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On the Definition of the Prompt Neutron Lifetime

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1 Introduction

There are myriad definitions for the mean neutron lifetime, neutron generation time, and other related quantities to characterize a neutron and its progeny's propagation through an assembly. In this document, we consider lifetime definitions that are associated with eigenvalue forms of the neutron transport equation. Specifically, we focus on the two most widely used eigenvalues: the k eigenvalue and the α eigenvalue. These eigenvalues are used more often than other eigenvalues due to the physical phenomenon they capture. The k eigenvalue allows for the ability to determine if the system can sustain a chain reaction, while the α eigenvalue captures the asymptotic time dependent behavior of the system. An advantage of using these eigenvalues resides in the biorthogonality of the solutions with their adjoint counterpart. This feature allows us to simplify the expressions and to obtain appropriately weighted definitions that highlight important physics and regions of an assembly.

The earliest known definition for a neutron lifetime appears in the Frisch-Peierls memorandum [1] from 1940, in which the authors assert that the lifetime is inversely proportional to the neutron absorption rate. Serber rigorously defined the neutron lifetime as the average time between fission events in an infinite medium without parasitic capture [2], followed by a Feynman extension of this notion to the time between any absorption event for an ideal $1/v$-absorber [3]. As generalizations continued, the preciseness of the language became more imprecise from one work to the next. It was with the intent of clarifying the language of this subject that Spriggs and Busch wrote a LANL report [4] to lessen the quandarous burden. We will follow their conventions in this document.

We may first define the neutron generation time, Λ, as the average time between successive generations in a fission chain. A generation is initiated when a neutron is birthed from fission, streams through the medium, and eventually collides with a nucleus inducing another production process, thereby ending the generation and commencing a new generation via the emission of more neutrons. The generation time can then be defined as the total neutron population, N, divided by the rate of neutron production, $\mathcal{P}$,

$$\Lambda = \frac{\text{Total Population}}{\text{Production Rate}} = \frac{N}{\mathcal{P}}. \tag{1}$$

For a supercritical system, N will increase and thus $\mathcal{P}$ will also be increasing, and the opposite is true for a subcritical system; therefore, Λ is a measure of the change in the population and its consequent production rate. The neutron lifetime is defined as the average amount of time between a neutron's birth and its demise. The lifetime, ℓ, can be determined as the total population divided

by the loss rate, $\mathcal{L}$,

$$\ell = \frac{\text{Total Population}}{\text{Loss Rate}} = \frac{N}{\mathcal{L}}. \tag{2}$$

We will rigorously define N, $\mathcal{P}$, and $\mathcal{L}$ for the aforementioned eigenvalue formulations; all of which stem from the neutron transport equation (NTE). We note that these are integral quantities (i.e., a single value) that characterize the entire system or subdomain and thus are used in many settings as an essential metric. For this reason, it is crucial to understand the varying definitions, and their subtle non-negligible differences, found throughout the literature.

The goal of this document is to demonstrate several methodologies for defining the neutron generation time and lifetime. The primary idea behind the methods presented is to project the adjoint neutron flux onto the NTE. This is a well-established concept and is done so as to weight the NTE solution by its importance at given phase-space points to 'filter-out' unnecessary neutrons that are unimportant to the fission process. This process reduces the NTE to a point-kinetics equation (i.e., only a function of time), from which we may define our desired integral quantities: Λ and ℓ. The document is outlined as follows: we define the necessary forms and operators of the NTE in Sec. 2, then demonstrate the process of defining Λ and ℓ for the k-eigenvalue in Sec. 3, then perform the steps for the α-eigenvalue in Sec. 4. Finally, we compare the definitions from both methods in Sec. 5.

2 Forms of the Neutron Transport Equation

Consider a convex system of volume V with a general distributed neutron source $Q(\vec{r}, E, \hat{\Omega}, t)$. We allow for neutrons to induce fission which will produce prompt neutrons and we neglect effects of delayed neutrons. The neutron transport equation satisfied by the neutron angular flux, $\psi(\vec{r}, E, \hat{\Omega}, t)$, is then:

$$\left[\frac{1}{v(E)}\frac{\partial}{\partial t} + L + T\right]\psi(\vec{r}, E, \hat{\Omega}, t) = \Big[S + \chi(E)F\Big]\psi(\vec{r}, E', \hat{\Omega}', t) + Q(\vec{r}, E, \hat{\Omega}, t), \tag{3}$$

with the initial and boundary conditions:

$$\psi(\vec{r}, E, \hat{\Omega}, t = t_o) = \psi_x(\vec{r}, E, \hat{\Omega}), \quad \text{for } \vec{r} \in V, E \in \mathbb{R}^+, \hat{\Omega} \in \mathbb{S}^2, \tag{4a}$$

$$\psi(\vec{r}, E, \hat{\Omega}, t) = f(E, \hat{\Omega}, t), \quad \text{for } \vec{r} \in \partial V, E \in \mathbb{R}^+, \hat{n}\cdot\hat{\Omega} < 0,\ t \geq t_o \tag{4b}$$

where $f(E, \hat{\Omega}, t)$ is an arbitrary problem-specific function. In the above, we have defined the operators:

$$L\left[\psi(\vec{r}, E, \hat{\Omega}, t)\right](\vec{r}, E, \hat{\Omega}, t) = \hat{\Omega}\cdot\nabla\psi(\vec{r}, E, \hat{\Omega}, t) \tag{5a}$$

$$T\left[\psi(\vec{r}, E, \hat{\Omega}, t)\right](\vec{r}, E, \hat{\Omega}, t) = \Sigma_t(\vec{r}, E, t)\psi(\vec{r}, E, \hat{\Omega}, t) \tag{5b}$$

$$S\left[\psi(\vec{r}, E', \hat{\Omega}', t)\right](\vec{r}, E, \hat{\Omega}, t) = \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi}\Sigma_s(\vec{r}, E' \to E, \hat{\Omega}'\cdot\hat{\Omega}, t)\psi(\vec{r}, E', \hat{\Omega}', t) \tag{5c}$$

$$F\left[\psi(\vec{r}, E', \hat{\Omega}', t)\right](\vec{r}, t) = \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi}\overline{\nu}\Sigma_f(\vec{r}, E', t)\psi(\vec{r}, E', \hat{\Omega}', t), \tag{5d}$$

which are appropriately referred to as the streaming, total collision, scatter, and fission operators, respectively. The adjoint neutron transport equation, with a solution given by the adjoint neutron

flux, $\psi^\dagger(\vec{r}, E, \hat{\Omega}, t)$, is:

$$\left[-\frac{1}{v(E)}\frac{\partial}{\partial t} + L^\dagger + T^\dagger\right]\psi^\dagger(\vec{r}, E, \hat{\Omega}, t) = \left[S^\dagger + \overline{\nu}\Sigma_f(\vec{r}, E, t)F^\dagger\right]\psi^\dagger(\vec{r}, E', \hat{\Omega}', t) + Q^\dagger(\vec{r}, E, \hat{\Omega}, t), \quad (6)$$

with final and boundary conditions:

$$\psi^\dagger(\vec{r}, E, \hat{\Omega}, t = t_f) = \psi_x^\dagger(\vec{r}, E, \hat{\Omega}), \quad \text{for } \vec{r} \in V, E \in \mathbb{R}^+, \hat{\Omega} \in \mathbb{S}^2, \quad (7a)$$

$$\psi^\dagger(\vec{r}, E, \hat{\Omega}, t) = f^\dagger(E, \hat{\Omega}, t), \qquad \text{for } \vec{r} \in \partial V, E \in \mathbb{R}^+, \hat{n}\cdot\hat{\Omega} > 0,\ t \le t_f \quad (7b)$$

where $f^\dagger(E, \hat{\Omega}, t)$ is an arbitrary problem-specific function. The above adjoint operators are defined as:

$$L^\dagger\left[\psi^\dagger(\vec{r}, E, \hat{\Omega}, t)\right](\vec{r}, E, \hat{\Omega}, t) = -\hat{\Omega}\cdot\nabla\psi^\dagger(\vec{r}, E, \hat{\Omega}, t) \quad (8a)$$

$$T^\dagger\left[\psi^\dagger(\vec{r}, E, \hat{\Omega}, t)\right](\vec{r}, E, \hat{\Omega}, t) = \Sigma_t(\vec{r}, E, t)\psi^\dagger(\vec{r}, E, \hat{\Omega}, t) \quad (8b)$$

$$S^\dagger\left[\psi^\dagger(\vec{r}, E, \hat{\Omega}, t)\right](\vec{r}, E, \hat{\Omega}, t) = \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi}\Sigma_s(\vec{r}, E \to E', \hat{\Omega}\cdot\hat{\Omega}', t)\psi^\dagger(\vec{r}, E', \hat{\Omega}', t) \quad (8c)$$

$$F^\dagger\left[\psi^\dagger(\vec{r}, E, \hat{\Omega}, t)\right](\vec{r}, t) = \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi}\chi(E')\psi^\dagger(\vec{r}, E', \hat{\Omega}', t). \quad (8d)$$

In the operator definitions, Eqs. 5 and 8, we make clear the functional dependence after performing the operation, but we only do so in the definitions.

We will make use of the k- and α-eigenvalue forms of the forward and adjoint transport equation with respective fundamental-mode solutions, $\psi_x(\vec{r}, E, \hat{\Omega})$ and $\psi_x^\dagger(\vec{r}, E, \hat{\Omega})$, whose associated forward and adjoint eigenvalues are $x = \{k, \alpha\}$ and $x^\dagger = \{k^\dagger, \alpha^\dagger\}$. The forward and adjoint k eigenvalue equations are, respectively:

$$0 = \left[L + T\right]\psi_k(\vec{r}, E, \hat{\Omega}) - \left[S + \frac{1}{k}\chi(\vec{r}, E)F\right]\psi_k(\vec{r}, E', \hat{\Omega}') \quad (9a)$$

$$0 = \left[L^\dagger + T^\dagger\right]\psi_k^\dagger(\vec{r}, E, \hat{\Omega}) - \left[S^\dagger + \frac{1}{k^\dagger}\overline{\nu}\Sigma_f(\vec{r}, E)F^\dagger\right]\psi_k^\dagger(\vec{r}, E', \hat{\Omega}') \quad (9b)$$

and the forward and adjoint α-eigenvalue equations are:

$$0 = \left[\frac{\alpha}{v(E)} + L + T\right]\psi_\alpha(\vec{r}, E, \hat{\Omega}) - \left[S + \chi(\vec{r}, E)F\right]\psi_\alpha(\vec{r}, E', \hat{\Omega}') \quad (10a)$$

$$0 = \left[\frac{\alpha^\dagger}{v(E)} + L^\dagger + T^\dagger\right]\psi_\alpha^\dagger(\vec{r}, E, \hat{\Omega}) - \left[S^\dagger + \overline{\nu}\Sigma_f(\vec{r}, E)F^\dagger\right]\psi_\alpha^\dagger(\vec{r}, E', \hat{\Omega}'). \quad (10b)$$

We note the time derivative for the $k, k^\dagger$ case disappears because we assume the system is in a steady-state configuration and the fission production term can be scaled such that $\overline{\nu} \to \overline{\nu}/k$, balancing the losses with the gains. For the $\alpha, \alpha^\dagger$ case, we assume separability of the time-dependence of the solution $\psi, \psi^\dagger$ and we insert the expressions, $\psi(\vec{r}, E, \hat{\Omega}, t) = \psi_\alpha(\vec{r}, E, \hat{\Omega})\,\mathrm{e}^{+\alpha t}$ and $\psi^\dagger(\vec{r}, E, \hat{\Omega}, t) = \psi_\alpha^\dagger(\vec{r}, E, \hat{\Omega})\,\mathrm{e}^{-\alpha^\dagger t}$, into Eqs. 3 and 6, respectively[1]. We note that, for a given eigenvalue in the

[1]From the expression $\psi^\dagger(\vec{r}, E, \hat{\Omega}, t) = \psi_\alpha^\dagger(\vec{r}, E, \hat{\Omega})\,\mathrm{e}^{-\alpha^\dagger t}$, we see why the adjoint time derivative becomes positive in Eq. 10b. The importance in a supercritical system decreases with time while it increases with time in a subcritical system [5]. This is consistent with the definition of neutron importance: in a supercritical system, an earlier neutron has more importance because it has additional time to multiply and contribute more while a neutron in a subcritical system will become more important if it exists closer to the time of observation as it will more likely vanish for earlier times.

spectrum, the following holds: $k_i = k_i^\dagger$ [6] and also $\mathrm{Re}\,(\alpha_i) = \mathrm{Re}(\alpha_i^\dagger)$ [7] (in fact, the entire α spectrum are simply complex conjugates of one another) and thus we will drop the adjoint notation when possible.

3 Definitions from the k-Eigenvalues

3.1 Terminology

The k-eigenvalue is equivalent to the commonly used *effective multiplication factor*, i.e. $k = k_{\mathrm{eff}}$, but we omit the subscript for brevity. Just as the generation time and lifetime are ambiguously defined over several sources, the term "multiplication factor" is as well, and the reader must be alerted to each author's definition. Fermi originally used the term *neutron reproduction number* or *neutron reproduction factor* for k to symbolize the ratio between either the number of neutrons in two successive fission generations or the neutron production and loss rates [8, 9], which we write here as:

$$k = k_{\mathrm{eff}} = \frac{\text{Neutrons in generation } i}{\text{Neutrons in generation } i-1} = \frac{\mathcal{P}}{\mathcal{L}}. \tag{11}$$

Other sources follow Fermi's nomenclature and exclusively use "reproduction factor" for k [11, 14]. The reason Keepin [14] gives for not using "multiplication factor" for k is to distinguish it from the quantity of *neutron multiplication in a subcritical assembly* given by $M = 1/(1-k)$, which is the formula for the converged geometric series (i.e. $k+k^2+k^3+\cdots$, where the n^{th} term in the series gives the contribution to the neutron population from the n^{th} generation). This expression was originally given by Serber [10] where $k < 1$ is required for the formula to be applicable whence a geometric series converges. Another source, Glasstone and Edlund [15], mention the terms "reproduction factor" and "multiplication factor" as having no distinction while many others do not even use the term "reproduction factor" and only use "multiplication factor" [16]. Lewis and Miller [6] refer to k as the "multiplication eigenvalue" while Duderstadt and Hamilton [17] additionally use the term "criticality eigenvalue", while Cacuci *et al.* [18] go so far as to specify k as the "effective fission multiplication factor". We also note that Henry [13] refers to the k-eigenvalue as the λ-eigenvalue, but still uses the terminology "effective multiplication factor". In this paper we shall call k_{eff} the neutron reproduction factor or the k-eigenvalue. Using these names for k_{eff} minimizes any confusion with the multiplication factor definition used by Serber.

3.2 To Define the Lifetime

A method to derive a lifetime formula is outlined by Hetrick [11] where the motivation is to reduce the phase-space neutron transport equation from a 7-dimensional integro-differential equation to a purely time-dependent kinetics equation for the neutron flux. We wish to follow this reduction process to outline the algorithm for obtaining different integral nuclear quantities related to the neutron lifetime.

We begin by assuming the neutron flux and its steady-state adjoint counterpart satisfy Eqs. 3 and, in this case, Eq. 9b and together satisfy the boundary condition [12]:

$$\int \mathrm{d}\vec{r} \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi}\, \hat{\Omega}\cdot\nabla \left[\psi_k^\dagger(\vec{r},E,\hat{\Omega})\psi(\vec{r},E,\hat{\Omega},t)\right] = 0, \tag{12}$$

for all $t \geq t_o$. Next we use the above to reduce the transport equation to a kinetics equation. To do so, we first assume that the cross sections appearing in the adjoint equation, Eq. 9b, are the

same cross sections appearing in Eq. 3 (i.e., they are time-dependent and the system appropriately possesses ψ and $\psi_k^\dagger$ as solutions). Next, we multiply Eq. 3 by $\psi_k^\dagger(\vec{r}, E, \hat{\Omega})$ and integrate over all $\vec{r}, E$, and $\hat{\Omega}/4\pi$ and simultaneously multiply Eq. 9b by $\psi(\vec{r}, E, \hat{\Omega}, t)$ and integrate over $\vec{r}, E$, and $\hat{\Omega}/4\pi$, followed by subtracting one from the other. For brevity, we define the differential phase-space element: $\mathrm{d}\vec{p} = \mathrm{d}\vec{r}\,\mathrm{d}E\,\mathrm{d}\Omega/4\pi$, such that

$$\int \mathrm{d}\vec{p} = \int_V \mathrm{d}\vec{r} \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi}, \tag{13}$$

and we find:

$$\begin{aligned}\int \mathrm{d}\vec{p} \frac{\psi_k^\dagger}{v} \frac{\partial \psi}{\partial t} + \int \mathrm{d}\vec{p}\, \psi_k^\dagger (L+T)\psi - \int \mathrm{d}\vec{p}\, \psi (L^\dagger + T^\dagger)\psi_k^\dagger = \int \mathrm{d}\vec{p}\, \psi_k^\dagger S\psi + \int \mathrm{d}\vec{p}\, \psi_k^\dagger \chi F\psi \\ + \int \mathrm{d}\vec{p}\, \psi_k^\dagger Q - \int \mathrm{d}\vec{p}\, \psi S^\dagger \psi_k^\dagger - \frac{1}{k}\int \mathrm{d}\vec{p}\, \psi \overline{\nu}\Sigma_f F^\dagger \psi_k^\dagger.\end{aligned} \tag{14}$$

We can simplify this equation by analyzing some of the terms on the left-hand side. Clearly, the total collision operators $T = T^\dagger$ and thus $\int \mathrm{d}\vec{p}\, \psi_k^\dagger T\psi = \int \mathrm{d}\vec{p}\, \psi T^\dagger \psi_k^\dagger$, which cancel each other. Next, we have the streaming operators:

$$\begin{aligned}\int \mathrm{d}\vec{p} \left[\psi_k^\dagger L\psi - \psi L^\dagger \psi_k^\dagger\right] &= \int \mathrm{d}\vec{p} \left[\psi_k^\dagger \left(\hat{\Omega}\cdot\nabla\psi\right) - \psi\left(-\hat{\Omega}\cdot\nabla\psi_k^\dagger\right)\right] \\ &= \int \mathrm{d}\vec{p}\, \hat{\Omega}\cdot\left[\psi_k^\dagger\nabla\psi + \psi\nabla\psi_k^\dagger\right] \\ &= \int \mathrm{d}\vec{p}\, \hat{\Omega}\cdot\nabla\left[\psi_k^\dagger\psi\right] \\ &= 0\end{aligned} \tag{15}$$

where we have used Eq. 12 for the last step in the above. A similar cancellation occurs for the scattering terms and Eq. 14 reduces to:

$$\int \mathrm{d}\vec{p} \frac{\psi_k^\dagger}{v} \frac{\partial \psi}{\partial t} = \int \mathrm{d}\vec{p}\, \psi_k^\dagger \chi F\psi - \frac{1}{k}\int \mathrm{d}\vec{p}\, \psi \overline{\nu}\Sigma_f F^\dagger \psi_k^\dagger + \int \mathrm{d}\vec{p}\, \psi_k^\dagger Q. \tag{16}$$

The reader is referred to Appendix A.1 for details on the scattering term cancellation.

We may further simplify Eq. 16 by first recognizing:

$$\int \mathrm{d}\vec{p}\, \psi_k^\dagger \chi F\psi = \int_V \mathrm{d}\vec{r} \left(F^\dagger \psi_k^\dagger\right)(F\psi) \tag{17a}$$

$$\int \mathrm{d}\vec{p}\, \psi \overline{\nu}\Sigma_f F^\dagger \psi_k^\dagger = \int_V \mathrm{d}\vec{r} \left(F^\dagger \psi_k^\dagger\right)(F\psi) \tag{17b}$$

to find:

$$\frac{\partial}{\partial t}\int \mathrm{d}\vec{p} \frac{\psi_k^\dagger \psi}{v} = \frac{k-1}{k}\int_V \mathrm{d}\vec{r} \left[F^\dagger \psi_k^\dagger\right] F\psi + \int \mathrm{d}\vec{p}\, \psi_k^\dagger Q. \tag{18}$$

We note that another way to arrive at Eq. 18 would be to multiply Eq. 3 by $\psi_k^\dagger$ and integrate over $\vec{p}$. We may then relate the integrated terms to the definition of the adjoint operator, i.e. $\int \mathrm{d}\vec{p}\, \psi_k^\dagger L\psi = \int \mathrm{d}\vec{p}\, \psi L^\dagger \psi_k^\dagger = 0$ and similarly for all other non-eigenvalue-scaled terms, to demonstrate they cancel out by default.

We next introduce a formal identity:

$$\psi(\vec{p},t) = n(t)\phi(\vec{p},t), \tag{19}$$

where we see that both n, the amplitude function, and ϕ, the shape function, have a dependence on time and therefore this is not an assumption on the functional separability of the flux, per se. It is assumed that ϕ is weakly time-dependent and therefore slowly varies in time. This assumption applies to most stationary systems. Inserting this identity into Eq. 18 and solving for $\mathrm{d}n/\,\mathrm{d}t$ yields:

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \left(\frac{k-1}{k}\right)\left[\frac{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger\right]F\phi}{\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi}\right]n(t) + \frac{\int_V \mathrm{d}\vec{p}\,\psi_k^\dagger Q}{\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi} - \frac{\frac{\partial}{\partial t}\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi}{\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi}n(t). \tag{20}$$

We are now prepared to define the following integral nuclear quantities:

$$\frac{\rho}{\Lambda}(t) = \left(\frac{k-1}{k}\right)\frac{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger(\vec{p})\right]F\phi(\vec{p},t)}{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\phi(\vec{p},t)} \tag{21a}$$

$$q(t) = \frac{\int_V \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})Q(\vec{p},t)}{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\phi(\vec{p},t)} \tag{21b}$$

$$0 = \frac{\frac{\partial}{\partial t}\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi}{\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi} = \frac{\partial}{\partial t}\ln\int \mathrm{d}\vec{p}\,\psi_k^\dagger\frac{1}{v}\phi \tag{21c}$$

where we have also required the last term to vanish as a means of maintaining consistency with the typical point-kinetics equation in the absence of delayed neutrons:

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \frac{\rho}{\Lambda}(t)n(t) + q(t). \tag{22}$$

Here we note that Eq. 21a does not provide an explicit formula for the neutron generation time, Λ, nor the reactivity index, ρ. It is argued by Henry [11, 13] that these definitions are arbitrary and need not even be separated (i.e., we could call $\frac{\rho}{\Lambda}(t) = y(t)$). Keepin [14] suggests specifically solving for the reactivity index $\rho = (k-1)/k$ and using a perturbation-theory approximation, but this in general would remove meaning from the more fundamental definitions we have above. We outline Keepin's Method below to formally extract a definition of the generation time.

Keepin performs the same process as above up until Eq. 20, where he instead suggests calculating the reactivity index $(k-1)/k$ separately using a result of a first-order perturbation analysis:

$$\rho = \frac{k-1}{k} \equiv -\frac{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\phi(\vec{p},t)\overline{\delta\Sigma}}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger\right]F\phi} \tag{23}$$

where

$$\overline{\delta\Sigma} = \delta\Sigma_t - \overline{\delta\Sigma_s} - \overline{\chi(E)}\overline{\nu}\delta\Sigma_f. \tag{24}$$

Here, we have defined $\delta\Sigma_x = \Sigma_x - \Sigma_{x,o}$ as being the difference between the cross-sections of a perturbed system (with Σ_x cross-sections) and an unperturbed, steady-state system (with $\Sigma_{x,o}$ cross-sections). We have further defined:

$$\overline{f(E)} = \frac{1}{\psi_k^\dagger(\vec{p})}\int_0^\infty \mathrm{d}E' \int_{4\pi}\frac{\mathrm{d}\Omega}{4\pi}\psi_k^\dagger(\vec{r},E',\hat{\Omega}')f(E',\hat{\Omega}') \tag{25}$$

Note that Hetrick also presents the suggestion (citing Henry and Keepin, amongst others) of keeping $\frac{\rho}{\Lambda}(t)$ as one quantity and therefore Eq. 21a should be viewed as a fundamental nuclear quantity. With that in mind, $\rho(t)$ and $\Lambda(t)$ should only be calculated separately for practical purposes.

In proceeding, we note some fundamental quantities that are present in Eq. 20. First, we recognize that the neutron population about an element of phase space is given by $\frac{1}{v(E)}\psi(\vec{p},t)\,\mathrm{d}\vec{p}$ and thus the total neutron population in the phase space domain is given by $\int \mathrm{d}\vec{p}\,\frac{1}{v(E)}\psi(\vec{p},t)/v(E)$. Thus, the quantity appearing in Eq. 20 as $\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\psi(\vec{p},t)$ is referred to as the adjoint-weighted neutron population, N_k:

$$N_k(t) = \int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\psi(\vec{p},t), \tag{26}$$

where we append a k subscript to signify that it is weighted by the adjoint k-eigenvalue solution.

Similarly, we may assign meaning to the other integral quantities. We may define the fission neutron production rate about a point in phase space as $\overline{\nu}\Sigma_f(\vec{r},E,t)\psi(\vec{p},t)\,\mathrm{d}\vec{p}$ and therefore over the entire phase space domain we have the total neutron fission production rate: $\int \mathrm{d}\vec{p}\,\overline{\nu}\Sigma_f(\vec{r},E,t)\psi(\vec{p},t) = \int_V \mathrm{d}\vec{r}\,F\psi$. Also, note that the quantity $F^\dagger\psi^\dagger$ can be interpreted as the adjoint fission emission rate (see Eq. 8d), i.e. $F^\dagger\psi^\dagger$ informs of the importance of a neutron appearing at the point $\vec{r}$ with energy E following a fission event. Comparing these with the integral appearing in the numerator of the first term of Eq. 20, we see the natural emergence of the total adjoint-emission-rate-weighted fission neutron production rate:

$$\mathcal{P}_k(t) = \int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger(\vec{p})\right]F\psi(\vec{p},t), \tag{27}$$

where again the subscript k denotes the weighting functions are of the k-eigenvalue. For completion, we note that Eq. 21b needs no alteration or "rethinking" as it is simply the adjoint-weighted total source normalized to the neutron population at a given time.

Noting the heuristic definition of the generation time given by Eq. 1, we take the ratio of Eq. 26 to Eq. 27 to find Keepin's expression:

$$\Lambda_k(t) = \frac{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\phi(\vec{p},t)}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger(\vec{p})\right]F\phi(\vec{p},t)}, \tag{28}$$

where the k subscript signifies the weighting function type. Thus, it immediately follows that the point-kinetics equation is:

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \frac{1}{\Lambda_k(t)}\left[\frac{k-1}{k}\right]n(t) + q(t). \tag{29}$$

The prompt neutron lifetime does not explicitly present itself in Eq. 20, nor its reduction given by Eq. 29 above. We may construct an expression using the heuristic definition of Eq. 2 and following the same logic we used in defining Eqs. 26 and 27 to define the loss rate $\mathcal{L}$. To do so, we first note that the loss mechanisms, about a given point in phase space, present in the NTE (Eq. 3) are the streaming operator and a portion of the total collision operator. To clarify, scattering interactions are not necessarily a complete loss mechanism[2] and we must therefore stratify the total collision operator as: $\Sigma_t(\vec{r},E,t) = \Sigma_a(\vec{r},E,t) + \Sigma_s(\vec{r},E,t)$, where $\Sigma_a = \Sigma_c + \Sigma_f$ is the macroscopic absorption cross section. Thus, the loss rate about $\vec{p}$ can be inferred as $[L + \Sigma_a(\vec{r},E,t)]\psi(\vec{p},t)\,\mathrm{d}\vec{p}$ and the total loss rate in the entire system is then $\int \mathrm{d}\vec{p}[L + \Sigma_a(\vec{r},E,t)]\psi(\vec{p},t)$. It is at this point that interpretations become less rigorous in defining the weighted loss rate, $\mathcal{L}$, because it does not

[2]If one were interested in determining the neutron loss rate from the specific energy E, then we would keep the $\Sigma_s(E)$ term.

present itself in the kinetics equation derivation. Thus, we may define $\mathcal{L}$ however we please, and we could simply insert a weighting function, $w(\vec{p})$, of our choice:

$$\mathcal{L}(t) = \int \mathrm{d}\vec{p}\, w(\vec{p})[L + \Sigma_a(\vec{r}, E, t)]\psi(\vec{p}, t), \tag{30}$$

where w can be set to unity, or the adjoint flux $\psi^\dagger(\vec{p})$, or the adjoint k-flux $\psi_k^\dagger(\vec{p})$, or the adjoint loss rate $[L^\dagger + \Sigma_a]\psi_k^\dagger$, and so on. Note that we will only append the k subscript to $\mathcal{L}$ if $w = f(\psi_k^\dagger)$. From this, we may define the neutron lifetime as:

$$\ell_{(k,w)}(t) = \frac{\int \mathrm{d}\vec{p}\, \psi_k^\dagger(\vec{p}) \frac{1}{v(E)} \phi(\vec{p}, t)}{\int \mathrm{d}\vec{p}\, w(\vec{p})[L + \Sigma_a(\vec{r}, E, t)]\phi(\vec{p}, t)}, \tag{31}$$

where we are signifying the weighting function type in the numerator and denominator. Alternatively, Keepin observes that the k-eigenvalue can be thought of as the ratio of the production rate to the loss rate (i.e., $k = \mathcal{P}/\mathcal{L}$) and therefore the definition is posed:

$$\ell_k(t) = k\Lambda_k(t), \tag{32}$$

which implies that the production rates cancel out from k and Λ_k, leaving the ratio of N_k to $\mathcal{L}$. Also, note that k is obtained by solving either form of Eq. 9 (we have presumably solved Eq. 9b for $\psi_k^\dagger$ and therefore have $k^\dagger$ available). We will present a more in-depth discussion on the differences between Eqs. 31 and 32 in Sec. 5.

The full-fidelity calculation of $\Lambda_k(t)$ proceeds by solving the adjoint k-eigenvalue equation (Eq. 9b) to obtain k and $\psi_k^\dagger(\vec{p})$. Obtaining $\phi(\vec{p}, t)$ requires solving the forward time-dependent transport equation, Eq. 3, for $\psi(\vec{p}, t)$ as well as the kinetics equation, Eq. 29 for $n(t)$; from which we may find $\phi(\vec{p}, t) = \frac{\psi(\vec{p}, t)}{n(t)}$ from Eq. 19. Herein lies the issue: the kinetics equation requires $\Lambda(t)$, which requires $\phi(\vec{p}, t)$, which requires $n(t)$. Thus, an iteration on these equations is required to converge on a value of $\Lambda_k(t)$ (note that $\psi(\vec{p}, t)$ does not need to be iterated on in this scheme). As a first approximation for $\Lambda_k(t)$ (either to simply make a rough calculation or to initiate the iteration process), one may assume $\phi(\vec{p}, t) \approx \psi_k(\vec{p})$, the forward k-eigenfunction. Depending on the system, this may be an acceptable approximation because we implicitly assumed $n(t)$ contains most of the time-dependent information of the system and therefore $\phi(\vec{p}, t)$ is slowly varying in time (as evidenced by the assumption given by Eq. 21c).

We summarize the process that both Hetrick and Keepin perform in a series of steps, where the forefront of the process is the same:

1. Define neutron transport equation (NTE) with solution $\psi(\vec{p}, t)$ and identify adjoint eigenvalue neutron transport equation (AENTE) with the fundamental eigenfunction solution $\psi_k^\dagger(\vec{p})$ with the associated k-eigenvalue.
2. Multiply NTE by $\psi_k^\dagger(\vec{p})$ and integrate: $\int \mathrm{d}\vec{p}\,\big[\psi_k^\dagger(\vec{p}) \times \mathrm{NTE}\big]$.
3. Multiply AENTE by $\psi(\vec{p}, t)$ and integrate: $\int \mathrm{d}\vec{p}\,\big[\psi(\vec{p}, t) \times \mathrm{AENTE}\big]$.
4. Assume the cross sections of the AENTE are the same as those of the NTE (allow them to be time-dependent).
5. Subtract Step 3 from Step 2 and use boundary condition given by Eq. 12 to simplify.
6. Assume the solution has the form: $\psi(\vec{p}, t) = n(t)\phi(\vec{p}, t)$.

7. Enforce Eq. 21c.

8. Isolate $\mathrm{d}n(t)/\,\mathrm{d}t$ and accordingly define coefficients.

In Step 1, it is assumed that a perturbed nuclear system (satisfying the NTE) may be hedged against an unperturbed system whose solution is regarded as the importance of a neutron with phase-space coordinate $\vec{p}$ (the AENTE). Following through the steps and arriving at a kinetics equation, the coefficients are then regarded as the adjoint-weighted (i.e., importance-weighted) integral nuclear quantities normalized by the integrated statistical weight. The integrated statistical weight is the quantity $\int \mathrm{d}\vec{p}\,\psi(\vec{p},t)\psi_k^\dagger(\vec{p})/v(E)$. It is in Step 8 that Keepin and Hetrick differ in their derivations.

In the derivation of these kinetic parameters, the perturbation of the nuclear system limits the quantitative validity of the parameters to a small region around $k = 1$. Far from critical ($k = 1$), the perturbation approximations are no longer valid and it is not clear what the quantitative evaluation of the k-eigenvalue actually means. The magnitude of the neutron reproduction factor, k-eigenvalue, better represents the capability for the system to sustain a chain reaction and the degree to which it deviates from the critical state rather than represents an actual physical parameter. As an example, a neutron reproduction factor of 0.8 does not mean the neutron population decreases by 20% from one generation to the next– that is what the multiplication factor k means in the expression for the neutron multiplication in a subcritical assembly. A k-eigenvalue of 0.8 is further from critical than an assessment of 0.98. Likewise, a neutron reproduction factor of 1.2 is not increasing the neutron population by 20% in each generation. The k-eigenvalue of 1.4 is more supercritical than the k-eigenvalue of 1.01. It can be stated that the only neutron reproduction factor that has physical significance is $k = 1$. All other k-eigenvalue numbers are just relative distances from the "just critical" state and therefore give a relative quantification of how well the system can sustain a chain reaction.

4 Definitions from the α-Eigenvalues

We now wish to derive integral nuclear quantities in terms of the α-eigenvalue and eigenfunctions. For characterizing time-evolving systems, the α-eigenvalue is considered superior to the k-eigenvalue because the α-eigenvalue equation contains a remnant of the time derivative, $\frac{\alpha^\dagger}{v(E)}$, whereas the k-eigenvalue equation simply ignores the time derivative (compare Eqs. 9b and 10b) by assuming there is a scaling factor k of the fission source (traditionally the scaling $\overline{\nu} \to \frac{\overline{\nu}}{k}$ is done) which would make the system steady-state. Conversely, the k-eigenvalue is preferred over the α-eigenvalue when determining if a system can sustain a chain reaction.

4.1 Terminology

Fortunately, the α-eigenvalue is far less plagued by ambiguity within the literature. We point out that there are, however, a few naming conventions for the quantity, such as Bell and Glasstone [5] referring to α as the "multiplication rate eigenvalue", while Cacuci *et al.* [18] uses the term "asymptotic inverse reactor period", and Duderstadt and Hamilton [17] call it the "time absorption eigenvalue". The α-eigenvalue is also referred to as the "static α" to distinguish it from the "dynamic α", α_{dyn}, which is computed by solving the NTE, Eq. 3, for $N(t)$ and using the formula: $\alpha_{\mathrm{dyn}}(t) = \frac{1}{N(t)}\frac{\mathrm{d}N}{\mathrm{d}t}$. The α-eigenvalue for the "static α" represents the asymptotic time behavior of the neutrons for a static geometry, while the dynamic α expression represents the relative rate at which the neutron population is changing at an instance of time.

A common assumption is to equate the static α to $(k-1)/\ell_k$ (see Eq. 29 and the to-be-derived Eq. 35) without further thought. The k and ℓ_k of Eq. 29 are obtained by solving Eq. 9 for k and performing the required integrals of Eq. 28 to apply Eq. 32. These quantities are all obtained in terms of the k-eigenvalue formulation very near to exact-critical and are in no way related to solutions to the α-eigenvalue equation, Eq. 10, except for the special case of a system at exactly critical (and therefore the eigenvalue equations coalesce). This approximation quickly becomes invalid as the system departs from critical. We will find a similar-in-appearance formula that may be used with this nuance and distinction in mind (see Eq. 44, rearranged).

4.2 Applying Hetrick's Algorithm

Following the process outlined in Sec. 3, we may also reduce the transport equation to a point-kinetics equation using the α-eigenvalue equation. If we then multiply Eq. 3 by $\psi^\dagger_\alpha(\vec{p})$ and integrate over $\vec{p}$, simultaneously multiply Eq. 10b by $\psi(\vec{p},t)$ and integrate over $\vec{p}$, then subtract the two equations from each other, we obtain:

$$\int \mathrm{d}\vec{p}\frac{\psi^\dagger_\alpha}{v}\frac{\partial\psi}{\partial t} + \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha(L+T)\psi - \alpha\int \mathrm{d}\vec{p}\frac{\psi\psi^\dagger_\alpha}{v} - \int \mathrm{d}\vec{p}\,\psi(L^\dagger+T^\dagger)\psi^\dagger_\alpha = \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha S\psi + \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha \chi F\psi + \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha Q - \int \mathrm{d}\vec{p}\,\psi S^\dagger\psi^\dagger_\alpha - \int \mathrm{d}\vec{p}\,\psi\overline{\nu}\Sigma_f F^\dagger\psi^\dagger_\alpha. \tag{33}$$

Enforcing the boundary condition:

$$\int \mathrm{d}\vec{p}\,\hat{\Omega}\cdot\nabla\left[\psi^\dagger_\alpha(\vec{p})\psi(\vec{p},t)\right] = 0,$$

the LHS terms containing the streaming operators cancel (see Eq. 15 for the k analog). Furthermore, the terms containing the collision, scattering, and fission operators cancel out, simplifying Eq. 33 to:

$$\int \mathrm{d}\vec{p}\frac{\psi^\dagger_\alpha}{v}\frac{\partial\psi}{\partial t} - \alpha\int \mathrm{d}\vec{p}\frac{\psi\psi^\dagger_\alpha}{v} = \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha Q. \tag{34}$$

Next we make the assumption that $\psi(\vec{p},t) = n(t)\phi(\vec{p},t)$, insert into the above and enforce Eq. 21c (except we replace k with α) to yield:

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \alpha n(t) + q(t), \tag{35}$$

where $q(t)$ is appropriately defined by Eq. 21b.

As can be seen, there was no convenient emergence of some integral quantities that we could assign physical definitions, such as a lifetime or generation time. This is due simply to the fact that we are projecting the forward and adjoint transport equations onto each other and the only term that is scaled by an eigenvalue is the time derivative. For this reason, the projections (and subsequent subtraction) cancel out all non-scaled homogeneous terms of the equations and we are left with the artifacts of the time-derivative and the inhomogeneous source term in Eq. 35. In the next section, we demonstrate a process for which α-weighted physical quantities may be formally derived.

4.3 To Define the Lifetime

An alternative novel method for arriving at an equation that allows us to define integral quantities, specifically the neutron generation time, is now demonstrated. We begin with the forward transport equation with constant-in-time cross sections and assume the neutron flux is phase space-time separable and behaves exponentially in time: $\psi(\vec{p},t) = \psi_\alpha(\vec{p})\,\mathrm{e}^{\alpha t}$. This ansatz is a statement that the system has reached *asymptotic behavior*. A system is said to be behaving asymptotically when the static and dynamic components are well within agreement. For the α-eigenvalue problem, we would determine a convergence to asymptotic behavior whence the static and dynamic α values agree to within some prescribed threshold, i.e., $\alpha \sim \alpha_{\mathrm{dyn}}$; this will be discussed further in Sec. 5. Inserting this ansatz into Eq. 3 gives the static forward α-eigenvalue transport equation given by Eq. 10a and written here for convenience:

$$\left[\frac{\alpha}{v(E)} + H\right]\psi_\alpha(\vec{p}) = \chi(E)F\psi_\alpha, \tag{36}$$

where we have defined the effective removal operator:

$$H\left[\psi_\alpha(\vec{p})\right](\vec{p}) = \Big[L + T - S\Big]\psi_\alpha(\vec{p}). \tag{37}$$

Next, we multiply Eq. 36 by the adjoint α-eigenfunction, $\psi^\dagger_\alpha(\vec{p})$, and integrate over all phase-space:

$$\begin{aligned}\int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha(\vec{p})\left[\frac{\alpha}{v(E)} + H\right]\psi_\alpha(\vec{p}) &= \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha\chi(E)F\psi_\alpha \\ &= \int_V \mathrm{d}\vec{r}\left[F^\dagger\psi^\dagger_\alpha\right]F\psi_\alpha,\end{aligned} \tag{38}$$

where we invoked the α-analog of Eq. 17a in going from the first to the second line of the above.

If we divide Eq. 38 by $\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi^\dagger_\alpha\right]F\psi_\alpha$, we find:

$$\alpha\frac{\int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi^\dagger_\alpha\right]F\psi_\alpha} + \frac{\int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha(\vec{p})H\psi_\alpha(\vec{p})}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi^\dagger_\alpha\right]F\psi_\alpha} = 1. \tag{39}$$

We now have an equation in terms of integral nuclear quantities, which we may assign meaning to. First, we note that the adjoint-weighted neutron population is given by:

$$N_\alpha = \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p}), \tag{40}$$

which is the α-equivalent to Eq. 26 (but we now have $\psi_\alpha(\vec{p})$ instead of $\psi(\vec{p},t)$), and the adjoint-fission-emission-weighted fission neutron production rate is given by:

$$\mathcal{P}_\alpha = \int_V \mathrm{d}\vec{r}\,\left[F^\dagger\psi^\dagger_\alpha(\vec{p})\right]F\psi_\alpha(\vec{p}). \tag{41}$$

Finally, we may define an adjoint-weighted total loss rate as

$$\mathcal{L}_\alpha = \int \mathrm{d}\vec{p}\,\psi^\dagger_\alpha(\vec{p})H\psi_\alpha(\vec{p}). \tag{42}$$

From this, we can define the generation time and the instantaneous neutron reproduction factor respectively as:

$$\Lambda_\alpha = \frac{N_\alpha}{\mathcal{P}_\alpha} = \frac{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})}{\int_V \mathrm{d}\vec{r}\,\left[F^\dagger\psi_\alpha^\dagger(\vec{p})\right]F\psi_\alpha(\vec{p})} \tag{43a}$$

$$k_{\text{inst}} = \frac{\mathcal{P}_\alpha}{\mathcal{L}_\alpha} = \frac{\int_V \mathrm{d}\vec{r}\,\left[F^\dagger\psi_\alpha^\dagger(\vec{p})\right]F\psi_\alpha(\vec{p})}{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})H\psi_\alpha(\vec{p})}, \tag{43b}$$

and Eq. 39 reduces to:

$$\alpha\Lambda_\alpha + \frac{1}{k_{\text{inst}}} = 1. \tag{44}$$

Another unfortunate use of legacy notation is used here; the reader is cautioned not to confuse the instantaneous reproduction, k_{inst}, with k_{eff} (the k-eigenvalue), which is obtained by solving Eq. 9a or 9b. The obvious main difference is that we are using α-eigenfunctions to define the production and loss rates and we will make clear some other distinctions in Sec. 5.

Next, by defining the α-importance reactivity as:

$$\varrho = 1 - \frac{1}{k_{\text{inst}}}, \tag{45}$$

we may eliminate k_{inst} and find a relationship between the adjoint-weighted α eigenvalue quantities of interest:

$$\varrho = \alpha\Lambda_\alpha. \tag{46}$$

This method provides a definition of the mean generation time that the Keepin method of Sec. 3 could not provide using the α-eigenvalues, as evidenced at the beginning of this section. The most restricting issue with these quantities is that they assume the system has reached an asymptotic state and therefore none of the quantities are time-dependent. One option is to solve the forward and adjoint α-eigenvalue equations for a given time step of an evolving system, and from there one may construct the asymptotic generation time temporal profile for all time steps of interest. In a sense, then, the quantities are parameterized by time and are a direct function of the cross sections at an instance in time. We emphasize again that the forward and adjoint calculations provide an asymptotic solution to a system at a given time; thus, if the system is quickly evolving, the asymptotic solution between time steps may be non-negligibly different and the above calculations may provide inaccurate results. This is likely to have a dramatic effect on solutions for systems near critical for which the relative magnitude of α and ψ_α differ significantly for slight deviations. Finally, we note that the defined generation times of the Keepin method, Eq. 28, and this method, Eq. 43a, are quite similar (barring the explicit time-dependence) and simply have different weighting functions.

The neutron lifetime may now be calculated as $\ell_\alpha = \frac{N_\alpha}{\mathcal{L}_\alpha} = k_{\text{inst}}\Lambda_\alpha$, computed by determining k_{inst} and Λ_α, or directly as:

$$\ell_\alpha = \frac{N_\alpha}{\mathcal{L}_\alpha} = \frac{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})}{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})H\psi_\alpha(\vec{p})}. \tag{47}$$

In order to avoid the calculation of the denominator due to the streaming and scattering integrals, we can use Eq. 38 to replace the denominator with:

$$\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})H\psi_\alpha(\vec{p}) = \int_V \mathrm{d}\vec{r}F^\dagger\psi_\alpha^\dagger F\psi_\alpha - \alpha\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger\frac{1}{v}\psi_\alpha. \tag{48}$$

This allows us to rewrite k_{inst} and ℓ as:

$$k_{\text{inst}} = \frac{1}{1-\alpha\Lambda_\alpha} = \frac{1}{1-\varrho} \tag{49a}$$

$$\ell_\alpha = \frac{\Lambda_\alpha}{1-\alpha\Lambda_\alpha} = \frac{\Lambda_\alpha}{1-\varrho} \tag{49b}$$

where, of course, Eq. 49a is simply a rearranging of Eq. 44.

5 Comparing Definitions

We now compare the different definitions of the generation time and lifetime from Secs. 3 and 4 for the k-eigenvalue method and the α-eigenvalue method. First, let us note some general definitions: the generation time is given by the total neutron population divided by the fission neutron production rate (see Eq. 1); the neutron lifetime is the total neutron population divided by the total loss rate (see Eq. 2); and the reproduction factor is the ratio of the production rate to the loss rate. We may also add a physical interpretation to α by rearranging Eq. 44, $\alpha = \frac{k_{\text{inst}}-1}{k_{\text{inst}}\Lambda_\alpha}$ as the reactivity insertion per generation or, when written as $\alpha = \frac{k_{\text{inst}}-1}{\ell_\alpha}$, as the excess multiplication introduced per neutron lifetime.

With these definitions in mind, let us state Keepin's k-method definitions:

$$\Lambda_k(t) = \frac{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\phi(\vec{p},t)}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger(\vec{p})\right]F\phi(\vec{p},t)} \tag{50a}$$

$$\begin{aligned}\ell_k(t) &= k\frac{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\phi(\vec{p},t)}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_k^\dagger(\vec{p})\right]F\phi(\vec{p},t)} \\ &\equiv k\Lambda_k(t)\end{aligned} \tag{50b}$$

and the α-method definitions:

$$\Lambda_\alpha = \frac{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_\alpha^\dagger(\vec{p})\right]F\psi_\alpha(\vec{p})} \tag{51a}$$

$$\begin{aligned}\ell_\alpha &= \frac{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})}{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})H\psi_\alpha(\vec{p})} \\ &= \frac{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})}{\int_V \mathrm{d}\vec{r}\left[F^\dagger\psi_\alpha^\dagger(\vec{p})\right]F\psi_\alpha(\vec{p}) - \alpha\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})\frac{1}{v(E)}\psi_\alpha(\vec{p})} \\ &\equiv k_{\text{inst}}\Lambda_\alpha.\end{aligned} \tag{51b}$$

Comparing the generation times, Eqs. 50a and 51a, we see that the largest distinction is the explicit time-dependence on $\Lambda_k(t)$ while Λ_α is more-or-less only parametrized by time (if one were to keep time-dependence on the cross sections). The k-method allows one to keep explicit time-dependence because it is an instantaneous quantity whereas the α-method has assumed the solution has settled into an asymptotic fundamental mode and obeys: $\psi(\vec{p},t) = \psi_\alpha(\vec{p})\,\mathrm{e}^{\alpha t}$. One may assess whether

the assumption that the system is behaving asymptotically (at a given moment in time if the cross sections are time-dependent) by comparing the values of the static α-eigenvalue (obtained by solving Eq. 36) and the dynamic α (obtained by solving the transport equation and computing the quantity: $\alpha_{\text{dyn}}(t) = \frac{1}{N(t)}\frac{\mathrm{d}N(t)}{\mathrm{d}t}$, where N is the appropriately defined neutron population). If $\alpha(;t)/\alpha_{\text{dyn}}(t) \sim 1$ (where we are treating α as parameterized by time), within some prescribed tolerance, then one may say the system has reached instantaneous asymptotic behavior. An explicit relationship has been devised by Ahrens and Larsen [19], demonstrating that the quantities converge for supercritical systems with a large enough neutron population.

Additionally, Λ_k has the function $\phi(\vec{p},t)$ embedded within it. As mentioned in Sec. 3, there is no supplemental equation for ϕ; it must be obtained by iteration or replaced by an assumed form. Typically, the assumed form is the fundamental forward k-eigenfunction, $\psi_k(\vec{p})$. With that, the generation time equations could arguably be treated interchangeably insofar as the operators are equivalent and the operands are either the forward and adjoint k- or α-eigenfunctions.

For the neutron lifetime definitions, they both conform to the word definitions mentioned above; the key difference between the two (aside from the weighting functions and operands being calculated from the respective k or α equation) is in the form of the expression. For the α-method, k_{inst} is calculated by solving the forward and adjoint α-eigenvalue equation and then evaluating the expression given by Eq. 43b. This is clearly not the same quantity as the k-eigenvalue, which is calculated by solving the forward k-eigenvalue equation and may be expressed with the formula:

$$\begin{aligned} k &= \frac{\int \mathrm{d}\vec{p}\,\chi(E)F\psi_k(\vec{p})}{\int \mathrm{d}\vec{p}\,H\psi_k(\vec{p})} \\ &= \frac{\int_V \mathrm{d}\vec{r}\,F\psi_k(\vec{p})}{\int \mathrm{d}\vec{p}\,[L+\Sigma_a(\vec{r},E)]\psi_k(\vec{p})}, \end{aligned} \tag{52}$$

where we have taken advantage of Eq. A.8 in the second line and, because $F\psi_k(\vec{p})$ is only a function of $\vec{r}$, the $\chi(E)$ is integrated out due to normalization. This formula differs from the definition of k_{inst} from Eq. 43b:

$$k_{\text{inst}} = \frac{\int_V \mathrm{d}\vec{r}\,\left[F^\dagger\psi_\alpha^\dagger\right]F\psi_\alpha}{\int \mathrm{d}\vec{p}\,\psi_\alpha^\dagger(\vec{p})H\psi_\alpha(\vec{p})}$$

because the numerator is the adjoint-emission-spectrum-rate-weighted (i.e., $F^\dagger\psi_\alpha^\dagger$) fission production rate and the numerator cannot be further simplified due to the adjoint weighting (see Eq. A.6 for proof). This is why we have taken care to make a distinction between k and k_{inst} because, although they are qualitatively the same (they are both a ratio of the production rate to the loss rate), they are calculated quite differently and from characteristically different eigenvalue equations. From this, the k-method lifetime is determined as:

$$\ell_k(t) = \frac{\int_V \mathrm{d}\vec{r}\,F\psi_k(\vec{p})}{\int \mathrm{d}\vec{p}\,[L+\Sigma_a(\vec{r},E)]\psi_k(\vec{p})} \cdot \frac{\int \mathrm{d}\vec{p}\,\psi_k^\dagger(\vec{p})\frac{1}{v(E)}\psi_k(\vec{p})}{\int_V \mathrm{d}\vec{r}\,\left[F^\dagger\psi_k^\dagger(\vec{p})\right]F\psi_k(\vec{p})} \tag{53}$$

where we made the replacement $\phi(\vec{p},t) \to \psi_k(\vec{p})$ and the α-method lifetime is given by Eq. 51b. Equation 53 is equivalent to both Eqs. 32 and 50b. Depending on the physics one wishes to highlight, one may insert their choice of the weighting function into Eq. 31. However, the only way to make Eqs. 31 and 32 equivalent is to set $w \equiv 1$, $\phi(\vec{p},t) = \psi_k(\vec{p})$, and to enforce $F^\dagger\psi_k^\dagger \equiv 1$. The latter statement, $F^\dagger\psi_k^\dagger \equiv 1$, is a suggestion that every neutron emitted from fission has equal importance.

6 Conclusions

The goal of this report was to outline two methods for defining the prompt neutron generation time and lifetime, Λ and ℓ, respectively. We demonstrated the k-eigenvalue method and the α-eigenvalue method to which one can perform manipulations to the neutron transport equation to derive fundamental integral quantities of the system and relationships between these quantities. For the k-eigenvalue method, we outlined the steps given by Hetrick to manipulate the NTE into a reduced-form kinetics equation which resulted in a set of natural definitions for Λ and the reactivity, ρ. Additional steps must be taken, as shown by Keepin, to discern more practical definitions for Λ and ρ, which can then be used to define ℓ. For the α-eigenvalue method, it was shown that the 'Hetrick Algorithm' does not provide the desired Λ and ℓ definitions due to the nature of the eigenvalue equation (the only terms that remain at the end of the Hetrick Algorithm are the nonhomogeneous source terms and those that are scaled by the eigenvalue). We proposed a method for manipulating the NTE and its adjoint counterpart to arrive at a set of relationships between Λ, α, and the reactivity, ϱ. We then compared the definitions of the generation times and lifetimes and made observations on their meaning.

Additional work would be to perform similar manipulations to the NTE with delayed neutron precursors included, and to analyze other eigenvalue forms of the NTE, specifically the γ and δ forms that Cacuci [18] investigated.

Appendix A Some Scattering Function Proofs

The macroscopic double-differential scattering cross section may be defined as:

$$\Sigma_s(E \to E', \hat{\Omega} \cdot \hat{\Omega}') = \Sigma_s(E) p(E \to E', \hat{\Omega} \cdot \hat{\Omega}') \tag{A.1}$$

where we are suppressing $\vec{r}, t$ dependence for clarity, $\Sigma_s(E)$ is the total scattering cross section at energy E, and $p(E \to E', \hat{\Omega} \cdot \hat{\Omega}')$ is the post-collision probability distribution of scattering from $E, \hat{\Omega} \to E', \hat{\Omega}'$. Noting p is normalized as $1 = \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} p(E \to E', \hat{\Omega} \cdot \hat{\Omega}')$, the total scattering cross section at energy E is:

$$\Sigma_s(E) = \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(E \to E', \hat{\Omega} \cdot \hat{\Omega}'). \tag{A.2}$$

Appendix A.1 Cancellation of Adjoint-Forward Projections

In both the k- and α-eigenvalue problems, we projected the adjoint eigenvalue solution onto the forward equation and the forward solution onto the adjoint eigenvalue equation and took the difference of the two. For the scattering terms, we find the difference and perform the necessary manipulations:

$$\begin{aligned}
\int \mathrm{d}\vec{p} \left[\psi_x^\dagger S\psi - \psi S^\dagger \psi_x^\dagger\right] &= \int \mathrm{d}\vec{p} \Bigg\{ \psi_x^\dagger(\vec{p}) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \cdot \hat{\Omega}, t) \psi(\vec{r}, E', \hat{\Omega}', t) \\
&\qquad - \psi(\vec{p}, t) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E \to E', \hat{\Omega} \cdot \hat{\Omega}', t) \psi_x^\dagger(\vec{r}, E', \hat{\Omega}') \Bigg\} \\
&= \int_V \mathrm{d}\vec{r} \Bigg\{ \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi} \psi_x^\dagger(\vec{r}, E, \hat{\Omega}) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \cdot \hat{\Omega}, t) \psi(\vec{r}, E', \hat{\Omega}', t)
\end{aligned}$$

$$- \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi} \psi(\vec{r}, E, \hat{\Omega}, t) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E \to E', \hat{\Omega} \cdot \hat{\Omega}', t) \psi_x^\dagger(\vec{r}, E', \hat{\Omega}') \Bigg\}. \tag{A.3}$$

Now if we simply switch the dummy variable labels of the second integral, we can rearrange said integral to obtain:

$$\begin{aligned}
\int \mathrm{d}\vec{p} \left[\psi_x^\dagger S\psi - \psi S^\dagger \psi_x^\dagger\right] &= \int_V \mathrm{d}\vec{r} \Bigg\{ \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi} \psi_x^\dagger(\vec{r}, E, \hat{\Omega}\,) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \cdot \hat{\Omega}, t) \psi(\vec{r}, E', \hat{\Omega}', t) \\
&\quad - \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \psi(\vec{r}, E', \hat{\Omega}', t) \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \cdot \hat{\Omega}, t) \psi_x^\dagger(\vec{r}, E, \hat{\Omega}) \Bigg\} \\
&= \int_V \mathrm{d}\vec{r} \Bigg\{ \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi} \psi_x^\dagger(\vec{r}, E, \hat{\Omega}\,) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \cdot \hat{\Omega}, t) \psi(\vec{r}, E', \hat{\Omega}', t) \\
&\quad - \int_0^\infty \mathrm{d}E \int_{4\pi} \frac{\mathrm{d}\Omega}{4\pi} \psi_x^\dagger(\vec{r}, E, \hat{\Omega}) \int_0^\infty \mathrm{d}E' \int_{4\pi} \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(\vec{r}, E' \to E, \hat{\Omega}' \cdot \hat{\Omega}, t) \psi(\vec{r}, E', \hat{\Omega}', t) \Bigg\} \\
&= 0.
\end{aligned} \tag{A.4}$$

Appendix A.2 Adjoint Projection onto the H Operator

One issue with the lifetime and generation time definitions we find from the α-method, outlined in Sec. 4.3, is the presence of the H operator. In the k-eigenvalue method, we were able to remove the streaming and scattering terms because of the projections of the forward and adjoint solutions onto the opposite equations and taking the difference between the two (see Eq. 15 and the process outlined in Sec. Appendix A.1 for proof). This is not a possibility with the α-method for reasons shown at the forefront of Sec. 4, and we are then unable to define the desired lifetime by following the k-method steps. Here, we define an adjoint-weighted removal cross section that, at a minimum, adds physical meaning to the irreducible H operator.

Take the adjoint weighted inner product of $H\psi_\alpha$:

$$\int \mathrm{d}\vec{p}\, \psi_\alpha^\dagger H \psi_\alpha = \int \mathrm{d}\vec{p}\, \psi_\alpha^\dagger \left[L + T - S\right] \psi_\alpha. \tag{A.5}$$

Let us focus on the scattering terms from the $T - S$ operators to vie for a simplification. Noting that $T = \Sigma_t(\vec{r}, E) = \Sigma_a(\vec{r}, E) + \Sigma_s(\vec{r}, E)$, we have for the scattering terms of the above (we drop $\vec{r}$ dependence and integration limits for clarity):

$$\begin{aligned}
\int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha^\dagger \left[\Sigma_s(E) - S\right] \psi_\alpha &= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha^\dagger \Bigg\{ \Sigma_s(E) \psi_\alpha(E, \hat{\Omega}) - \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(E' \to E, \hat{\Omega}' \cdot \hat{\Omega}) \psi_\alpha(E', \hat{\Omega}') \Bigg\} \\
&= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha^\dagger(E, \hat{\Omega}) \Sigma_s(E) \psi_\alpha(E, \hat{\Omega}) \\
&\quad - \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha^\dagger(E, \hat{\Omega}) \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(E' \to E, \hat{\Omega}' \cdot \hat{\Omega}) \psi_\alpha(E', \hat{\Omega}') \\
&= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha^\dagger(E, \hat{\Omega}) \psi_\alpha(E, \hat{\Omega}) \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(E \to E', \hat{\Omega} \cdot \hat{\Omega}')
\end{aligned}$$

$$
\begin{aligned}
&- \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha^\dagger(E,\hat{\Omega}) \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(E' \to E, \hat{\Omega}' \cdot \hat{\Omega}) \psi_\alpha(E', \hat{\Omega}') \\
&= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \Bigg\{ \psi_\alpha^\dagger(E,\hat{\Omega}) \psi_\alpha(E,\hat{\Omega}) \Sigma_s(E \to E', \hat{\Omega} \cdot \hat{\Omega}') \\
&\qquad - \psi_\alpha^\dagger(E,\hat{\Omega}) \psi_\alpha(E',\hat{\Omega}') \Sigma_s(E' \to E, \hat{\Omega}' \cdot \hat{\Omega}) \Bigg\}
\end{aligned}
\tag{A.6}
$$

Here we have used Eq. A.2 to expand $\Sigma_s(E)$. Unfortunately, there is no way to isolate the second Σ_s in the last line because either $\psi_\alpha^\dagger(E,\hat{\Omega})$ or $\psi_\alpha(E',\hat{\Omega}')$ will be involved, thus no simplification can be made to Eq. A.6.

Setting $\psi_\alpha^\dagger(\vec{r},E,\hat{\Omega}) \equiv 1$, we find

$$
\begin{aligned}
\int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \left[\Sigma_s(E) - S\right] \psi_\alpha &= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha(E,\hat{\Omega}) \Sigma_s(E) - \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \Sigma_s(E' \to E, \hat{\Omega}' \cdot \hat{\Omega}) \psi_\alpha(E', \hat{\Omega}') \\
&= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha(E,\hat{\Omega}) \Sigma_s(E) - \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \psi_\alpha(E',\hat{\Omega}') \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \Sigma_s(E' \to E, \hat{\Omega}' \cdot \hat{\Omega}) \\
&= \int \mathrm{d}E \int \frac{\mathrm{d}\Omega}{4\pi} \psi_\alpha(E,\hat{\Omega}) \Sigma_s(E) - \int \mathrm{d}E' \int \frac{\mathrm{d}\Omega'}{4\pi} \psi_\alpha(E',\hat{\Omega}') \Sigma_s(E') \\
&= 0,
\end{aligned}
\tag{A.7}
$$

which is simply a testament to the conservation of particle mass. From this, an unweighted inner product of $H\psi_\alpha$ is given by:

$$
\int \mathrm{d}\vec{p}\, H\psi_\alpha(\vec{p}) = \int \mathrm{d}\vec{p} \left[\hat{\Omega} \cdot \nabla \psi_\alpha(\vec{p}) + \Sigma_a \psi_\alpha(\vec{p})\right], \tag{A.8}
$$

where the volume integral of the streaming term may be further simplified via the Divergence Theorem.

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