Nonlocal operators with variable coefficient

SAND No. XXXXXXXXX

SAND2021-0398C

Motivation

- Fractional-order operators have long history in subsurface applications.
- Encode self-similar material at the macro level.
- Commonly used constant coefficient kernels cannot adequately model material layers.

Variable coefficient fractional-type kernels

$$\text{p. v.} \int_{\Omega\cup\Omega_I} (u(\mathbf{x}) - u(\mathbf{y}))\gamma(\mathbf{x},\mathbf{y}) d\mathbf{y} = f(\mathbf{x}) \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^d,$$

$$u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Omega_I = (\Omega + B_\delta) \setminus \Omega.$$

with kernel

$$\gamma(\mathbf{x},\mathbf{y}) = \frac{\phi(\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+2s(\mathbf{x},\mathbf{y})}} \mathcal{X}_{|\mathbf{x}-\mathbf{y}|\le\delta}$$

Well-posedness

Let $\gamma_s(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\gamma(\mathbf{x},\mathbf{y}) + \gamma(\mathbf{x},\mathbf{y}))$ and set

$$\|v\|_\gamma^2 := \iint_{(\Omega\cup\Omega_I)^2} (v(\mathbf{x}) - v(\mathbf{y}))^2 \gamma_s(\mathbf{x},\mathbf{y})\, d\mathbf{x}\, d\mathbf{y},$$

$$H_\Omega(\Omega\cup\Omega_I;\gamma) := \left\{ v \in L^2(\Omega\cup\Omega_I), \|v\|_\gamma < \infty, v|_{\Omega_I} = 0 \right\},$$

Assume existence of constants $\underline{s}, \overline{s}$, and $\underline{\phi}, \overline{\phi}$ such that

$$0 < \underline{s} \le s(\mathbf{x},\mathbf{y}) \le \overline{s} < 1,$$

$$0 < \underline{\phi} \le \phi(\mathbf{x},\mathbf{y}) \le \overline{\phi} < \infty.$$

Let

$$\beta(r) := \sup_{|\mathbf{x}-\mathbf{y}|\le r} |s(\mathbf{x},\mathbf{y}) - s(\mathbf{y},\mathbf{x})|, \quad \alpha(r) := \sup_{|\mathbf{x}-\mathbf{y}|\le r} |\phi(\mathbf{x},\mathbf{y}) - \phi(\mathbf{y},\mathbf{x})|.$$

Require

$$\int_0^1 dr\, \frac{(\beta(r)\,|\log r|)^2}{r^{1+2\overline{s}}} < \infty$$

and that α is $(\overline{s}+\varepsilon)$-Hölder continuous for $\varepsilon > 0$ arbitrarily small.
Then the problem is well-posed.

Two domain examples

$\Omega = [-1,1]$, $\delta = 1$, $f = \mathcal{X}_{[0.2,0.4]}$, $\mathbb{P}_1$ finite elements

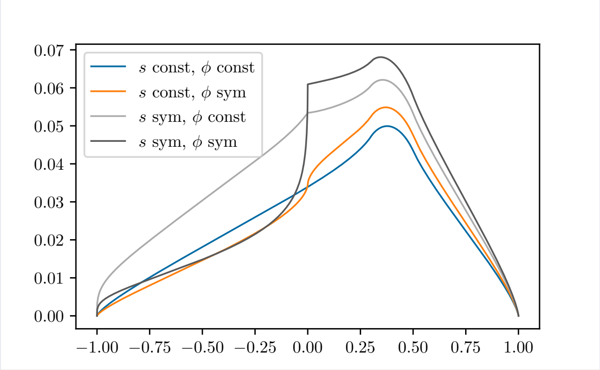


Figure: Solutions for symmetric kernels, piecewise on $(\mathbb{R}_+, \mathbb{R}_-)$

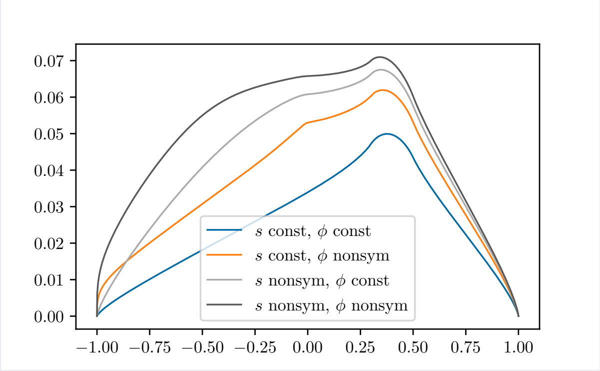


Figure: Solutions for non-symmetric kernels with $s = s(\mathbf{x})$, $\phi = \phi(\mathbf{x})$ smooth

- Constant s leads to differentiable solutions at the interface.
- Non-symmetric kernels require (even more) specialized quad

Convergence

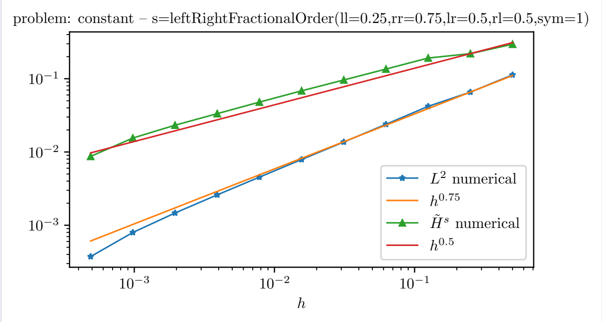
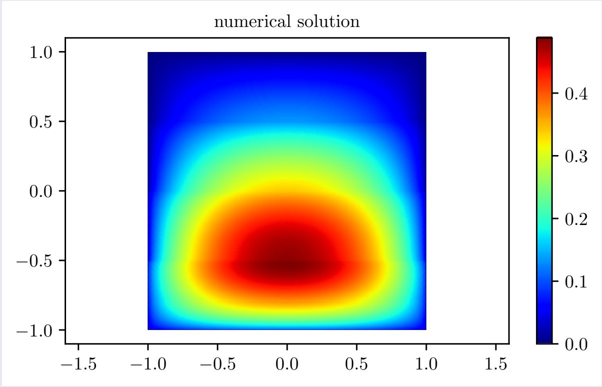


Figure: Convergence in L^2 and energy norm matches equivalent result for constant coefficient kernels.

2D example

- $\Omega = [-1,1]^2$, $\delta = 1/2$, $f \equiv 1$
- $\mathcal{H}$-matrices for efficient assembly & solve
- 4 horizontal layers with increasing fractional order



Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.