Analysis of Anisotropic Nonlocal Diffusion Models: Well-posedness of Fractional Problems fo

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Transport

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Nonlocal modeling

- Nonlocal models have become a preferred modeling choice for scientific and engineering applications featuring global behavior that is affected by small scales.
- In particular, nonlocal models can capture effects that classical partial differential equations (PDEs) fail to describe.
- These effects include multiscale behavior and anomalous transport such as superdiffusion and subdiffusion.
- A nonlocal equation is characterized by integral operators acting on a lengthscale or "horizon", describing long-range forces and reducing the regularity requirements on the solutions.
- Engineering applications include surface or subsurface transport, fracture mechanics, turbulence, image processing, and stochastic processes.

Model for anisotropic anomalous transport

- We focus mainly on nonlocal operators of fractional type, of the form

$$\mathcal{L}_{\omega;A} = \mathcal{D}_\omega \left(A(\boldsymbol{x}) \mathcal{G}_\omega \right), \text{ with } \omega \propto |\boldsymbol{x} - \boldsymbol{y}|^{-n-s} \tag{1}$$

- $A(\boldsymbol{x})$ denotes a diffusion tensor and $\mathcal{D}_\omega$ and $\mathcal{G}_\omega$ denote weighted nonlocal divergence and gradient operators, respectively, with weight function ω.
- Modeling subsurface transport is challenging due to the heterogeneities of the media which generate, at the continuum scale, diffusion processes exhibiting transport rates that may be "faster" or "slower" than those described by the classical integer-order diffusion equation.
- At the smaller scales a local PDE model can accurately describe diffusion processes by explicitly embedding the heterogeneities in the model parameters. At the continuum scale, such models may fail to do so.
- A fractional-order model using an diffusion operator of the form (1) may act as a homogenized model that encodes the heterogeneities of the medium in the integral operator itself.

Summary

- We analyze the well-posedness of an anisotropic, nonlocal diffusion equation involving the operator $\mathcal{L}_{\omega;A}$ in the previous slide.
- We establish an equivalence between weighted and unweighted anisotropic nonlocal diffusion operators in the vein of *unified nonlocal vector calculus*. This allows us to utilize well-posedness for unweighted nonlocal diffusion problems, which are much simpler.
- We apply our analysis to a class of fractional-order operators and present rigorous estimates for the solution of the corresponding anisotropic anomalous diffusion equation.
- We extend our analysis to the anisotropic diffusion-advection equation and prove well-posedness for fractional orders $s \in [0.5, 1)$. We also present an application of the advection-diffusion equation to anomalous transport of solutes.

Unweighted Nonlocal Vector Calculus Operators

Let $\boldsymbol{\alpha} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2, 3$, be an anti-symmetric two-point vector function. For $\mathbf{v} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the nonlocal *unweighted divergence* $\mathcal{D}\mathbf{v} : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathcal{D}\mathbf{v}(\boldsymbol{x}) := \int_{\mathbb{R}^n} (\mathbf{v}(\boldsymbol{x}, \boldsymbol{y}) + \mathbf{v}(\boldsymbol{y}, \boldsymbol{x})) \cdot \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y}. \tag{2}$$

For $u : \mathbb{R}^n \to \mathbb{R}$ the nonlocal *unweighted gradient*, $\mathcal{G}u : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, the negative adjoint of (2) is defined as

$$\mathcal{G}u(\boldsymbol{x}, \boldsymbol{y}) = (u(\boldsymbol{y}) - u(\boldsymbol{x}))\boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y}). \tag{3}$$

The nonlocal *unweighted Laplacian* is defined as the composition of unweighted nonlocal divergence and gradient, i.e.

$$\mathcal{L}u(\boldsymbol{x}) = \mathcal{D}\mathcal{G}u(\boldsymbol{x}) = 2 \int_{\mathbb{R}^n} (u(\boldsymbol{y}) - u(\boldsymbol{x}))\gamma(\boldsymbol{x}, \boldsymbol{y}) d\boldsymbol{y}, \tag{4}$$

where the nonnegative kernel γ is given by $\gamma = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$.

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The diffusion problem

- To define a diffusion problem in a bounded domain $\Omega \subset \mathbb{R}^n$, by definition of $\mathcal{L}u(\boldsymbol{x})$, it is necessary to evaluate $u(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathbb{R}^n \setminus \Omega$. We refer to conditions on u in the exterior of the domain as exterior conditions or volume constraints.
- With this in mind, the strong form of an unweighted nonlocal diffusion problem is given by: for $f : \Omega \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ and $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$, find u such that

$$\begin{cases} \partial_t u(\boldsymbol{x}, t) = \mathcal{L}u(\boldsymbol{x}, t) + f(\boldsymbol{x}, t), & (\boldsymbol{x}, t) \in \Omega \times (0, T) \\ u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), & (\boldsymbol{x}, t) \in \mathbb{R}^n \setminus \Omega \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \end{cases} \tag{5}$$

  where the second condition in (5) is the nonlocal counterpart of a Dirichlet boundary condition for PDEs and it is referred to as *Dirichlet volume constraint* , and is required to guarantee the well-posedness of (5).
- For simplicity and without loss of generality we analyze the homogeneous case $g = 0$; all the results below can be extended to the non-homogeneous case using “lifting” arguments.

Nonlocal Green's identity

To obtain the variational form of equation (5), we apply the following nonlocal form of the first Green's identity, introduced in Du et al. (2013):

$$\int_{\Omega} -\mathcal{L}u(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{G}u(\boldsymbol{x},\boldsymbol{y}) \cdot \mathcal{G}v(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} + \int_{\mathbb{R}^n\setminus\Omega} \mathcal{D}(\mathcal{G}u)(\boldsymbol{x})\, v(\boldsymbol{y})\, d\boldsymbol{x}. \tag{6}$$

Multiplying (5) by a test function v such that $v = 0$ on $\mathbb{R}^n \setminus \Omega$ and integrating over the domain Ω yields, for all $t \geq 0$,

$$0 = \int_{\Omega} (\partial_t u(\boldsymbol{x},t) - \mathcal{L}u(\boldsymbol{x},t) - f(\boldsymbol{x},t))\, v(\boldsymbol{x})\, d\boldsymbol{x} \tag{7}$$

$$= \int_{\Omega} \partial_t u(\boldsymbol{x},t)\, v(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{G}u(\boldsymbol{x},\boldsymbol{y},t) \cdot \mathcal{G}v(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x} \tag{8}$$

$$+ \int_{\mathbb{R}^n\setminus\Omega} \mathcal{D}(\mathcal{G}u)(\boldsymbol{x},t)\, v(\boldsymbol{x})\, d\boldsymbol{x} - \int_{\Omega} f(\boldsymbol{x},t)\, v(\boldsymbol{x})\, d\boldsymbol{x},$$

where the integral over $\mathbb{R}^n \setminus \Omega$ on the right-hand side is zero due to the properties of v.

Weak/variational formulation

- Given a function space S with norm $\|\cdot\|$, we define the space $L^2(0,T;S)$ as follows

  $L^2(0,T;S) = \{w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \text{ such that } w(\cdot,t) \in S \; \forall\, t \geq 0, \text{ and } \|w(\cdot,t)\|_S \in L^2(0,T)\}.$

- Then, the weak form of the nonlocal diffusion problem reads as follows. For $f \in L^2(0,T;V'_\Omega(\mathbb{R}^n))$, find $u \in L^2(0,T;V_\Omega(\mathbb{R}^n))$ such that

$$(\partial_t u, v) + \mathcal{B}(u,v) = \mathcal{F}(v), \;\; \forall\, v \in V_\Omega(\mathbb{R}^n), \tag{9}$$

  where $(\cdot,\cdot)$ indicates the L^2 inner product over Ω, and

$$\begin{aligned} \mathcal{B}(u,v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathcal{G}u(\boldsymbol{x},\boldsymbol{y}) \cdot \mathcal{G}v(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x}, \mathcal{F}(v) &= \int_\Omega f(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x}, \\ V_\Omega(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : |||v||| < \infty \text{ and } v|_{\mathbb{R}^n\setminus\Omega} = 0\}. \end{aligned} \tag{10}$$

  Here, $|||v|||^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{G}v(\boldsymbol{x},\boldsymbol{y})|^2\, d\boldsymbol{y}\, d\boldsymbol{x}$, and the space V'_Ω is the dual space of V_Ω.

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Well-posedness of the unweighted/standard diffusion problem

- Note that the bilinear form $\mathcal{B}(\cdot,\cdot)$ defines an inner product on $V_\Omega(\mathbb{R}^n)$ and that $|||u|||^2 = \mathcal{B}(u,u)$.
- This fact implies that the bilinear form is coercive and, hence, weakly coercive.
- Together with the continuity of $\mathcal{B}$ and $\mathcal{F}$, this yields the well-posedness of the weak form (9).

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Weighted operators and corresponding volume-constrained problems

- We let $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a nonnegative, symmetric scalar function known as the *weight* function.
- For $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$, the nonlocal *$\omega$-weighted divergence* $\mathcal{D}_\omega \mathbf{v} : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\begin{aligned} \mathcal{D}_\omega \mathbf{v}(\boldsymbol{x}) &:= \mathcal{D}(\omega(\boldsymbol{x},\boldsymbol{y})\mathbf{v}(\boldsymbol{x})) \\ &= \int_{\mathbb{R}^n} (\omega(\boldsymbol{x},\boldsymbol{y})\mathbf{v}(\boldsymbol{x}) + \omega(\boldsymbol{y},\boldsymbol{x})\mathbf{v}(\boldsymbol{y})) \cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y}. \end{aligned} \tag{11}$$

- For $u : \mathbb{R}^n \to \mathbb{R}$, the nonlocal *$\omega$-weighted gradient* $\mathcal{G}_\omega u : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\begin{aligned} \mathcal{G}_\omega u(\boldsymbol{x}) &:= \int_{\mathbb{R}^n} \omega(\boldsymbol{x},\boldsymbol{y}) \mathcal{G} u(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y} \\ &= \int_{\mathbb{R}^n} \omega(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{y}) - u(\boldsymbol{x}))\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) d\boldsymbol{y}. \end{aligned} \tag{12}$$

- $\mathcal{G}_\omega$ is the negative adjoint of the divergence $\mathcal{D}_\omega$.

Weighted nonlocal diffusion operator

As in the unweighted case, we define the nonlocal *ω-weighted Laplacian* as the composition of (11) and (12), i.e.,

$$\begin{aligned}\mathcal{L}_\omega u(\boldsymbol{x}) &= \mathcal{D}_\omega \mathcal{G}_\omega u(\boldsymbol{x}) \\ &= \int_{\mathbb{R}^n} \Big[\omega(\boldsymbol{x},\boldsymbol{y}) \int_{\mathbb{R}^n} \omega(\boldsymbol{x},\boldsymbol{z})(u(\boldsymbol{z})-u(\boldsymbol{x}))\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{z})d\boldsymbol{z} \\ &\qquad + \omega(\boldsymbol{y},\boldsymbol{x}) \int_{\mathbb{R}^n} \omega(\boldsymbol{y},\boldsymbol{z})(u(\boldsymbol{z})-u(\boldsymbol{y}))\boldsymbol{\alpha}(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}\Big] \cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}.\end{aligned} \tag{13}$$

Using the symmetry of ω, we can further write

$$\begin{aligned}\mathcal{L}_\omega u(\boldsymbol{x}) = \int_{\mathbb{R}^n} \omega(\boldsymbol{x},\boldsymbol{y}) \Big[\int_{\mathbb{R}^n} \omega(\boldsymbol{x},\boldsymbol{z})(u(\boldsymbol{z})-u(\boldsymbol{x}))\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{z})d\boldsymbol{z} \\ + \int_{\mathbb{R}^n} \omega(\boldsymbol{y},\boldsymbol{z})(u(\boldsymbol{z})-u(\boldsymbol{y}))\boldsymbol{\alpha}(\boldsymbol{y},\boldsymbol{z})d\boldsymbol{z}\Big] \cdot \boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y}.\end{aligned} \tag{14}$$

Tensor valued weight functions

- We have reviewed the definitions (11), (12), and (13) assuming that ω is a symmetric, scalar valued function.
- Later on, in the main sections, we will consider the case when ω is a nonsymmetric tensor.
- Definitions (11), (12), and (13), but not the simplification (14), may be utilized for this case with products of ω and vectors being interpreted as matrix-vector multiplication.

Weighted nonlocal diffusion problems

As for the unweighted case, problems defined on bounded domains involving these operators require a volume constraint on the exterior of Ω. We introduce the strong form of a weighted, nonlocal diffusion problem with homogeneous volume constraints. For $f : \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$, find u such that

$$\begin{cases} \partial_t u(\boldsymbol{x}, t) = \mathcal{L}_\omega u(\boldsymbol{x}, t) + f(\boldsymbol{x}, t), & (\boldsymbol{x}, t) \in \Omega \times (0, T] \\ u(\boldsymbol{x}, t) = 0, & (\boldsymbol{x}, t) \in \mathbb{R}^n \setminus \Omega \times (0, T] \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \end{cases} \tag{15}$$

where the second condition in (15) is still referred to as Dirichlet volume constraint. Next, by multiplying (15) by a test function $v : \mathbb{R}^n \to \mathbb{R}$ such that

$$v = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \tag{16}$$

and integrating over the domain Ω, we have the following weak form:

$$\int_\Omega (\partial_t u(\boldsymbol{x}, t) - \mathcal{L}_\omega u(\boldsymbol{x}, t) - f(\boldsymbol{x}, t))\, v(\boldsymbol{x})\, d\boldsymbol{x} = 0, \text{ for all } t > 0. \tag{17}$$

Weighted nonlocal Green's identity

We introduce the following weighted nonlocal Green's first identity

$$\int_{\Omega} -\mathcal{L}_\omega u(\boldsymbol{x}) v(\boldsymbol{x}) \; d\boldsymbol{x} = \int_{\mathbb{R}^n} \mathcal{G}_\omega u(\boldsymbol{x}) \cdot \mathcal{G}_\omega u(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^n \setminus \Omega} \mathcal{D}_\omega \mathcal{G}_\omega u(\boldsymbol{x}) v(\boldsymbol{x}) \; d\boldsymbol{x}. \qquad (18)$$

By substituting the latter in (17), we obtain

$$\int_{\Omega} \partial_t u(\boldsymbol{x}, t)\, v(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\mathbb{R}^n} \mathcal{G}_\omega u(\boldsymbol{x}, t) \cdot \mathcal{G}_\omega u(\boldsymbol{x}, t) d\boldsymbol{x} + \int_{\mathbb{R}^n \setminus \Omega} \mathcal{D}_\omega \mathcal{G}_\omega u(\boldsymbol{x}, t) v(\boldsymbol{x}) \; d\boldsymbol{x} - \int_{\Omega} f(\boldsymbol{x}, t)\, v(\boldsymbol{x})\, d\boldsymbol{x} = 0. \qquad (19)$$

By (16), the integral over $\mathbb{R}^n \setminus \Omega$ on the left-hand side is zero.

Weak form of weighted nonlocal diffusion problems

Thus, the weak form of the nonlocal diffusion problem reads as follows. For $f \in L^2(0,T;(V^\omega_\Omega)'(\mathbb{R}^n))$, and $u_0 \in V^\omega_\Omega(\mathbb{R}^n)$, find $u \in L^2(0,T;V^\omega_\Omega(\mathbb{R}^n)$ such that

$$(\partial_t u, v) + \mathcal{B}_\omega(u,v) = \mathcal{F}(v), \;\; \forall\, v \in V^\omega_\Omega(\mathbb{R}^n), \tag{20}$$

where

$$\begin{aligned} \mathcal{B}_\omega(u,v) &= \int_{\mathbb{R}^n} \mathcal{G}_\omega u(\boldsymbol{x}) \cdot \mathcal{G}_\omega v(\boldsymbol{x}) d\boldsymbol{x}, \\ V^\omega_\Omega(\mathbb{R}^n) &= \{v \in L^2(\mathbb{R}^n) : \left|\!\left|\!\left| v \right|\!\right|\!\right|_\omega < \infty \text{ and } v|_{\mathbb{R}^n \setminus \Omega} = 0\}, \end{aligned} \tag{21}$$

and where the *weighted energy* is defined as

$$\left|\!\left|\!\left| v \right|\!\right|\!\right|^2_\omega = \int_{\mathbb{R}^n} |\mathcal{G}_\omega v(\boldsymbol{x})|^2 \, d\boldsymbol{x}. \tag{22}$$

The well-posedness of problem (20) follows when an equivalence relationship can be established between weighted and unweighted operators, as we summarize in the next section.

The equivalence kernel

The equivalence theorem between weighted and unweighted operators provides an equivalence kernel γ_{eq} that, for given $\boldsymbol{\alpha}$ and ω, guarantees that $\mathcal{L} = \mathcal{L}_\omega$. In what follows, we summarize the main result and its consequences.

Theorem

Let $\mathcal{D}_\omega$ and $\mathcal{G}_\omega$ be the operators associated with the symmetric scalar weight function ω and the anti-symmetric function $\boldsymbol{\alpha}$. For the equivalence kernel γ_{eq} defined by

$$\begin{aligned} 2\gamma_{\mathrm{eq}}(\boldsymbol{x},\boldsymbol{y};\omega,\boldsymbol{\alpha}) = \int_{\mathbb{R}^n} & [\omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \cdot \omega(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{z}) \\ & + \omega(\boldsymbol{z},\boldsymbol{y})\boldsymbol{\alpha}(\boldsymbol{z},\boldsymbol{y}) \cdot \omega(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{y}) \\ & + \omega(\boldsymbol{z},\boldsymbol{y})\boldsymbol{\alpha}(\boldsymbol{z},\boldsymbol{y}) \cdot \omega(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\alpha}(\boldsymbol{x},\boldsymbol{z})]d\boldsymbol{z}, \end{aligned} \tag{23}$$

the weighted operator $\mathcal{L}_\omega = \mathcal{D}_\omega\mathcal{G}_\omega$ and the unweighted Laplacian operator $\mathcal{L}$ with kernel γ_{eq} are equivalent, i.e. $\mathcal{L} = \mathcal{L}_\omega$.

Variational equivalence between weighted and unweighted problems

This result, and the weighted nonlocal Green's first identity, imply the following important variational equivalence.

Theorem

For $\gamma_{\rm eq}(\boldsymbol{x},\boldsymbol{y};\omega,\boldsymbol{\alpha})$ defined as in (23), the variational forms associated with weighted and unweighted nonlocal operators are equivalent. That is, for all $v=0$ in $\mathbb{R}^n\setminus\Omega$,

$$\mathcal{B}(u,v)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\mathcal{G}u(\boldsymbol{x},\boldsymbol{y})\cdot\mathcal{G}v(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}\,d\boldsymbol{x}=\int_{\mathbb{R}^n}\mathcal{G}_\omega u(\boldsymbol{x})\cdot\mathcal{G}_\omega v(\boldsymbol{x})\,d\boldsymbol{x}=\mathcal{B}_\omega(u,v). \qquad (24)$$

An immediate consequence of this theorem is the equivalence of weighted and unweighted energies, i.e.

$$|||v|||^2=\mathcal{B}(v,v)=\mathcal{B}_\omega(v,v)=|||v|||_\omega^2. \qquad (25)$$

More importantly, the variational equivalence allows us to extend the unweighted well-posedness results to the weighted case, anytime the equivalence kernel $\gamma_{\rm eq}$ induces an unweighted coercive bilinear form $A(\cdot,\cdot)$.

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The special case of fractional-order operators

- We specify the choices of $\boldsymbol{\alpha}$ and ω for which the weighted fractional Laplacian is equivalent to the standard fractional Laplacian.
- The (Riesz) fractional Laplacian is defined as

$$(-\Delta)^s u = C_{n,s} \int_{\mathbb{R}^n} \frac{u(\boldsymbol{x}) - u(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}} d\boldsymbol{y}, \tag{26}$$

where

$$C_{n,s} = \frac{4^s \Gamma\left(s + \frac{n}{2}\right)}{\pi^{n/2} |\Gamma(-s)|}. \tag{27}$$

- The weighted fractional gradient and divergence operators are defined as

$$\begin{aligned} \operatorname{grad}^s u(\boldsymbol{x}) &= \int_{\mathbb{R}^n} [u(\boldsymbol{x}) - u(\boldsymbol{y})] \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{n+s}} d\boldsymbol{y}, \\ \operatorname{div}^s \mathbf{v}(\boldsymbol{x}) &= \int_{\mathbb{R}^n} [\mathbf{v}(\boldsymbol{x}) - \mathbf{v}(\boldsymbol{y})] \cdot \frac{\boldsymbol{x} - \boldsymbol{y}}{|\boldsymbol{x} - \boldsymbol{y}|} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{n+s}} d\boldsymbol{y}. \end{aligned} \tag{28}$$

Theorem

Let $\mathbf{v} \in \mathbf{H}^s(\mathbb{R}^d)$ *and* $u \in H^s(\mathbb{R}^d)$*. For the weight function and kernel*

$$\omega = C_\omega |\boldsymbol{x} - \boldsymbol{y}|^{-(n+s)}, \quad \boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\boldsymbol{y} - \boldsymbol{x}}{|\boldsymbol{y} - \boldsymbol{x}|}, \tag{29}$$

the fractional divergence and gradient operators can be identified with the weighted nonlocal operators,

$$\mathrm{div}^s \mathbf{v}(\boldsymbol{x}) = \mathcal{D}_\omega \mathbf{v}(\boldsymbol{x}), \quad \mathrm{grad}^s u(\boldsymbol{x}) = \mathcal{G}_\omega u(\boldsymbol{x}). \tag{30}$$

Furthermore, $\boldsymbol{\alpha}(\boldsymbol{x}, \boldsymbol{y})\omega(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y} - \boldsymbol{x})|\boldsymbol{y} - \boldsymbol{x}|^{-(n+s+1)}$*, implies that*

$$\gamma_{\mathrm{eq}}(\boldsymbol{x}, \boldsymbol{y}) = \gamma_{FL}(\boldsymbol{x}, \boldsymbol{y}) = -\frac{C_{n,s}}{2} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s}}, \tag{31}$$

where FL stands for "fractional Laplacian" and $C_{n,s}$ *is the defined as in (27). Then, for* $u \in H^{2s}(\mathbb{R}^n)$*,*

$$\mathcal{L}u = \mathcal{L}_\omega u = -(-\Delta)^s u. \tag{32}$$

In words, the fractional gradient and divergence are special instances of weighted gradient and divergence operators, for special choices of $\boldsymbol{\alpha}$ and ω, and their composition is equivalent to the standard fractional Laplacian operator.

Well-posedness for fractional problems

- The corresponding weighted and unweighted diffusion problems are both well-posed in $L^2(0,T;H^s_\Omega(\mathbb{R}^n))$ where $H^s_\Omega(\mathbb{R}^n)=\{v\in H^s(\mathbb{R}^n): v|_{\mathbb{R}^n\setminus\Omega}=0\}$.
- This follows from the coercivity of $\mathcal{B}(\cdot,\cdot)$ for $\gamma=\gamma_{FL}$ and from the variational equivalence in Theorem 2.
- More precisely, on one hand, the fact that the bilinear form $\mathcal{B}(\cdot,\cdot)$ associated with γ_{FL} defines an inner product on $H^s_\Omega(\mathbb{R}^n)$ guarantees the well-posedness of the unweighted parabolic problem.
- On the other hand, the variational equivalence guarantees that the weighted bilinear form $\mathcal{B}_\omega(\cdot,\cdot)$ associated with ω and $\boldsymbol{\alpha}$ defined as in (29) is equivalent to $\mathcal{B}(\cdot,\cdot)$.
- This fact implies that the weighted parabolic problem is also well-posed in $L^2(0,T;H^s_\Omega(\mathbb{R}^n))$.

Anisotropic diffusion tensor can be exchanged for anisotropic weight

- We first introduce the anisotropic diffusion tensor and operator: let
$$A : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \text{ be bounded, measurable, symmetric and elliptic,} \tag{33}$$
i.e. there exist $0 < \lambda_{\min} \leq \lambda_{\max} < \infty$ such that for all $\mathbf{v} \in \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$,
$$\lambda_{\min}|\mathbf{v}|^2 \leq \mathbf{v} \cdot A(\boldsymbol{x})\mathbf{v} \leq \lambda_{\max}|\mathbf{v}|^2. \tag{34}$$
- This implies the existence of a tensor-valued function $A^{\frac{1}{2}}(\boldsymbol{x})$ such that $A^{\frac{1}{2}}(\boldsymbol{x})A^{\frac{1}{2}}(\boldsymbol{x}) = A(\boldsymbol{x})$.
- We define the anisotropic nonlocal weighted Laplacian as
$$\mathcal{L}_{\omega;A}u(\boldsymbol{x}) = \mathcal{D}_\omega(A(\boldsymbol{x})\mathcal{G}_\omega u(\boldsymbol{x})). \tag{35}$$

Lemma

Let A satisfy (33) and $\boldsymbol{\alpha}$ and ω be an anti-symmetric vector function and a symmetric scalar function respectively. Then, for $\widetilde{\omega} = A^{\frac{1}{2}}\omega$,

$$\mathcal{L}_{\omega;A}u(\boldsymbol{x}) := \mathcal{D}_\omega(A(\boldsymbol{x})\mathcal{G}_\omega u(\boldsymbol{x})) = \mathcal{L}_{\widetilde{\omega}}u(\boldsymbol{x}). \tag{36}$$

Equivalence kernel for *nonsymmetric* weight functions

- Having established that the use of a space-dependent diffusion tensor corresponds to having a nonsymmetric weight function in the nonlocal Laplacian operator (13), we show that the corresponding weighted Laplacian still admits a symmetric equivalence kernel.
- Note that the arguments below hold also when ω is a tensor.

Lemma

Let the weight function ω be two-point function, not necessarily symmetric, i.e. $\omega(\boldsymbol{x},\boldsymbol{y}) \neq \omega(\boldsymbol{y},\boldsymbol{x})$. Then, there exists a symmetric equivalence kernel $\gamma_{\rm eq}$ such that

$$\mathcal{D}_\omega(\mathcal{G}_\omega u(\boldsymbol{x})) = 2\int_{\mathbb{R}^n} (u(\boldsymbol{y}) - u(\boldsymbol{x}))\gamma_{\rm eq}(\boldsymbol{x},\boldsymbol{y};\omega)\, d\boldsymbol{x}, \quad (37)$$

where $\gamma_{\rm eq}(\boldsymbol{x},\boldsymbol{y};\omega,\boldsymbol{\alpha})$ is a symmetric function of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Nonlocal anisotropic Poisson problem and Green's identity

- We now introduce the anisotropic nonlocal Poisson equation. For $f : \Omega \to \mathbb{R}$, we seek $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} -\mathcal{L}_{\omega;A} u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), & \boldsymbol{x} \in \Omega \\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{38}$$

- As usual, a form of Green's first identity is required to introduce a weak form for the equation above.

Theorem

Let $\mathcal{L}_{\omega;A}$ be defined as in (35). Then,

$$-\int_{\Omega} \mathcal{L}_{\omega;A} u(\boldsymbol{x}) v(\boldsymbol{x}) \, d\boldsymbol{x} = \int_{\mathbb{R}^n} \mathcal{G}_\omega v(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^n \setminus \Omega} \mathcal{D}_\omega(A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x})) v(\boldsymbol{x}) \, d\boldsymbol{x} \tag{39}$$

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Weak form of anisotropic Poisson problem

- Utilizing the results of the previous subsection, we can formulate the weak form of equation (38) and show that the corresponding energy is equivalent to an unweighted nonlocal energy.
- We multiply (38) by a test function $v = 0$ in $\mathbb{R}^n \setminus \Omega$ and integrate over the domain Ω; we have

$$\int_\Omega \left(-\mathcal{L}_{\omega;A} u(\boldsymbol{x}) - f(\boldsymbol{x})\right) v(\boldsymbol{x})\, d\boldsymbol{x} = 0. \tag{40}$$

- The anisotropic weighted nonlocal Green's first identity (39) then implies

$$\int_{\mathbb{R}^n} \mathcal{G}_\omega u(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x}) d\boldsymbol{x} + \int_{\mathbb{R}^n \setminus \Omega} \mathcal{D}_\omega (A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x})) v(\boldsymbol{x}) \ d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x} = 0.$$

Weak form of anisotropic Poisson problem

- Thus, the weak form of the nonlocal Poisson problem reads as follows. For $f \in V'_A(\mathbb{R}^n)$, find $u \in V_\Omega^A(\mathbb{R}^n)$ such that

$$\mathcal{B}_{\omega;A}(u,v) = \mathcal{F}(v), \quad \forall\, v \in V_A(\Omega \cup \Omega_I), \tag{41}$$

where

$$\begin{aligned} \mathcal{B}_{\omega;A}(u,v) &= \int_{\mathbb{R}^n} \mathcal{G}_\omega u(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x}) d\boldsymbol{x}, \\ V_\Omega^A(\mathbb{R}^n) &= \{v \in L^2(\mathbb{R}^n) : |||v|||_A < \infty \text{ and } v|_{\mathbb{R}^n \setminus \Omega} = 0\}, \end{aligned} \tag{42}$$

and where the *anisotropic energy* is defined as

$$|||v|||_A^2 = \int_{\mathbb{R}^n} \mathcal{G}_\omega v(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega v(\boldsymbol{x})\, d\boldsymbol{x}. \tag{43}$$

- The existence of the equivalence kernel guaranteed by Lemma 5, allows us to establish an equivalence relationship between the anisotropic weighted bilinear form $\mathcal{B}_{\omega;A}$ defined above and the unweighted bilinear form $\mathcal{B}$ given in (9), where the latter is associated to the equivalence kernel $\gamma_{\text{eq}}(\boldsymbol{x}, \boldsymbol{y}; A^{\frac{1}{2}}\omega, \boldsymbol{\alpha})$, as shown in the following lemma.

Lemma

Let A be a bounded, measurable and elliptic tensor, ω be a symmetric scalar function and $\boldsymbol{\alpha}$ an anti-symmetric vector function. Then, the following identity holds:

$$\mathcal{B}_{\omega;A}(u,v) = \int_{\mathbb{R}^n} \mathcal{G}_\omega v(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x})\, d\boldsymbol{x} = \mathcal{B}(u,v), \quad \forall\, u,v \in V_\Omega^A(\mathbb{R}^n), \tag{44}$$

where $\mathcal{B}(\cdot,\cdot)$ is the unweighted bilinear form defined in (10) associated to the symmetric equivalence kernel $\gamma_{\rm eq}(\boldsymbol{x},\boldsymbol{y}; A^{\frac{1}{2}}\omega, \boldsymbol{\alpha})$.

The proof follows from Lemmas 5 and 6. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{G}_\omega v(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x})\, d\boldsymbol{x} &= -\int_\Omega \mathcal{D}_\omega (A(\boldsymbol{x}) \mathcal{G}_\omega u)(\boldsymbol{x}) v(\boldsymbol{x})\, d\boldsymbol{x} && \text{(weighted Green's identity)} \\
&= -\int_\Omega \mathcal{D}\mathcal{G} u(\boldsymbol{x}) v(\boldsymbol{x})\, d\boldsymbol{x} && \text{(Equivalence kernel)} \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{G} u(\boldsymbol{x},\boldsymbol{y}) \cdot \mathcal{G} v(\boldsymbol{x},\boldsymbol{y})\, d\boldsymbol{y}\, d\boldsymbol{x}. && \text{(unweighted Green's identity)}
\end{aligned}$$

Conditions for well-posedness

- The theorem above is not enough to guarantee the well-posedness of problem (41).
- One approach to obtaining the existence and uniqueness of solutions involves establishing certain properties of γ_{eq}.
- However, thanks to the ellipticity property of A, the well-posedness of the anisotropic problem follows from the well-posedness of the weighted problem associated with the corresponding isotropic weighted bilinear form $\mathcal{B}_\omega$.
- In fact, $\mathcal{B}_{\omega;A}$ is coercive and continuous with respect to the energy induced by of $\mathcal{B}_\omega$, as we show in the following lemma.

Lemma

The bilinear form $\mathcal{B}_{\omega;A}(u,v)$ defined as in (42) is continuous and coercive in $V_\Omega^\omega(\mathbb{R}^n)$, i.e.

$$\begin{aligned} |\mathcal{B}_{\omega;A}(u,v)| &\le \lambda_{\max} |||u|||_\omega |||v|||_\omega \\ \mathcal{B}_{\omega;A}(u,u) &\ge \lambda_{\min} |||u|||_\omega^2, \end{aligned} \tag{45}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of A over $\mathbb{R}^n$, respectively.

The anisotropic fractional problem

- We apply the analysis of the nonlocal anisotropic problem to the case of fractional operators.
- That is, we consider ω and $\boldsymbol{\alpha}$ defined as in (29) and s show that the corresponding anisotropic problem is well-posed in the usual fractional Sobolev space.
- We only need to show that the bilinear form $\mathcal{B}_{\omega;A}$ is coercive and continuous with respect to the fractional Sobolev norm.
- In fact, Theorem (3) states that the equivalence kernel associated with the weight and kernel functions in (29) is the fractional Laplacian kernel γ_{FL}.
- Variational equivalence implies that the corresponding weighted energy space V^{ω}_{Ω} is equivalent to H^s_{Ω} and that the weighted energy $|||\cdot|||_{\omega}$ is equivalent to the H^s norm.

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- Thus, Lemma 8 implies the continuity and coercivity of $\mathcal{B}_{\omega;A}$ in H^s_Ω and the well-posedness of problem (41) is immediate, as stated in the folling lemma.

Lemma

Let A satisfy (33), and let ω and $\boldsymbol{\alpha}$ be defined as in (29). Then, the corresponding bilinear form $\mathcal{B}_{\omega;A}$ defined as in (42) is coercive and continuous in $H^s_\Omega(\mathbb{R}^n)$ with coercivity and continuity constants

$$C_{\text{coer}} = \frac{C_{n,s}}{2}\lambda_{\min} \quad \text{and} \quad C_{\text{cont}} = \frac{C_{n,s}}{2}\lambda_{\max}, \tag{46}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of A in $\mathbb{R}^n$, respectively. Furthermore, problem (41) is well-posed.

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The anisotropic fractional problem

- Note that for the fractional case and for a class of tensors satisfying (33) that we specify below, we can characterize the equivalence kernel.
- In particular, the equivalence kernel is such that the corresponding unweighted bilinear form is a *Dirichlet form*, as we show in the following lemma.

Lemma

Let I be the identity tensor in $\mathbb{R}^n$ and let $A(\boldsymbol{x}) = a(\boldsymbol{x})I$ satisfy (33) for $a : \mathbb{R}^n \to \mathbb{R}$, i.e. there exist two positive constants such that $0 < \underline{a} \le a(\boldsymbol{x}) \le \overline{a} < \infty$. Then, the equivalence kernel $\gamma_{\rm eq}(\boldsymbol{x}, \boldsymbol{y}; a^{\frac{1}{2}}\omega, \boldsymbol{\alpha})$ is such that

$$\frac{\underline{a}C_{n,s}}{2}\,|\boldsymbol{x}-\boldsymbol{y}|^{-n-2s} \le \gamma_{\rm eq}(\boldsymbol{x}, \boldsymbol{y}; a^{\frac{1}{2}}\omega, \boldsymbol{\alpha}) \le \frac{\overline{a}C_{n,s}}{2}\,|\boldsymbol{x}-\boldsymbol{y}|^{-n-2s}.$$

- Lemma 10 implies that the equivalence kernel is positive; in addition to symmetry, this property guarantees that the corresponding unweighted bilinear form $\mathcal{B}$ is a Dirichlet form
- We point out that the class of tensors in Lemma 10 corresponds to a space dependent isotropic diffusion as the intensity of the diffusion is the same in all directions.

Well-posedness of a parabolic equation with anisotropic nonlocal diffusion

The above results allow us to analyze the anisotropic parabolic problem. In fact, the coercivity of the bilinear form $\mathcal{B}_{\omega;A}$ implies the well-posedness of the corresponding parabolic problem, for which weak coercivity would be sufficient. We introduce the strong form of the anisotropic parabolic equation and, by using the Green's identity (39), we formulate the corresponding weak problem and state a well-posedness result.
For $f : \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$, we seek u such that

$$\begin{cases} \partial_t u(\boldsymbol{x},t) = \mathcal{D}_\omega(A(\boldsymbol{x})\mathcal{G}_\omega u(\boldsymbol{x},t)) + f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Omega \times (0,T] \\ u(\boldsymbol{x},t) = 0, & (\boldsymbol{x},t) \in \mathbb{R}^n \setminus \Omega \times (0,T] \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \end{cases} \tag{47}$$

By multiplying (47) by a test function $v = 0$ in $\mathbb{R}^n \setminus \Omega$, integrating over the domain Ω, and using the anisotropic Green's identity (39), we have

$$\begin{aligned} 0 &= \int_\Omega (\partial_t u(\boldsymbol{x},t) - \mathcal{L}_\omega u(\boldsymbol{x},t) - f(\boldsymbol{x},t))\, v(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int_\Omega \partial_t u(\boldsymbol{x},t)\, v(\boldsymbol{x})\, d\boldsymbol{x} + \int_{\mathbb{R}^n} \mathcal{G}_\omega u(\boldsymbol{x}) \cdot A(\boldsymbol{x}) \mathcal{G}_\omega u(\boldsymbol{x}) d\boldsymbol{x} - \int_\Omega f(\boldsymbol{x},t)\, v(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{48}$$

Weak form

For $f \in L^2(0,T;(V_\Omega^A)'(\mathbb{R}^n))$, and $u_o \in V_\Omega^A(\mathbb{R}^n)$, find $u \in L^2(0,T;V_\Omega^A(\mathbb{R}^n)$ such that

$$(\partial_t u, v) + \mathcal{B}_{\omega;A}(u,v) = \mathcal{F}(v), \ \ \forall\, v \in V_\Omega(\mathbb{R}^n). \tag{49}$$

When the equivalence kernel $\gamma_{\rm eq}(\boldsymbol{x},\boldsymbol{y};A^{\frac{1}{2}}\omega)$ associated with A is such that the unweighted bilinear form $\mathcal{B}$ is coercive, problem (49) is well-posed, as we state in the following theorem.

Theorem

For $f \in L^2(0,T;V'_\Omega(\mathbb{R}^n))$, $u_0 \in V_\Omega^A$ and $\mathcal{B}_{\omega;A}(\cdot,\cdot)$ such that the corresponding $\gamma_{\rm eq}(\boldsymbol{x},\boldsymbol{y};A^{\frac{1}{2}}\omega,\boldsymbol{\alpha})$ induces a weakly coercive and continuous unweighted for $\mathcal{B}(\cdot,\cdot)$, the problem (49) has a unique solution $u^* \in L^2(0,T;V_\Omega(\mathbb{R}^n))$, where $V_\Omega(\mathbb{R}^n)$ is the energy space associated with the bilinear form $\mathcal{B}(\cdot,\cdot)$. Furthermore, if $\mathcal{B}(\cdot,\cdot)$ is coercive and the associated energy norms satisfies a Poincaré inequality with constant C_p, that solution satisfies the a priori estimate

$$\|u^*(\cdot,t)\|^2_{L^2(\Omega)} + C_{\rm coer}\int_0^t |||u^*(\cdot,s)|||^2\, ds \le \|u_0\|^2_{L^2(\Omega)} + \frac{C_p^2}{2C_{\rm coer}}\int_0^t \|f(\cdot,s)\|^2_{V'_\Omega}\, ds \quad \forall\, t>0, \tag{50}$$

where $\|\cdot\|_{V'_\Omega}$ indicates the standard operator norm in the dual space of $V_\Omega(\mathbb{R}^n)$ and $C_{\rm coer}$ is the coercivity constant of the bilinear form $\mathcal{B}(\cdot,\cdot)$.

Well-posedness

- We have shown that for a tensor A satisfying (33), and for ω and $\boldsymbol{\alpha}$ as in (29), the equivalence kernel associated with $\mathcal{D}_\omega(A(\boldsymbol{x})\mathcal{G}_\omega)$ induces an unweighted bilinear form $\mathcal{B}(\cdot,\cdot)$, whose energy norm is equivalent to the H^s-norm. This implies that $\mathcal{B}(\cdot,\cdot)$ is coercive and continuous on $H^s_\Omega(\mathbb{R}^n)$.
- Thus, Theorem 11 can be immediately applied to the special case of fractional operators, as we show in the following corollary. Note that, in this case, the unweighted energy norm corresponds to the H^s norm for which the Poincaré inequality is satisfied for all $u \in H^s_\Omega(\mathbb{R}^n)$.

Corollary

Let A be a tensor satisfying (33), and let ω and $\boldsymbol{\alpha}$ be defined as in (29). For $f \in L^2(0,T;(H^s_\Omega(\mathbb{R}^n))')$ and $u_0 \in H^s_\Omega(\mathbb{R}^n)$, the problem

$$(\partial_t u, v) + \mathcal{B}_{\omega;A}(u,v) = \mathcal{F}(v), \ \ \forall\, v \in H^s_\Omega(\mathbb{R}^n), \tag{51}$$

has a unique solution $u^* \in L^2(0,T;H^s_\Omega(\mathbb{R}^n))$ that satisfies the estimate (50) for $|||\cdot||| = \|\cdot\|_{H^s_\Omega}$ and $C_{\rm coer}$ as in (46).

The anisotropic anomalous transport equation and its well-posedness

- We extend the anisotropic fractional diffusion model to an advection-diffusion model that takes into account the presence of drift.
- We assume the advection field to be a given solenoidal field $\mathbf{v}$; in general, such a field is the solution of Darcy's equation.
- Let A be a bounded, measurable and elliptic tensor and $\mathbf{v}$ be a bounded, solenoidal vector, i.e. $\|\mathbf{v}\|_{L^\infty(\Omega)} \le C_v < \infty$ and $\nabla \cdot \mathbf{v} = 0$. For ω and $\boldsymbol{\alpha}$ defined as in (29), $f : \Omega \to \mathbb{R}$, $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}^n$, the strong form of the anomalous transport problem is defined as follows

$$\begin{cases} \partial_t u(\boldsymbol{x},t) = \mathcal{D}_\omega(A(\boldsymbol{x})\mathcal{G}_\omega u(\boldsymbol{x},t)) - \mathbf{v}(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x},t) + f(\boldsymbol{x},t), & (\boldsymbol{x},t) \in \Omega \times (0,T] \\ u(\boldsymbol{x},t) = g, & (\boldsymbol{x},t) \in \mathbb{R}^n \setminus \Omega \times (0,T] \\ u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega \end{cases} \tag{52}$$

- The anisotropic Green's first identity allows us to write the weak formulation of (52).

Weak formulation

- For the sake of simplicity, we analyze the weak form for homogeneous volume-constraints, i.e., $g \equiv 0$.
- We restrict the fractional order to $s \in [0.5, 1)$, in order to guarantee the coercivity of the problem in presence of advection.
- For $s \in [0.5, 1)$, $f \in L^2(0, T; H^{-s}(\mathbb{R}^n))$, and $u_0 \in H^s_\Omega(\mathbb{R}^n)$, we seek $u \in L^2(0, T; H^s(\mathbb{R}^n))$ such that

$$(\partial_t u, v) + \mathcal{B}_{\omega;A}(u, v) + (\mathbf{v} \cdot \nabla u, v) = \mathcal{F}(v), \quad \forall\, v \in H^s_\Omega(\mathbb{R}^n), \tag{53}$$

  where $\mathcal{B}_{\omega;A}$ is the bilinear form defined in (42).
- The following lemma shows that the bilinear form $\mathcal{B}_{\omega;A}(u, v) + (\mathbf{v} \cdot \nabla u, v)$ is coercive. Its proof is a combination of equation (45) and Proposition 3 in Bonito et al. (2020).

Coercivity of bilinear form with advection

Lemma

Let the fractional order $s \in [0.5, 1)$ and $\lambda_{\min}$ be the smallest eigenvalue of the tensor A over $\mathbb{R}^n$. If the advection field $\mathbf{v}$ is bounded and solenoidal, then the bilinear form $\mathcal{B}'(u, v) = \mathcal{B}_{\omega;A}(u, v) + (\mathbf{v} \cdot \nabla u, v)$ is coercive. In particular,

$$\mathcal{B}'(u, u) = \mathcal{B}_{\omega;A}(u, u) \geq \frac{C_{n,s}}{2} \lambda_{\min} \|u\|^2_{H^s_\Omega}.$$

- Arguments similar to Corollary 12 imply the well-posedness of problem (53).

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- Thank you for your time and attention!