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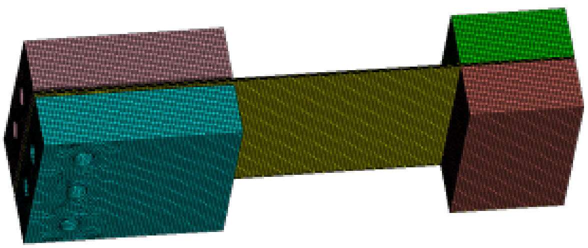
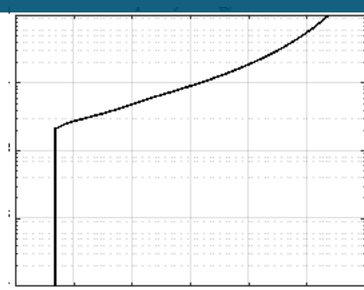
Multi-Harmonic Balance with Preconditioned Iterative Solver





Presented at IMAC XXXIX in February 2021

Presented by

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Motivation: Nonlinear Normal Modes

Nonlinear Normal Modes provide theoretical foundation for modal analysis in the presence of nonlinear physics

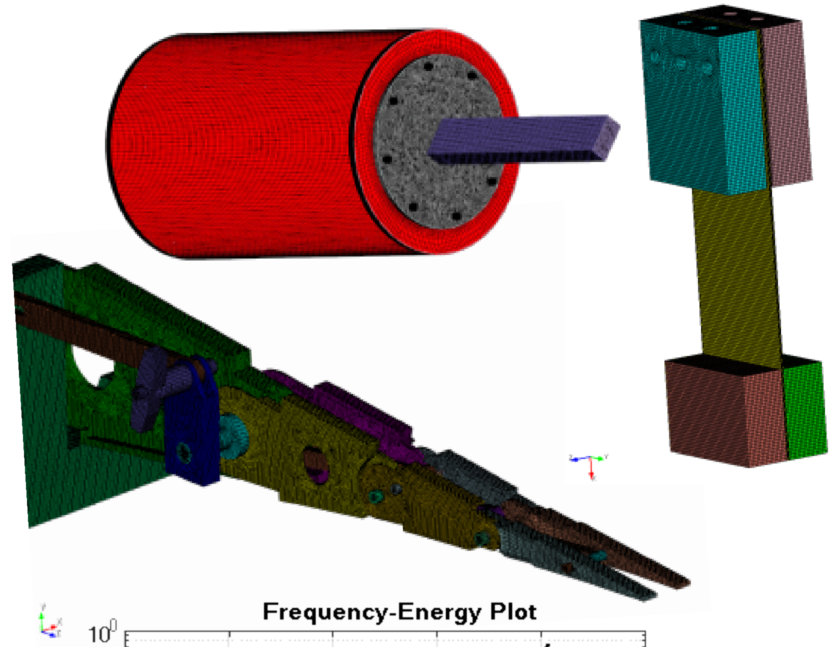
- Useful framework when linear(ized) modal analysis no longer valid

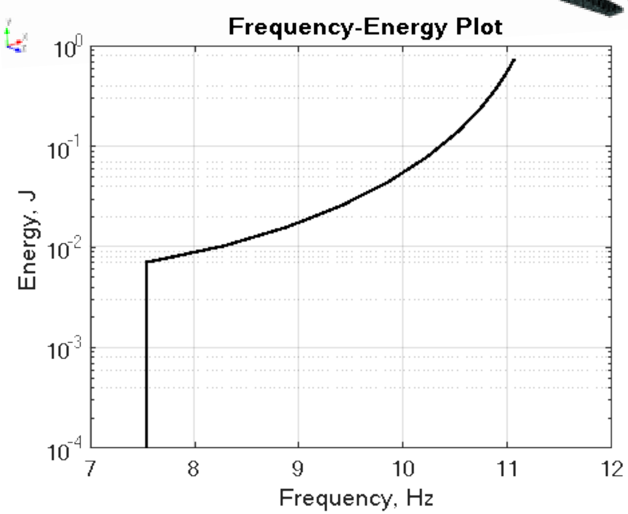
Several algorithms in literature to solve NNMs for mechanical systems

- E.g. see review by Renson et al. [1]

Remaining challenge is to address issue of scalability to large-order systems arising in computational mechanics/dynamics

- Seek to utilize iterative solvers within multi-harmonic balance to speed up and parallelize inversion of large algebraic systems





[1] L. Renson, G. Kerschen, and B. Cochelin, "Numerical computation of nonlinear normal modes in mechanical engineering," *Journal of Sound and Vibration,* vol. 364, pp. 177-206, 2016.

Nonlinear Normal Mode Definition

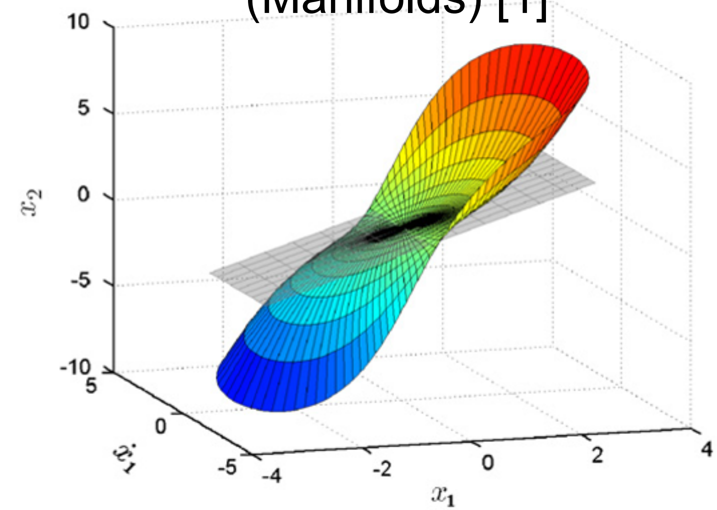
Many definitions exist for either damped or undamped systems [1, 2]

For a conservative (undamped) system, a nonlinear normal mode (NNM) is defined as a ***not necessarily synchronous periodic response of the undamped nonlinear system***

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{f}_{nl}\big(\mathbf{x}(t)\big) = \mathbf{f}_{pre}$$

For an MDOF system, there exists *N* NNM solution branches that are extensions of linear normal modes at low energy [1]

Nonlinear Mode Shapes (Manifolds) [1]



[1] Kerschen, G., et al., *Nonlinear normal modes. Part I. A useful framework for the structural dynamicist.* Mechanical Systems and Signal Processing, 2009.
[2] Haller, G., Ponsioen, S., *Nonlinear normal modes and spectral submanifolds: Existence, uniqueness, and use in model reduction*. Nonlinear Dynamics, 2016.

Multi-harmonic Balance for Periodic Orbits

Assume truncated Fourier series for the periodic response and nonlinear restoring force

$$\mathbf{x}(t) = \frac{\mathbf{c}_0^x}{\sqrt{2}} + \sum_{k=1}^{N_h} [\mathbf{s}_k^x \sin(k\omega t) + \mathbf{c}_k^x \cos(k\omega t)]$$

$$\mathbf{f}_{nl}(\mathbf{x}) = \frac{\mathbf{c}_0^f}{\sqrt{2}} + \sum_{k=1}^{N_h} [\mathbf{s}_k^f \sin(k\omega t) + \mathbf{c}_k^f \cos(k\omega t)]$$

After substitution and Galerkin projection onto orthogonal periodic functions

$$\mathbf{r}(\mathbf{z}, \omega) = \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}(\mathbf{z}) - \mathbf{b}_{pre} = \mathbf{0}$$

Unknowns: vector $\mathbf{z}$ (collection of Fourier coefficients) and scalar ω (fundamental frequency)

See references [1] and [2] for details and derivation of MHB for mechanical systems

[1] T. Detroux, L. Renson, L. Masset, and G. Kerschen, "The harmonic balance method for bifurcation analysis of large-scale nonlinear mechanical systems,“ *Computer Methods in Applied Mechanics and Engineering*, vol. 296, pp. 18-38, 2015.
[2] M. Krack and J. Gross, *Harmonic Balance for Nonlinear Vibration Problems*, 1st ed. Springer International Publishing, 2019.

Predictor-Corrector Methods for Tracing Curves

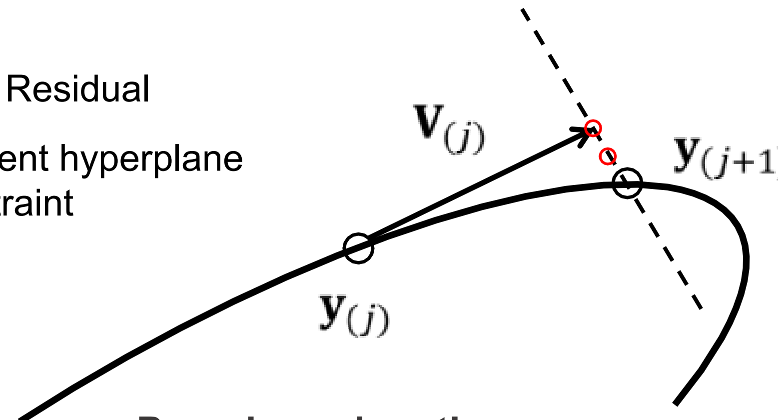
Pseudo-arclength continuation used to trace periodic solution of MHB equations

$$\mathbf{R}(\mathbf{y}) = \begin{bmatrix} \mathbf{A}(\omega)\mathbf{z} + \mathbf{b}(\mathbf{z}) - \mathbf{b}_{pre} \\ \mathbf{V}^{\mathrm{T}}\left(\mathbf{y} - \mathbf{y}^{(k=1)}\right) \end{bmatrix}$$

MHB Residual

Tangent hyperplane constraint

$$\mathbf{y} = [\mathbf{z}^{\mathrm{T}} \quad \omega]^{\mathrm{T}}$$



Pseudo-arclength continuation

Truncating the Taylor series expansion of above equations results in a system of equations to iteratively solve for corrections

$$\begin{bmatrix} \mathbf{r}_{\mathbf{z}}\left(\mathbf{y}^{(k)}\right) & \mathbf{r}_{\omega}\left(\mathbf{y}^{(k)}\right) \\ \mathbf{V}_{\mathbf{z}}^{\mathrm{T}} & V_{\omega} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}^{(k)} \\ \Delta\omega^{(k)} \end{bmatrix} = -\mathbf{R}\left(\mathbf{y}^{(k)}, \mathbf{V}\right)$$

Requires solution to potentially large-scale, sparse linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ that scales as

N ≈ 2 · (# Harmonics) · (# DOF)

System of equations scales linearly with the assumed number of harmonics functions in the Fourier basis – issues realized when inverting large matrices such as those arising in high-fidelity FEA

Iterative Solver for MHB

Starting with an inexact Newton update [1], find an approximate update $\Delta\mathbf{y}^{(k)}$ satisfying,

$$\left\|\mathbf{R}\left(\mathbf{y}^{(k)}\right)+\mathbf{R}_{\mathbf{y}}\left(\mathbf{y}^{(k)}\right)\Delta\mathbf{y}^{(k)}\right\| \leq \eta_k \left\|\mathbf{R}\left(\mathbf{y}^{(k)}\right)\right\| \qquad 0 \leq \eta_k < 1$$

(Forcing Term)

Utilize Krylov subspace iterative methods to solve large-scale and sparse linear system system until inexact Newton condition satisfied

$$\mathbf{R}_{\mathbf{y}}\left(\mathbf{y}^{(k)}\right)\Delta\mathbf{y}^{(k)} = -\mathbf{R}\left(\mathbf{y}^{(k)}\right)$$

This work uses the "restarted" Generalized Minimal Residual (GMRES) method [2]

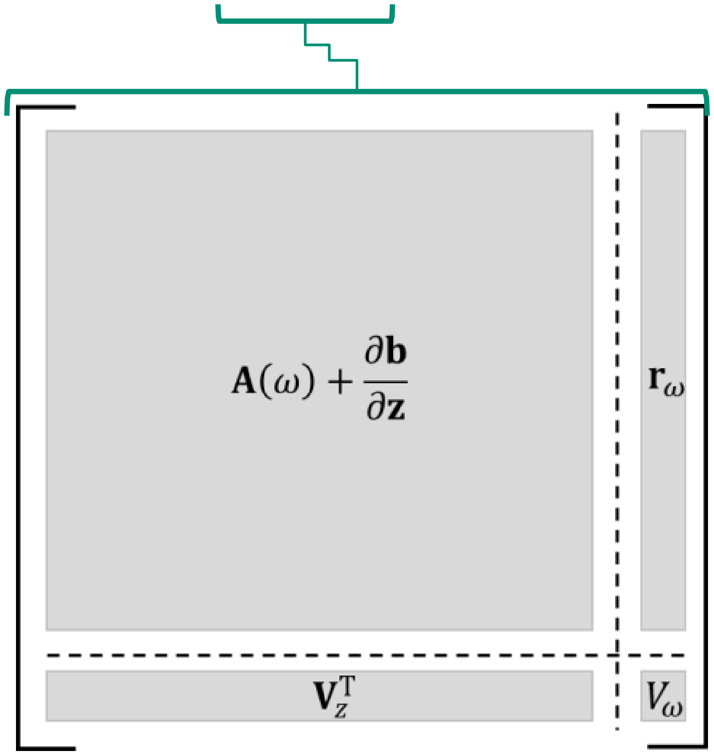
- Nonsymmetric and indefinite matrices
- Convergence and computational cost drastically influenced by the choice of the (left) preconditioner $\mathbf{M}$

$$\left[\mathbf{M}^{-1}\mathbf{R}_{\mathbf{y}}\left(\mathbf{y}^{(k)}\right)\right]\Delta\mathbf{y}^{(k)} = -\ \mathbf{M}^{-1}\ \mathbf{R}\left(\mathbf{y}^{(k)}\right)$$

[1] R. S. Dembo, S. C. Eisenstat, and T. Steihaug, "Inexact newton methods," *SIAM Journal on Numerical analysis,* vol. 19, no. 2, pp. 400-408, 1982.
[2] Y. Saad and M. H. Schultz, "GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems," *SIAM Journal on scientific and statistical computing,* vol. 7, no. 3, pp. 856-869, 1986.

Iterative Solver for Sparse Large-Scale Systems

$$\mathbf{R}_{\mathbf{y}}\left(\mathbf{y}^{(k)}\right)\Delta\mathbf{y}^{(k)} = -\mathbf{R}\left(\mathbf{y}^{(k)}\right)$$



Ingredients for a good preconditioner $\mathbf{M}$

- Close to the inverse of $\mathbf{R}_{\mathbf{y}}\left(\mathbf{y}^{(k)}\right)$ for reduced spectral condition number
- Inexpensive to perform inverse several times

Known properties of the large matrix $\mathbf{R}_{\mathbf{y}}\left(\mathbf{y}^{(k)}\right)$

- Nonsymmetric
- Block-bordered matrix
- Dominated by $\mathbf{r}_{\mathbf{z}}\left(\mathbf{y}^{(k)}\right)$ which is sparse, symmetric (for most nonlinearities) and has a block diagonal form for the linear portion

$$\mathbf{A}(\omega) = \begin{bmatrix} \mathbf{K} & & & & & \\ & \mathbf{K}-(\omega)^2\mathbf{M} & 0 & & & \\ & 0 & \mathbf{K}-(\omega)^2\mathbf{M} & & & \\ & & & \ddots & & \\ & & & & \mathbf{K}-(N_h\omega)^2\mathbf{M} & 0 \\ & & & & 0 & \mathbf{K}-(N_h\omega)^2\mathbf{M} \end{bmatrix}$$

$$\frac{\partial \mathbf{b}}{\partial \mathbf{z}} = \begin{bmatrix} \dfrac{\partial \mathbf{b}_0}{\partial \mathbf{z}_0} & \cdots & \dfrac{\partial \mathbf{b}_{cN_h}}{\partial \mathbf{z}_0} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \mathbf{b}_0}{\partial \mathbf{z}_{cN_h}} & \cdots & \dfrac{\partial \mathbf{b}_{cN_h}}{\partial \mathbf{z}_{cN_h}} \end{bmatrix}$$

Preconditioner for GMRES

Proposed preconditioner for MHB for undamped NNM computations:

$$\mathbf{M} = \begin{bmatrix} \mathbf{A}(\omega) + \widetilde{\frac{\partial \mathbf{b}}{\partial \mathbf{z}}} & \mathbf{r}_\omega\left(\mathbf{y}^{(k)}\right) \\ \mathbf{V}_\mathrm{z}^\mathrm{T} & V_\omega \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{r}}_\mathbf{z}\left(\mathbf{y}^{(k)}\right) & \mathbf{r}_\omega\left(\mathbf{y}^{(k)}\right) \\ \mathbf{V}_\mathrm{z}^\mathrm{T} & V_\omega \end{bmatrix}$$

where

$$\tilde{\mathbf{r}}_\mathbf{z}\left(\mathbf{y}^{(k)}\right) = \begin{bmatrix} \mathbf{K} + \frac{\partial \mathbf{b}_0}{\partial \mathbf{z}_0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{K} - (\omega)^2\mathbf{M} + \frac{\partial \mathbf{b}_{s1}}{\partial \mathbf{z}_{s1}} & & \vdots \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{K} - (N_h\omega)^2\mathbf{M} + \frac{\partial \mathbf{b}_{cN_h}}{\partial \mathbf{z}_{cN_h}} \end{bmatrix}$$

Eliminates off-diagonal terms in $\frac{\partial \mathbf{b}}{\partial \mathbf{z}}$ to allow for preconditioner to have block-diagonal and block-bordered form

- Size still dominated by the upper left quadrant
- Eliminates coupling between nonlinear force and response harmonics

Preconditioner for GMRES

Inverse of block-bordered preconditioner becomes

$$\mathbf{M}^{-1} = \begin{bmatrix} \tilde{\mathbf{r}}_{\mathbf{z}} & \mathbf{r}_{\omega} \\ \mathbf{V}_{\mathrm{Z}}^{\mathrm{T}} & V_{\omega} \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1} + \tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1}\mathbf{r}_{\omega}\beta^{-1}\mathbf{V}_{\mathrm{Z}}^{\mathrm{T}}\tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1} & -\tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1}\mathbf{r}_{\omega}\beta^{-1} \\ -\beta^{-1}\mathbf{V}_{\mathrm{Z}}^{\mathrm{T}}\tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1} & \beta^{-1} \end{bmatrix}$$

where

$$\beta^{-1} = V_{\omega} - \mathbf{V}_{\mathrm{Z}}^{\mathrm{T}}\tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1}\mathbf{r}_{\omega}$$

The computational intensive portions of this equation are

- $\tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1}\mathbf{r}_{\omega}$
- $\tilde{\mathbf{r}}_{\mathbf{z}}{}^{-1}\mathbf{b}_1$ when computing linear system of form $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$

Now the inversion of the block-diagonal matrix $\tilde{\mathbf{r}}_{\mathbf{z}}$ can be computed by inverting **2 · (# Harmonics)** linear systems of order **(# DOF)**!

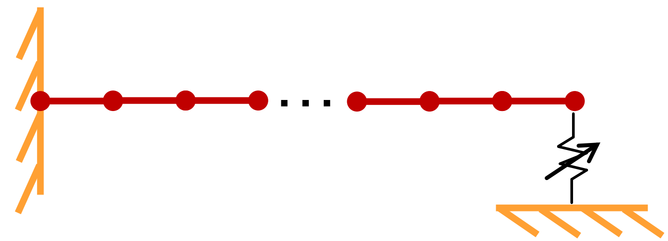
- Compared to a single inversion of a **2 · (# Harmonics) · (# DOF)** system

"Embarrassingly Parallel" Inversion of Preconditioner

Numerical Examples using Full-FEMs

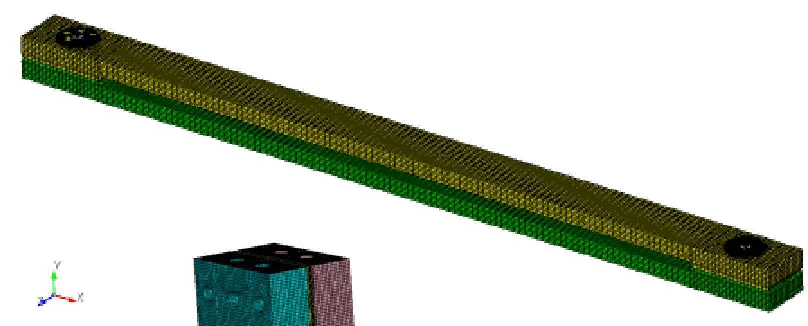
Cantilever beam with cubic spring

- Verification of algorithm with traditional direct solver and Newton iterations



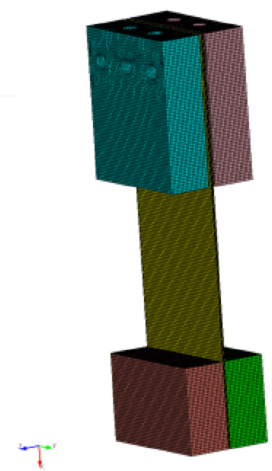
Nonlinear C-Beam FEM

- Evaluation of cost savings and performance



Nonlinear pylon FEM

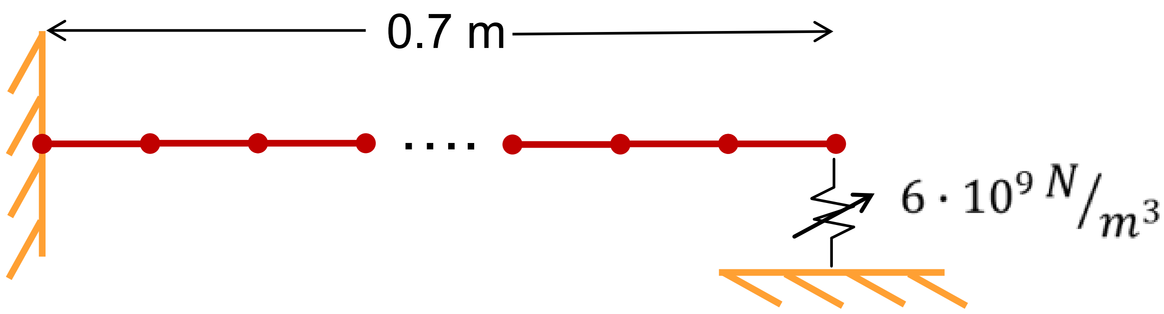
- Demonstration of scalability on large-scale model



Nonlinear Cantilever Beam Example

Finite element model created in MATLAB

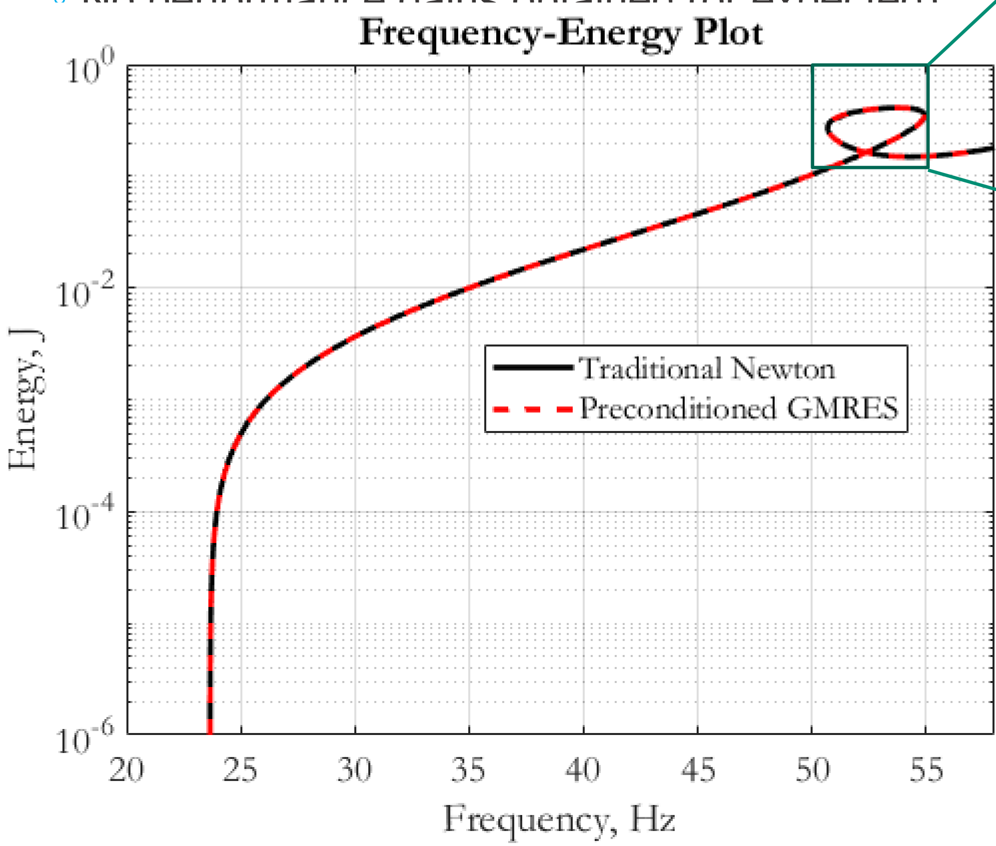
- 57 DOF (19 elements)
- Cubic spring at tip
- Structural steel properties
- Assumed 5 harmonics in solution $k\omega$ with $k = [1, 3, 5, 7, 9]$

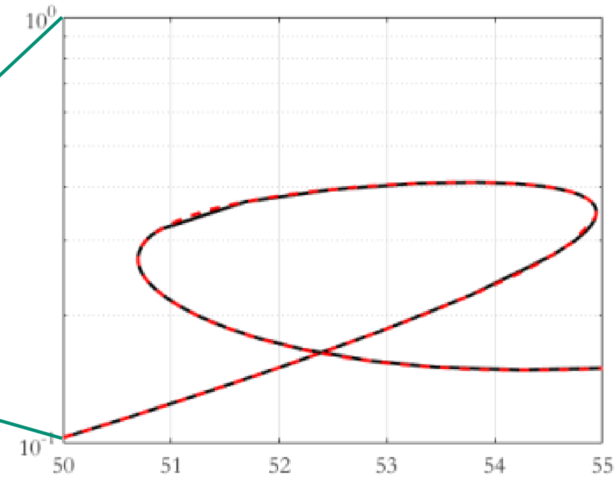


NNM 1 of Nonlinear Cantilever Beam

Frequency-energy plots from both solvers agree well

- Verifies preconditioned GMRES
- No performance gains obtained (or expected)





Preconditioned GMRES generally requires more corrections per solution due to forcing term $\eta_k = 0.2$

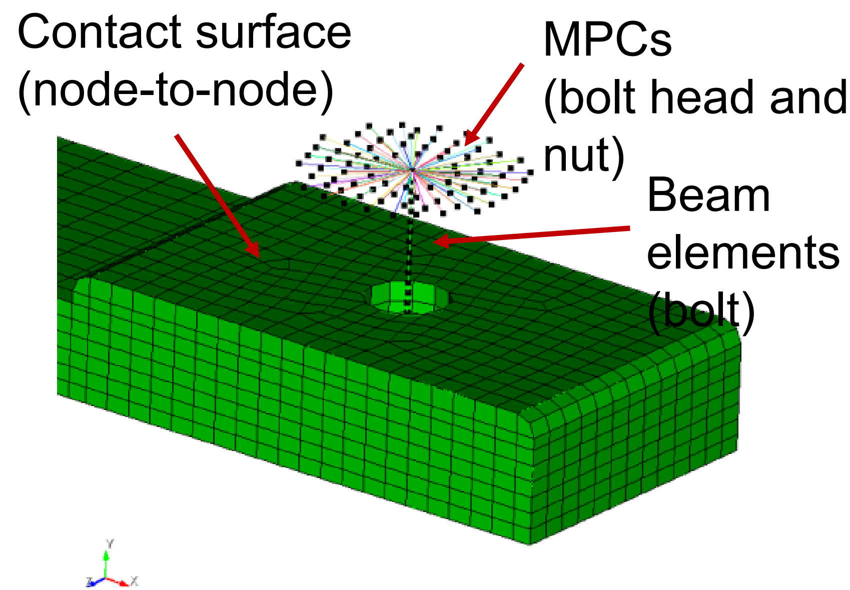
Number of corrections performed for the calculation of 200 points on the NNM 1 branch

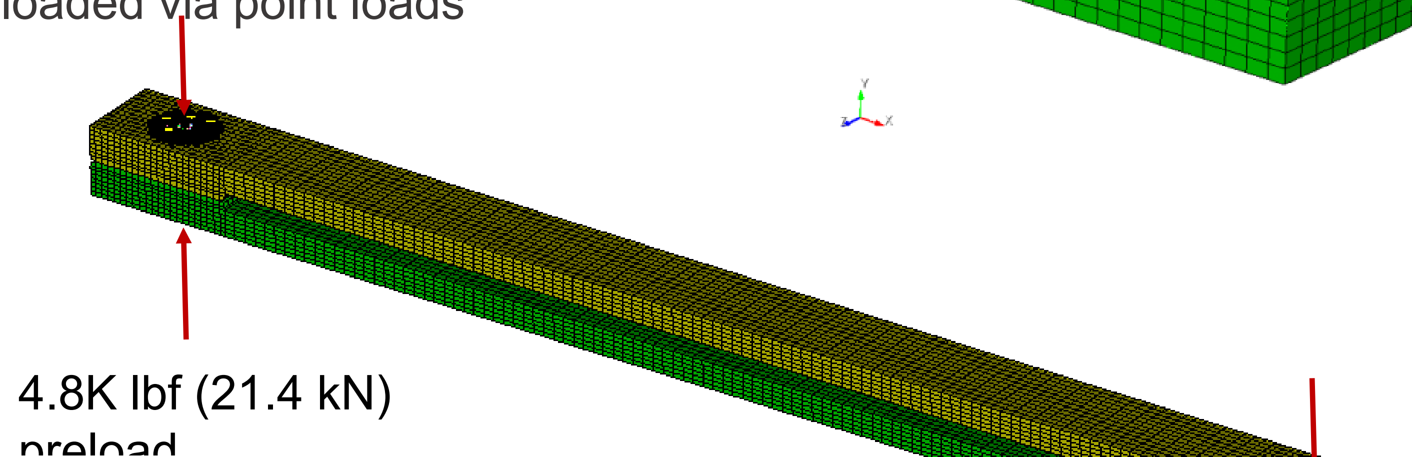
- Traditional Newton – 579 corrections
- Preconditioned GMRES –926 corrections

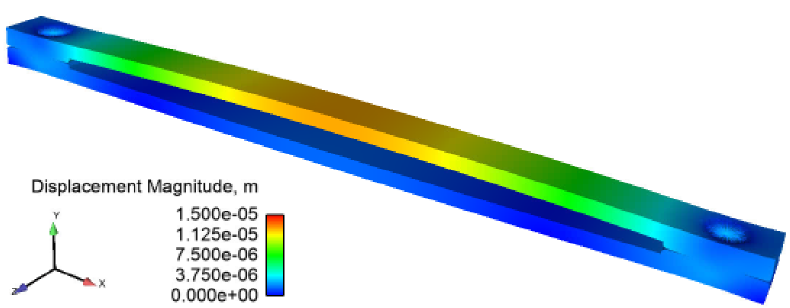
C-Beam Example

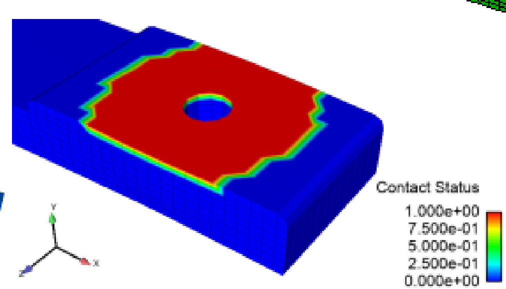
Full-order finite element model exported to MATLAB

- 100K DOF (~25K elements)
- Triaxial penalty springs (stick and gap behavior but no slip)
- Preloaded via point loads









Computational Costs with NNM 2 of C-Beam

Table shows comparison of computational cost of single inversion for
- Preconditioner within GMRES iterative solves
- Full Jacobian matrix in traditional Newton with serial direct solves

Preconditioned GMRES cost dominated by size of FEM - **(# DOF)** – and does not grow with additional harmonics (assuming processors available for computations)
- Typically GMRES requires more iterations and matrix inversions (per iteration) compared to traditional Newton scheme

2 · (# Harmonics) + 1*	**3**	**5**	**7**	**9**
Preconditioned GMRES**	15.1 s	16.0 s	16.4 s	15.7 s
Direct solve Newton	163 s	391 s	1020 s	1290 s

*Must include DC term to account for static preload

MATLAB implementation uses **2 · (# Harmonics)+1 workers in parallel computations when inverting preconditioner (i.e. each subblock inverted using *parfor* loops)

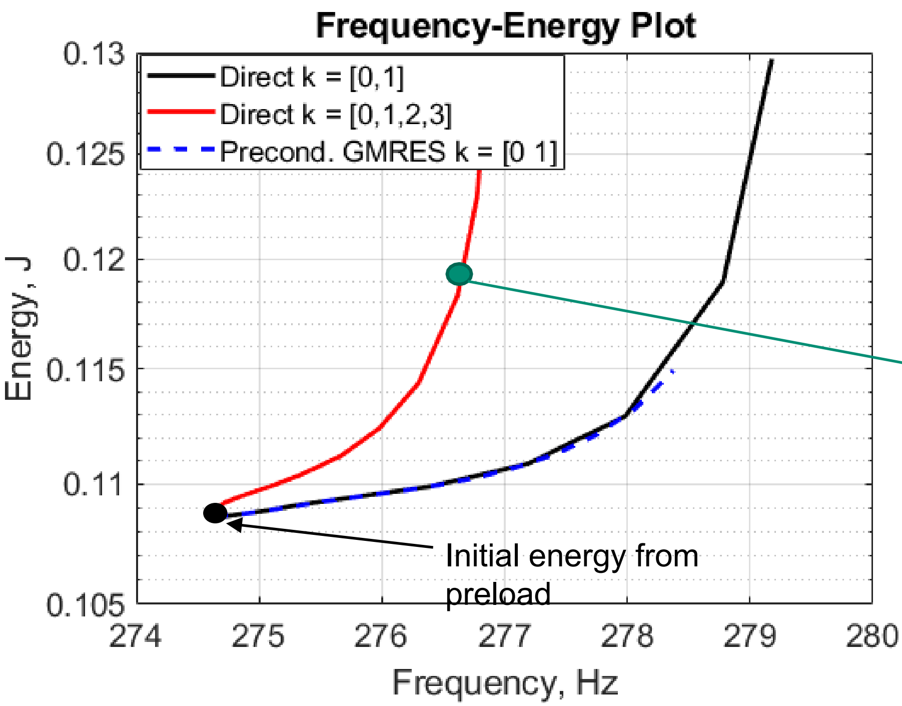
C-Beam Example

NNM 2 initiated from linearized out-of-phase first bending mode (274.6 Hz)

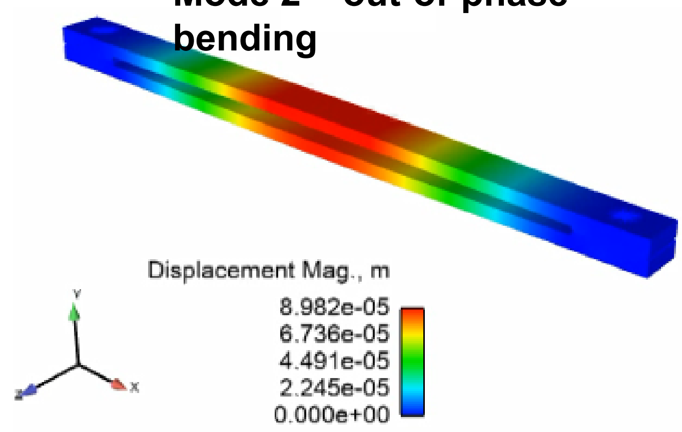
Additional harmonics (2ω and 3ω) have significant influence on backbone

Preconditioned GMRES shown to accurately replicate curves from serial direct solves with Newton method

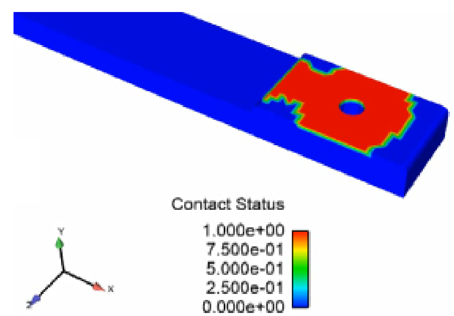
- Sensitive to settings for tolerances, forcing term, etc..

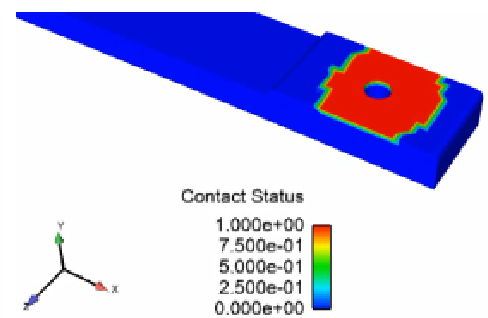


Mode 2 – out-of-phase bending



Example: Snapshot of contact status (red – in contact) for NNM 2

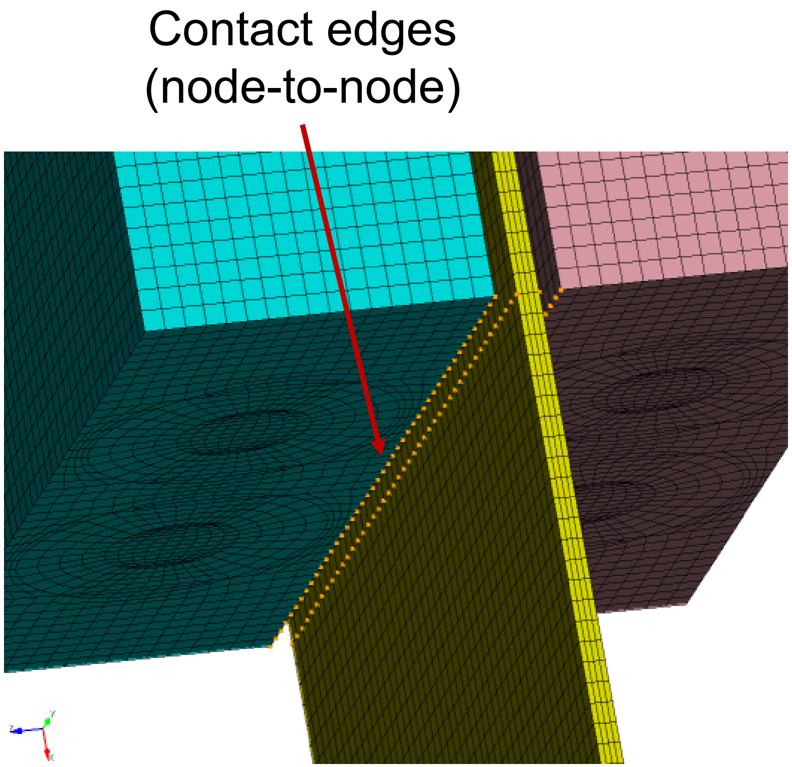


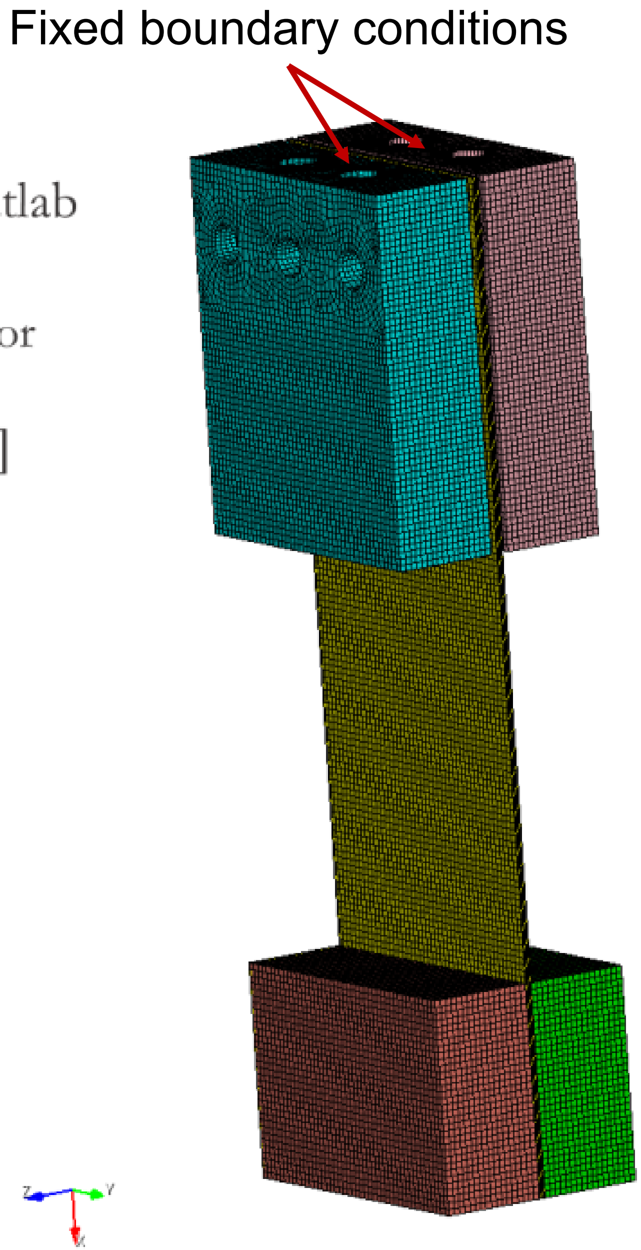


Nonlinear Pylon Example

Full-order finite element model exported to Matlab

- 670K DOF (~202K elements)
- Uniaxial penalty springs (gap behavior but no slip or stick)
- Assumed 1 harmonic in solution $k\omega$ with $k = [1]$





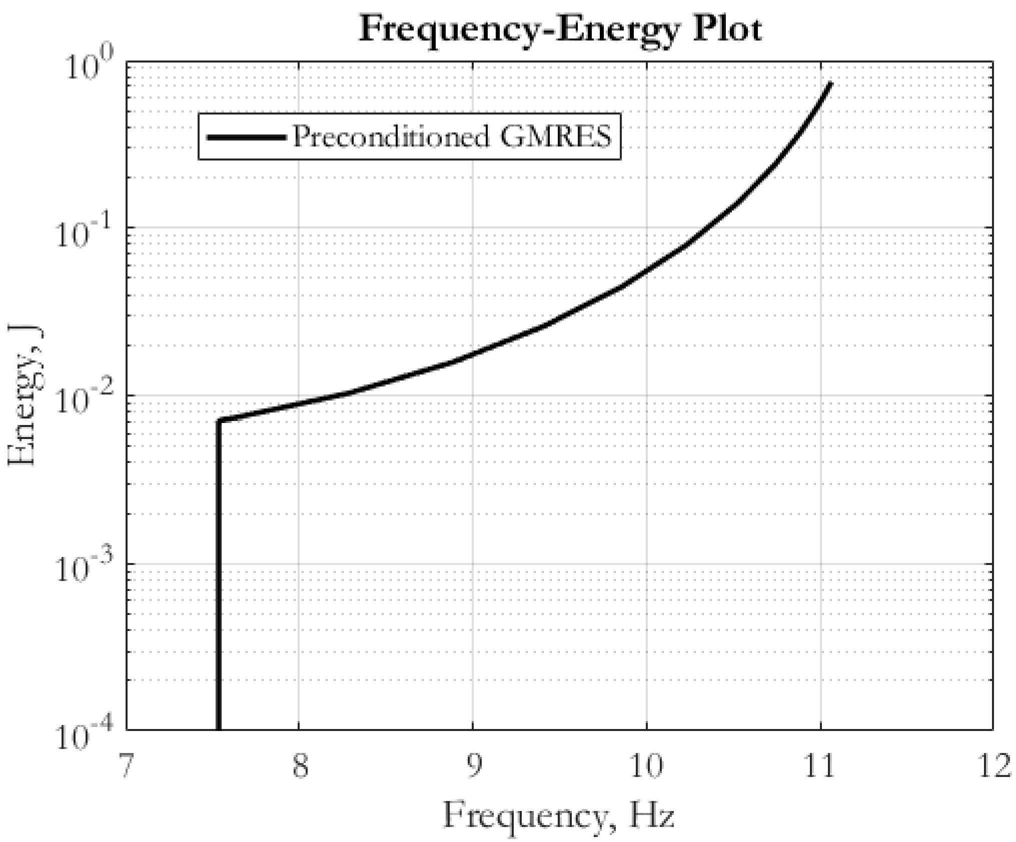
NNM 1 of Nonlinear Pylon

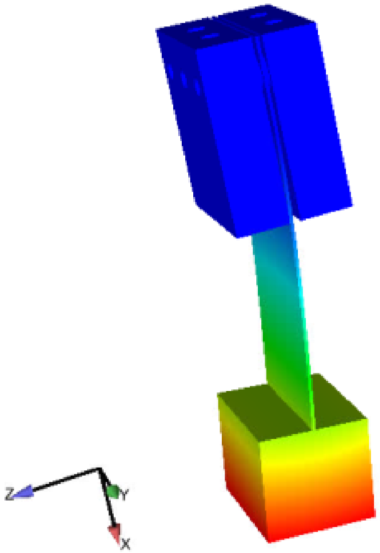
First mode associated with pylon swinging

Direct solve for Newton approach prohibitive for this example

Stiffening effect once thin strip impacts the contact block

- Linear frequency starts at 7.54 Hz





Conclusions

Developed a multi-harmonic balance solver to compute periodic orbits (i.e. NNMs) of potentially large-scale models

Preconditioned GMRES + inexact Newton updates allow for parallel computations of matrix inversions

- Potential for significant computational speedups for large models with many harmonics

Examples reveal accuracy of preconditioned GMRES compared to traditional Newton corrections with direct solvers

Future work seeks to combine algorithm with domain decomposition methods to further speed up inverse calculations of the FEM

Any Questions?

Contact information

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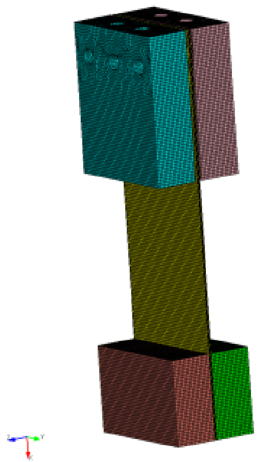
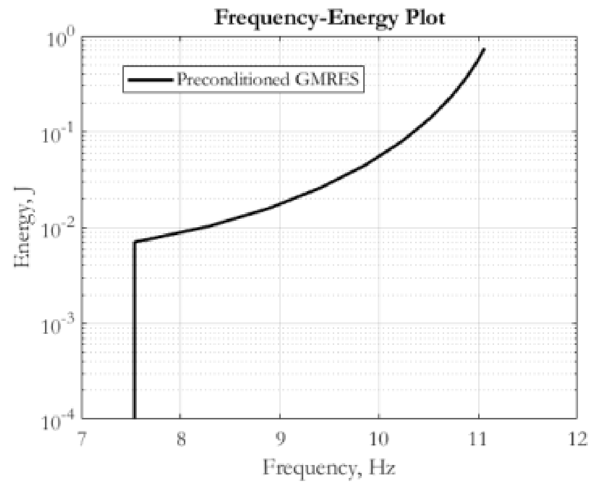
- Andrew Steyer, asteyer@sandia.gov





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