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Addressing Challenges on Compressible Turbulence Simulations using Entropy Stable Scheme

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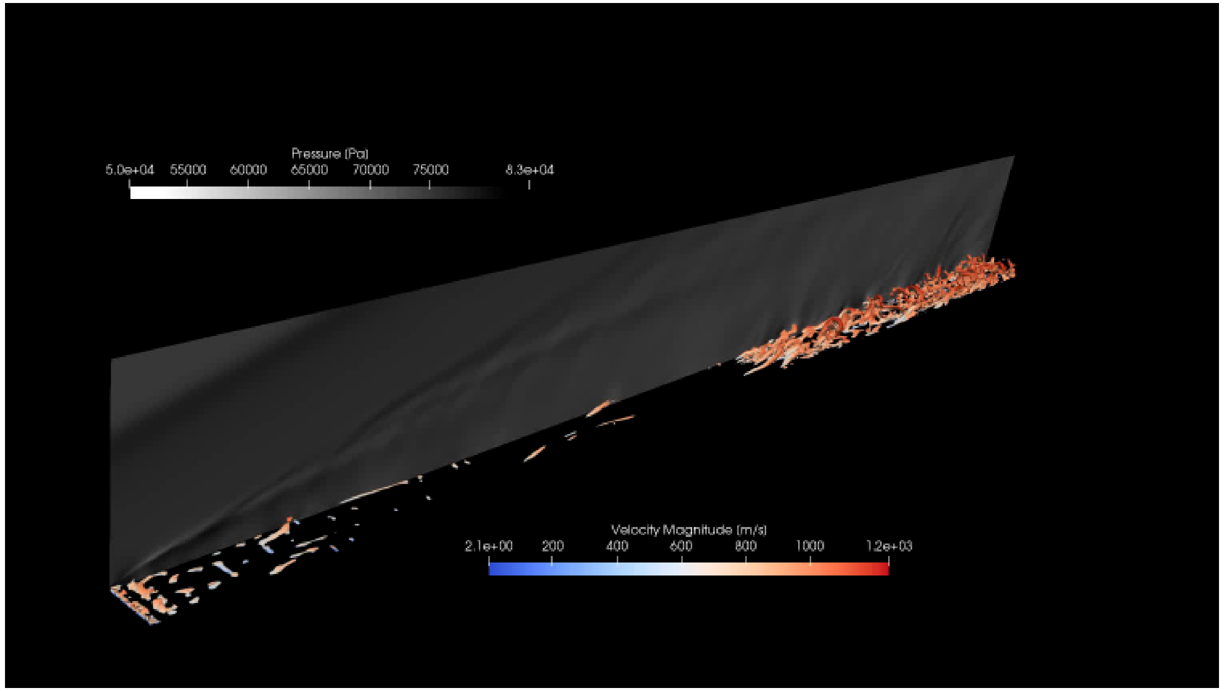
Challenges in Compressible Turbulence Simulations

Growing computational power => Higher engineering fidelity simulations possible

- Large-Eddy Simulations
- Direct Numerical Simulations

Challenges

- Resolving turbulence in under-resolved mesh
- Shock capturing
- Explicit time integration



Why Entropy Stable Scheme

For nonlinear governing equations

$$(\mathbf{u})_t + (\mathbf{f})_{x_k} = (\mathbf{f}^V)_{x_k}, x_k \in \Omega$$

- Generalized Summation-by-Parts operators[1] for conservation

$$\mathbf{u}_t + \mathcal{P}_k^{-1}\mathcal{Q}_k\mathbf{f}_k(\mathbf{u}) = \mathcal{P}_k^{-1}\mathbf{g}_k^{Int} + \mathcal{P}_k^{-1}\mathbf{g}_k^{Bnd} + \mathcal{P}_k^{-1}\mathcal{Q}_k c_{kj}\mathcal{Q}_j\mathbf{u}$$

$$\mathbf{u}_t + \mathcal{P}^{-1}[2\mathcal{Q}\circ\mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\mathbf{g}^{Int} + \mathcal{P}^{-1}\mathbf{g}^{Bnd} + \mathcal{P}^{-1}\mathcal{Q}_k c_{kj}\mathcal{Q}_j\mathbf{u}$$

$$\mathcal{D} = \mathcal{P}^{-1}\mathcal{Q}, \quad \mathcal{P} = \mathcal{P}^T, \quad \boldsymbol{\xi}^T\mathcal{P}\boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \neq 0$$

$$\mathcal{Q}^T = \mathcal{B} - \mathcal{Q}. \qquad \mathcal{B} = \mathbf{b}_1\mathbf{b}_1^T - \mathbf{b}_{-1}\mathbf{b}_{-1}^T,$$

- Provable stability
  - Two-point entropy stable inviscid flux
  - Entropy stable viscous flux

1. DCCRey Fernandez, JCP 2014

Entropy Stable Schemes

Entropy stable high-order finite difference

- Summation-by-Parts (SBP) method
- Multi-block structured mesh
- Generalized SBP operator
  - Cell-centered scheme
    - Offers stronger inter-block coupling
  - Node-centered scheme
- Dissipation mechanisms for shock capturing
  - Hybrid WENO scheme
  - Artificial viscosity
  - Artificial dissipation

Entropy stable spectral collocation method

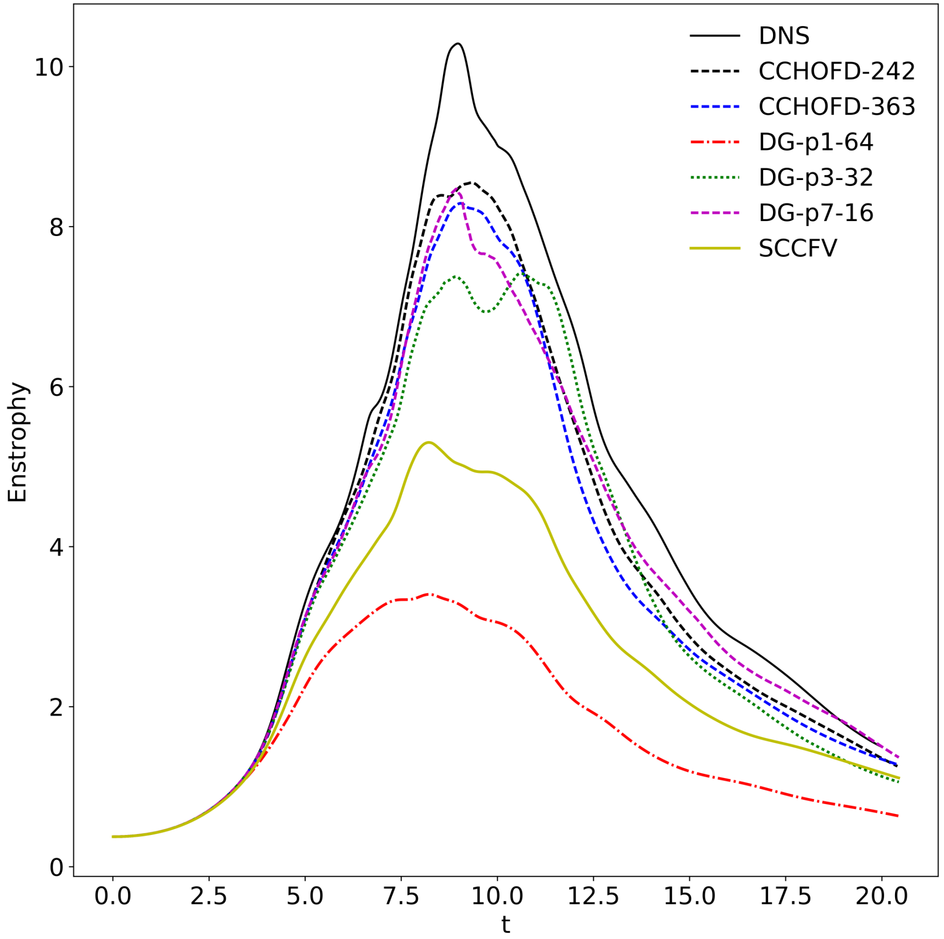
- Summation-by-Parts (SBP) method
- Unstructured mesh
- Tensor product elements
  - Legendre-Gauss
  - Legendre-Gauss-Lobatto solution points
- Dissipation mechanisms for shock capturing
  - Inter-element penalty
  - Artificial viscosity

Taylor Green Vortex

3D incompressible (M=0.1) Taylor Green vortex

- Turbulent kinetic energy dissipation comparison by discretization

- DOF
  - SCCFV and CCHOFD: 128^3 elements
  - DG p1: 64^3 elements
  - DG p3: 32^3 elements
  - DG p7: 16^3 elements
  - Reference solution: DNS 512^3 spectral method



Shock Capturing Scheme

Entropy stable shock capturing scheme

- Entropy stable high-order finite difference scheme
    - Weighted Essentially Non-Oscillatory (WENO) scheme
    - Artificial viscosity
- Entropy stable spectral collocation scheme
    - Artificial viscosity

Finite volume scheme

- Low-dissipation Subbareddy Candler scheme
    - 2nd order TVD limiter
    - 4th order central dissipation with various switches

Shock Capturing Scheme

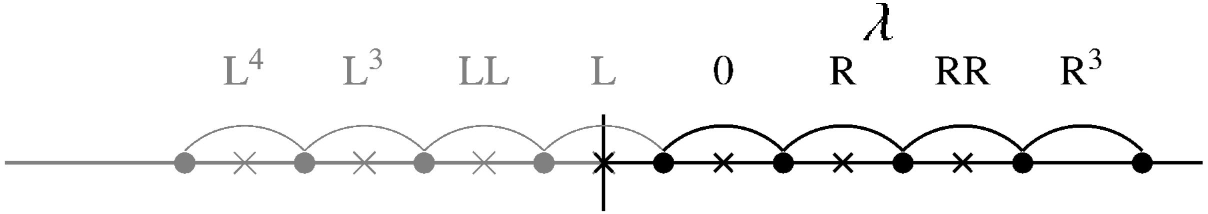
Entropy stable WENO high-order finite difference scheme[1]

$$\bar{f}_i^W = \sum_{l=1}^{n_s} \bar{\omega}_i^l \bar{f}_i^{S_l}. \qquad \bar{\omega}_i^l = \frac{\bar{\alpha}_i^l}{\sum_j \bar{\alpha}_i^j}, \quad \bar{\alpha}_i^l = \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), \quad l = 1, \ldots, n_s$$

$$\bar{f}_i^{SSW} = \bar{f}_i^W + \delta(\bar{f}_i^S - \bar{f}_i^W), \quad \delta = \frac{\sqrt{b^2 + c^2} - b}{\sqrt{b^2 + c^2}}, \quad b = (w_{i+1} - w_i)^T (\bar{f}_i^S - \bar{f}_i^W), \quad c = 10^{-12}$$

$$(w_{i+1} - w_i)^T (\bar{f}_i^{SSW} - \bar{f}_i^S) \le 0, \quad 0 \le i \le N - 1,$$

- WENO across multi-block interface for generalized HOFD



  - Cell-centered SBP operator gives a strong coupling between blocks
  - WENO target flux, weight, candidate stencil based on non-dissipative interface operator
  - Need a different biasing due to larger stencil width

$$\bar{\alpha}_i^l = \begin{cases} \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), & \text{if } l \in [0, R, RR, R^3] \\ \gamma \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), & \text{if } l \in [L, LL, L^3, L^4] \end{cases}, \quad l = 1, \ldots, n_s,$$

1. “High-order entropy stable finite difference schemes for nonlinear conservation laws: Finite domains”, T. Fisher, M. Carpenter. JCP 2013

Shock Capturing Scheme

Hybrid entropy stable scheme

- Dilatation based shock sensor[1] for detecting compressible region

$$\epsilon_{\mathrm{ss}} = -\nabla \cdot \mathbf{v} - \max\left(5\sqrt{\omega \cdot \omega},\ \alpha_{\mathrm{fs}} \frac{c}{h}\right) > 0$$

- Entropy stable artificial dissipation
  - Entropy stable HOFD is non-dissipative
    - May require dissipation for flows with discontinuities and/or under-resolved flows

$$\mathbf{u}_t + \mathcal{P}_k^{-1} \Delta_k \bar{\mathbf{f}}_k = \mathcal{P}_k^{-1} \mathbf{g}_k^{int} + \mathcal{P}_k^{-1} \Delta_k \bar{\mathbf{f}}_k^{ad}$$

$$\mathcal{P}_k^{-1} \Delta_k \bar{\mathbf{f}}_k^{ad} = \mathcal{D}_2 \left|\Lambda\right| \frac{\partial \mathbf{u}}{\partial \mathbf{w}} \mathcal{D}_2 \mathbf{w} \quad \text{where } \mathcal{D}_2 = \Delta \Delta^T$$

1. “Wall-modeled Large-Eddy Simulations of the HIFiRE-2 Scramjet”, I. Bermejo-Moreno, J. Larsson, J. Bodart, R. Vicquelin, CTR 2013

Shock Capturing Scheme

Entropy stable artificial viscosity[1]

$$\mathbf{u}_t + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k = \mathcal{P}_k^{-1}\mathbf{g}_k^{int} + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad} + \mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{av}$$

$$\mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{av} = \mathcal{D}_i g_{ij}\frac{\partial \mathbf{u}}{\partial \mathbf{w}}\hat{\mu}\mathcal{D}_j\mathbf{w}$$

$$\hat{\mu} = \max\left[\left(\frac{(L\mathbf{u})^T\mathbf{w}_\mathbf{u}(L\mathbf{u})}{\phi + (\mathbf{w}_{x_i})^T g_{ij}\mathbf{u}_\mathbf{w}(\mathbf{w}_{x_j})}\right)^{1/2}, c_{ref}\frac{|u|+c}{h}\right], \quad L\mathbf{u} = \frac{\partial \mathbf{f}_k}{\partial \mathbf{v}}\frac{\partial \mathbf{v}}{\partial x_k} - \frac{\partial \mathbf{f}_k}{\partial x_k}$$

- Computes artificial viscosity based on linearized and nonlinearized residual differences
- Used in both entropy stable schemes
- Additional tuning for different flow regions

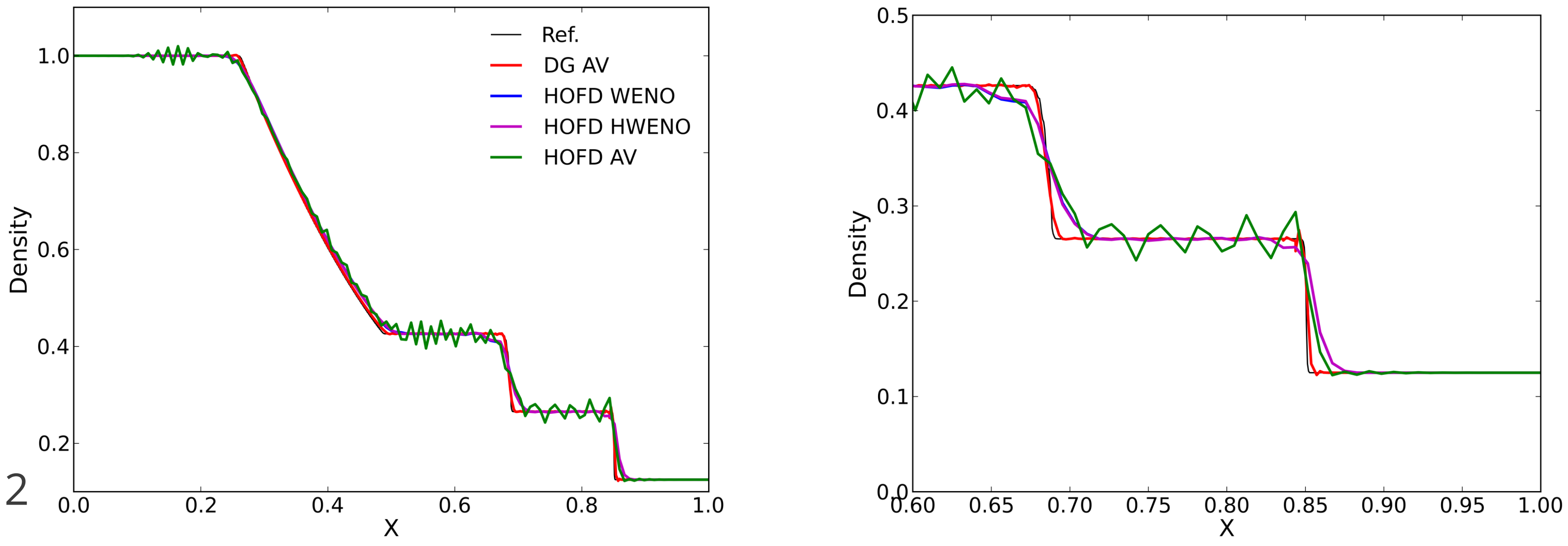
$$\mu_{\text{tuned}} = \alpha_{\text{shock}}\epsilon_{\text{ss}}\mu + \alpha_{\text{non-shock}}(1-\epsilon_{\text{ss}})\mu$$

1. A new finite element formulation for computational fluid dynamics: X. The compressible Euler and Navier-Stokes equations. F. Shakib, TJR. Hughes, Z. Johan CM 1991

Entropy Stable Shock Capturing Schemes - Assessment

1D Sod shock tube problem

- HOFD Hybrid WENO, WENO and DG AV resolves all flow features
- HOFD AV without artificial dissipation can cause unwanted oscillations
    - Further tuning of AD & AV can alleviate

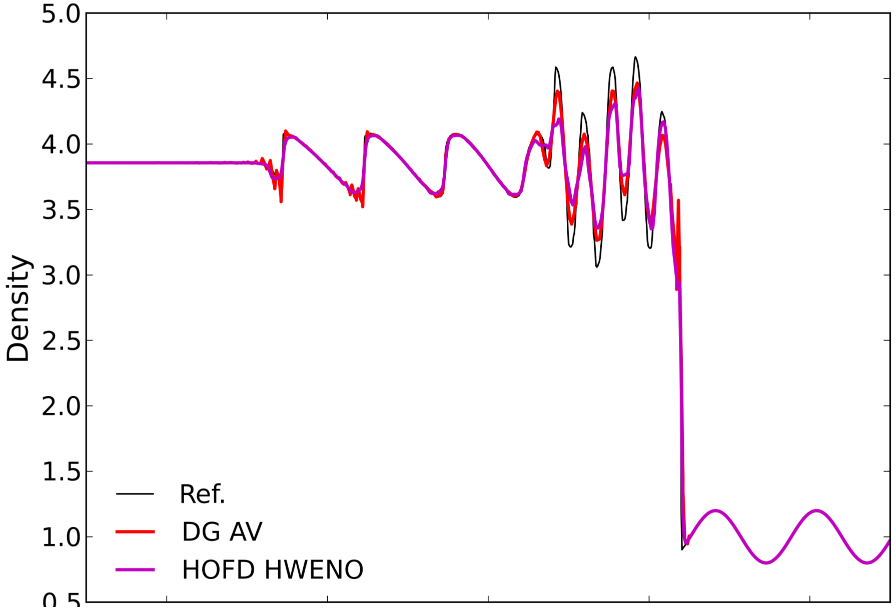


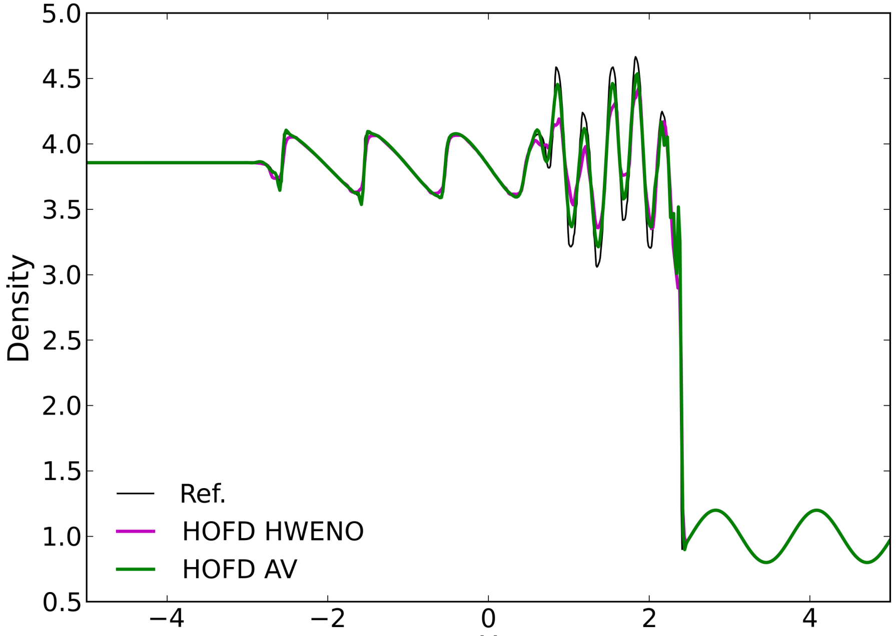
242 HOFD-N512
DG P3-N128

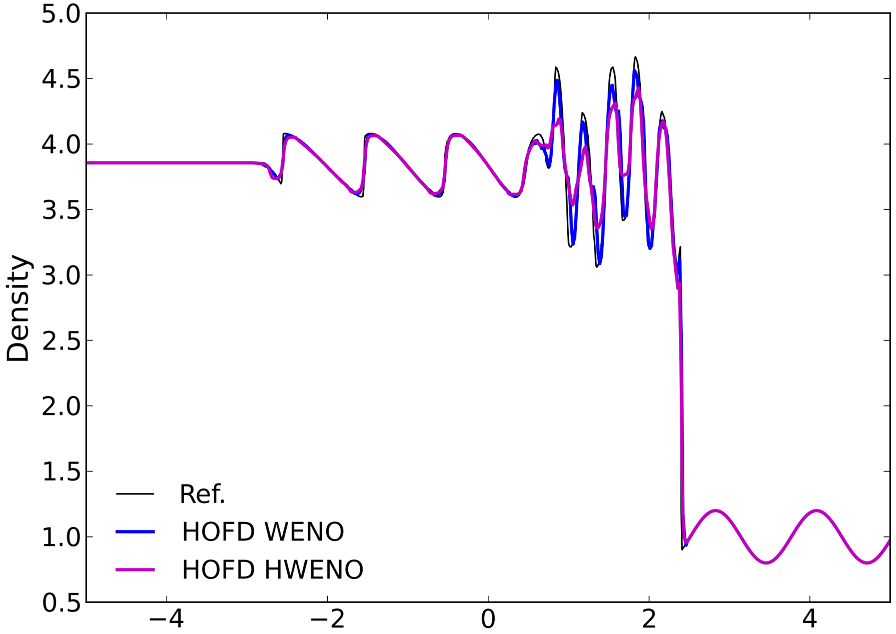
Entropy Stable Shock Capturing Schemes - Assessment

1D Shu Osher problem

- Trade-offs
  - HOFD WENO captures shocks well at the expense of reduced resolution in shock-entropy wave interaction
  - Both HOFD AV and DG AV exhibit some oscillation with increased resolution in shock-entropy wave interaction







242 HOFD-N512
DG P3-N128

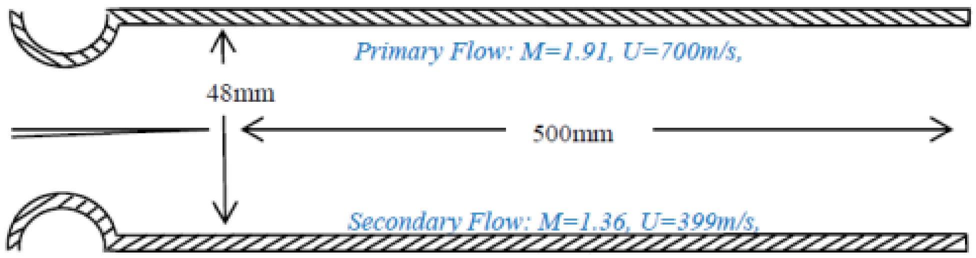
3D Compressible Mixing Layer

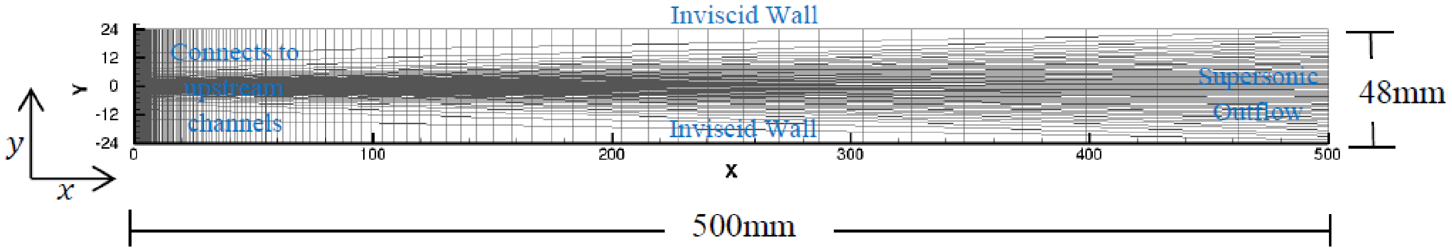
Compressible mixing layer simulation

- Assess implicit LES with various discretization schemes

Problem setup

- Flow configuration: convective Mach 0.46
  - Primary flow Mach 1.91
  - Secondary flow Mach 1.36
- Synthetic turbulence inflow
  - For inflow, SST RANS precursor provides:
    - Reynolds stress tensor
    - Mean flow velocity
    - Length scale
- Mesh
  - Domain: [500, 48, 6] mm
  - Mesh dimension
    - Fine: [513, 213, 17]
    - Coarse: [257, 107, 11]

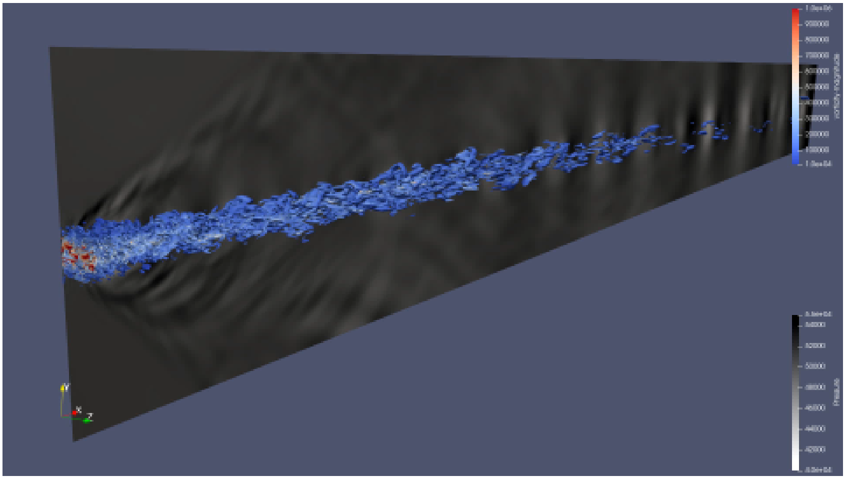




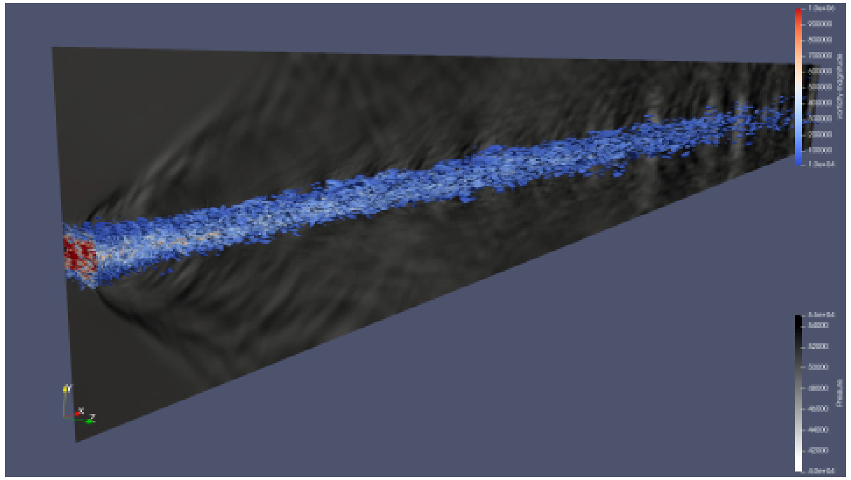
3D Compressible Mixing Layer – Qualitative Assessment

Turbulence simulation with various schemes

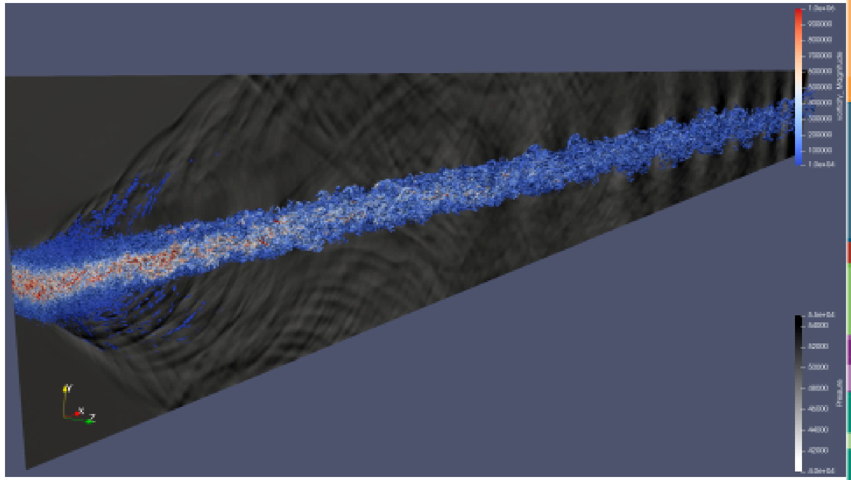
- Contour: Q criterion 1e9, colored by vorticity magnitude
- Hybrid WENO HOFD with AD



Low dissipation SCCFV



Hybrid WENO HOFD

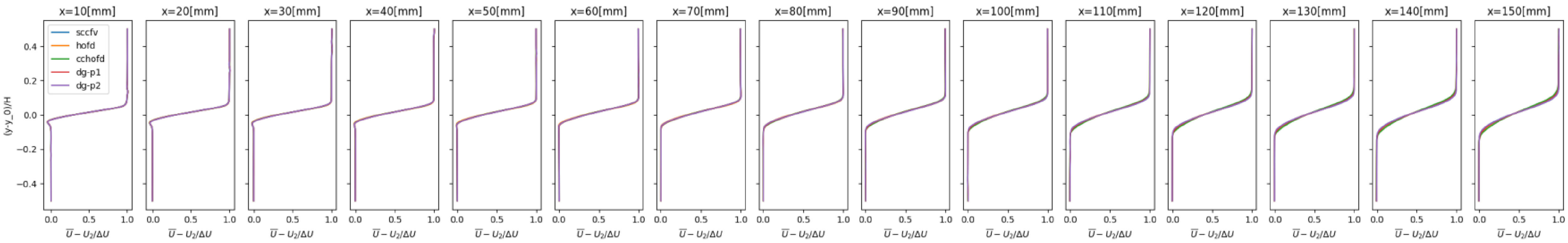


DG LG p2 AV

3D Compressible Mixing Layer – Mean Velocity and Stresses

Mean velocity profile

- Converged for all discretization



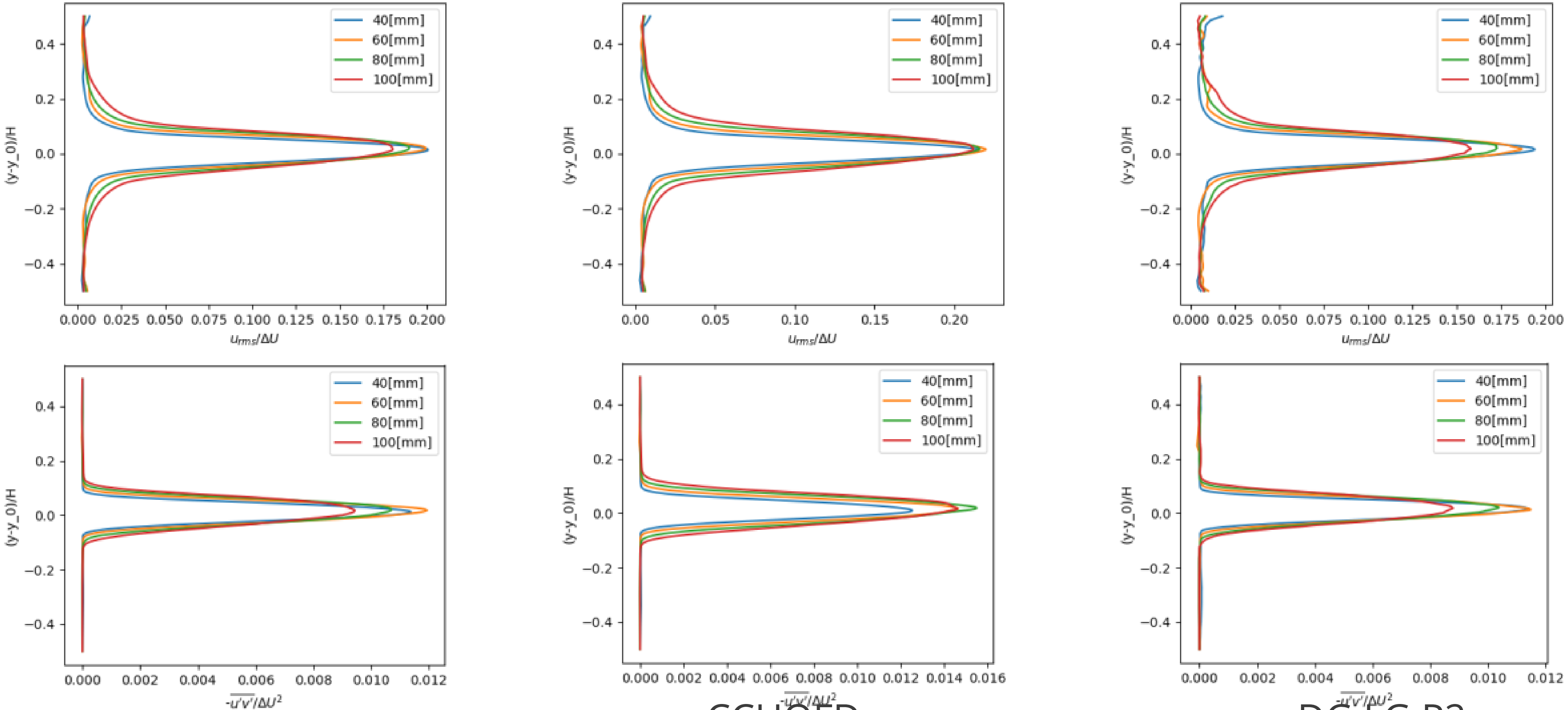
3D Compressible Mixing Layer – Mean Velocity and Stresses

Stresses

- Accurate prediction of turbulent statistics varies

Normal stress

Shear stress



SCCFV

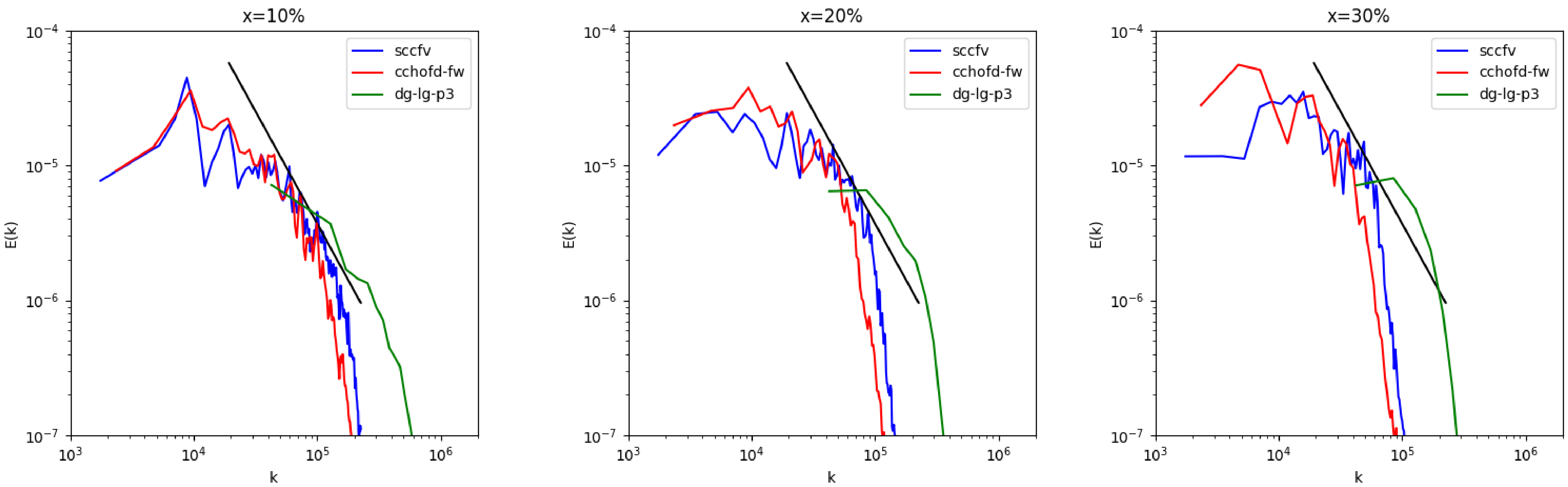
CCHOFD

DG LG P2

3D Compressible Mixing Layer – Turbulent Spectra

Comparison by discretization

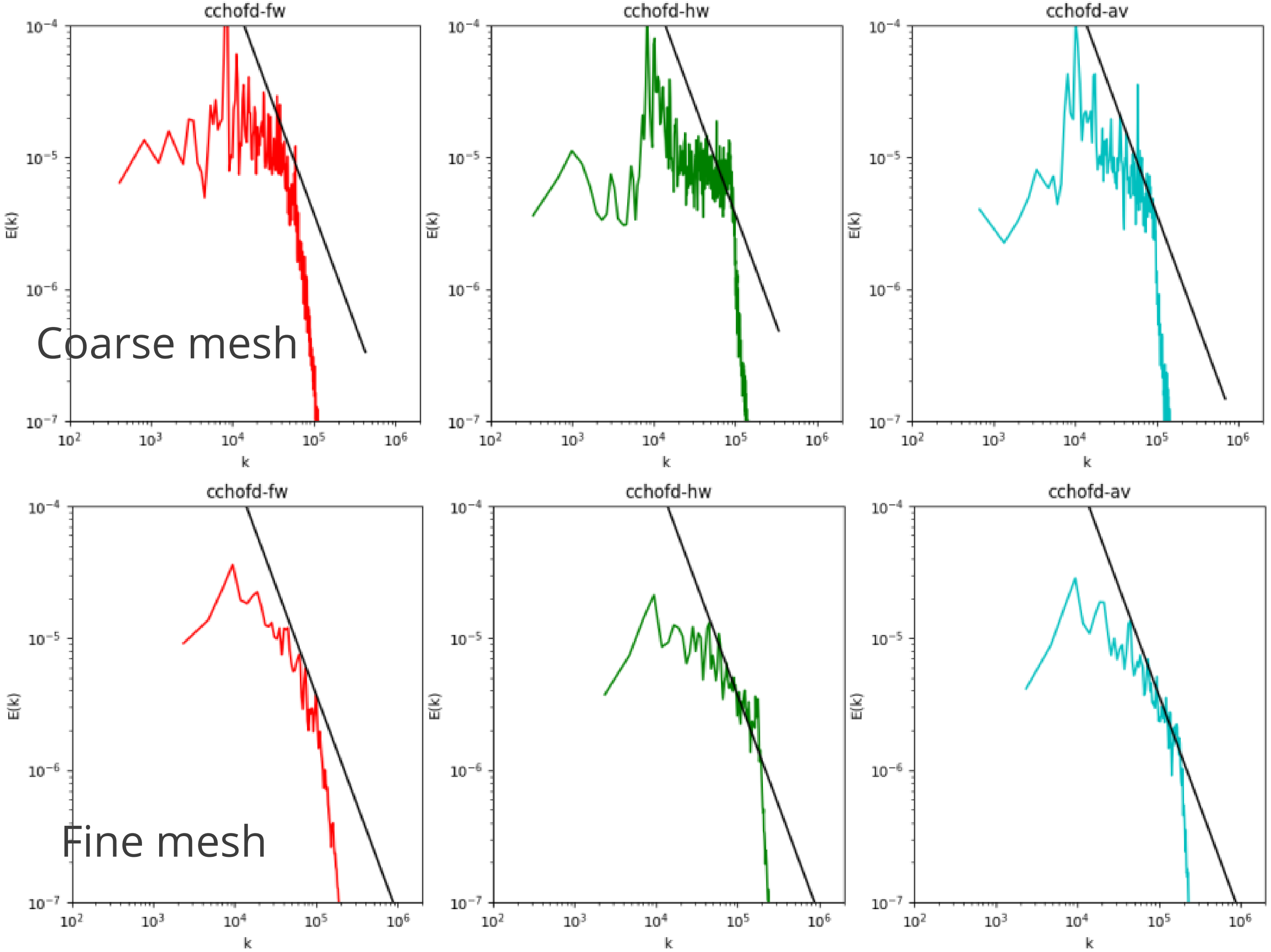
- For ILES, both cell-centered methods capture inertial subrange well
- DG resolves spectra well in high-frequency range
- While DG holds up well up to 40%, both cell-centered schemes fail to accurately predict turbulent kinetic energy spectra past 30% of domain



3D Compressible Mixing Layer – Turbulent Spectra

Comparison of CCHOFD dissipation methods, x = 10%

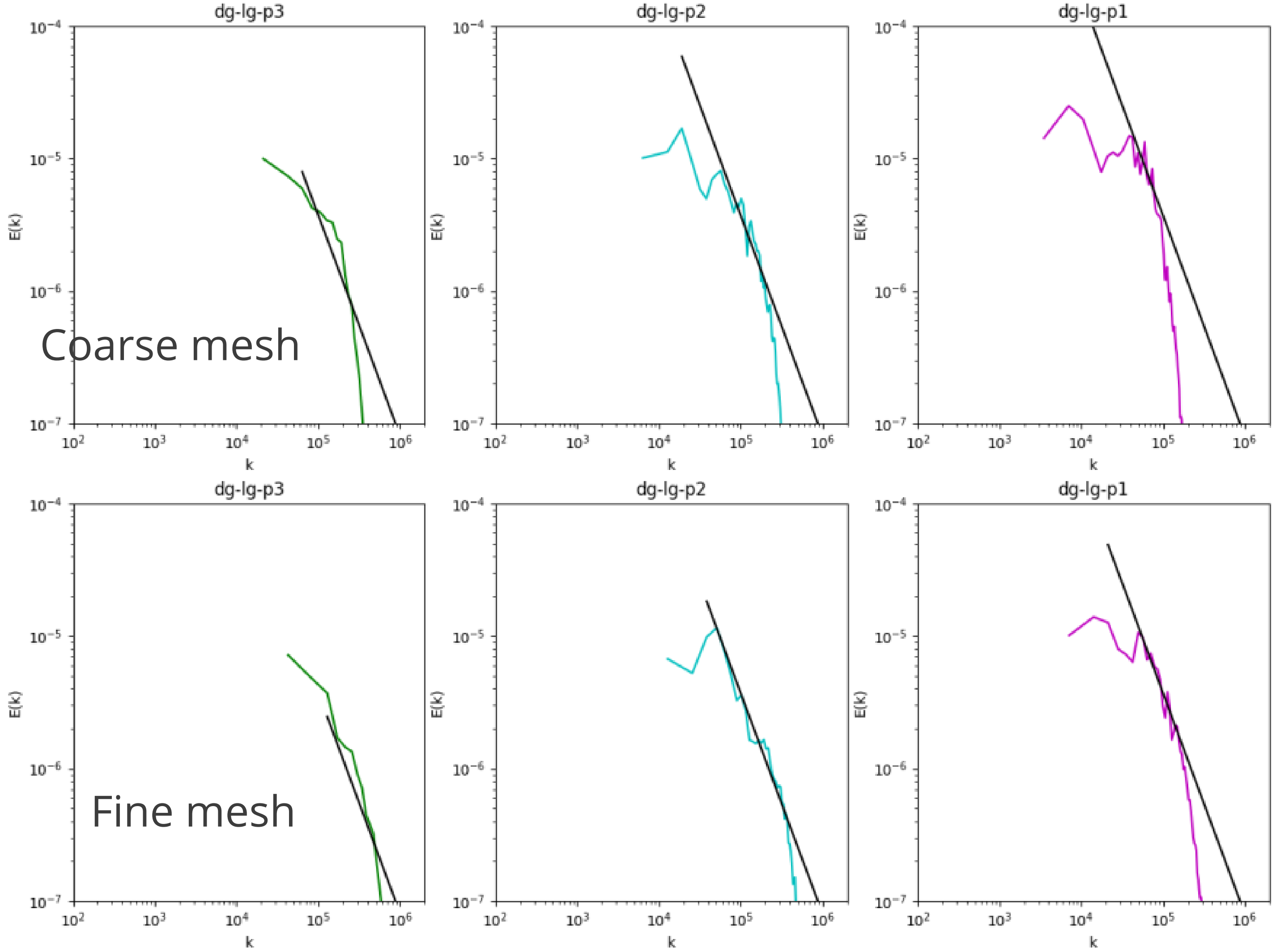
- WENO
  - Higher wave number range is well not resolved
- Hybrid WENO and AV
  - As mesh resolution increases spectra captures inertial subrange



3D Compressible Mixing Layer – Turbulent Spectra

Comparison of DG spatial order, x = 10%

- Resolvable wave number
  - Lower order resolution lacks resolution in inertial subrange
  - Higher order resolves higher spectrum range better



Summary

For compressible turbulent scale-resolving simulations

- Entropy stable schemes can provide a few options for shock capturing
  - WENO, Hybrid WENO, AV
- Varying nature of problems can decide the choice of nonlinear stabilization
  - Tuning can affect shock capturing and turbulence resolution
  - Further parametric study is needed to fully assess
- Simulating compressible turbulent flows
  - Low dissipation finite volume scheme suitable for ILES
    - However increase in grid resolution may be needed
  - High-order finite difference WENO appropriate for ILES
    - Other methods of stabilization exhibit energy pile up
  - Spectral collocation scheme can resolve inertial subrange for all orders

Challenges and Future Work

Various nonlinear stabilization methods are available for HO methods

- Tuning makes HO schemes challenging

Implicit LES with HOFD needs more numerical evaluation

- Further research of appropriate dissipation scheme for turbulent flows

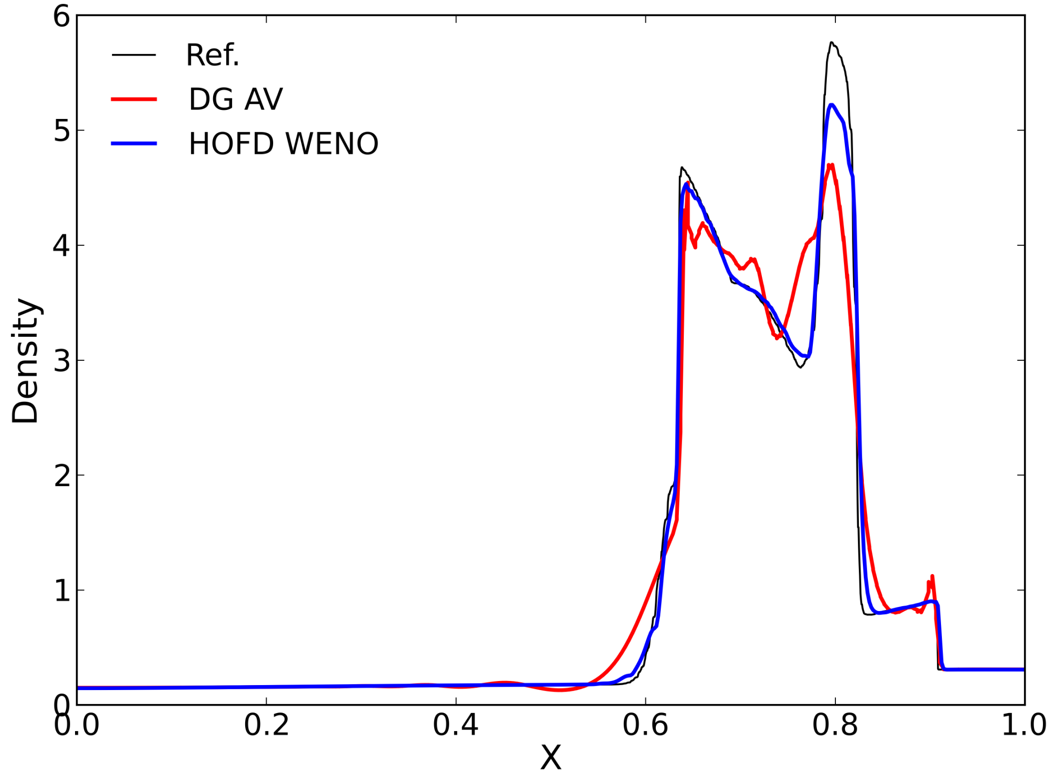
Entropy stable spectral collocation's bottleneck

- With increased tensor product order comes greater cost of time-step restriction

Entropy Stable Shock Capturing Schemes - Assessment

1D Blast wave problem

- Two strong shock wave interaction
- HOFD AV and DG AV issues
  - Shock sensing
  - AV regularization



242 HOFD-N512
DG P1-N256

Cell-Centered Framework via Generalized Summation-by-Parts

Nonlinear conservation laws

$$\mathbf{u}_t + (\mathbf{f}_k)_{x_k} = 0,\ x_k \in \Omega,\ t \in [0,\infty),$$

$$\mathbb{B}(\mathbf{u}) = \mathbf{g}^{bnd},\ x_k \in \partial\Omega,\ t \in [0,\infty),$$

$$\mathbf{u}(x,0) = \mathbf{g}_0(x_k),\ x_k \in \Omega,$$

$$\mathbf{u}_t + \mathcal{D}_k \mathbf{f}_k = \mathcal{P}_k^{-1}\mathbf{g}_k^{int} + \mathcal{P}_k^{-1}\mathbf{g}_k^{bnd}, \quad k = 1,2,3$$

Generalized summation-by-parts (SBP) operator (Del Rey Fernandez, JCP 2014)

- Discrete analogue of **integration-by-parts**

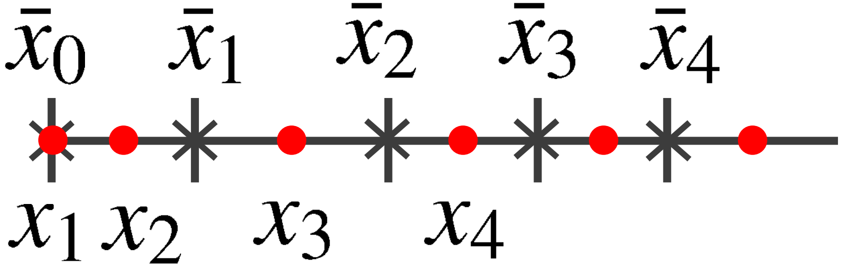
$$\mathcal{D} = \mathcal{P}^{-1}\mathcal{Q}, \quad \mathcal{P} = \mathcal{P}^T, \quad \boldsymbol{\xi}^T\mathcal{P}\boldsymbol{\xi} > 0, \quad \boldsymbol{\xi} \neq 0$$

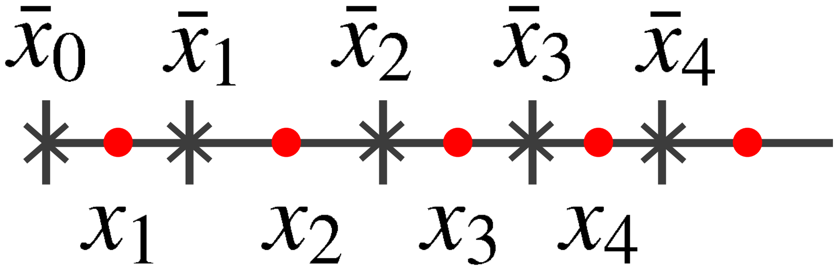
$$\mathcal{Q}^T = \mathcal{B} - \mathcal{Q}. \qquad \mathcal{B} = \mathbf{b}_1\mathbf{b}_1^T - \mathbf{b}_{-1}\mathbf{b}_{-1}^T,$$

- Generalization of boundary solution points

$$\mathbf{b}_{-1} = (1,0,0,\ldots,0)^T$$

$$\mathbf{b}_{-1} = \left(\frac{35}{16}, -\frac{35}{16}, \frac{21}{16}, -\frac{5}{16}, 0, \ldots, 0\right)^T$$





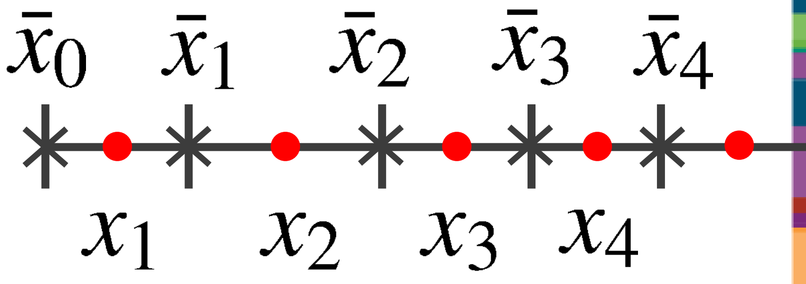
Entropy Stable Cell-Centered High-Order Finite Difference

$$\mathbf{u}_t + \mathcal{P}^{-1}[2Q \circ \mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\mathbf{g}^{int}$$

Complementary grid enables us to recast gradient form to **flux form**

- Important for entropy stable WENO flux (Fisher, JCP 2013)

$$\mathbf{f}(\mathbf{u})_x = \mathcal{P}^{-1}[2Q \circ \mathcal{F}]\mathbf{1} = \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} \quad \Rightarrow \quad \mathbf{u}_t + \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} = \mathcal{P}^{-1}\mathbf{g}^{int}$$



$$\mathcal{P}_{ii} = \bar{x}_{i+1} - \bar{x}_i$$

Entropy stable two-point nonlinear flux

$$\frac{d}{dt}\mathbf{1}^T \mathcal{P} S + \sum_{k=1}^{N}\sum_{l=1}^{N} b_{1,k}b_{1,l}\bar{F}(u_l,u_k) - b_{-1,k}b_{-1,l}\bar{F}(u_l,u_k) = 0$$

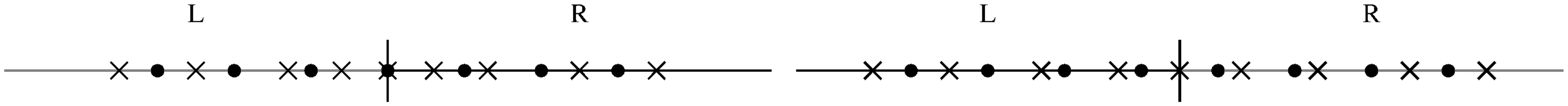
$$\frac{d}{dt}\mathbf{1}^T \mathcal{P} S + \bar{F}|_1 - \bar{F}|_{-1} = 0$$

$$\bar{f}_i^S = \sum_{k=i}^{N}\sum_{l=1}^{i} 2\hat{q}_{(l,k)}\bar{f}(u_l,u_k) + \sum_{k=i+1}^{N}\sum_{l=1}^{N} -b_{-1,l}b_{-1,k}\bar{f}(u_l,u_k) + \sum_{k=1}^{i}\sum_{l=1}^{N} b_{1,l}b_{1,k}\bar{f}(u_l,u_k), \quad 1 \le i \le N-1,$$

Benefits of cell-centered approach

- Similar to finite volume and satisfies telescoping flux property
- Stronger coupling across multi-block interface
- Better shock capturing

Generalized Entropy Stable Interface Penalty



Conventional　　　　Cell-centered

Two-domain finite difference in flux form

$$\mathbf{u}_t + \mathcal{P}^{-1}\Delta\bar{\mathbf{f}} = \mathcal{P}^{-1}\mathbf{g}^{int}$$

$$\Delta\bar{\mathbf{f}} = (\mathcal{Q} + \mathcal{G})\begin{bmatrix}\mathbf{f}_L \\ \mathbf{f}_R\end{bmatrix} \qquad \mathcal{Q} = \begin{bmatrix} Q_L & 0 \\ 0 & Q_R \end{bmatrix} \quad \mathcal{G} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1^L\mathbf{b}_1^{L^T} & \frac{1}{2}\mathbf{b}_1^L\mathbf{b}_{-1}^{R^T} \\ -\frac{1}{2}\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T} & \frac{1}{2}\mathbf{b}_{-1}^R\mathbf{b}_{-1}^{R^T} \end{bmatrix}$$

Generalized entropy stable interface penalty

$$\begin{aligned}
\mathbf{g}^{int} = & \left\{\left(\mathbf{b}_1^L\mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u}) - \mathbf{b}_1^L\mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u})\right)\right. \\
& \left. -\frac{1}{2}\mathbf{b}_1^L\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_1^{L^T}\mathbf{w} - \mathbf{b}_{-1}^{R^T}\mathbf{w}\right)\right\} \\
& - \left\{\left(-\mathbf{b}_{-1}^R\mathbf{b}_1^{L^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u}) + \mathbf{b}_{-1}^R\mathbf{b}_{-1}^{R^T} \circ \mathbf{f}(\mathbf{u},\mathbf{u})\right)\right. \\
& \left. -\frac{1}{2}\mathbf{b}_{-1}^R\mathbf{R}|\Lambda|\mathbf{R}^T\left(\mathbf{b}_{-1}^{R^T}\mathbf{w} - \mathbf{b}_1^{L^T}\mathbf{w}\right)\right\}
\end{aligned}$$

Shock Capturing with Weighted Essentially Non-Oscillatory

Entropy stable WENO (Fisher and Carpenter, JCP 2013)

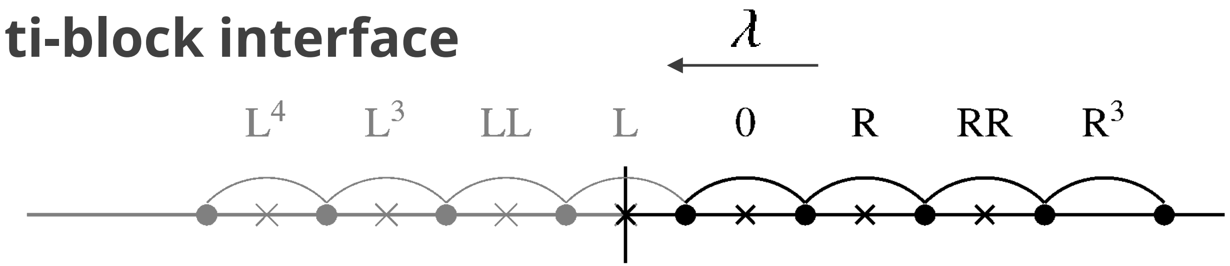
$$\bar{f}_i^W = \sum_{l=1}^{n_s} \bar{\omega}_i^l \bar{f}_i^{S_l}. \qquad \bar{\omega}_i^l = \frac{\bar{\alpha}_i^l}{\sum_j \bar{\alpha}_i^j}, \quad \bar{\alpha}_i^l = \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), \quad l = 1, \ldots, n_s$$

$$\bar{f}_i^{SSW} = \bar{f}_i^W + \delta(\bar{f}_i^S - \bar{f}_i^W), \quad \delta = \frac{\sqrt{b^2 + c^2} - b}{\sqrt{b^2 + c^2}}, \quad b = (w_{i+1} - w_i)^T (\bar{f}_i^S - \bar{f}_i^W), \quad c = 10^{-12}$$

- Entropy stability condition is satisfied with entropy stable WENO

$$(w_{i+1} - w_i)^T (\bar{f}_i^{SSW} - \bar{f}_i^S) \leq 0, \quad 0 \leq i \leq N - 1,$$

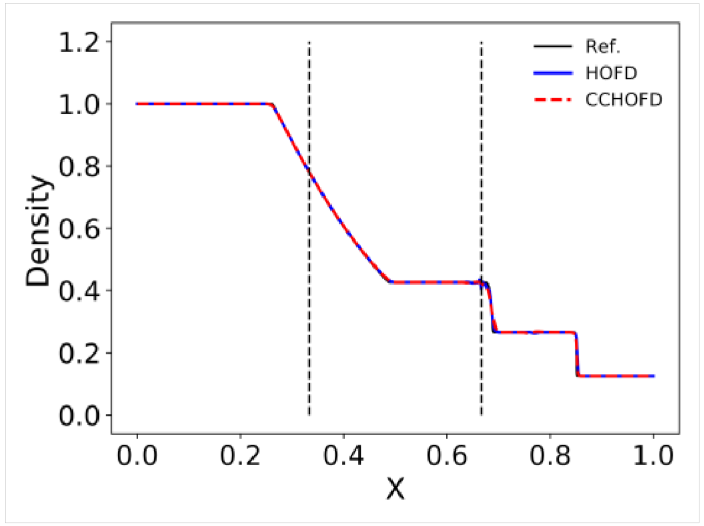
WENO across multi-block interface



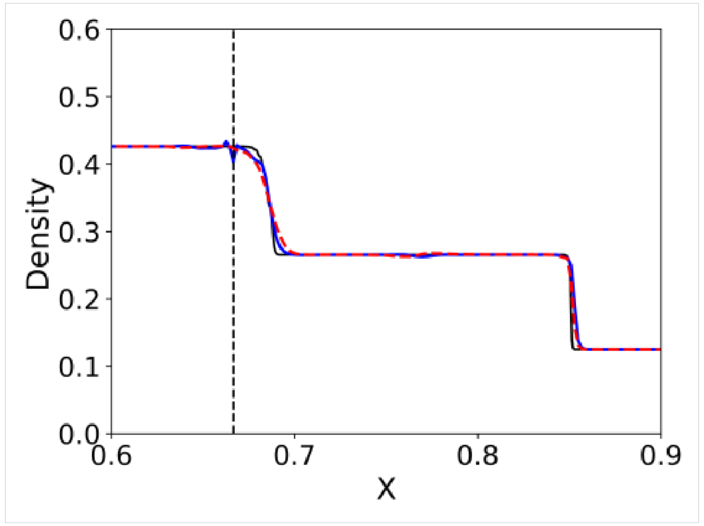
- Cell-centered SBP operator gives a strong coupling between blocks
- WENO target flux, weight, candidate stencil based on non-dissipative interface operator
- Need a different biasing due to larger stencil width

$$\bar{\alpha}_i^l = \begin{cases} \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), & \text{if } l \in [0, R, RR, R^3] \\ \gamma \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), & \text{if } l \in [L, LL, L^3, L^4] \end{cases}, \quad l = 1, \ldots, n_s,$$

Multi-Block Shock Capturing WENO: 1D Shock Examples



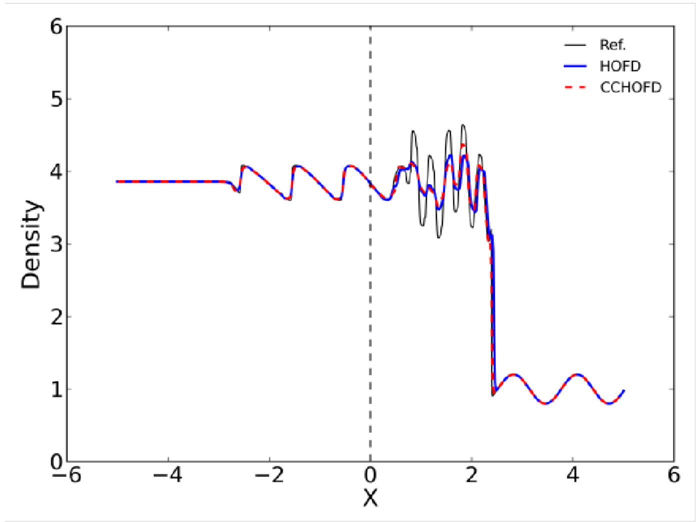
(a) Density



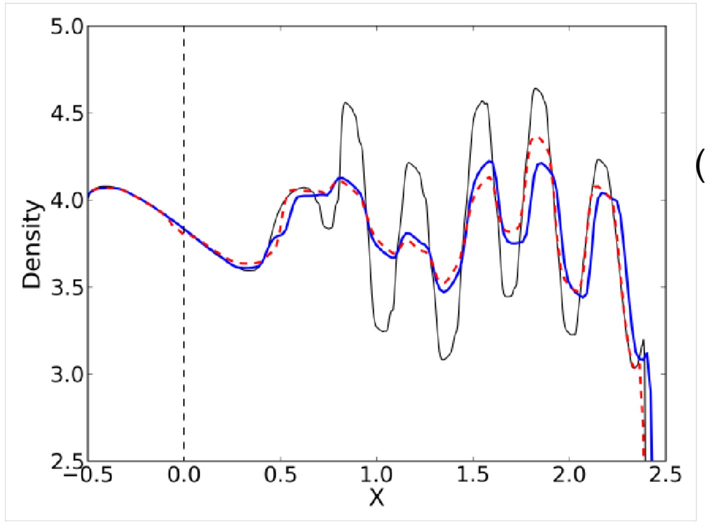
(b) Density close up near the contact discontinuity

$$(\rho, u, p) = \begin{cases} (1, 0, 1), & \text{if } x < 0.5 \\ (0.125, 0, 0.1), & \text{if } x \geq 0.5, \end{cases}$$

N=512, t_f = 0.25
Three-block



(a) Density

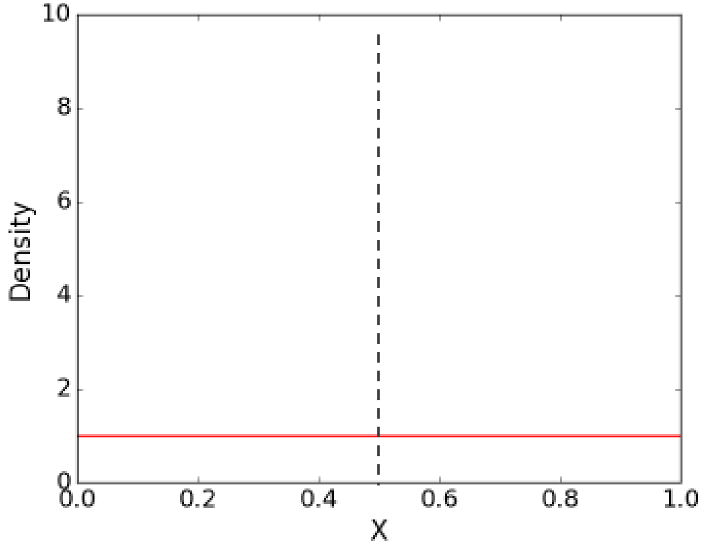


(b) Density close up

$$(\rho, u, p) = \begin{cases} (3.857143, 2.629369, 10.3333), & \text{if } x < -4.0 \\ (1 + 0.2\sin(5x), 0, 1), & \text{if } x \geq -4.0. \end{cases}$$

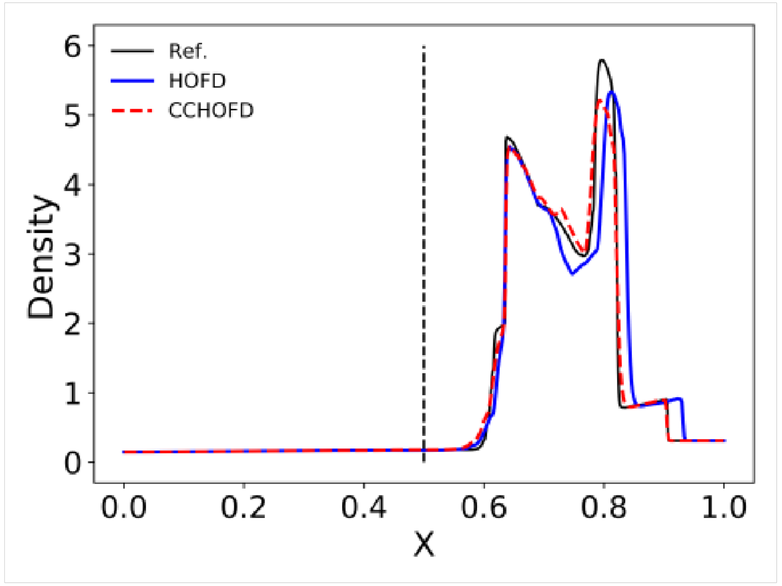
N=512, t_f = 1.8, N_{ref} = 2000
Two-block

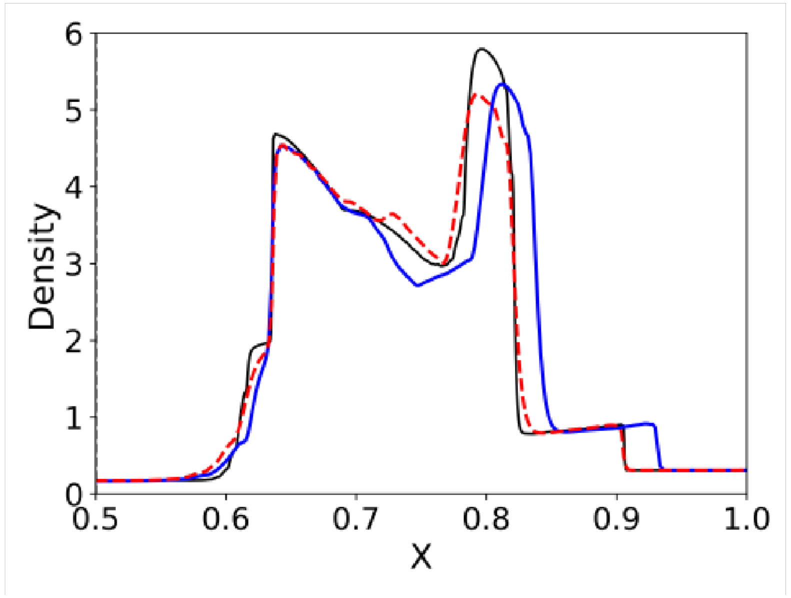
Strong Shock Across Interface: Woodward Colella



Initial condition

$$(\rho, u, p) = \begin{cases} (1, 0, 1000), & \text{if } x < 0.1 \\ (1, 0, 100), & \text{if } 0.1 \leq x < 0.9 \\ (1, 0, 0.01), & \text{if } x \geq 0.9. \end{cases}$$





N=512, t_f = 0.04, N_{ref} = 2000
Two-block

SSWENO for HOFD

Entropy stable WENO [6]

$$\bar{f}_i^W = \sum_{l=1}^{n_s} \bar{\omega}_i^l \bar{f}_i^{S_l}. \qquad \bar{\omega}_i^l = \frac{\bar{\alpha}_i^l}{\sum_j \bar{\alpha}_i^j}, \quad \bar{\alpha}_i^l = \bar{d}_i^l \left(1 + \frac{\bar{\tau}_i^l}{\bar{\beta}_i^l + \bar{\epsilon}_i}\right), \quad l = 1, \ldots, n_s$$

$$\bar{f}_i^{SSW} = \bar{f}_i^W + \delta(\bar{f}_i^S - \bar{f}_i^W), \quad \delta = \frac{\sqrt{b^2 + c^2} - b}{\sqrt{b^2 + c^2}}, \quad b = (w_{i+1} - w_i)^T (\bar{f}_i^S - \bar{f}_i^W), \quad c = 10^{-12}$$

- Entropy stability condition is satisfied with entropy stable WENO

$$(w_{i+1} - w_i)^T (\bar{f}_i^{SSW} - \bar{f}_i^S) \leq 0, \quad 0 \leq i \leq N - 1,$$

  - Provably stable in conventional SBP HOFD
  - Not so much for generalized SBP
- Hybridization requires AD
  - Because SSHOFD is nondissipative

Artificial Dissipation for HOFD

HOFD is non-dissipative

Extend Mattsson's AD operators [5] using entropy variables

- 4^{th} order

$$\mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad} = \mathcal{D}_2|\Lambda|\frac{\partial \mathbf{u}}{\partial \mathbf{w}}\mathcal{D}_2\mathbf{w} \quad \text{where} \quad \mathcal{D}_2 = \Delta\Delta^T$$

- We also have one for 6^{th} order

Entropy Stable Artificial Viscosity

Formulation from Shakib [7] using entropy variables

$$\hat{\mu} = \max\left[\mu,\ \alpha_{\text{ref}}\frac{|u|+c}{\sqrt{g_{ij}g_{ji}}}\right] \quad \text{where} \quad \mu = \left(\frac{(L\mathbf{u})^T \mathbf{w}_{\mathbf{u}}(L\mathbf{u})}{\phi + (\mathbf{w}_{x_i})^T g_{ij}\mathbf{u}_{\mathbf{w}}(\mathbf{w}_{x_j})}\right)^{1/2}, \quad L\mathbf{u} = \frac{\partial \mathbf{f}_k}{\partial \mathbf{v}}\frac{\partial \mathbf{v}}{\partial x_k} - \frac{\partial \mathbf{f}_k}{\partial x_k}$$

$$\mathcal{P}_k^{-1}\Delta_k \bar{\mathbf{f}}_k^{av} = \mathcal{D}_i g_{ij}\frac{\partial \mathbf{u}}{\partial \mathbf{w}}\hat{\mu}\mathcal{D}_j\mathbf{w}$$

- Artificial viscosity is *tuned* for shock and non-shock regions

$$\mu_{\text{tuned}} = \alpha_{\text{shock}}\epsilon_{\text{ss}}\mu + \alpha_{\text{non-shock}}(1-\epsilon_{\text{ss}})\mu$$

Idea 1: Adaptive Artificial Dissipation

Improve AD operator

$$\mathcal{P}_k^{-1}\Delta_k\bar{\mathbf{f}}_k^{ad} = \mathcal{D}_2|\Lambda|\frac{\partial \mathbf{u}}{\partial \mathbf{w}}\mathcal{D}_2\mathbf{w} \quad \text{where} \quad \mathcal{D}_2 = \Delta\Delta^T$$

- Smoothness of key primitive variables
  - Modify wave speed

$$\Lambda = \alpha_u|u| + \alpha_c c$$

- Has potential for improving hybrid HOFD-SSWENO
  - Not so sure if it can be extended to AV application

Idea 2: Pressure-Based Shock Sensor

Based on HPCMP CREATE-AV COFFE solver's artificial diffusion flux [3]

$$\epsilon_{\text{ss}} = \psi \tanh(10\psi) \quad \text{where} \quad \psi = \frac{\xi}{\xi + \kappa\tilde{p}} \quad \text{and} \quad \xi = \sqrt{\nabla\tilde{p}^T \left(g_{ij} g_{ji}\right) \nabla\tilde{p}}, \quad \tilde{p} = \frac{p}{0.5\rho\mathbf{u}^2}$$

- Shock sensor is passive if grid resolution can support pressure gradient
- Otherwise switch activates the artificial diffusion flux in SUPG

Easy to implement in current SPARC

Idea 3: Characteristics-Based Modal Shock Sensor [1,2]

Characteristics convey information about waves

- Entropy and acoustic waves

Under-resolved region can be identified by modal energy decay

- Attractive for high-order methods, particularly DG

High-risk high-reward (potentially)

- None of the previous sensors utilize *high-order* contents of solution
- However, using physical information can be robust and makes sense
  - E.g. Dilatation, pressure, vorticity, enstrophy, etc.

Idea 3: Characteristics-Based Modal Shock Sensor [1,2]

Characteristics in *k* direction

$$[\hat{\mathbf{w}}]^i = \mathbf{R}^{-1}(\bar{\mathbf{u}})\,\mathbf{u}_k, \quad i = 1, \dots, n$$

Modes of characteristics using Vandermonde matrix of Legendre polynomial

$$[\hat{\mathbf{w}}]^m = \sum_{i=1}^{n} V^p_{mi}\,[\hat{\mathbf{w}}]^i, \quad m = 1, \dots, n$$

Shock sensor

- Choose entropic and acoustic wave components based on interest

$$\epsilon_{\text{ss}}([\hat{\mathbf{w}}_\alpha]) = \log\left(\frac{[\hat{\mathbf{w}}^2_\alpha]^n}{\sum_{m=1}^{n}[\hat{\mathbf{w}}^2_\alpha]^m}\right), \quad \alpha = 1, \dots, 5$$