Outperforming QAOA on MaxCut with Fast Classical Hyperplane Rounding Algorithms

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MaxCut

- Given a graph $G = (V, E)$, where $n = |V|$ and $m = |E|$, the unweighted MaxCut problem can be described by the following linear program,

$$\max \frac{1}{2} \sum_{(i,j)\in E} (1 - z_i \cdot z_j)$$

$$\text{such that } z_i \in \{-1, 1\} \; \forall i \in V$$

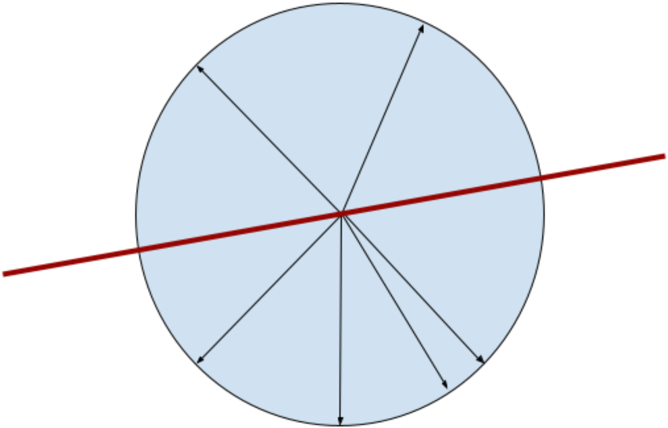
MaxCut SDP Relaxation

- This problem can be relaxed to the following semidefinite program (SDP),

$$\max \frac{1}{2} \sum_{(i,j)\in E} (1 - v_i \cdot v_j)$$

$$\text{such that } v_i \in S^n \; \forall i \in V$$

Hyperplane Rounding



Goemans-Williamson (GW) Algorithm

- Given any feasible solution to the SDP (i.e. any set of unit vectors, $\{v_i\}_{i=1}^{n} \subset S^n$) and a uniformly random vector $r \in S^n$
- Consider the cut given by, $A = \{i \mid v_i \cdot r > 0\}$ and $A = V \backslash A$
- The expected weight of this cut is,

$$E[W] = \sum_{(i,j)\in E} \frac{\arccos(v_i \cdot v_j)}{\pi}$$
$$\geq \alpha \cdot \frac{1}{2} \sum_{(i,j)\in E} (1 - v_i \cdot v_j)$$

- Using an optimal solution gives, $E[W] \geq \alpha \cdot SDP \geq \alpha \cdot OPT$ [1]

Alternative Bounding

- If we instead consider a particular solution, specifically such that $v_i \cdot v_j \geq \sigma \ \forall (i,j) \in E$

$$E[W] = \sum_{(i,j)\in E} \frac{\arccos(v_i \cdot v_j)}{\pi}$$
$$\geq m \cdot \left(\frac{1}{2} + \frac{\sigma}{\pi}\right)$$

- Bounds how well the solution does over random
- Useful for analyzing graphs with added restrictions, such as regular graphs in our case

QAOA for MaxCut

- The expectation of QAOA_1 [3] on the MaxCut problem has been well studied
- From [4], for a d-regular triangle free graph, $G = (V, E)$, where $m = |E|$, the expected cut is,

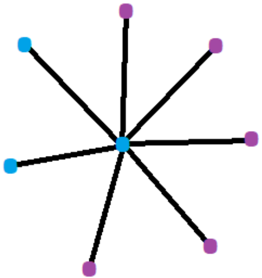
$$E[W] \geq m \cdot \left(\frac{1}{2} + \frac{0.3032}{\sqrt{d}} \right)$$

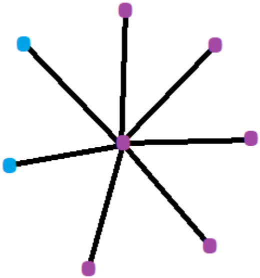
- As expected, it has been shown the QAOA_2 performs better than QAOA_1 [5]

A Classical Thresholding Algorithm

- In [6], a local classical algorithm is given which beats QAOA_1 for MaxCut on a d-regular triangle-free graph
- Consider a random cut on the graph, and switch the sign of any vertex which agrees with sufficiently many of its neighbors
- The threshold value is optimized for different values of d, and the guaranteed expectations are shown to beat QAOA_1
- In [5], another local thresholding algorithm is shown which beats QAOA_2

Thresholding Algorithm Example





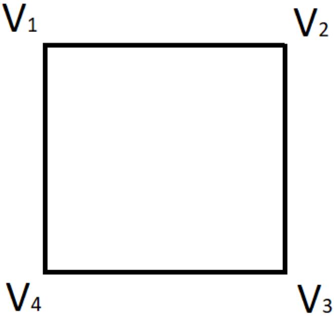
Another Classical Algorithm: Our Inspiration

- Consider the following example from [2]
- On a D-regular, triangle-free graph, define for every $i, j \in V$:

$$v_{ij} = \begin{cases} \frac{1}{\sqrt{2}} & i = j \\ \frac{-1}{\sqrt{2D}} & (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- Let $\mathbf{v}_i = [v_{i1}, ..., v_{in}]^T$.

Inspiration Example



$$\mathbf{v}_1^T = \left[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2d}}, 0, \frac{-1}{\sqrt{2d}}\right]$$

$$\mathbf{v}_2^T = \left[\frac{-1}{\sqrt{2d}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2d}}, 0\right]$$

$$\|\mathbf{v}_1\|_2 = \|\mathbf{v}_2\|_2 = 1$$

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \frac{-1}{\sqrt{d}}$$

Inspiration, Cont.

- Since $\mathbf{v}_i \in S^n$, the set $\{\mathbf{v}_i\}_{i=1}^n$ is a feasible solution to the above semidefinite program
- Using the GW hyperplane rounding scheme gives a cut with expected size

$$\begin{aligned} E[W] &= \frac{1}{\pi} \sum_{(i,j)\in E} \arccos(\mathbf{v}_i \cdot \mathbf{v}_j) \\ &\geq \frac{m}{2} - \frac{1}{\pi} \sum_{(i,j)\in E} (\mathbf{v}_i \cdot \mathbf{v}_j) \qquad\qquad \mathbf{v}_i \cdot \mathbf{v}_j \leq 0 \\ &= \left(\frac{1}{2} + \frac{1}{\pi\sqrt{D}}\right) \cdot m \end{aligned}$$

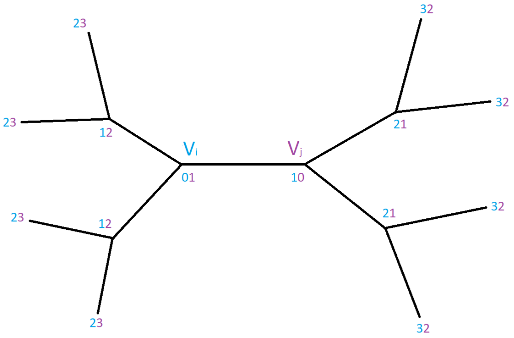
Extended Explicit Vector Solution

- Consider a D-regular graph, $G = (V, E)$, where $n = |V|$ and $m = |E|$, with graph girth at least $2k$
- For each vertex, we define a vector $\mathbf{v}_i$ for all $i \in V$, where $\mathbf{v}_i = [v_{i1}, ..., v_{in}]^T$, and

$$v_{ij} = \begin{cases} \alpha_0 & i = j \\ \alpha_\ell & d(i,j) = \ell < k \\ 0 & \text{otherwise} \end{cases}$$

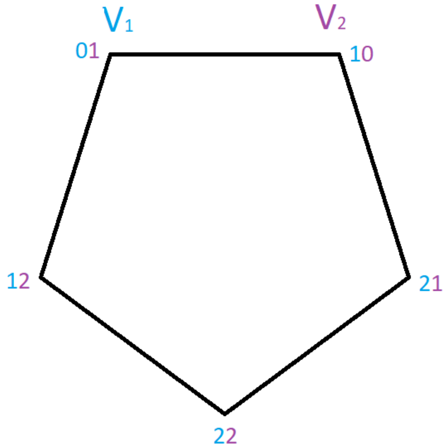
- Here, $d(i,j)$ is the shortest path between i and j, and α_ℓ are some to-be-calculated values.

Vector Solution Example



$$\|\mathbf{v}_i\|_2 = \alpha_0^2 + 3\alpha_1^2 + 6\alpha_2^2 \quad \mathbf{v}_i \cdot \mathbf{v}_j = 2\alpha_0\alpha_1 + 4\alpha_1\alpha_2 + 8\alpha_2\alpha_3$$

Girth Requirement Example



Definition of α_i's

- The optimal choice of α_ℓ can be found as the minimum eigenvector of a $k \times k$ matrix
- Our goal is to minimize $\mathbf{v}_i \cdot \mathbf{v}_j$ for $(i, j) \in E$, where the vectors are restricted to being unit length
- This is captured by the following optimization problem

$$\min 2\alpha_0\alpha_1 + 2(D-1)\alpha_1\alpha_2 + \cdots + 2(D-1)^{k-2}\alpha_{k-2}\alpha_{k-1}$$
$$\text{such that } \alpha_0^2 + D\alpha_1^2 + D(D-1)\alpha_2^2 + \cdots + D(D-1)^{k-2}\alpha_{k-1}^2 = 1$$

Finding the Optimal α_i's

- By scaling the α_ℓ's, the problem becomes a minimization of a quadratic function over the k-dimensional sphere:

$$\min_{\|\boldsymbol{\beta}\|_2=1} \frac{2}{\sqrt{D}}\beta_0\beta_1 + \frac{2\sqrt{D-1}}{D}\beta_1\beta_2 + \cdots + \frac{2\sqrt{D-1}}{D}\beta_{k-2}\beta_{k-1}$$

$$= \min_{\|\boldsymbol{\beta}\|_2=1} \boldsymbol{\beta}^T A_k \boldsymbol{\beta}$$

$$\boldsymbol{\beta} = [\beta_0, \ldots, \beta_{k-1}]^T$$

$$\beta_\ell = \begin{cases} \alpha_0 & \ell = 0 \\ \alpha_\ell\sqrt{D(D-1)^{\ell-1}} & \ell > 0 \end{cases}$$

Finding the Optimal α_i's, cont.

- The optimal value is then the minimum eigenvalue of the following $k \times k$ matrix:

$$A_k = \begin{bmatrix} 0 & a & 0 & 0 & 0 & \cdots \\ a & 0 & b & 0 & 0 & \cdots \\ 0 & b & 0 & b & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & b & 0 & b \\ 0 & 0 & \cdots & 0 & b & 0 \end{bmatrix} \quad a = \frac{1}{\sqrt{D}} \quad b = \frac{\sqrt{D-1}}{D}$$

The actual solution is then the corresponding eigenvector.

Bounding the Minimum Eigenvalue

- Now consider the following $k \times k$ matrix:

$$B_k = \begin{bmatrix} 0 & b & 0 & 0 & 0 & \cdots \\ b & 0 & b & 0 & 0 & \cdots \\ 0 & b & 0 & b & 0 & \cdots \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & b & 0 & b \\ 0 & 0 & \cdots & 0 & b & 0 \end{bmatrix}$$

- It is straightforward to show, that $\lambda_{\min}(A_k) \leq \lambda_{\min}(B_k)$

Bounding the Minumum Eigenvalue, cont.

- Therefore, since arcosine is strictly decreasing,

$$
\begin{aligned}
E[W] &= \sum_{(i,j)\in E} \frac{\arccos(\mathbf{v}_i \cdot \mathbf{v}_j)}{\pi} \\
&= \frac{m}{\pi} \arccos(\lambda_{\min}(A)) \\
&\geq \frac{m}{\pi} \arccos(\lambda_{\min}(B)) \\
&= \frac{m}{\pi} \arccos(\rho_D \cdot \cos \frac{\pi}{k+1})
\end{aligned}
$$

Main Result - Guaranteed Cut Bound

Theorem: *Given a finite D-regular graph $G = (V, E)$ where $m = |E|$ with girth at least $2k$, there is an edge cut of G with size*

$$E[W] \geq \frac{m}{\pi} \cdot \arccos\left(\rho_D \cdot \cos\frac{\pi}{k+1}\right) \geq m \cdot \left(\frac{1}{2} - \frac{\rho_D}{\pi} \cdot \cos\frac{\pi}{k+1}\right)$$

where $\rho_D = -\frac{2\sqrt{D-1}}{D}$.

Gaussian Wave Comparision

- In [7][8], the authors consider a Gaussian process on an infinite tree, which is then approximated to be used to find independent sets [7] and max cuts [8] on regular graphs with high girth

$$X_i = \sum_{j \in V} \alpha_{d(i,j)} Z_j = \sum_{k=0}^{\infty} \sum_{j \text{ s.t. } d(i,j)=k} \alpha_k Z_j$$

Gaussian Wave Comparision, cont.

- Consider a vector assignment given by our algorithm, $\{\mathbf{v}_i\}_{i=1}^n$, and let $V = [\mathbf{v}_1 ... \mathbf{v}_n]$
- Hyperplane rounding can be thought of as sampling from the n-variable Gaussian distribution, $\mathbf{x} \sim \mathcal{N}(0^n, V^T V)$ and taking the sign of each variable
- any multivariate Gaussian distribution can be written as a combination of i.i.d. Gaussian distributions; i.e., $\mathbf{x} = V\mathbf{z}$, where $\mathbf{z}$ are n i.i.d. Gaussian random variables
- Thus, our algorithm is exactly a simplified version of the linear factor of i.i.d. processes presented in [7][8]

Gaussian Wave Comparision, cont.

Theorem: Given a finite D-regular graph $G = (V, E)$ with girth at least $2k$, the explicit vector solution has a higher performance guarantee than that of the Gaussian wave algorithm in [8] for all $k \geq 3$ and $D \geq 3$.

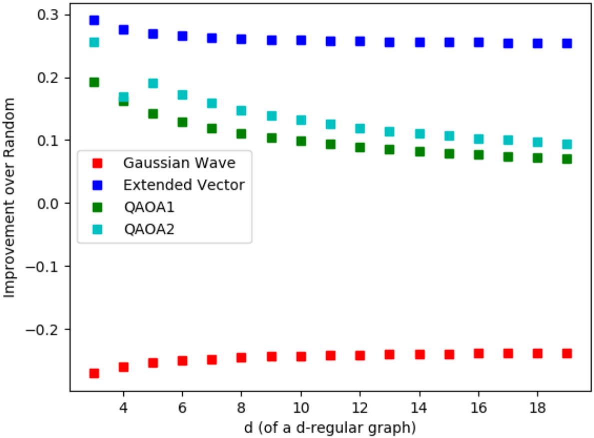
Gaussian Wave Comparision, cont.

- First consider this form of the expected cut fraction:

$$E[W]/m = \frac{1}{2} - \frac{\rho_D}{\pi}X$$

- By the first theorem, the explicit vector solution has relative expectation $A = cos(\pi/(k+1))$
- The algorithm in [8] has relative expectation $B = (1 + \frac{D-1}{D(k-1)})^{-1}$
- It can be shown that $A \geq B$ for all $k \geq 3$ and $D \geq 3$.

Experimental Results



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