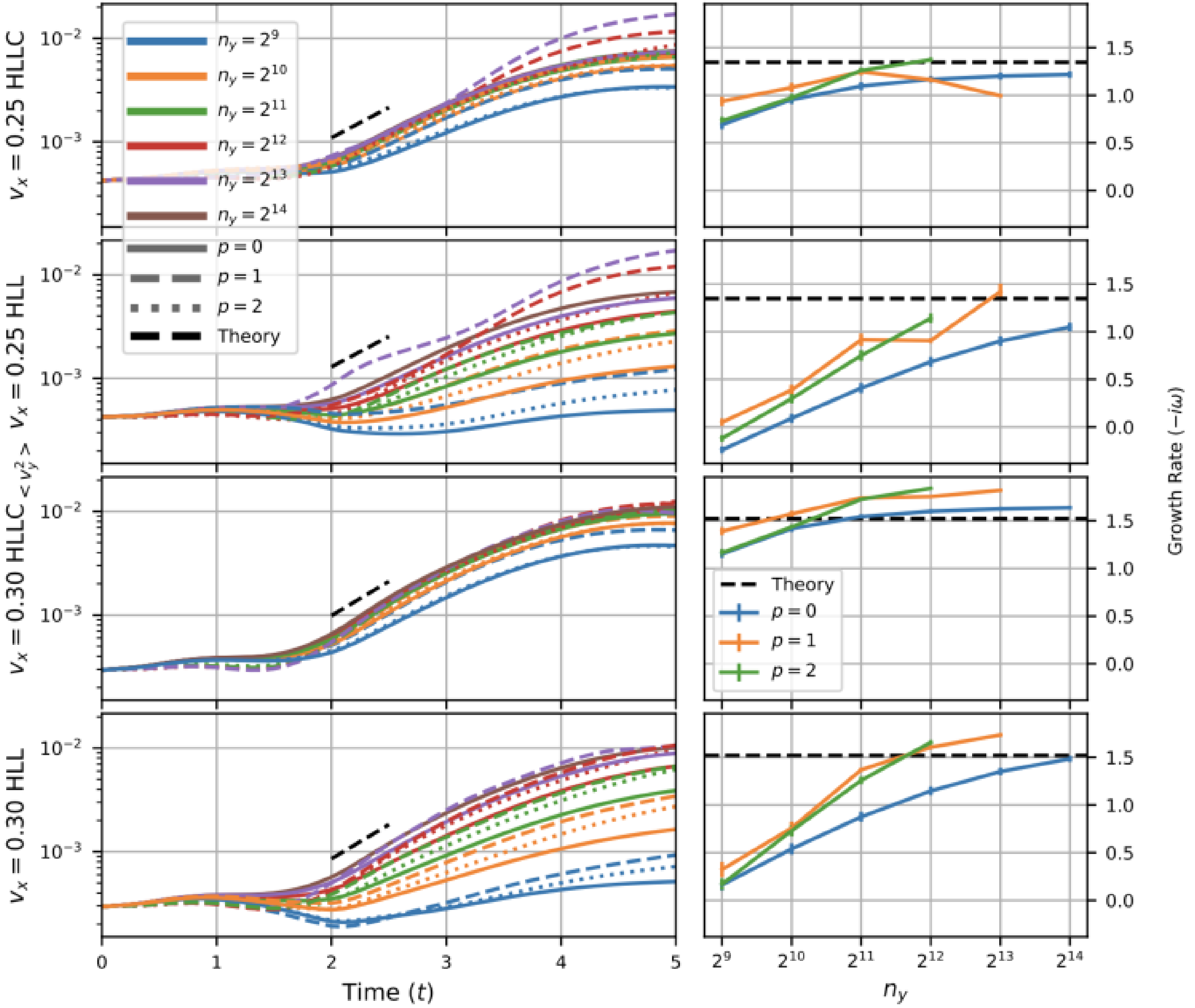
Kelvin Helmholtz Instability





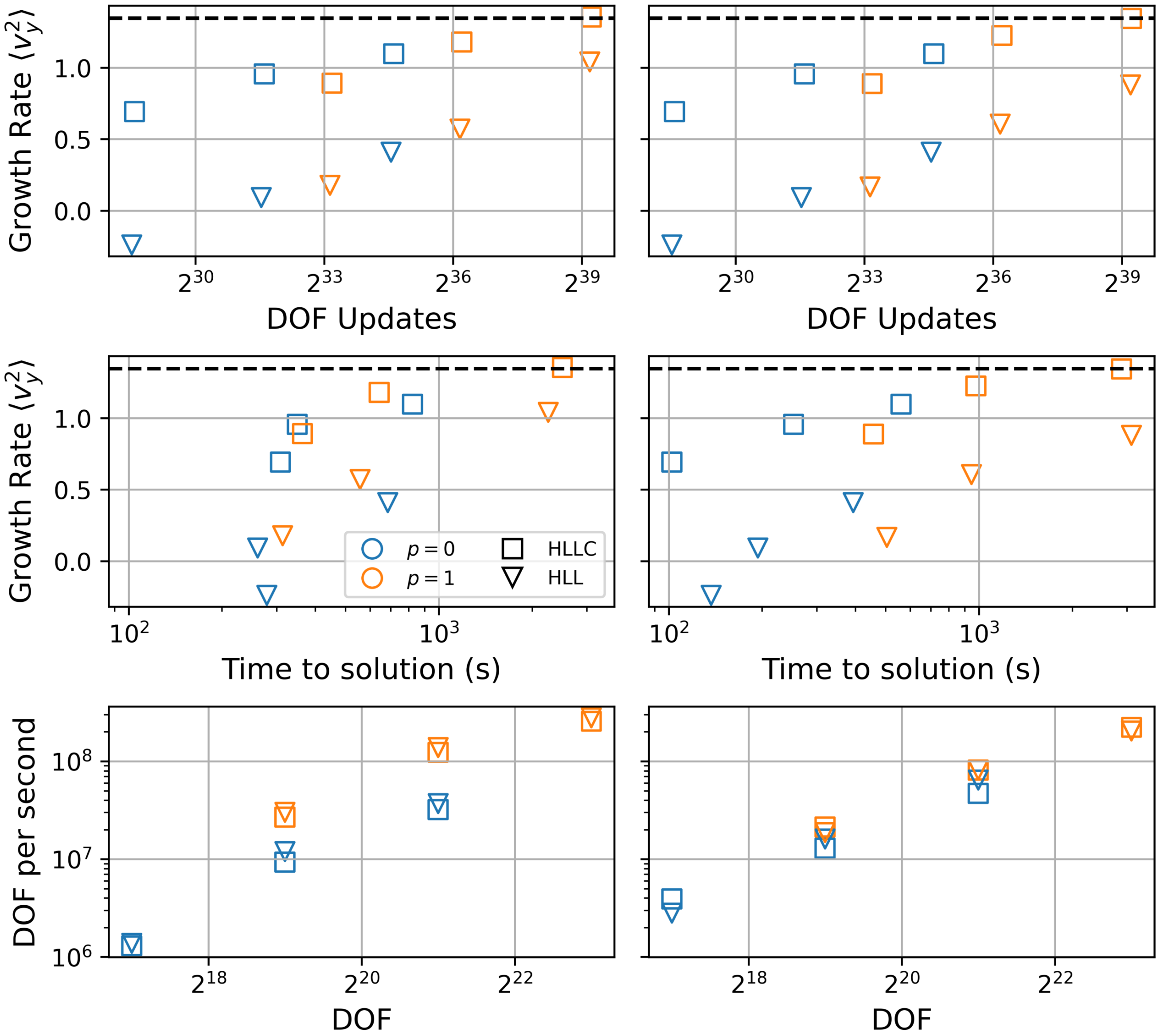
For KHI, Riemann Solver Beats Resolution or Order

- Suite of Kelvin-Helmholtz simulations (using Bodo 2004)
  - Probing resolution, method order, Reimann solver
  - For different shear velocities
- Compare to analytic growth rate
- Riemann solver makes the biggest difference
- Basis order, resolution makes a smaller difference



Are lower order bases more efficient for KHI?

- More cells increases growth rate slightly more than higher order
- Implementation specific
  - Are all basis orders equally computationally efficient on all architectures?
  - For our implementation on GPUs yes, for CPUs no



Extending to Two-Fluids (and beyond)

Relativistic Hydrodynamics

Coupling Source Terms

$$\frac{\partial}{\partial t}(\rho_s\gamma_s) + \nabla\cdot(\rho_s\mathbf{u}_s) = 0$$

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$$\frac{\partial}{\partial t}(w_s\gamma_s^2 - p_s) + \nabla\cdot(w_s\gamma_s\mathbf{u}_s) = \mu_s\rho_s\mathbf{u}_s\cdot\mathbf{E} + R_s^0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{E} - \nabla\times\mathbf{B} = -\frac{\mathbf{J}}{c}$$

$$\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B} + \nabla\times\mathbf{E} = 0$$

$$\nabla\cdot\mathbf{E} = \varrho$$

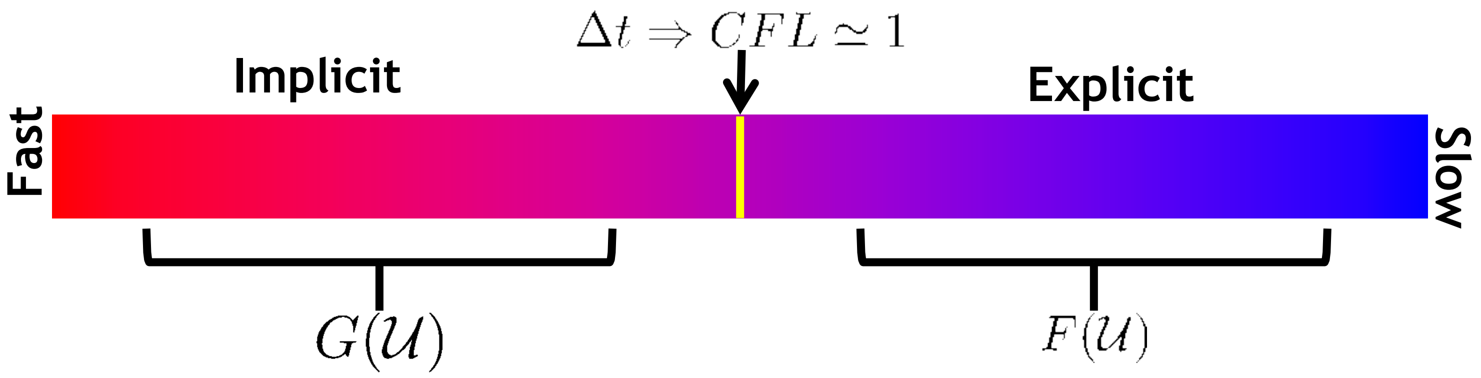
$$\nabla\cdot\mathbf{B} = 0$$

Electrodynamics

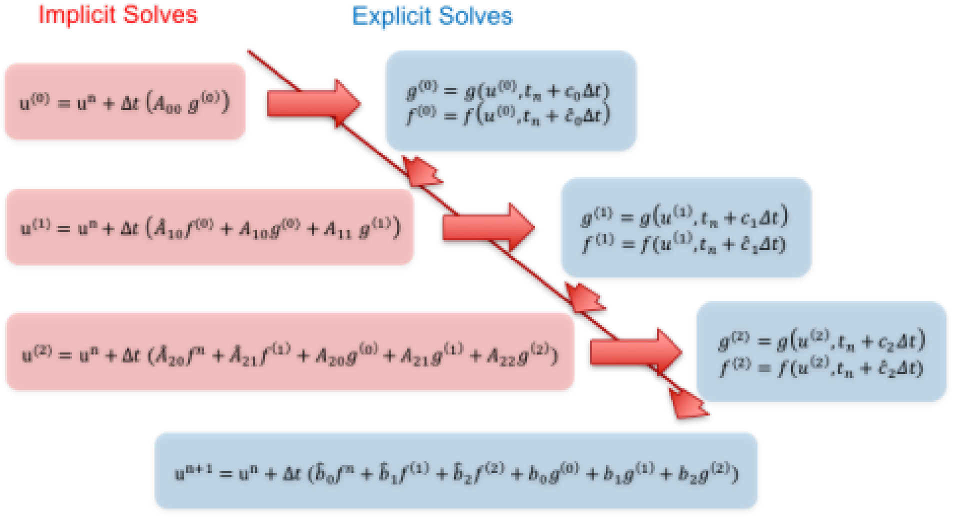
- Explicit Hydrodynamics, Electric Fields
- Implicit Source terms

Implicit-Explicit (IMEX) Time Integration & stiff modes in relativistic plasmas

- IMEX methods split fast and slow modes
  - Implicit terms solve for stiff modes (plasma oscillation, collisions, cyclotron frequency)
  - Explicit terms are accurately resolved (all of CoM physics)
  - IMEX assumes an additive decomposition: $\dot{\mathcal{U}} + F(\mathcal{U}) + G(\mathcal{U}) = 0$



3 Stage IMEX-RK Algorithm



Stiff Modes:

- (blue) Plasma., Oscillation
- (green) Collisions
- (red) Cyclotron frequency

$$\frac{\partial}{\partial t}\left[\sum_s \mu_s \rho_s \gamma_s\right] + \nabla \cdot \left[\sum_s \mu_s \rho_s \mathbf{u}_s\right] = 0$$

$$\frac{\partial}{\partial t}\left[\sum_s \mu_s \rho_s h_s \gamma_s \mathbf{u}_s\right] + \nabla \cdot \left[\sum_s \left(\mu_s \rho_s h_s \mathbf{u}_s \mathbf{u}_s + \mu_s P_s \mathbb{I}\right)\right] = \omega_P^2 \left(\gamma \mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) + \sum_s \mu_s \mathbf{R}_s$$

$$\frac{\partial}{\partial t}\left[\sum_s \left(\mu_s \rho_s h_s \gamma_s^2 - \mu_s P_s\right)\right] + \nabla \cdot \left[\sum_s \mu_s \rho_s h_s \gamma_s \mathbf{u}_s\right] = \omega_P^2 \mathbf{u} \cdot \mathbf{E} + \sum_s \mu_s R_s^0$$

$$\frac{\partial \mathbf{E}}{\partial t} = c \nabla \times \mathbf{B} - \sum_s \mu_s \rho_s \mathbf{u}_s$$

Current Work

- Multi-Fluid IMEX method with separates fluids
  - Minimize error-sensitive conserved to primitive conversions
- Apply relativistic two-fluid electrodynamics methods to relativistic jets

References

Beckwith, K., & Stone, J. M. 2011, The Astrophysical Journal Supplement Series, 193, 6

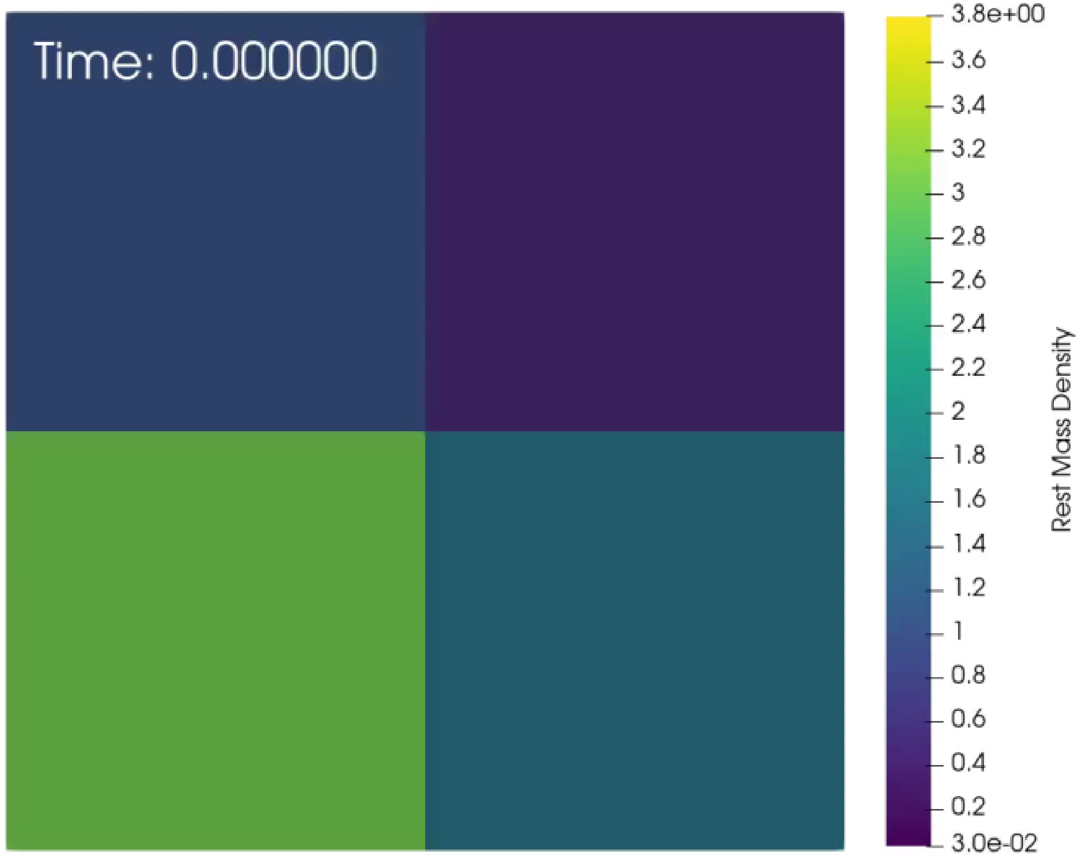
Bodo, G., Mignone, A., & Rosner, R. 2004, PHYSICAL REVIEW E, 4

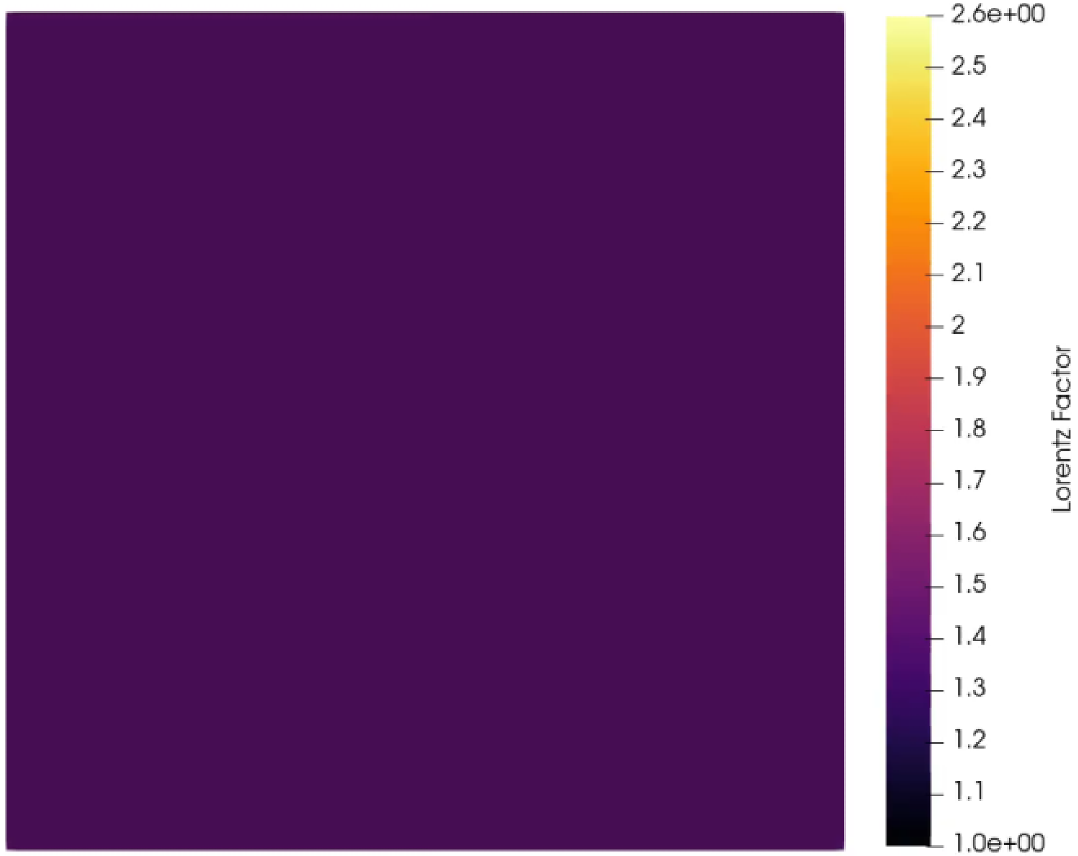
Moe, S. A., Rossmanith, J. A., & Seal, D. C. 2015, arXiv:150703024 [math], http://arxiv.org/abs/1507.03024

Núñez-de la Rosa, J., & Munz, C.-D. 2018, Computer Physics Communications, 222, 113

Backup Slides

2D Riemann Problems





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- Implementation specific
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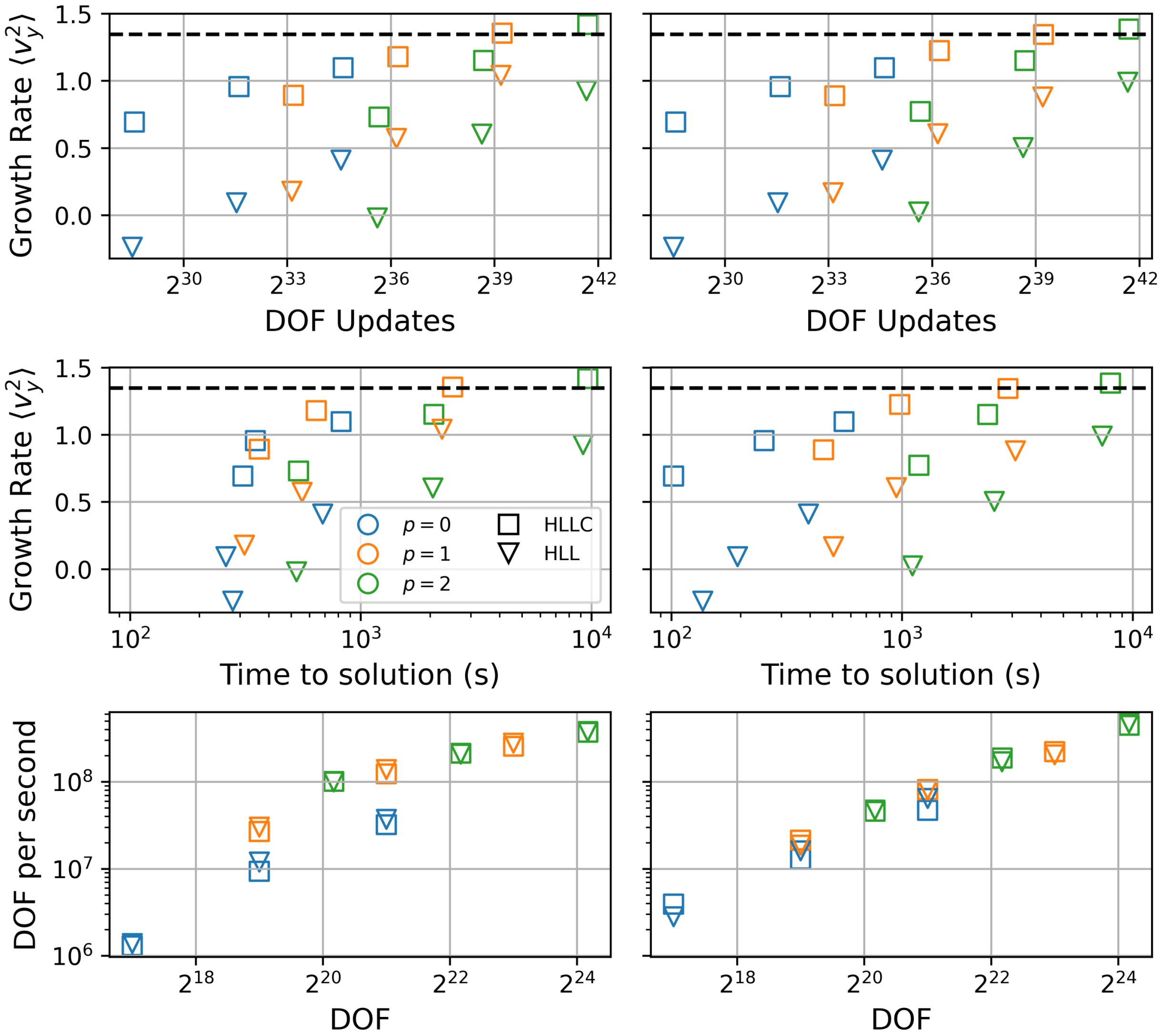
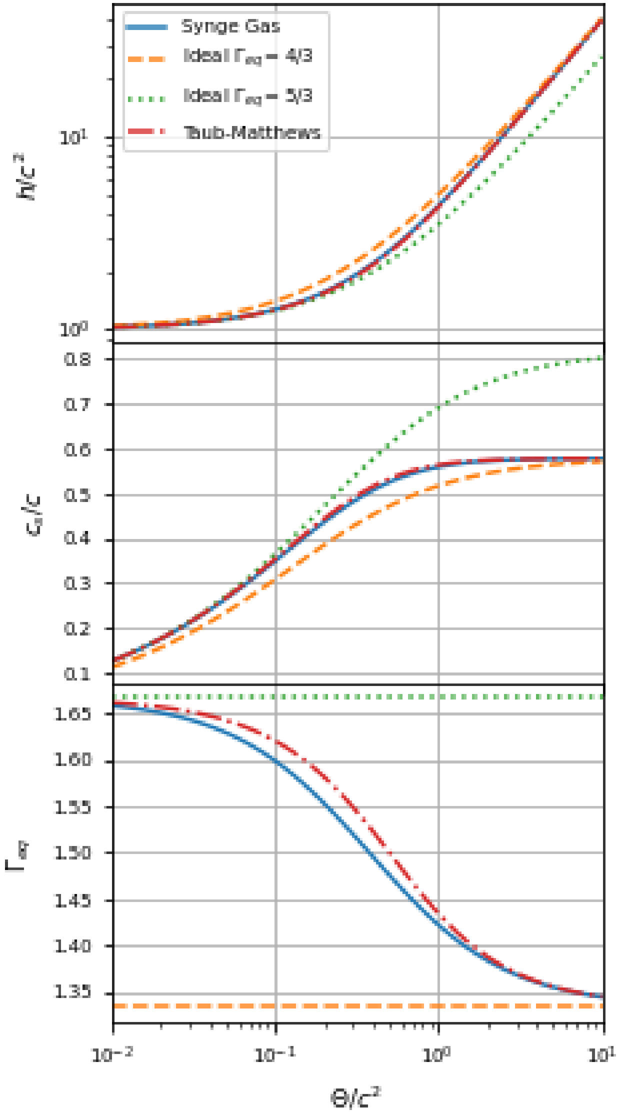
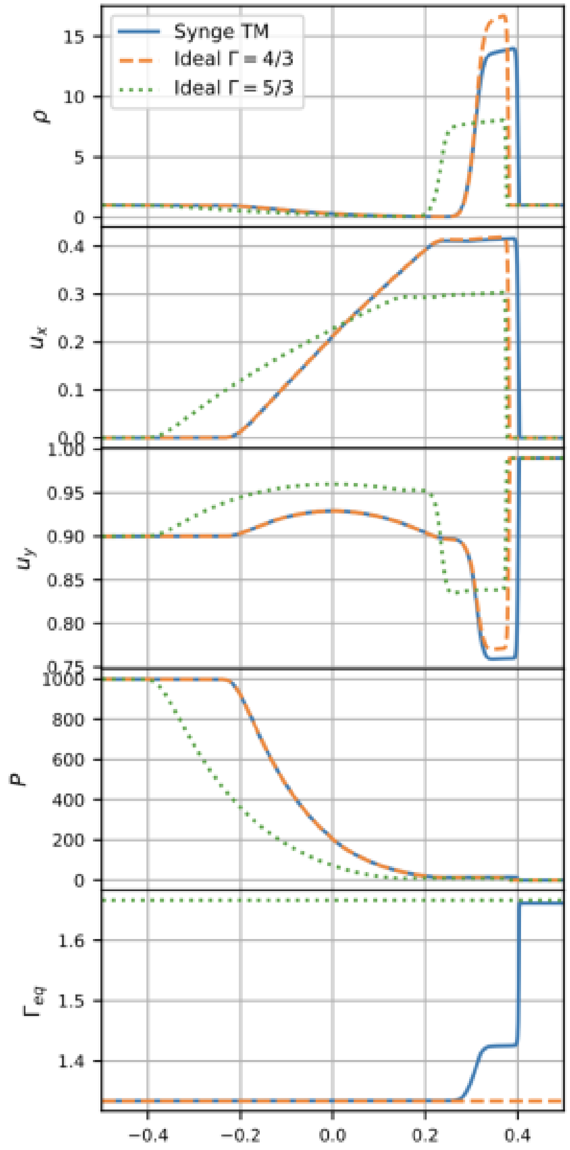
  - For our implementation on GPUs yes, for CPUs no
- Minmod limiter incompatible with basis order 2



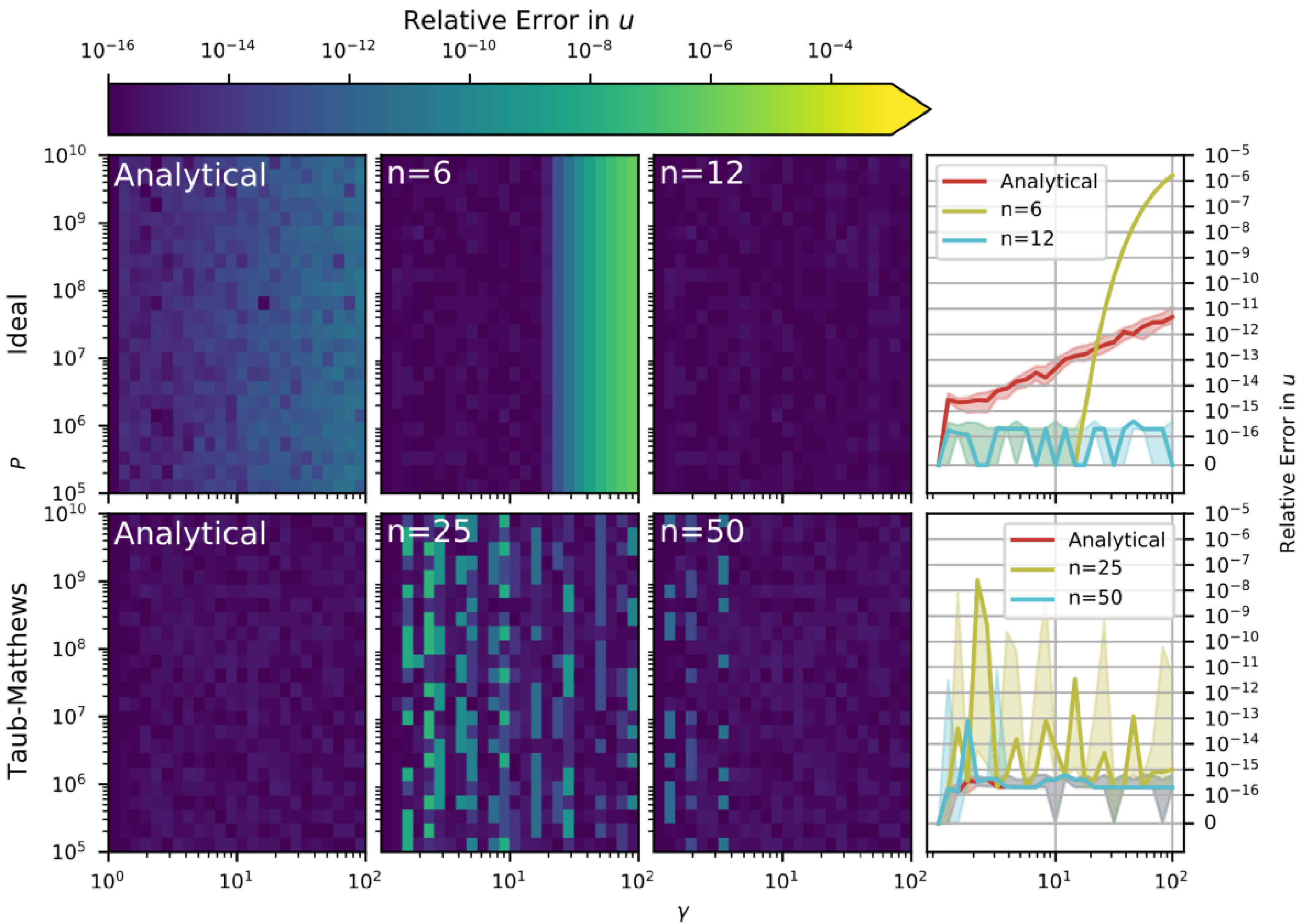
Synge Gas

- Adiabatic index of a perfect gas varies from 5/3 to 4/3 for sub-relativistic to relativistic temperatures
- Synge gas correctly models perfect gas
  - Requires Bessel functions, Inverse Bessel functions
- Taub-Matthews approximates Synge Gas





Ideal and Taub-Matthews Solver Accuracy



Synge Gas Performance