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# A framework to evaluate IMEX schemes for atmospheric models

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Based on our GMD paper,
https://gmd.copernicus.org/articles/13/6467/2020/gmd-13-6467-2020.html

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# Formulation of nonhydrostatic HOMME (theta-l nonhydrostatic dycore) is suitable for horizontally explicit, vertically implicit time integration

$$\mathbf{u}_t + (\nabla_s \times \mathbf{u} + 2\Omega) \times \mathbf{u} + \frac{1}{2}\nabla_s \mathbf{u}^2 + \dot{s}\mathbf{u}_s + \frac{1}{\kappa}\theta_v \nabla_s \Pi + \mu \nabla_s \phi = 0, \text{ horizontal momentum}$$

$$w_t + \mathbf{u} \cdot \nabla_s w + \dot{s} w_s \boxed{+ g(1-\mu)} = 0, \text{ vertical momentum}$$

$$\phi_t + \mathbf{u} \cdot \nabla_s \phi + \dot{s}\phi_s \boxed{- gw} = 0, \text{ geopotential}$$

$$\Theta_t + \nabla_s \cdot (\mathbf{u}\Theta) + (\dot{s}\Theta)_s = 0, \text{ potential temperature}$$

$$(\pi_s)_t + \nabla_s \cdot (\mathbf{u}\pi_s) + (\dot{s}\pi_s)_s = 0, \text{ pseudodensity}$$

$$\pi - \text{hydrostatic pressure}, p - \text{nonhydrostatic pressure},$$

$$EOS: \phi_s = -\Theta \frac{\Pi}{p}, \Theta = \pi_s \theta_v, \mu = \frac{p_s}{\pi_s}, \text{ equation of state}$$

Terms in red frames are the ones solved 'implicitly', since they are responsible for acoustic waves in vertical direction and lead to a very restrictive CFL.

We use Implicit-Explicit Runge-Kutta timestepping methods to integrate in time.

# Motivation: A simple tool to evaluate IMEX timestepping schemes for nonhydrostatic HOMME

- Nonhydrostatic HOMME needs new IMplicit-EXplicit (IMEX) Runge-Kutta methods to integrate acoustic waves in vertical
- We develop an offline tool to investigate stability, dispersion, and dissipation of IMEX
- The tool is also used to find the optimal IMEX schemes

Previously, offline tools for IMEX were based on an idealized setup, a **2D acoustic system**:

$$
\begin{aligned}
u_t &= -\frac{\partial p}{\partial x} \\
w_t &= -\frac{\partial p}{\partial z} \\
p_t &= -c_s^2 \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right)
\end{aligned}
$$

We made a tool based on **system of normal modes**, also idealized, but with more complexity:

$$
\begin{aligned}
u_t &= fv - \left( \frac{1}{\rho^r} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} \right) \\
v_t &= -fu - \left( \frac{1}{\rho^r} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial y} \right) \\
w_t &= -g \frac{\sigma}{\sigma^r} - \frac{1}{\sigma^r} \frac{\partial p}{\partial \theta} \\
\phi_t &= gw \\
\sigma_t &= -\sigma^r \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{aligned}
$$

# The new tool is more complex than the previously used, incorporating all waves

**Previous 2D acoustic system:**
Contains only 2 acoustic modes and 1 artificial gravity mode; is linearized around constant density profile; thermodynamics as in dynamical cores is not supported.

$$u_t = -\frac{\partial p}{\partial x}$$
$$w_t = -\frac{\partial p}{\partial z}$$
$$p_t = -c_s^2 \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right)$$

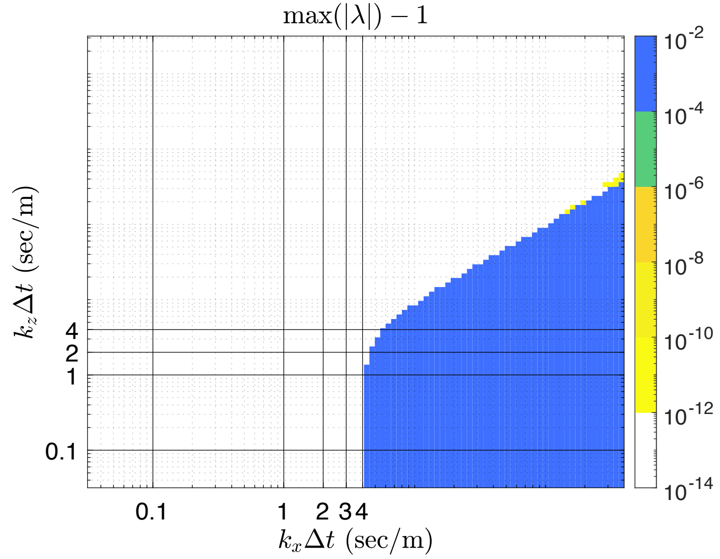
**New system of normal modes:**
Contains full set of atmospheric modes: 2 acoustic, 2 gravity, 1 Rossby; is linearized around steady hydrostatic profile with const T and variable pressure; allows various realistic boundary conditions at the top of the model; contains Coriolis terms; allows any set of thermodynamic variables.

$$\begin{aligned}
u_t &= fv - \left( \frac{1}{\rho^r} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} \right) \\
v_t &= -fu - \left( \frac{1}{\rho^r} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial y} \right) \\
w_t &= -g \frac{\sigma}{\sigma^r} - \frac{1}{\sigma^r} \frac{\partial p}{\partial \theta} \\
\phi_t &= gw \\
\sigma_t &= -\sigma^r \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{aligned}$$

implicit terms

# Stability plots and their axes

More traditional plot, axes are Courant numbers or similar. White region is stable, it depends on pairs (k_x, k_z).




$$u_t = -\frac{\partial p}{\partial x}$$

$$w_t = -\frac{\partial p}{\partial z}$$

$$p_t = -c_s^2 \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right)$$

System supports solution

$$\mathbf{y} = \mathbf{y}_0 e^{i(k_x x + k_z z - \omega t)}$$

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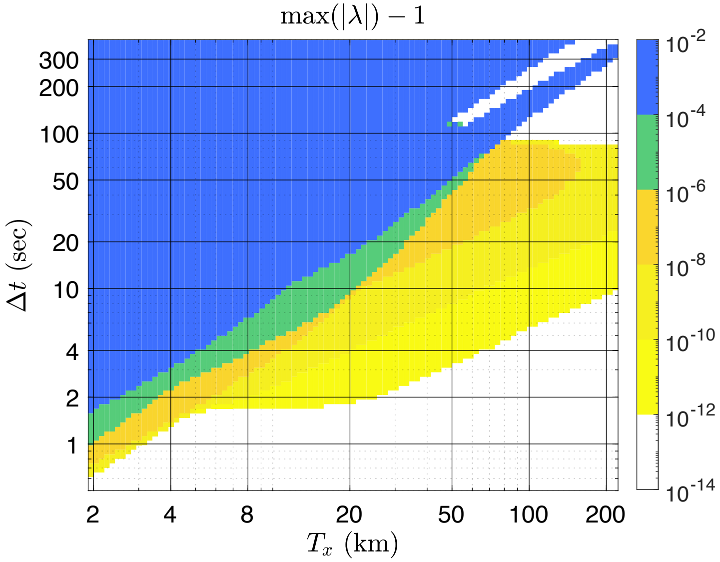
In our plot, each k_x corresponds to not one but a a set of k_z.
For axes, we use wavelength T_x = 2 pi / k_x and timestep dt.




$$
\begin{aligned}
u_t &= fv - \left( \frac{1}{\rho^r} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} \right) \\
v_t &= -fu - \left( \frac{1}{\rho^r} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial y} \right) \\
w_t &= -g \frac{\sigma}{\sigma^r} - \frac{1}{\sigma^r} \frac{\partial p}{\partial \theta} \\
\phi_t &= gw \\
\sigma_t &= -\sigma^r \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\end{aligned}
$$

In both plots, white regions are stable, non-white are unstable. Plotted are max(|λ|) - 1 of timestepping operator.

# The new tool is more selective than the previously used

We pick a reasonable IMEX scheme based on one of low storage, high CFL Kinnmark-Grey explicit Runge-Kutta methods.
**Old framework diagnoses this method as stable** for all resolutions and reasonable timesteps (approx. defined by explicit table).

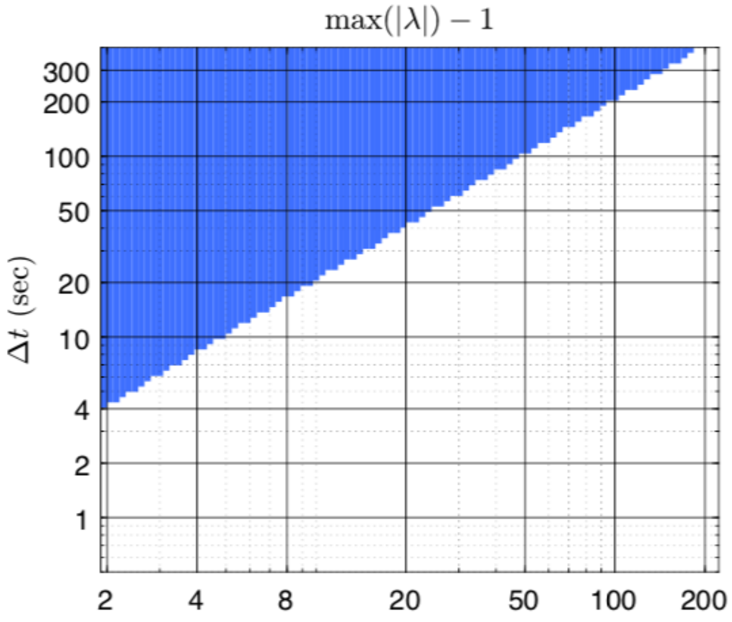
| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1/5 | 1/5 | 0 | 0 | 0 | 0 | 0 |
| 1/5 | 0 | 1/5 | 0 | 0 | 0 | 0 |
| 1/3 | 0 | 0 | 1/3 | 0 | 0 | 0 |
| 1/2 | 0 | 0 | 0 | 1/2 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 1 | 0 |

explicit table

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1/5 | 0 | 1/5 | 0 | 0 | 0 | 0 |
| 1/5 | 0 | 0 | 1/5 | 0 | 0 | 0 |
| 1/3 | 0 | 0 | 0 | 1/3 | 0 | 0 |
| 1/2 | 0 | 0 | 0 | 0 | 1/2 | 0 |
| 1 | 5/18 | 5/18 | 0 | 0 | 0 | 8/18 |
| | 5/18 | 5/18 | 0 | 0 | 0 | 8/18 |

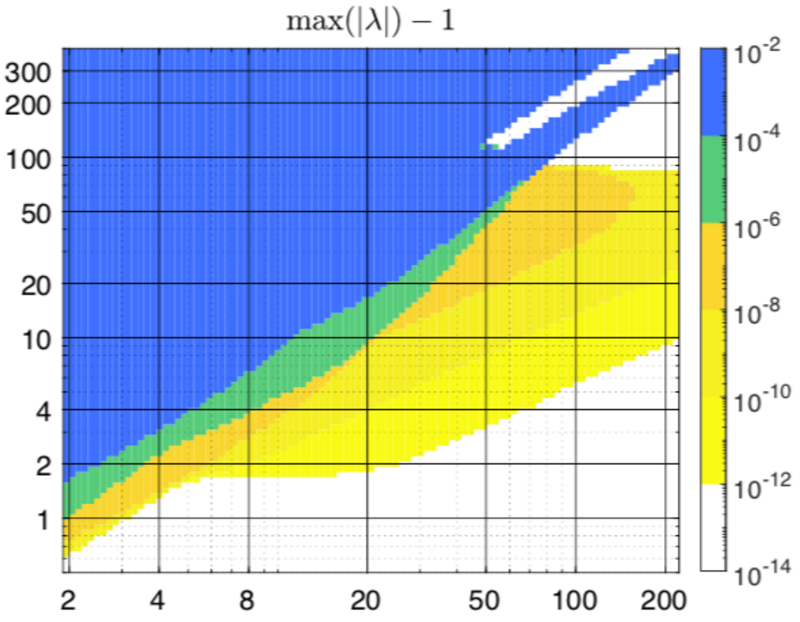
implicit table

## Old stability diagram




horizontal wavelength in km

## New stability diagram




horizontal wavelength in km

To analyze stability we form a matrix that represents one time step and look at the matrix's spectrum.

Stable regions are white.
Plotted is maximum by abs. value eigenvalue minus 1, to highlight magnitudes of eigenvalues bigger than 1+ tolerance.
**New framework shows that the above method is largely unstable.**

# Use the new tool to develop IMEX schemes

We pick a second order, low storage, high CFL explicit Runge-Kutta method and construct an implicit table for it. The last row of the implicit table is allowed to vary.

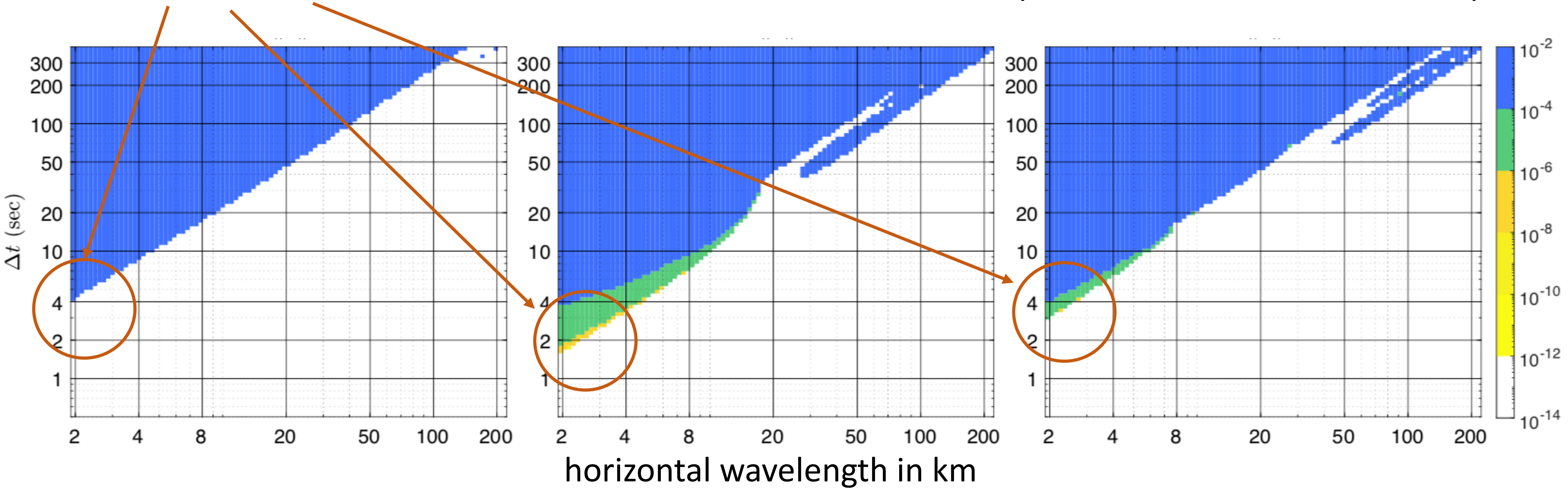
**Based on different coefficients *d* stability properties change significantly.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1/4 | 1/4 | 0 | 0 | 0 | 0 | 0 |
| 1/6 | 0 | 1/6 | 0 | 0 | 0 | 0 |
| 3/8 | 0 | 0 | 3/8 | 0 | 0 | 0 |
| 1/2 | 0 | 0 | 0 | 1/2 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 1 | 0 |

explicit table

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1/4 | 0 | 1/4 | 0 | 0 | 0 | 0 |
| 1/6 | 0 | 0 | 1/6 | 0 | 0 | 0 |
| 3/8 | 0 | 0 | 0 | 3/8 | 0 | 0 |
| 1/2 | 0 | 0 | 0 | 0 | 1/2 | 0 |
| 1 | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |

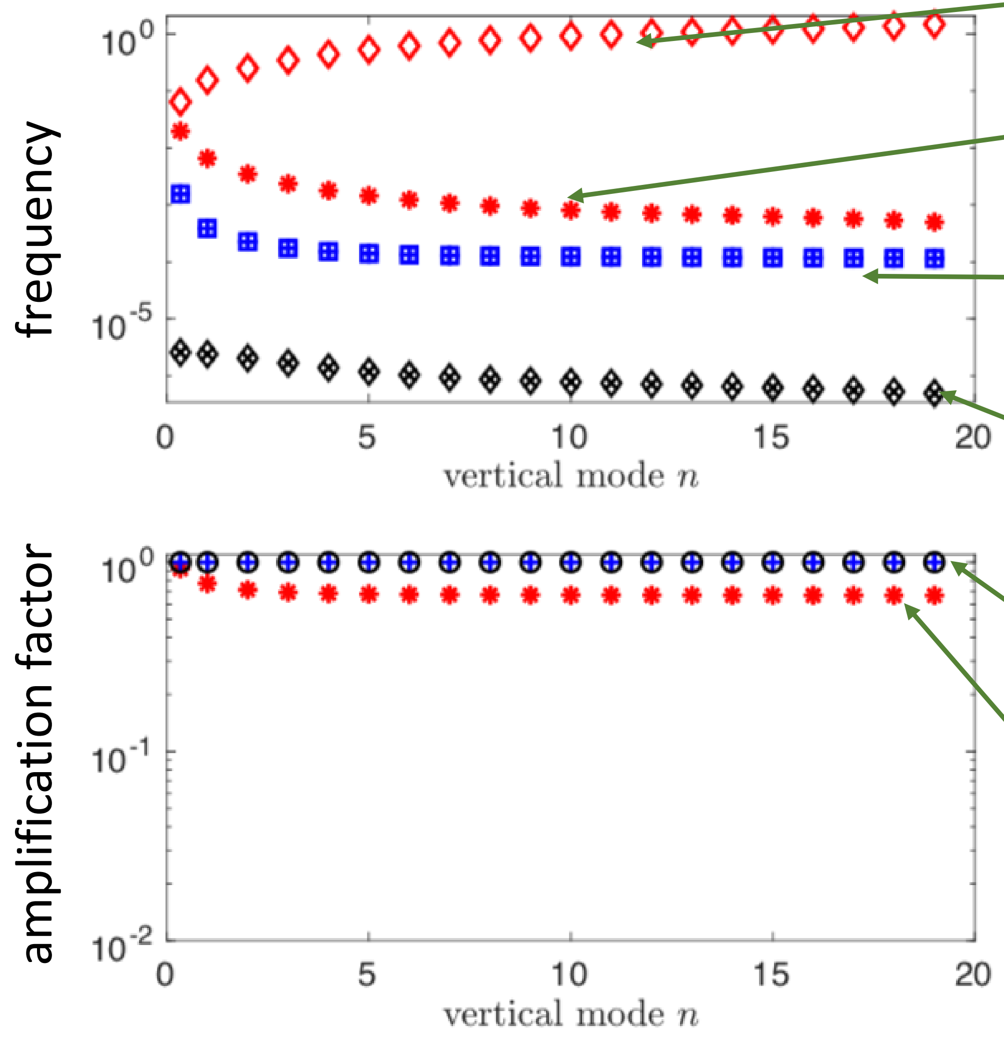
implicit table




**Stability is not the only property that varies ….**

# Dispersion and dissipation for all types of waves, example of diagnostics

Besides stability, we can recover numerical dispersion and dissipation for all 3 types of waves for a particular IMEX method.




'analytical' frequencies for **acoustics**

numerical frequencies for **acoustics**

'analytical' and numerical frequencies coincide for **gravity** waves

'analytical' and numerical frequencies coincide for **Rossby** waves

numerical amplification factors for **gravity and Rossby** waves are very close to 1

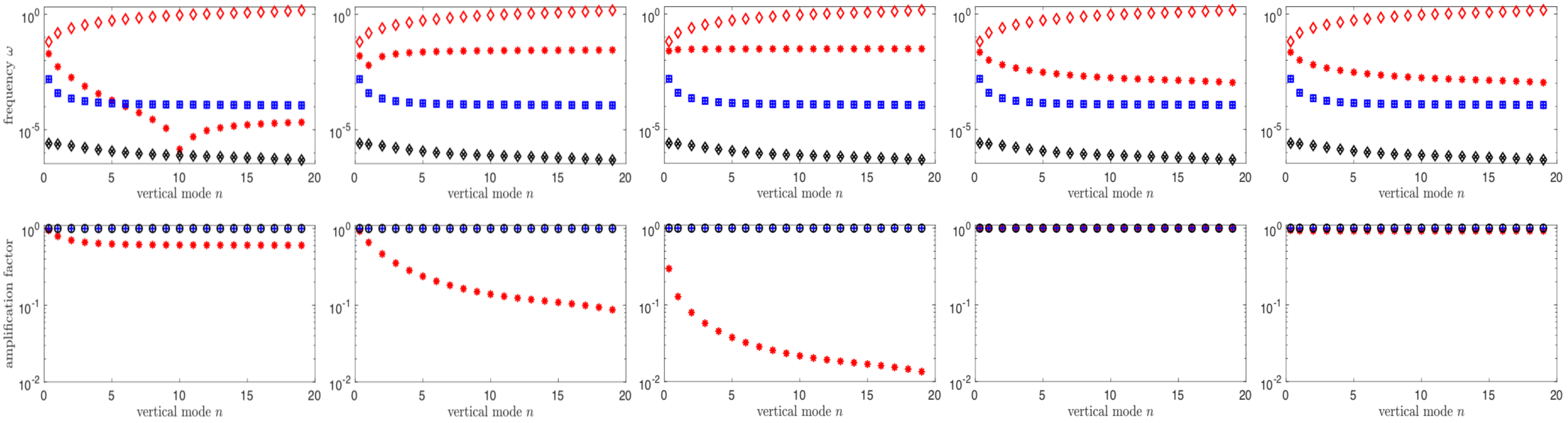
numerical amplification factors for **acoustic** waves measure dissipation of IMEX

# Dispersion and dissipation, there is a variety of properties in IMEX methods

Recall our method where the last row of the implicit table varies:

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1/4 | 1/4 | 0 | 0 | 0 | 0 | 0 |
| 1/6 | 0 | 1/6 | 0 | 0 | 0 | 0 |
| 3/8 | 0 | 0 | 3/8 | 0 | 0 | 0 |
| 1/2 | 0 | 0 | 0 | 1/2 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 1 | 0 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1/4 | 0 | 1/4 | 0 | 0 | 0 | 0 |
| 1/6 | 0 | 0 | 1/6 | 0 | 0 | 0 |
| 3/8 | 0 | 0 | 0 | 3/8 | 0 | 0 |
| 1/2 | 0 | 0 | 0 | 0 | 1/2 | 0 |
| 1 | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
| | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |




Moderately dissipative, bad dispersion of acoustics, 2nd order

Very dissipative, acceptable dispersion of acoustics, 2nd order

Most dissipative, good dispersion of acoustics, 1st order

Nondissipative, acceptable dispersion of acoustics, 2nd order

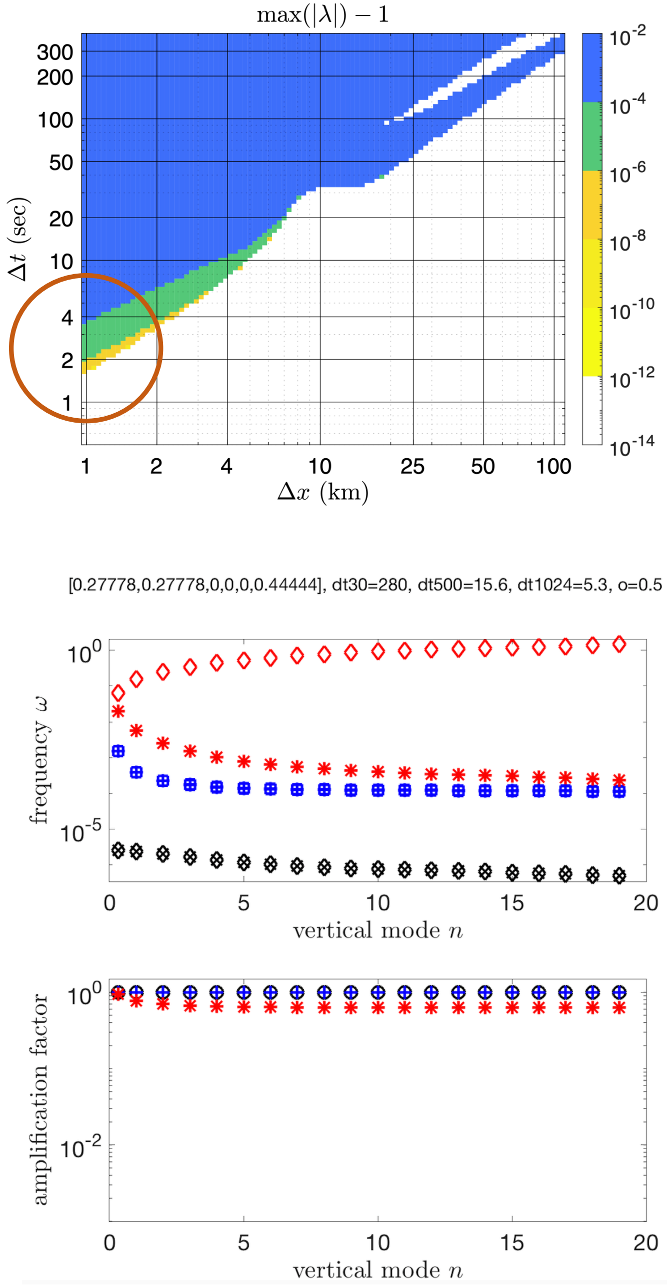
Almost nondissipative, acceptable dispersion of acoustics, 1st order

# Empirical search for the 3rd order explicit table

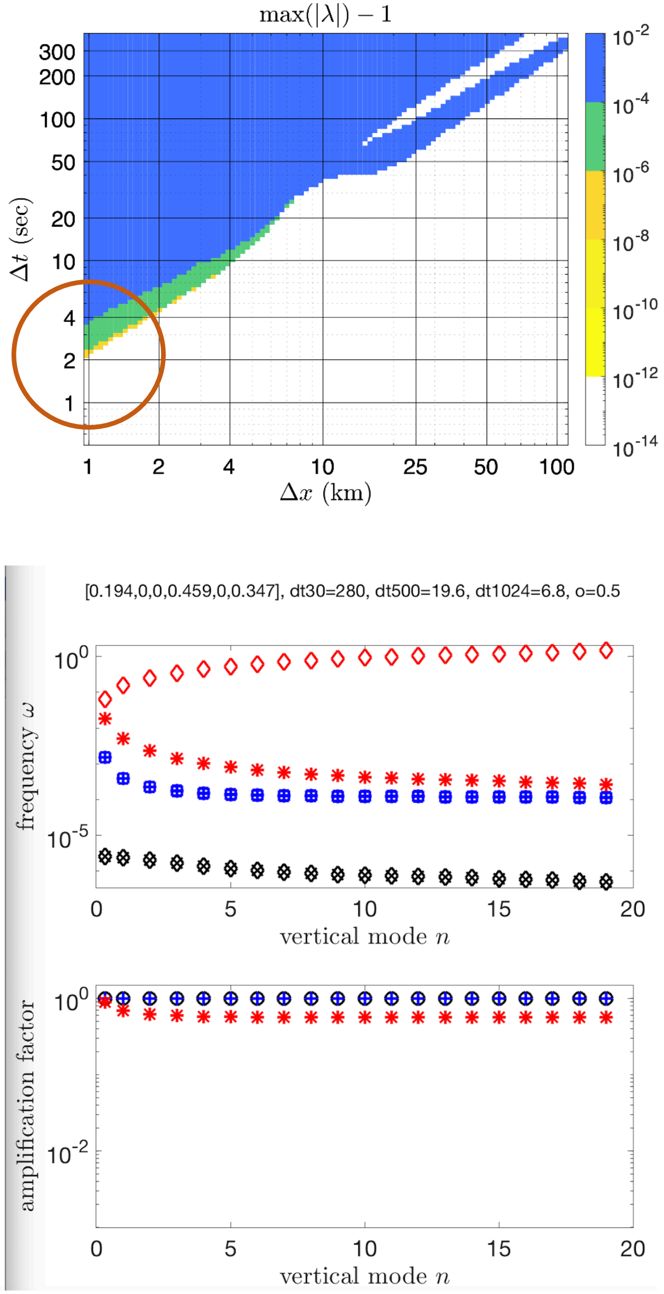
Using random search and quad programming we search for vector d with best properties.

Old vs new max stable dt for wavelength 6 km are 5.3 sec vs 6.8 sec, 28% difference.

Old: d=[5/18,5/18,0,0,0,8/18]




New: d=[.194,0,0,.459,0,.347]

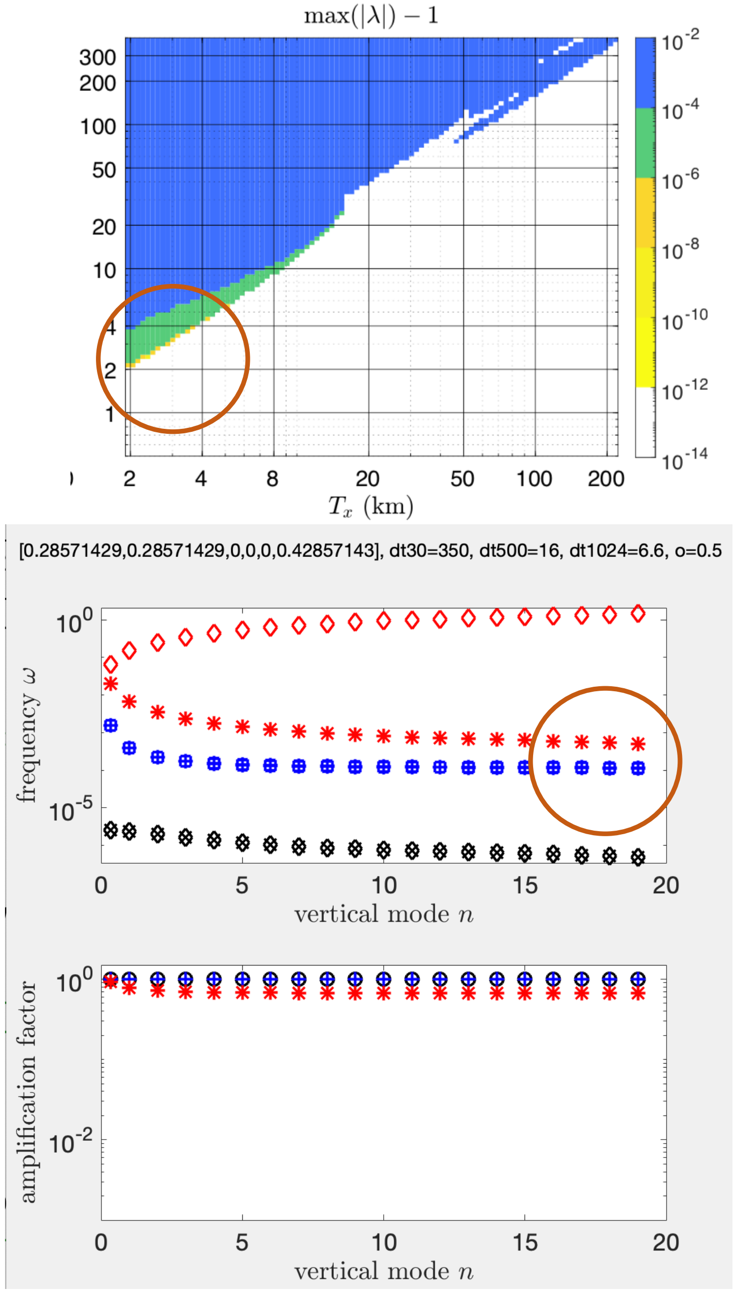


# Empirical search for the 2nd order explicit table

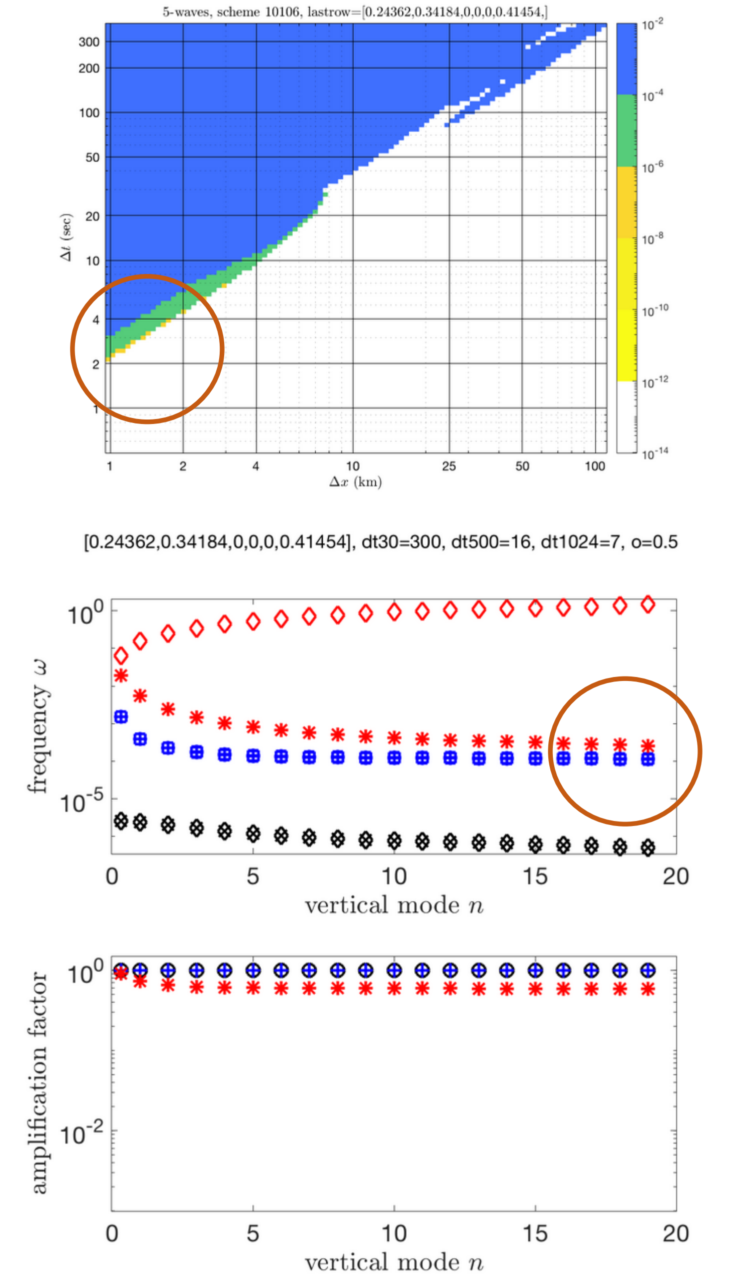
Old vs new max stable dt for wavelength 6 km are 6.6 sec vs 7 sec, 28% difference.

Not a big gain, but without search it is not clear if there is a better 2nd order method.

Old: d=[2/7,2/7,0,0,0,4/11]



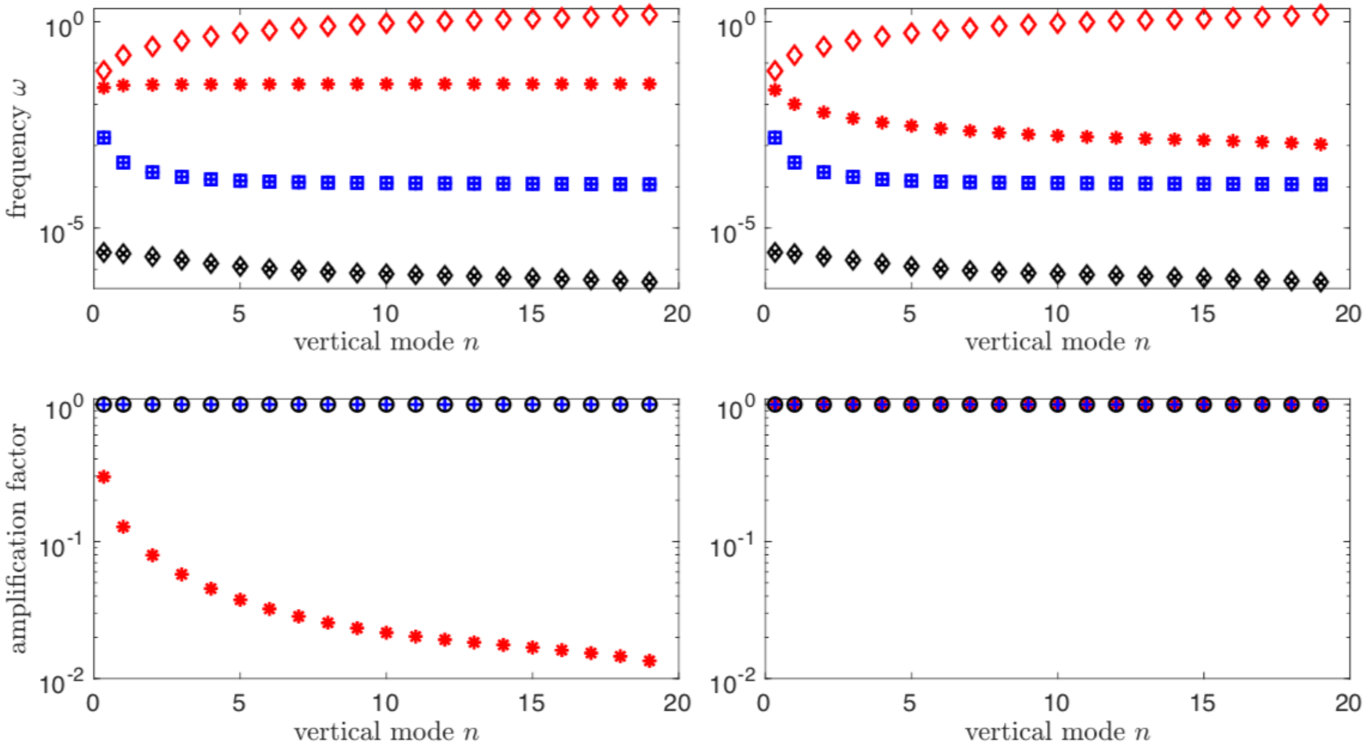

New: d=[.24362,.34184,0,0,0,.41454]



# How much does dispersion and dissipation matter for IMEX methods?

**Reasonable question:** Don't we want to completely dissipate acoustic waves?
**Our answer:** Probably not. All waves, including acoustic and gravity waves, need to be represented accurately to accurately restore hydrostatic balance (Thuburn 2012).

We are still searching for more realistic setup or a test case where we would see difference between IMEX methods with dramatically different dispersion and dissipation properties, like here:




LEFT PANEL: most dissipative for acoustics and most stable, 1st order, good dispersion.
RIGHT PANEL: nondissipative for acoustics, 2nd order, acceptable dispersion.

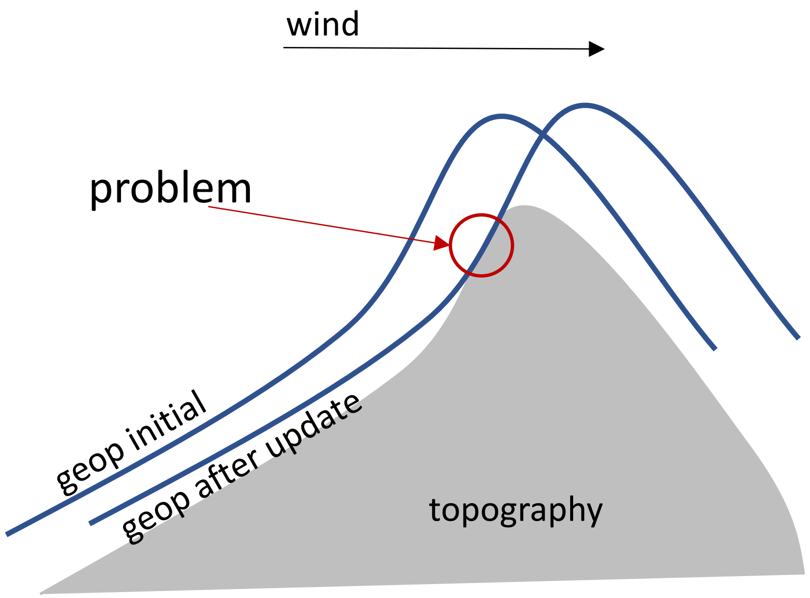
# Besides linear analysis, many other factors, for example, topography

Geopotential eqn is almost tracer eqn:

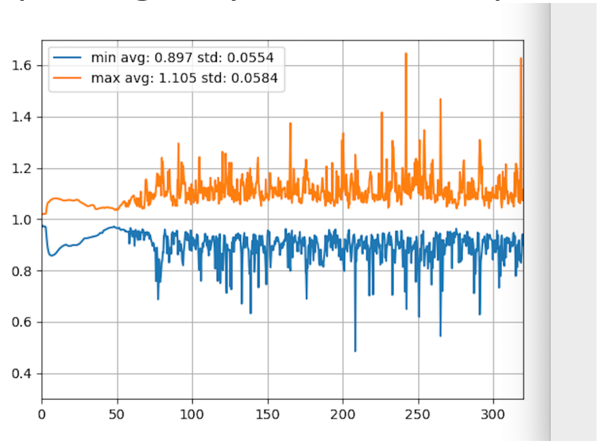
$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla_s \phi + \dot{s}\frac{\partial \phi}{\partial s} - gw = 0$$

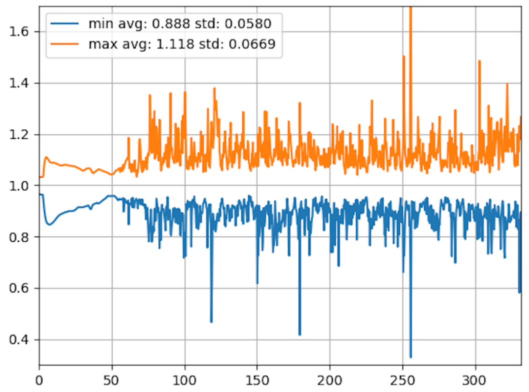
This required some sort of splitting:

$$\frac{\partial \phi}{\partial t} + \underbrace{\mathbf{u} \cdot \nabla_s \phi - \mathbf{u} \cdot \nabla_s \phi_{surf} + \dot{s}\frac{\partial \phi}{\partial s}}_{\text{explicit terms}} = \underbrace{gw - \mathbf{u} \cdot \nabla_s \phi_{surf}}_{\text{implicit terms}}$$




Splitting may affect IMEX performance:




Left is the less dissipative, less stable method compared to the right. Plotted is the measure of NH flow for 1 degree resolution in Held-Suarez test case with topography:

$$\mu = \frac{\partial p_{nonhydro}}{\partial p_{hydro}}$$

That a more dissipative method is 'less hydrostatic' in this case may or may not be a desirable result.