**Title:** Applying an Oriented Divergence Theorem to Swept Face Remap

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# Applying an Oriented Divergence Theorem to Swept Face Remap

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# Outline

Introduction

Theoretical Results
- Swept region divergence
- Parameterized divergence
- Independence of $\tau$

Numerical Results

Implementation Strategies and Generalizations

Conclusions

# ALE hydrodynamics

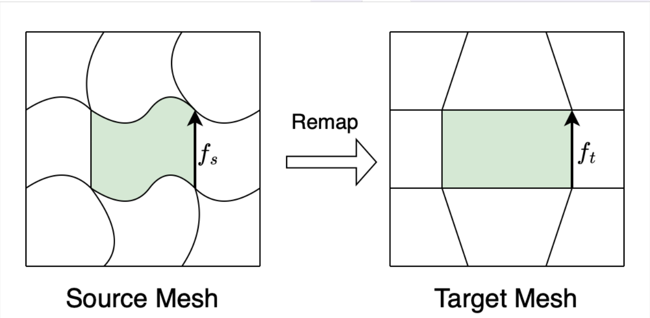
In arbitrary Langrangian-Eulerian hydrodynamics, we compute the physical behavior of a fluid over time with three phases:

1. The **Lagrangian phase**: the mesh distorts along with the fluid. May result in a tangled, unresolved mesh, so we pause at a geometrically valid source mesh.
2. The **rezone phase**: define a compatible, geometrically valid target mesh.
3. The **remap phase**: port the physical solution computed in the Lagrangian phase from the source mesh to the target mesh.

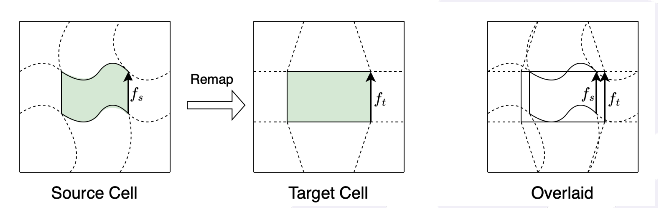
We will focus on the remap phase. Our domain is compact $\Omega \subset \mathbb{R}^n$, and we describe our physical solution with a scalar density function $\rho : \Omega \to \mathbb{R}$.

# Swept remap

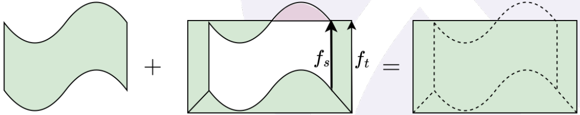
In swept remap, the source and target mesh have the same topology. Each source zone corresponds to a target zone, and each source face corresponds to a target face.



# Swept remap



We compute the mass of the target zone using the mass of the source zone, and the change in mass through each source face sweeping to a target face.



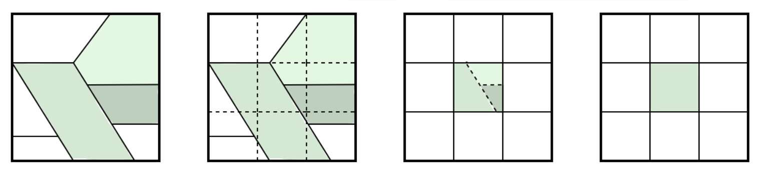
Source zone mass

Flux $f_s$ to $f_t$

Target zone mass

# Other types of remap

- **Intersection**: the source and target mesh may have different topologies, and the change in mass is computed by intersecting all source and target zones.
  This is challenging for curved meshes.



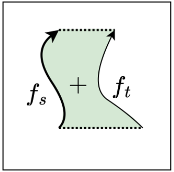
- **Dynamic process**: formulates remap as a dynamic process governed by a transport equation. The source and target mesh have the same topology, and we compute a pseudo velocity between them.
  Does not preserve solutions which are globally polynomials.

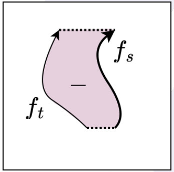
# Swept region

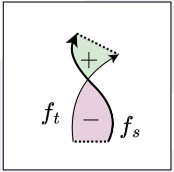
We assume that our faces are described with explicit, compatible parameterizations $f_s$ and $f_t$. Then we describe the swept region with a parameterization $\tau : [0,1]^n \to \mathbb{R}^n$.

$$\begin{aligned} \tau(t_1, \ldots, t_n) &: [0,1]^n \to \mathbb{R}^n \\ \tau(t_1, \ldots, t_{n-1}, 0) &= f_s(t_1, \ldots, t_{n-1}) \\ \tau(t_1, \ldots, t_{n-1}, 1) &= f_t(t_1, \ldots, t_{n-1}). \end{aligned}$$

We assume that $\tau$ is continuous and piecewise differentiable ($\mathcal{C}^1$). However, $\tau$ may not be injective or orientation-preserving, so it may overlap itself in complicated ways.





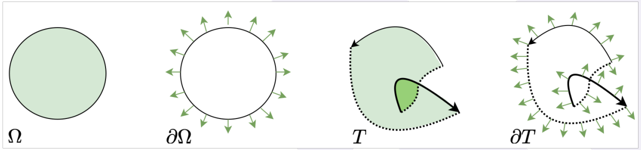


# Divergence theorem

### Theorem (Divergence)

*Let $\Omega \subset \mathbb{R}^n$ a compact subset with piecewise smooth boundary $\partial\Omega$. Let $\mathbf{v} \in H_{div}(\Omega)$ a vector field such that $\rho = \nabla \cdot \mathbf{v}$. Then*

$$\mathcal{F}_\rho(\Omega) \equiv \int_\Omega \rho dv = \oint_{\partial\Omega} \mathbf{v} \cdot d\mathbf{s}.$$



Our swept region $T$ may not fulfill the assumptions for $\Omega$.

# Novelty and roadmap

Though divergence theorem does not formally apply to swept regions, M. Shashkov and others have used it for swept remap. This project creates a solid theoretical foundation for this usage.

- We provide an explicit mathematical description for a swept region.
- We provide several characterizations of the flux through the swept region. The most practical of these is a volume integral of the unit cube, which makes use of a change of variables:

$$\mathcal{F}_\rho(T) = \int_{[0,1]^n} (\rho \circ \tau) j_\tau \, d\mathbf{t}.$$

- We validate the theoretical results numerically.
- We discuss implementation strategies for swept remap, and extend the results to polygonal mesh topologies.

We gain remap methods which preserve globally polynomial solutions.

# Theoretical Results

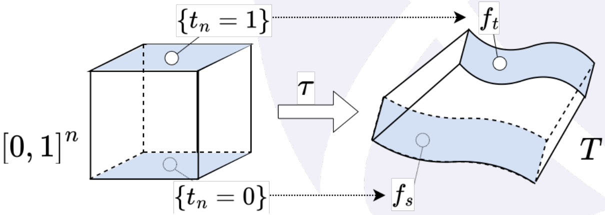
# Parameterizing our swept region

We choose a continuous, piecewise differentiable parameterization map from the *n*-cube:

$$\tau : [0,1]^n \to \mathbb{R}^n.$$

To relate $\tau$ to remap, we want it to sweep between our source and target faces:

$$f_s(t_1, \ldots, t_{n-1}) = \tau(t_1, \ldots, t_{n-1}, 0)$$
$$f_t(t_1, \ldots, t_{n-1}) = \tau(t_1, \ldots, t_{n-1}, 1).$$



# Parameterizing our swept region (notation)

We also use the following conventions for component functions and coordinates:

$$\begin{aligned} \tau &= (\tau_1, \ldots, \tau_n) \\ \mathbf{x} &= (x_1, \ldots, x_n) \in \mathbb{R}^n \\ \mathbf{t} &= (t_1, \ldots, t_n) \in [0,1]^n. \end{aligned}$$

Since $\tau$ is piecewise differentiable, we may define the Jacobian determinant $j_\tau$ almost everywhere on $[0,1]^n$ using the partial derivatives of component functions $\tau_i$:

$$j_\tau = \det \left( \frac{\partial \tau_i}{\partial t_j} \right)_{ij}.$$

We use $d\mathbf{x}$ and $d\mathbf{t}$ for both volume (dim n) and area (dim n-1) integrals.

# Features of swept regions

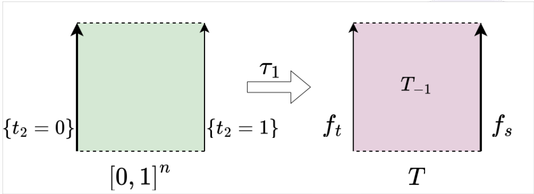
The parameterizations are continuous and piecewise differentiable, but not injective or orientation-preserving.

We consider the following features of their images:

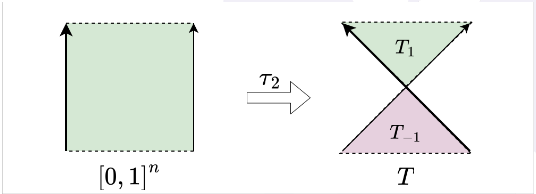
- **Orientation**: positive if $j_\tau > 0$, and negative if $j_\tau < 0$.
- **Overlaps**: constructive if they have the same orientation, and destructive if they have opposite orientations.

We define a swept region as the **signed, multiset image of** $\tau$ with an integer-valued characteristic function which has value $k$ on component sets $T_k$.

# 2D examples: $\tau_1$ and $\tau_2$

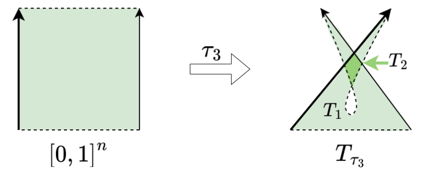


$$\tau_1(t_1, t_2) = (t_1, -t_2)$$

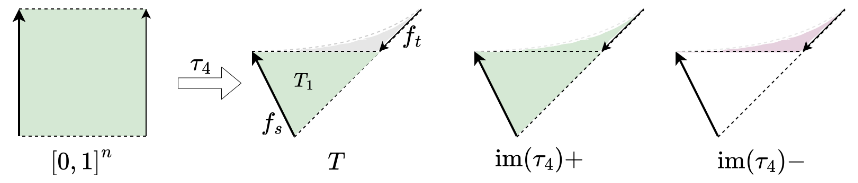


$$\tau_2(t_1, t_2) = (1 + t_1, 0 - t_1 - t_2 - 2t_1 t_2)$$

# 2D examples: $\tau_3$ and $\tau_4$



$$\tau_3(t_1,t_2) = (2.5t_2 - 9t_1t_2 + 9t_1^2t_2, 3t_1 + 2.8t_2 - 13t_1t_2 + 23t_1^2t_2 - 16t_1^3t_2)$$



$$\tau_4(t_1,t_2) = (t_2 - 2t_1t_2, -2t_1 + t_2)$$

# Swept region

### Definition

The **swept region** $T$ is a compact, oriented multiset with characteristic function

$$\chi_T : \mathbb{R}^n \to \mathbb{Z}$$
$$\chi_T(\mathbf{x}) = \sum_{\mathbf{t} \in \tau^{-1}(\mathbf{x})} sgn \circ j_\tau(\mathbf{t}).$$

As sets, the $\text{supp}(\chi_T) \subset \text{im}(\tau)$, but the subset may be strict.

# Flux through a swept region

We define component sets $T_k \subset \mathbb{R}^n$ using the level sets of $\chi_T$:

$$T_k = \chi_T^{-1}(k) \text{ for all } k \in \mathbb{Z}.$$

Each $T_k$ is pre-compact with compact, piecewise differentiable boundary $\partial T_k$.

### Definition

For a swept region $T$ and a scalar function $\rho : \mathbb{R}^n \to \mathbb{R}$, the **flux of $\rho$ through $T$** is

$$\mathcal{F}_\rho(T) \equiv \sum_{k \in \mathbb{Z}} k \int_{T_k} \rho(\mathbf{x}) d\mathbf{x} = \int_{\text{supp}(\chi_T)} \chi_T(\mathbf{x}) \rho(\mathbf{x}) d\mathbf{x}.$$

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# Swept region boundary

## Definition

For a swept region $T$ determined by $\tau$, the **boundary of $T$** is

$$\partial T \equiv \tau(\partial[0,1]^n) \supset \bigcup_{k\in\mathbb{Z}} \partial T_k.$$

The containment follows from the continuity of $\tau$ and the compactness of its domain. Only interfaces between distinct component sets will contribute to our contour integrals. However, $\partial T$ may not contain the boundary of the footprint $\mathrm{im}(\tau)$.

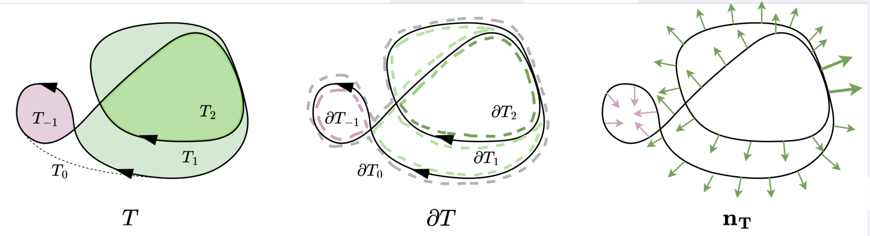
# Outward normal

## Definition

The **outward normal vector field $\mathbf{n_T}$** on $\partial T$ is given by

$$\mathbf{n_T}(\mathbf{x}) = \sum_{k\in\mathbb{Z}} k\mathbf{n_k}(\mathbf{x}) \text{ for } \mathbf{x} \in \partial T.$$

We use the outward normals $\mathbf{n_k}$ of the component sets $T_k$.



# Contour integral on $\partial T$

We demonstrate that $\mathbf{n_T}$ aligns with $T$ almost everywhere. More explicitly, it points from tangent component sets $T_i$ to $T_j$ with magnitude $i - j$ when $i < j$.

Then we may compute contour integrals around $\partial T$:

$$\oint_{\partial T} \mathbf{v} \cdot \mathbf{n_T} d\mathbf{x} = \sum_{k \in \mathbb{Z}} k \oint_{\partial T_k} \mathbf{v} \cdot \mathbf{n_k} d\mathbf{x}.$$

We want to show that divergence theorem applies to these swept regions.

# Swept region divergence theorem

## Theorem (Swept region divergence)

- *Let* $\tau : [0,1]^n \rightarrow \mathbb{R}^n$ *a continuous, piecewise differentiable function.*
- *Let* $T$ *the compact swept region determined by* $\tau$.
- *Let* $\partial T$ *be the boundary of* $T$ *with outward normal* $\mathbf{n_T}$.
- *Let* $\mathbf{v} \in H_{div}(\Omega)$ *a vector field with divergence* $\rho = \nabla \cdot \mathbf{v}$.

$$\text{Then } \mathcal{F}_\rho(T) = \oint_{\partial T} \mathbf{v} \cdot \mathbf{n_T} da.$$

# Sketch of proof

We first state our definition of flux of $\rho$ through $T$:

$$\mathcal{F}_\rho(T) = \sum_{k\in\mathbb{Z}} k \int_{T_k} \rho(\mathbf{x}) d\mathbf{x}.$$

Then we apply divergence theorem to each component set $T_k$ and its boundary $\partial T_k$:

$$\mathcal{F}_\rho(T) = \sum_{k\in\mathbb{Z}} k \oint_{\partial T_k} \mathbf{v} \cdot \mathbf{n_k} da.$$

Finally, we apply our definition for the contour integral of **V** around $\partial T$:

$$\mathcal{F}_\rho(T) = \oint_{\partial T} \mathbf{v} \cdot \mathbf{n_T} da.$$

# Change of variables

Assumption: $\tau$ is an orientation-preserving diffeomorphism.

For volume integrals, we precompose $\rho$ with $\tau$ and multiply by $j_\tau$ to account for the change in volume:

$$\mathcal{F}_\rho(T) = \int_T \rho d\mathbf{x} = \int_{[0,1]^n} (\rho \circ \tau) j_\tau d\mathbf{t}.$$

For contour integrals, we precompose **v** with $\tau$, but we need to account for both a change in outward direction and a change in area.

# Cross-product and pushforward

We use the **generalized cross-product** to compute an orthogonal vector from $n-1$ independent vectors $\mathbf{v}, \dots, \mathbf{w}$.

$$\mathbf{n} = \begin{vmatrix} \mathbf{x_1} & \dots & \mathbf{x_n} \\ v_1 & \dots & v_n \\ & \dots & \\ w_1 & \dots & w_n \end{vmatrix}.$$

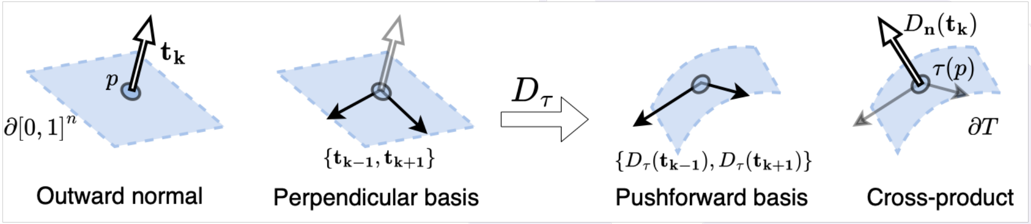
We use the **pushforward** $D_\tau$ to transport vectors along $\tau$. For a unit vector $\mathbf{t_k}$ at $p \in [0,1]^n$, then

$$D_\tau(p, \mathbf{t_k}) = (\frac{\partial \tau_1}{\partial t_k}(p), \dots, \frac{\partial \tau_n}{\partial t_k}(p)).$$

The pushforward is linear but does not maintain orthogonality.

# Contour integral change of variables

We compute an outward normal $\mathbf{N}_\tau$ along $\tau$. Its magnitude will be the change in area.



Finally, we complete our contour integral change of variables:

$$\mathcal{F}_\rho(T) = \oint_{\partial T} \mathbf{v} \cdot \mathbf{n} d\mathbf{x} = \oint_{\partial[0,1]^n} (\mathbf{v} \circ \tau) \cdot \mathbf{N}_\tau d\mathbf{t}.$$

# Revisiting assumptions: more general $\tau$

**Piecewise differentiable**: No change.
**Orientation-reversing**: For volume integrals, $j_\tau$ will be negative:

$$\mathcal{F}_\rho(T) = -\int_T \rho d\mathbf{x} = \int_{[0,1]^n} (\rho \circ \tau) j_\tau d\mathbf{t}.$$

For contour integrals, $\mathbf{N}_\tau$ will point inward with respect to im$(\tau)$:

$$\mathcal{F}_\rho(T) = -\oint_{\partial T} \mathbf{v} \cdot \mathbf{n} d\mathbf{x} = \oint_{\partial[0,1]^n} (\mathbf{v} \circ \tau) \cdot \mathbf{N}_\tau d\mathbf{t}.$$

**Degenerate**: If $j_\tau = 0$ or $N_\tau = 0$, then the corresponding integral is zero.

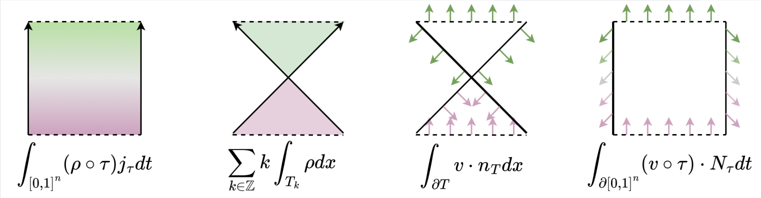
# Parameterized divergence theorem

Theorem (Parameterized divergence)

- *Let* $\tau : [0,1]^n \to \mathbb{R}^n$ *a continuous, piecewise differentiable function.*
- *Let* $T$ *the compact swept region determined by* $\tau$*.*
- *Let* $\mathbf{v} \in H_{div}(\Omega)$ *a vector field with divergence* $\rho = \nabla \cdot \mathbf{v}$*.*

$$\text{Then } \mathcal{F}_\rho(T) = \int_{[0,1]^n} (\rho \circ \tau) j_\tau d\mathbf{t} = \oint_{\partial[0,1]} (\mathbf{v} \circ \tau) \cdot \mathbf{N}_\tau d\mathbf{t}.$$

# Sketch of proof



We use the behavior of injective $\tau$ to generalize to non-injective $\tau$.

- Partition $[0,1]^n$ into injective domains with a positive orientation $U_i$ and injective domains with negative orientation $V_j$. We may ignore degenerate regions $W_k$.
- Since $\tau$ is continuous from a compact domain, then there are finitely many $U_i$ and $V_j$, so we can break up our integral.
- Using injectivity, we perform a change of variables in these restricted domains.
- Combine like domains in $\mathbb{R}^n$ to recover $\chi_T$ or $\mathbf{n_T}$.

# Independence of parameterization

### Theorem

*Let* $\tau, \omega : [0,1]^n \to \mathbb{R}^n$ *continuous, piecewise differentiable functions which agree for all* $\mathbf{t} \in \partial[0,1]^n$. *Let* $T, \Omega$ *the compact swept regions determined by* $\tau, \omega$ *respectively.*

*Then* $T = \Omega$.

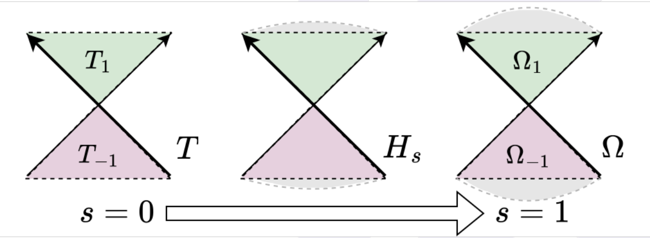
### Theorem

*Let* $T$ *the swept region determined by* $\tau$.
*Suppose* $\sigma : [0,1]^n \to [0,1]^n$ *a diffeomorphism which fixes the vertices of* $[0,1]^n$. *Let* $T_\sigma$ *the swept region determined by* $\tau \circ \sigma$.

*Then* $T = T_\sigma$.

# Sketch of proofs

- Define a map $h_s$ which sweeps between $\tau$ and $\omega$ with respect to $s$, and is constant along the boundary. It determines intermediate swept regions $H_s$.
- Since the boundary cannot move, then $\chi_{H_s}(\mathbf{x})$ must be constant along $s$.



Since $\sigma$ is a diffeomorphism which fixes vertices, then it preserves orientation. So then precomposing with $\sigma$ does not change the image of $\tau$ or the orientation of $\tau$.

# Summary of theoretical results

- Define a swept region $T$ as a signed multiset. Define the flux of $\rho$ through $T$, and contour integrals around $\partial T$.
- **Swept region divergence**: Compute the flux with a contour integral.
- Define notation to change variables from $T$ to $[0, 1]^n$.
- **Parameterized divergence**: Compute the flux with a change of variables.
- **Independence of $\tau$**: Our swept regions are an oriented, geometric object rather than an artifact of parameterization.

# Numerical Results

# 2D numerical models

We use the same parameterizations $\tau_1, \ldots, \tau_4$ from earlier. We choose $\rho$ a polynomial with the same degree, and take antiderivatives to determine a vector field **v** explicitly. We compute the flux of $\rho$ through $T$ using numerical approximations of

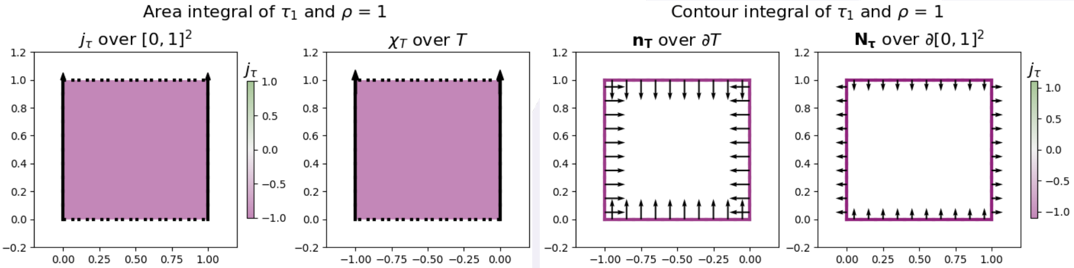
the area integral (A) and contour integral (C):

$$A = (res)^2 \sum\sum_{\text{squares in } [0,1]^2} (\rho \circ \tau)(j_\tau)(t_{\text{center}})$$

$$C = (res) \sum_{\text{segments in } \partial[0,1]^2} (\mathbf{v} \circ \tau)(\mathbf{N}_\tau)(t_{\text{center}}).$$

We use resolutions $res = 0.1$, 0.01, and 0.001.

# 2D graphs: $\tau_1$

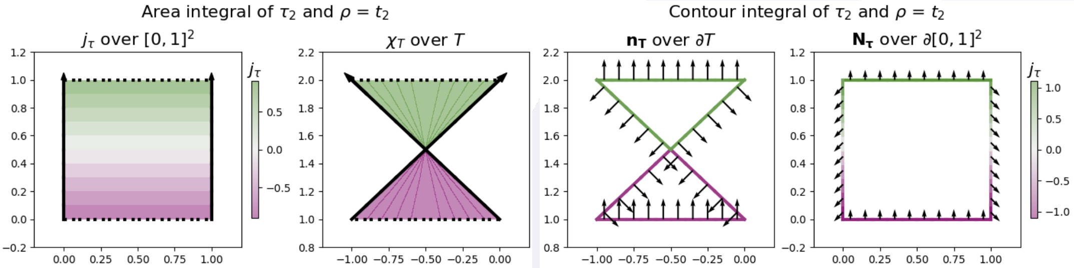


$$\tau_1(t_1, t_2) = (t_1, -t_2)$$

$$\rho(x_1, x_2) = 1$$

$$\mathbf{v}(x_1, x_2) = (\frac{1}{2}x_1, \frac{1}{2}x_2)$$

# 2D graphs: $\tau_2$

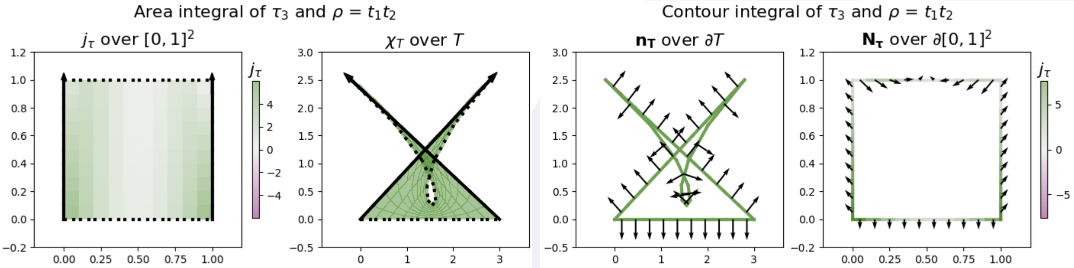


$$\tau_2(t_1, t_2) = (1 + t_1, 0 - t_1 - t_2 - 2t_1 t_2)$$

$$\rho(x_1, x_2) = x_2$$

$$\mathbf{v}(x_1, x_2) = (\frac{1}{2} x_1 x_2, \frac{1}{4} x_2^2)$$

# 2D graphs: $\tau_3$

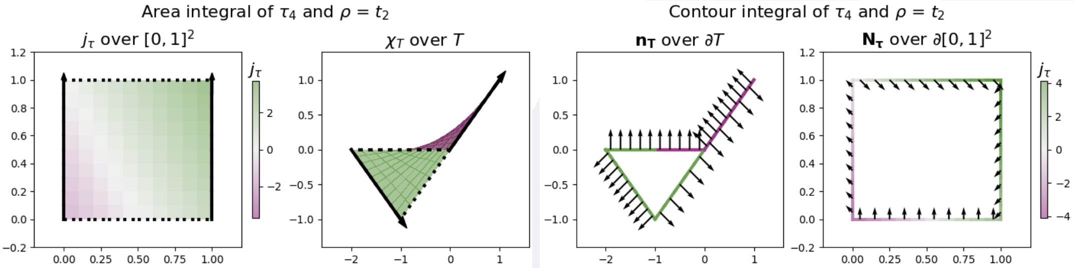


$$\tau_3(t_1, t_2) = (2.5t_2 - 9t_1 t_2 + 9t_1^2 t_2, 3t_1 + 2.8t_2 - 13t_1 t_2 + 23t_1^2 t_2 - 16t_1^3 t_2)$$

$$\rho(x_1, x_2) = x_1 x_2$$

$$\mathbf{v}(x_1, x_2) = (\frac{1}{4}x_1^2 x_2, \frac{1}{4}x_1 x_2^2)$$

# 2D graphs: $\tau_4$



Area integral of $\tau_4$ and $\rho = t_2$

Contour integral of $\tau_4$ and $\rho = t_2$

$j_\tau$ over $[0,1]^2$

$\chi_T$ over $T$

$\mathbf{n_T}$ over $\partial T$

$\mathbf{N_\tau}$ over $\partial[0,1]^2$

$$\tau_4(t_1, t_2) = (t_2 - 2t_1 t_2, -2t_1 + t_2)$$

$$\rho(x_1, x_2) = x_2$$

$$\mathbf{v}(x_1, x_2) = (\frac{1}{2}x_1 x_2, \frac{1}{4}x_2^2)$$

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# 2D convergence

We measure the discrepancy between area integral $A$ and contour integral $C$ with their difference $|C - A|$.

The discrepancy decreases by **two orders of magnitude** for each increase in resolution. The values A and C converge linearly.

| $\tau$ | $\rho$ | A | C | A | C | \|C-A\| | | |
|---|---|---|---|---|---|---|---|---|
| *res* | | 0.1 | | 0.001 | | 0.1 | 0.01 | 0.001 |
| $\tau_1$ | 1 | -1.0 | -1.0 | -1.0 | -1.0 | 0.0 | 0.0 | 0.0 |
| $\tau_2$ | $x_2$ | 0.165 | 0.162 | 0.167 | 0.167 | 2.6e-3 | 1.4e-5 | 1.2e-7 |
| $\tau_3$ | $x_1 x_2$ | 1.722 | 1.674 | 1.712 | 1.712 | 4.8e-2 | 5.2e-4 | 5.2e-6 |
| $\tau_4$ | $x_2$ | -0.330 | -0.335 | -0.333 | -0.333 | 5.0e-3 | 5.0e-5 | 5.0e-7 |

# 3D numerical models

We choose polynomial parameterizations $\tau_5, \tau_6, \tau_7$, then choose polynomial $\rho$ with the same degree and compute **v** explicitly. We compute the flux of $\rho$ through $T$ using

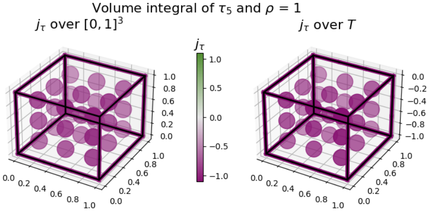
numerical approximations of the volume integral (V) and contour integral (C):

$$V = (res)^3 \underbrace{\sum\sum\sum}_{\text{cubes in } [0,1]^3} (\rho \circ \tau)(j_\tau)(t_{\text{center}})$$

$$C = (res)^2 \underbrace{\sum\sum}_{\text{squares in } \partial[0,1]^3} (\mathbf{v} \circ \tau)(\mathbf{N}_\tau)(t_{\text{center}}).$$

We use resolutions $res = 0.33$, 0.033, and 0.0033.

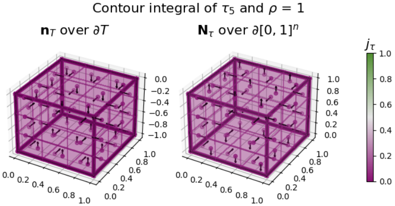
# 3D graphs: $\tau_5$



Volume integral of $\tau_5$ and $\rho = 1$

$j_\tau$ over $[0,1]^3$

$j_\tau$ over $T$



Contour integral of $\tau_5$ and $\rho = 1$

$\mathbf{n}_T$ over $\partial T$

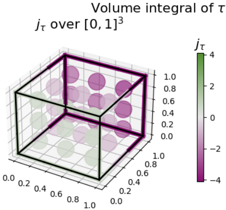
$\mathbf{N}_\tau$ over $\partial[0,1]^n$

$$t_5(t_1, t_2, t_3) = (t_1, t_2, t_3)$$

$$\rho(x_1, x_2, x_3) = 1$$

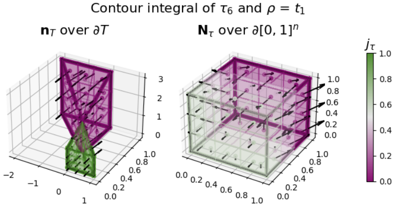
$$\mathbf{v}(x_1, x_2, x_3) = \frac{1}{3}(x_1, x_2, x_3)$$

# 3D graphs: $\tau_6$



Volume integral of $\tau_6$ and $\rho = t_1$

$j_\tau$ over $[0,1]^3$ $j_\tau$ over $T$



Contour integral of $\tau_6$ and $\rho = t_1$

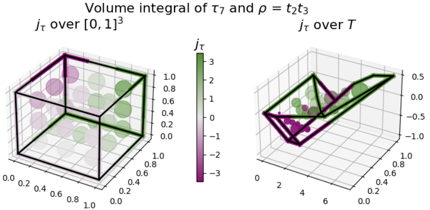
$\mathbf{n}_T$ over $\partial T$ $\mathbf{N}_\tau$ over $\partial[0,1]^n$

$$t_6(t_1, t_2, t_3) = (t_1 - 3t_1 t_2, t_2, t_3 + 2t_2 t_3)$$

$$\rho(x_1, x_2, x_3) = x_1$$

$$\mathbf{v}(x_1, x_2, x_3) = \frac{1}{3}(\frac{1}{2}x_1^2, x_1 x_2, x_1 x_3)$$

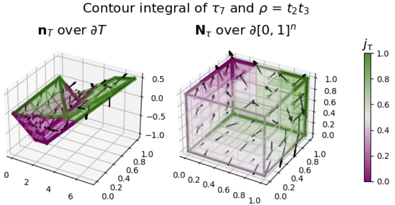
# 3D graphs: $\tau_7$



Volume integral of $\tau_7$ and $\rho = t_2 t_3$

$j_\tau$ over $[0,1]^3$

$j_\tau$ over $T$



Contour integral of $\tau_7$ and $\rho = t_2 t_3$

$\mathbf{n}_T$ over $\partial T$

$\mathbf{N}_\tau$ over $\partial[0,1]^n$

$$t_7(t_1, t_2, t_3) = (t_1 + 2t_3 + 4t_1 t_2 t_3,\, t_2,\, -2t_1 t_2 + t_1^2 t_2 - 0.5t_3 + 4t_1 t_2 t_3 - 2t_1^2 t_2 t_3)$$

$$\rho(x_1, x_2, x_3) = x_2 x_3$$

$$\mathbf{v}(x_1, x_2, x_3) = \frac{1}{3}(x_1 x_2 x_3,\, \frac{1}{2} x_2^2 x_3,\, \frac{1}{2} x_2 x_3^2)$$

# 3D convergence

We measure the discrepancy between volume integral $V$ and contour integral $C$ with their difference $|C - V|$.
The discrepancy decreases by two orders of magnitude for each increase in resolution. The values V and C converge.

| $\mathcal{T}$ | $\rho$ | V | C | V | C | $\lvert$C-V$\rvert$ | | |
|---|---|---|---|---|---|---|---|---|
| *res* | | 0.33 | | 0.0033 | | 0.33 | 0.033 | 0.0033 |
| $\mathcal{T}_5$ | 1 | -1.0 | -1.0 | -1.0 | -1.0 | 0.0 | 0.0 | 0.0 |
| $\mathcal{T}_6$ | $x_1$ | 1.139 | 1.291 | 1.250 | 1.250 | 1.5e-1 | 1.5e-3 | 1.5e-5 |
| $\mathcal{T}_7$ | $x_2 x_3$ | -0.135 | -0.148 | -0.135 | -0.135 | 1.3e-2 | 1.3e-4 | 1.3e-6 |

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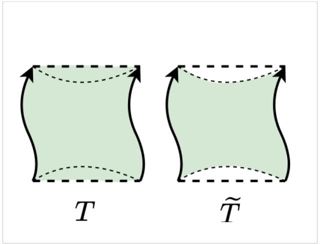
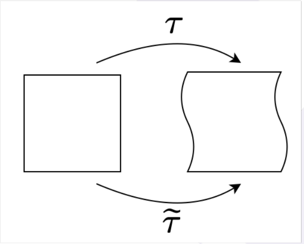
# Implementation Strategies and Generalizations

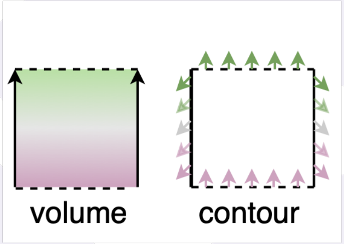
# Implementation Choices

To compute the flux through the region swept between a source and target face, we make several intermediate choices.

1. Between a source and target face, there are many possible swept regions.
2. For a given swept region, we have many possible parameterizations.
3. Finally, given a swept region and parameterization, we may compute the flux in several equivalent ways.

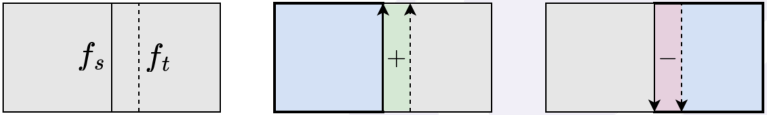




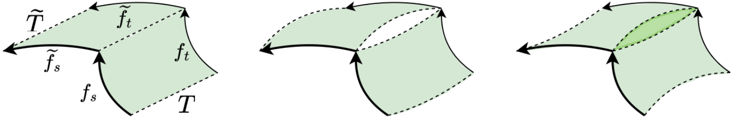


# Neighboring swept regions

The swept regions associated with a shared face should be computed once with opposite sign.



Neighboring swept regions must be contiguous without overlaps or gaps.



Good

Bad: gaps

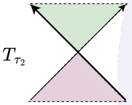
Bad: overlaps

# An efficient implementation strategy

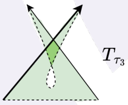
Choose a consistent mesh parameterization, then linearly sweep between $f_s$ and $f_t$.

$$\tau(t_1, \ldots, t_n) = f_s(t_1, \ldots, t_{n-1}) + t_n(f_t - f_s)(t_1, \ldots, t_{n-1}).$$

For computing numerical integrals, this adds only one degree to polynomial faces, and eliminates gaps and overlaps between neighboring zones.



$\tau_2$ sweeps linearly



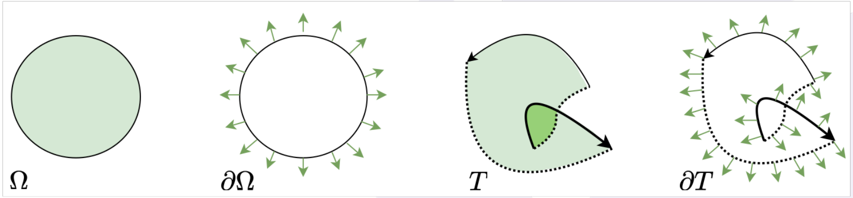
$\tau_3$ increases degree

To compute the flux, we recommend the parameterized volume integral:

$$\mathcal{F}_\rho(T) = \int_{[0,1]^n} (\rho \circ \tau) j_\tau \, d\mathbf{t}.$$

# Extensions to non-rectangular topologies

Our parameterization $\tau : [0, 1]^n \to \mathbb{R}^n$ maps from the unit cube. However, we could have used $\tau : \Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^n$ compact with piecewise differentiable boundary.



This gives us additional flexibility to non-rectangular mesh geometries. However, rectangular meshes are particularly convenient to parameterize, since we can use a map from a unit grid.

# Conclusions

# Summary of results and novelty

- We described our swept region mathematically, and provide several characterizations of the flux.
- We validate the theoretical results with numerical modeling.
- We discuss efficient strategies for parameterizing our swept regions and computing the flux.
- We consider non-rectangular mesh geometries.

This method for remap preserves globally polynomial solutions. It is continuous in that an unchanged mesh means no change in zone masses.

# Next steps

- If our physical solution $\rho$ is defined on source zones, how do we compute the flux without reconstructing a global $\rho$?
- How do we construct an efficient global mesh parameterization?
- How robust is this method to uncertainty in our measurements?
- Considering higher-order remap, can we create a flux integral directly from moments, rather than explicitly giving a parameterization?

# Acknowledgments

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