LA-UR-22-28273

Title: PinT 2022

Author(s): Southworth, Benjamin Scott
Rhebergen, Sander
Sivas, Abdullah
Manteuffel, Tom

Intended for: Parallel in time Conference, 2022-07-11/2022-07-15 (Marseille, France)

Issued: 2022-08-08

PinT 2022

Ben Southworth, **Sander Rhebergen, Abdullah Sivas, Tom Mantueffel**

11 July 2022

Los Alamos
NATIONAL LABORATORY

National Nuclear Security Administration
Managed by Triad National Security, LLC for the U.S. Department of Energy's NNSA

Two parts

Two-parts:

1. Convergence of two-level PinT
2. Nonsymmetric PDE solvers for space-time/all-at-once

Space-time operators

Space-time PDE:

$$u_t + \mathcal{L}(u, \mathbf{x}) = g(\mathbf{x}, t).$$

Space-time operators

Space-time PDE:

$$u_t + \mathcal{L}(u, \mathbf{x}) = g(\mathbf{x}, t).$$

- Discretize and linearize $\mathcal{L}$ in space
- $\mathbf{u}(t) = \mathbf{u}_{t_i}$ discrete solution in space at time t

$$\implies \qquad \mathbf{u}_t + \mathcal{L}(t)\mathbf{u} = \mathbf{g}(t).$$

Multigrid reduction in time (MGRiT)

Reduction-based multigrid for matrices

$$A\mathbf{u} := \begin{bmatrix} I & & & & \\ -\Phi_1 & I & & & \\ & -\Phi_2 & I & & \\ & & \ddots & \ddots & \\ & & & -\Phi_{N-1} & I \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{N-1} \end{bmatrix} = \mathbf{g},$$

Multigrid reduction in time (MGRiT)

Reduction-based multigrid for matrices

$$A\mathbf{u} := \begin{bmatrix} I & & & & \\ -\Phi_1 & I & & & \\ & -\Phi_2 & I & & \\ & & \ddots & \ddots & \\ & & & -\Phi_{N-1} & I \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_{N-1} \end{bmatrix} = \mathbf{g},$$

- Split time points into C-points/F-points
- Time discretization provides error w.r.t. continuous; want discrete error/residual $\rightarrow 0$ ($\mathbf{e} = \hat{\mathbf{u}} - \mathbf{u}$ and $\mathbf{r} = \mathbf{g} - A\mathbf{u}$)

Multigrid reduction in time (MGRiT)

Define $\Phi_i^j := \Phi_i \Phi_{i-1} ... \Phi_j$. Then

$$\begin{bmatrix} I \\ -\Phi_1 & I \\ & -\Phi_2 & I \\ & & \ddots & \ddots \\ & & & -\Phi_{N-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} I \\ \Phi_1 & I \\ \Phi_2^1 & \Phi_2 & I \\ \Phi_3^1 & \Phi_3^2 & \Phi_3 & I \\ \vdots & \vdots & & \ddots & \ddots \\ \Phi_{N-1}^1 & \Phi_{N-1}^2 & \cdots & \cdots & \Phi_{N-1} & I \end{bmatrix}.$$

Define $A_\Delta \approx B_\Delta :=$

$$\begin{bmatrix} I \\ -\Phi_k^1 & I \\ & -\Phi_{2k}^{k+1} & I \\ & & \ddots & \ddots \\ & & & -\Phi_{(N_c-1)k}^{(N_c-2)k+1} & I \end{bmatrix} \approx \begin{bmatrix} I \\ -\Psi_1 & I \\ & -\Psi_2 & I \\ & & \ddots & \ddots \\ & & & -\Psi_{N_c-1} & I \end{bmatrix}.$$

Error and residual propagation

$$\mathcal{E}_F^p := \begin{bmatrix} -A_{ff}^{-1}A_{fc} \\ I \end{bmatrix} \begin{bmatrix} \mathbf{0} & (I - B_\Delta^{-1}A_\Delta)^p \end{bmatrix},$$

$$\mathcal{E}_{FCF}^p := \begin{bmatrix} -A_{ff}^{-1}A_{fc} \\ I \end{bmatrix} \begin{bmatrix} \mathbf{0} & \Big((I - B_\Delta^{-1}A_\Delta)(I - A_\Delta)\Big)^p \end{bmatrix},$$

$$\mathcal{R}_F^p := \begin{bmatrix} \mathbf{0} \\ (I - A_\Delta B_\Delta^{-1})^p \end{bmatrix} \begin{bmatrix} -A_{cf}A_{ff}^{-1} & I \end{bmatrix},$$

$$\mathcal{R}_{FCF}^p := \begin{bmatrix} \mathbf{0} \\ \Big((I - A_\Delta B_\Delta^{-1})(I - A_\Delta)\Big)^p \end{bmatrix} \begin{bmatrix} -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

Restricted to C-points:

$$\widetilde{\mathcal{E}}_F := I - B_\Delta^{-1}A_\Delta, \qquad \widetilde{\mathcal{E}}_{FCF} := (I - B_\Delta^{-1}A_\Delta)(I - A_\Delta),$$

$$\widetilde{\mathcal{R}}_F := I - A_\Delta B_\Delta^{-1}, \qquad \widetilde{\mathcal{R}}_{FCF} := (I - A_\Delta B_\Delta^{-1})(I - A_\Delta).$$

F-relaxation

Let $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$ and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & \\ I & \mathbf{0} & \\ & \ddots & \ddots \end{bmatrix}, I_z = \begin{bmatrix} I & & \\ & \ddots & \\ & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \Phi_* - \Psi_{N_c-1} & \\ & & & \mathbf{0} \end{bmatrix}$$

F-relaxation

Let $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$ and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \end{bmatrix}, I_z = \begin{bmatrix} I & & \\ & \ddots & \\ & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \Phi_* - \Psi_{N_c-1} & \\ & & & \mathbf{0} \end{bmatrix}$$

- $I_L D = (B_\Delta - A_\Delta)$ and $I_L^T I_L = I_z$.

F-relaxation

Let $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$ and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \end{bmatrix}, I_z = \begin{bmatrix} I & & \\ & \ddots & \\ & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \Phi_* - \Psi_{N_c-1} & \\ & & & \mathbf{0} \end{bmatrix}$$

- $I_L D = (B_\Delta - A_\Delta)$ and $I_L^T I_L = I_z$.
- $\widetilde{\mathcal{R}}_F = I - A_\Delta B_\Delta^{-1} = (B_\Delta - A_\Delta) B_\Delta^{-1} = I_L D B_\Delta^{-1}$.

F-relaxation

Let $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$ and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \end{bmatrix}, I_z = \begin{bmatrix} I & & \\ & \ddots & \\ & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \Phi_* - \Psi_{N_c-1} & \\ & & & \mathbf{0} \end{bmatrix}$$

- $I_L D = (B_\Delta - A_\Delta)$ and $I_L^T I_L = I_z$.
- $\widetilde{\mathcal{R}}_F = I - A_\Delta B_\Delta^{-1} = (B_\Delta - A_\Delta) B_\Delta^{-1} = I_L D B_\Delta^{-1}$.

$$\|\mathcal{R}_F\|^2 = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\langle I_L D B_\Delta^{-1}\mathbf{x}, I_L D B_\Delta^{-1}\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\langle I_z D B_\Delta^{-1}\mathbf{x}, I_z D B_\Delta^{-1}\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}.$$

F-relaxation

Let $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$ and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \end{bmatrix}, I_z = \begin{bmatrix} I & & \\ & \ddots & \\ & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \Phi_* - \Psi_{N_c-1} & \\ & & & \mathbf{0} \end{bmatrix}$$

- $I_L D = (B_\Delta - A_\Delta)$ and $I_L^T I_L = I_z$.
- $\widetilde{\mathcal{R}}_F = I - A_\Delta B_\Delta^{-1} = (B_\Delta - A_\Delta)B_\Delta^{-1} = I_L D B_\Delta^{-1}$.

$$\|\mathcal{R}_F\|^2 = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\langle I_L D B_\Delta^{-1}\mathbf{x}, I_L D B_\Delta^{-1}\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\langle I_z D B_\Delta^{-1} I_z\mathbf{x}, I_z D B_\Delta^{-1} I_z\mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle}.$$

F-relaxation

Let $\Phi_* := \Phi_{(N_c-1)k}^{(N_c-2)k+1}$ and define the shift operators and block diagonal matrix

$$I_L = \begin{bmatrix} \mathbf{0} & & & \\ I & \mathbf{0} & & \\ & \ddots & \ddots & \end{bmatrix}, I_z = \begin{bmatrix} I & & \\ & \ddots & \\ & & \mathbf{0} \end{bmatrix}, D = \begin{bmatrix} \Phi_k^1 - \Psi_1 & & & \\ & \ddots & & \\ & & \Phi_* - \Psi_{N_c-1} & \\ & & & \mathbf{0} \end{bmatrix}$$

- $I_L D = (B_\Delta - A_\Delta)$ and $I_L^T I_L = I_z$.
- $\widetilde{\mathcal{R}}_F = I - A_\Delta B_\Delta^{-1} = (B_\Delta - A_\Delta) B_\Delta^{-1} = I_L D B_\Delta^{-1}$.

$$\|\mathcal{R}_F\|^2 = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\langle I_L D B_\Delta^{-1} \mathbf{x}, I_L D B_\Delta^{-1} \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle} = \| I_z D B_\Delta^{-1} I_z \|^2 = \| \widetilde{D} \widetilde{B}_\Delta^{-1} \|^2.$$

F-relaxation

Altogether:

$$\|\widetilde{\mathcal{R}}_F\| = \sigma_{\max}\left(\widetilde{D}\widetilde{B}_\Delta^{-1}\right) = \max_{\mathbf{x}\neq \mathbf{0}} \frac{\|\widetilde{D}\mathbf{x}\|}{\|\widetilde{B}_\Delta \mathbf{x}\|} = \frac{1}{\sigma_{\min}\left(\widetilde{B}_\Delta \widetilde{D}^{-1}\right)},$$

where $\widetilde{B}_\Delta \widetilde{D}^{-1} :=$

$$\begin{bmatrix} I & & & \\ -\Psi_1 & I & & \\ & \ddots & \ddots & \\ & & -\Psi_{N_c-2} & I \end{bmatrix} \begin{bmatrix} \left(\Phi_k^1 - \Psi_1\right)^{-1} & & \\ & \ddots & \\ & & \left(\Phi_* - \Psi_{N_c-1}\right)^{-1} \end{bmatrix}.$$

(similar results for $\widetilde{\mathcal{E}}_F, \widetilde{\mathcal{E}}_{FCF}$, and $\widetilde{\mathcal{R}}_{FCF}$).

Diagonalizable operators

Theorem 1.
Let Φ, Ψ be independent of time and simultaneously diagonalizable with eigenvectors U, eigenvalues $\{\lambda, \mu\}$, coarsening factor k, and N_c coarse-grid time points. Then,

$$\sup_i \frac{|\mu_i - \lambda_i^k|}{\sqrt{(1-|\mu_i|)^2 + \frac{\pi^2|\mu_i|}{N_c^2}}} \leq \|\mathcal{R}_F\|_{(UU^*)^{-1}} \leq \sup_i \frac{|\mu_i - \lambda_i^k|}{\sqrt{(1-|\mu_i|)^2 + \frac{\pi^2|\mu_i|}{6N_c^2}}},$$

$$\sup_i \frac{|\lambda_i^k||\mu_i - \lambda_i^k|}{\sqrt{(1-|\mu_i|)^2 + \frac{\pi^2|\mu_i|}{N_c^2}}} \leq \|\mathcal{R}_{FCF}\|_{(UU^*)^{-1}} \leq \sup_i \frac{|\lambda_i^k||\mu_i - \lambda_i^k|}{\sqrt{(1-|\mu_i|)^2 + \frac{\pi^2|\mu_i|}{6N_c^2}}}.$$

Time-independent operators

Definition 2 (Temporal approximation property).

Let $\Phi \sim$ fine-grid and $\Psi \sim$ coarse-grid time-stepping, independent of time, and with coarsening factor k. Φ satisfies an F-relaxation temporal approximation property (F-TAP) with respect to Ψ with constant φ_F, if, for all $\mathbf{v}$,

$$\|(\Psi - \Phi^k)\mathbf{v}\| \leq \varphi_F \left[\min_{x \in [0,2\pi]} \left\| (I - e^{\mathrm{i}x}\Psi)\mathbf{v} \right\| \right]. \tag{1}$$

Similarly, Φ satisfies an FCF-TAP with respect to Ψ with constant φ_{FCF}, if, for all $\mathbf{v}$,

$$\|(\Psi - \Phi^k)\mathbf{v}\| \leq \varphi_{FCF} \left[\min_{x \in [0,2\pi]} \left\| (\Phi^{-k}(I - e^{\mathrm{i}x}\Psi))\mathbf{v} \right\| \right]. \tag{2}$$

Time-independent operators

Theorem 2 (Necessary and sufficient conditions).

Suppose Φ and Ψ are linear, stable ($\|\Phi^p\|, \|\Psi^p\| < 1$ for some p), and independent of time; and that $(\Psi - \Phi^k)$ is invertible. Suppose Φ satisfies an F-TAP w.r.t. Ψ with constant φ_F, and Φ satisfies an FCF-TAP w.r.t. Ψ with constant φ_{FCF}. Then, worst-case convergence of residual is exactly bounded by

$$\frac{\varphi_F}{1 + O(1/\sqrt{N_c})} \leq \frac{\|\mathbf{r}_{i+1}^{(F)}\|}{\|\mathbf{r}_i^{(F)}\|} < \varphi_F,$$

$$\frac{\varphi_{FCF}}{1 + O(1/\sqrt{N_c})} \leq \frac{\|\mathbf{r}_{i+1}^{(FCF)}\|}{\|\mathbf{r}_i^{(FCF)}\|} < \varphi_{FCF}$$

for iterations $i > 1$ (i.e., not the first iteration).

Related work

Lemma 3.
Suppose Ψ is real-valued. Then,

$$\min_{x\in[0,2\pi]} \|(I - e^{\mathrm{i}x}\Psi)\mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\Psi\mathbf{v}\|^2 - 2\left|\langle\Psi\mathbf{v},\mathbf{v}\rangle\right|.$$

B. S. Southworth. *Necessary conditions and tight two-level convergence bounds for parareal and multigrid reduction in time*.

B. S. Southworth et al. *Tight two-level convergence of Linear Parareal and MGRIT: Extensions and implications in practice*.

S. Friedhoff and B. S. Southworth. *On "Optimal" h-independent convergence of Parareal and multigrid-reduction-in-time using Runge-Kutta time integration*.

Algebraic reduction (and advection)

Two main points:

- Most PinT methods struggle with strong advection (as do *p*-multigrid, and many block preconditioning methods!)
-

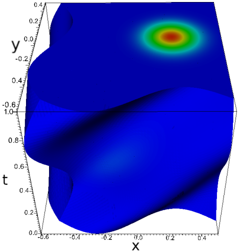
Table of Contents

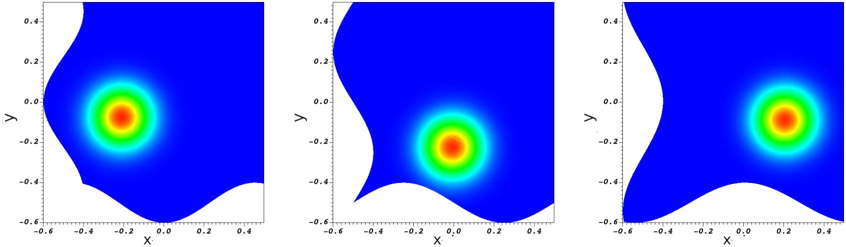
Advection-diffusion

$$\partial_t u + a \cdot \nabla u - \nu \nabla^2 u = f$$

$$\iff$$

$$\hat{a} \cdot \hat{\nabla} u - \nu \nabla^2 u = f$$





Conceptual basis for AIR

Partition (discontinuous) elements into C-elements and F-elements. Then in matrix form,

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{ff}^{-1}A_{fc} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

Conceptual basis for AIR

Partition (discontinuous) elements into C-elements and F-elements. Then in matrix form,

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{ff}^{-1}A_{fc} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

AIR preconditioner M^{-1} looks like:

$$M^{-1} = \left[\begin{array}{cc} I & \widehat{W} \\ 0 & I \end{array} \right] \left[\begin{array}{cc} \Delta_F & 0 \\ 0 & \mathcal{K}^{-1} \end{array} \right] \left[\begin{array}{cc} I & 0 \\ Z & I \end{array} \right].$$

Where $\mathcal{K} = RAP$. Want $\Delta_F \simeq A_{ff}^{-1}$, $Z \simeq -A_{cf}A_{ff}^{-1}$, etc.

Conceptual basis for AIR

Partition (discontinuous) elements into C-elements and F-elements. Then in matrix form,

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{ff}^{-1}A_{fc} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & \mathcal{S}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

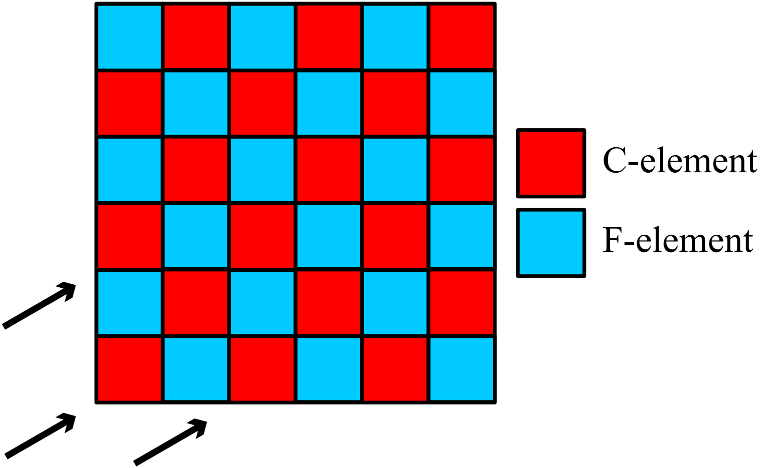
AIR preconditioner M^{-1} looks like:

$$M^{-1} = \begin{bmatrix} I & \widehat{W} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_F & 0 \\ 0 & \mathcal{K}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix}.$$

Where $\mathcal{K} = RAP$. Want $\Delta_F \simeq A_{ff}^{-1}$, $Z \simeq -A_{cf}A_{ff}^{-1}$, etc.

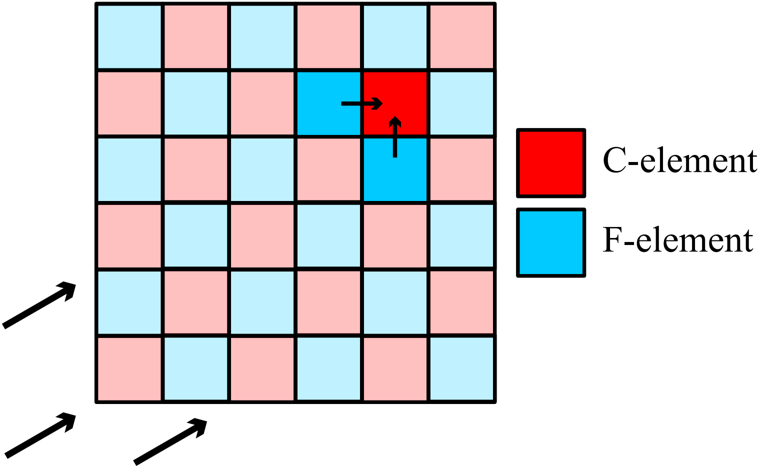
$\Longrightarrow$ **Can we approximate A_{ff}^{-1} well?**

Conceptual basis for AIR



Consider transport on structured 2d grid and partition elements into C-elements and F-elements.

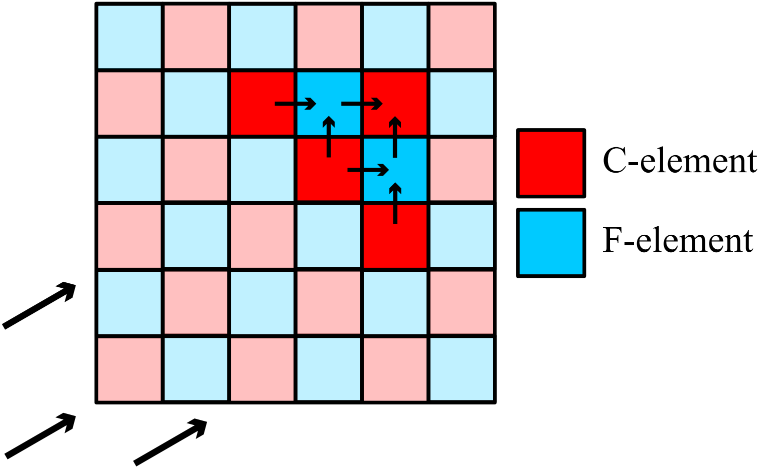
Conceptual basis for AIR



Notice that there are no C-C or F-F connections

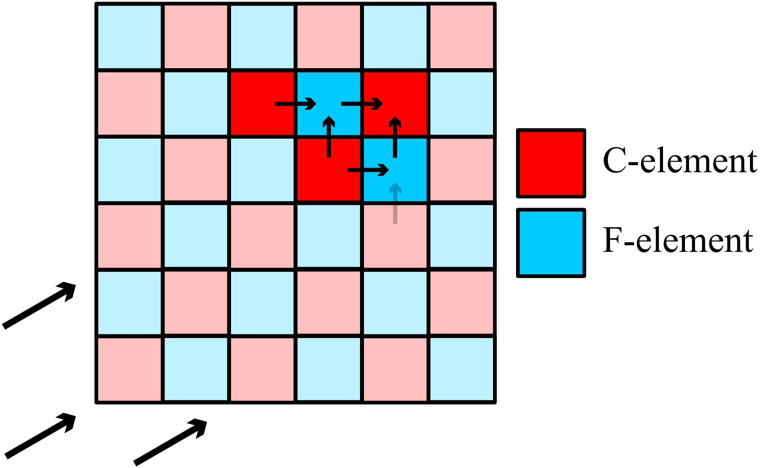
$$\implies A_{ff} = A_{cc} = I.$$

Conceptual basis for AIR



If $A_{ff} = I$, AMG coarse grid given by $A_{cc} - A_{cf}A_{fc} \iff$ all C-F-C connections.

Conceptual basis for AIR



If $A_{ff} = I$, AMG coarse grid given by $A_{cc} - A_{cf}A_{fc} \iff$ all C-F-C connections. **One of these connections is weak!**

Space-time coarsening

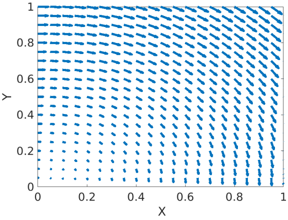


Fig.: Vel. field

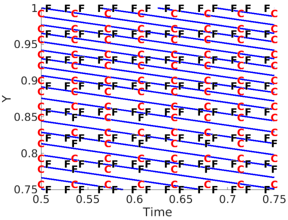


Fig.: $x = 0.1861$

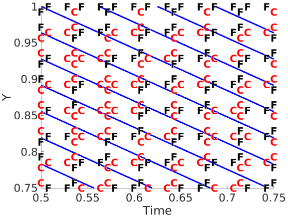


Fig.: $x = 0.5790$

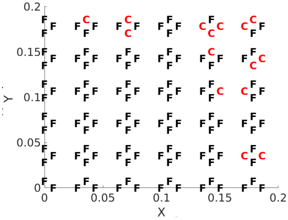


Fig.: $t = 0.3853$

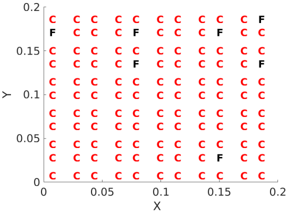


Fig.: $t = 0.3928$

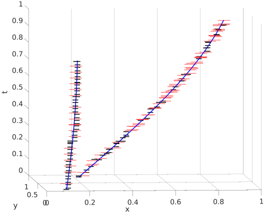


Fig.: Space-time char.

Space-time AMR

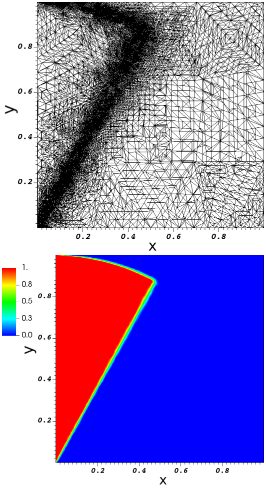


Fig.: Mesh/solution at $t = 0.5$

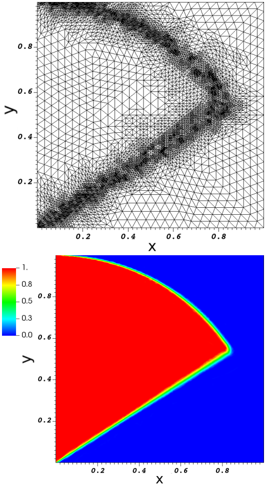


Fig.: Mesh/solution at $t = 1$

Results

- AIR-AMG applied to hybridized DG (EDG and HDG)
- BiCGSTAB iterations to relative residual 10^{-12}.

$p = 1$

n	ν				
	10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
136K	8	7	8	10	17
1M	8	8	10	13	54
8.5M	8	9	12	18	×

$p = 2$

n	ν				
	10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
272K	8	9	10	14	30
2.1M	9	11	13	18	46
17M	9	14	15	30	83

$p = 3$

n	ν				
	10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
454K	9	11	12	18	38
3.6M	9	13	15	25	73
28M	10	17	18	46	144

Results

- AIR-AMG applied to hybridized DG (EDG and HDG)
- BiCGSTAB iterations to relative residual 10^{-12}.

$p = 1$

n	ν = 10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
136K	8	7	8	10	17
1M	8	8	10	13	54
245K	**17**	**17**	**16**	**15**	**12**

$p = 2$

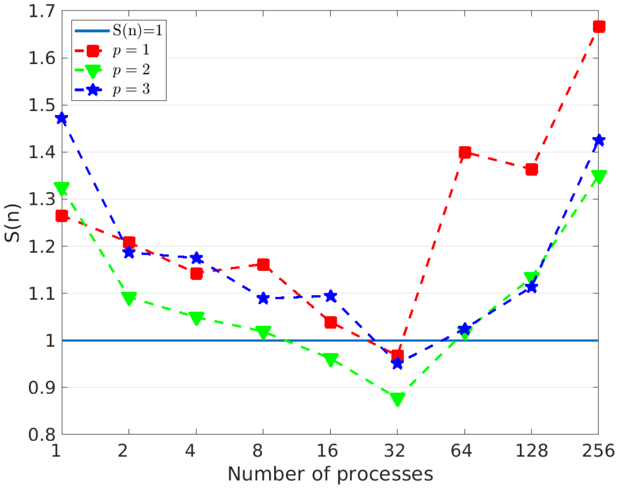
n	ν = 10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
272K	8	9	10	14	30
2.1M	9	11	13	18	46
1.9M	**17**	**16**	**19**	**21**	**30**

$p = 3$

n	ν = 10^{-6}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
454K	9	11	12	18	38
3.6M	9	13	15	25	73
6.5M	**19**	**17**	**14**	**16**	**41**

EDG

Speedup over sequential



Navier Stokes

Navier Stokes

- Working on block AIR for incompressible NS. Initial results for steady decent up to Reynolds 10^5.

Navier Stokes

- Working on block AIR for incompressible NS. Initial results for steady decent up to Reynolds 10^5.
- Working on AIR for shallow water/compressible NS/Euler. Haven't tested yet.

Navier Stokes

- Working on block AIR for incompressible NS. Initial results for steady decent up to Reynolds 10^5.

- Working on AIR for shallow water/compressible NS/Euler. Haven't tested yet.

- $\implies$ Add time to discretization, solve all at once!

Thank you!

Papers:

T. A. Manteuffel, J. Ruge, and B. S. Southworth. *Nonsymmetric Algebraic Multigrid Based on Local Approximate Ideal Restriction (ℓAIR)*.

A. A. Sivas, B. S. Southworth, and S. Rhebergen. *AIR algebraic multigrid for a space-time hybridizable discontinuous Galerkin discretization of advection (-diffusion)*.