SANDIA REPORT
SAND2022-0051
Printed January 2022

Sandia National Laboratories

Penetration Bounds For Azimuthal Slot On Infinite Cylinder With Finite Length Backing Cylindrical Cavity

Larry K. Warne, Salvatore Campione, Luis San Martin, Alden R. Pack, W. L. Langston, and Brian Zinser

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico
87185 and Livermore,
California 94550

Issued by Sandia National Laboratories, operated for the United States Department of Energy by National Technology & Engineering Solutions of Sandia, LLC.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone: (865) 576-8401
Facsimile: (865) 576-5728
E-Mail: reports@osti.gov
Online ordering: http://www.osti.gov/scitech

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5301 Shawnee Rd
Alexandria, VA 22312

Telephone: (800) 553-6847
Facsimile: (703) 605-6900
E-Mail: orders@ntis.gov
Online order: https://classic.ntis.gov/help/order-methods/

SAND2022-0051

Unlimited Release

Printed January 2022

Penetration Bounds For Azimuthal Slot On Infinite Cylinder With Finite Length Backing Cylindrical Cavity

L. K. Warne, S. Campione, L. San Martin, A. Pack, W. L. Langston, and B. Zinser
Electromagnetic Theory & Simulation Dept.
Sandia National Laboratories
P. O. Box 5800
Albuquerque, NM 87185-1152

Abstract

We examine coupling into azimuthal slots on an infinite cylinder with a finite length interior cavity operating both at the fundamental cavity modal frequencies, with small slots and a resonant slot, as well as higher frequencies. The coupling model considers both radiation on an infinite cylindrical exterior as well as a half space approximation. Bounding calculations based on maximum slot power reception and interior power balance are also discussed in detail and compared with the prior calculations. For higher frequencies limitations on matching are imposed by restricting the loads ability to shift the slot operation to the nearest slot resonance; this is done in combination with maximizing the power reception as a function of angle of incidence. Finally, slot power mismatch based on limited cavity load quality factor is considered below the first slot resonance.

This page left blank

Contents

List of Figures

This page left blank

1 INTRODUCTION

We discuss plane wave coupling to cylindrical cavities through a narrow azimuthal slot aperture as shown in Figure 1. The interior cavity radius is a and the exterior radius is b, with interior cavity height h_c, with azimuthal slot at height z_0 from the bottom of the cavity, and a narrow slot aperture with azimuthal length $\ell = 2h$. The exterior is treated rigorously as an infinite cylinder or approximately as a planar half space. Approximate treatments of such cavities have been done in rectangular geometry [1], [2] as well as cylindrical geometry [3], [4], including the case where intentional absorbing material is present [5], which can be treated by means of perturbation theory [6]. In this report the cylindrical coupling model is set up rigorously, and includes modifications of the slot voltage distribution from the cavity mode resonances. The usual method to treat azimuthal slots in a cylinder uses both types of Hertz potential [7]. In this report we use a different approach involving the full electric vector potential and scalar magnetic potential [8]. This alternative approach leads more directly to the desired integro-differential equations in the slot, which also facilitates the extraction of the dominant transmission line operator in the narrow slot; this form of the equation allows us to generalize the slot to have depth as well as wall losses and gaskets [9], [10]. A finite Fourier basis is used for the slot voltage distribution and the cylindrical exterior is compared to the planar exterior. Coupling to both cylindrical modes, Transverse Magnetic (TM) and Transverse Electric (TE) with respect to the cylinder axis, are treated.

Simplified analytic models are examined for the electrically short as well as resonant azimuthal slot apertures. A power balance bounding approach [11], [1], [3], [12] is then considered and compared with the rigorous cylindrical model.

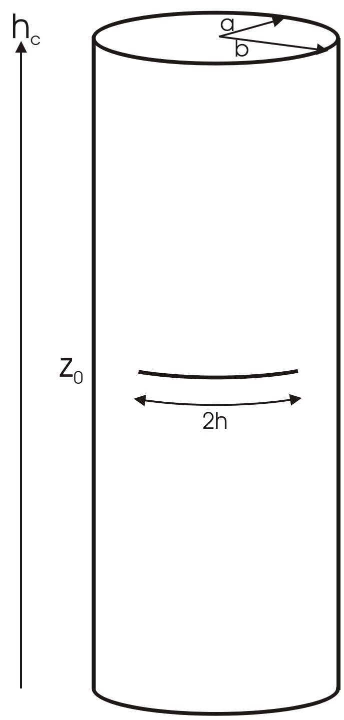

Figure 1: Geometry of cylindrical cavity with narrow azimuthal slot. The cavity has interior radius a and exterior radius b. The interior cavity height is h_c and the exterior cylinder is infinitely long. The slot has length $\ell = 2h$ and is positioned at height z_0 from the cavity bottom.

2 EXTERIOR PLANE WAVE DRIVE AND INFINITE CYLINDER CURRENT DENSITY

We first give the exterior cylindrical drives to the ports of entry.

2.1 Incident Plane Wave Fields

In the case of plane wave excitation we take a plane wave at oblique incidence with

$$\underline{k} = k\left(\underline{e}_x \cos\varphi_i \sin\theta_i + \underline{e}_y \sin\varphi_i \sin\theta_i + \underline{e}_z \cos\theta_i\right) = k\underline{e}_{r_i}$$

$$= k\left(\underline{e}_{\rho_i} \sin\theta_i + \underline{e}_z \cos\theta_i\right) = k_{\rho_i}\underline{e}_{\rho_i} + k_{z_i}\underline{e}_z$$

where the wavenumber is related to the magnetic permeability of free space μ_0 and electric permittivity of free space ε_0 (with time dependence $e^{-i\omega t}$ suppressed throughout this report)

$$k^2 = \omega^2\mu_0\varepsilon_0$$

On the cone at colatitude $\theta = \theta_i$ and azimuth $\varphi = \varphi_i$, the wavenumber is pointing in the $\underline{e}_{r_i}$ direction and we can take one perpendicular to the wavenumber direction to be $\underline{e}_{\theta_i}$ with the other being $\underline{e}_{\varphi_i}$, and thus we can write

$$\underline{E}_0 = E_0\left(\underline{e}_{\theta_i} \cos\varphi_p + \underline{e}_{\varphi_i} \sin\varphi_p\right)$$

$$= E_0\left[\underline{e}_x\left(\cos\theta_i\cos\varphi_i\cos\varphi_p - \sin\varphi_i\sin\varphi_p\right) + \underline{e}_y\left(\cos\theta_i\sin\varphi_i\cos\varphi_p + \cos\varphi_i\sin\varphi_p\right) - \underline{e}_z\sin\theta_i\cos\varphi_p\right]$$

$$= E_0\left(\underline{e}_{\rho_i}\cos\theta_i\cos\varphi_p + \underline{e}_{\varphi_i}\sin\varphi_p - \underline{e}_z\sin\theta_i\cos\varphi_p\right)$$

with

$$\underline{E}_0 \cdot \underline{k} = 0$$

where $\varphi_p = 0$ corresponds to the incident electric field alignment with $\underline{e}_{\theta_i}$ and $\varphi_p = \pi/2$ corresponds to the incident field alignment with $\underline{e}_{\varphi_i}$. The position vector is

$$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z = \rho\underline{e}_\rho + z\underline{e}_z$$

$$\underline{k}\cdot\underline{r} = k\rho\underline{e}_{\rho_i}\cdot\underline{e}_\rho + kz\cos\theta_i = k\rho\sin\theta_i\cos\left(\varphi - \varphi_i\right) + kz\cos\theta_i$$

The incident plane wave can then be written as

$$\underline{E}^{inc} = \underline{E}_0 e^{i\underline{k}\cdot\underline{r}} = E_0\left(\underline{e}_{\theta_i}\cos\varphi_p + \underline{e}_{\varphi_i}\sin\varphi_p\right)e^{ik\underline{e}_{r_i}\cdot\underline{r}}$$

$$= E_0\left(\underline{e}_{\rho_i}\cos\theta_i\cos\varphi_p + \underline{e}_{\varphi_i}\sin\varphi_p - \underline{e}_z\sin\theta_i\cos\varphi_p\right)e^{ik\rho\sin\theta_i\cos(\varphi-\varphi_i)+ikz\cos\theta_i}$$

$$= E_0\left(\underline{e}_{\rho_i}\cos\theta_i\cos\varphi_p + \underline{e}_{\varphi_i}\sin\varphi_p - \underline{e}_z\sin\theta_i\cos\varphi_p\right)e^{ik_{\rho_i}\rho\cos(\varphi-\varphi_i)+ik_{z_i}z}$$

The magnetic field is

$$i\omega\mu_0 \underline{H}^{inc} = i\underline{k} \times \underline{E}^{inc} = i\underline{k} \times \underline{E}_0 e^{i\underline{k}\cdot\underline{r}}$$

with vector amplitude

$$\underline{H}_0 = \frac{1}{\omega\mu_0}\underline{k} \times \underline{E}_0 = \frac{k}{\omega\mu_0}\underline{e}_{r_i} \times \underline{E}_0$$

$$= H_0 \underline{e}_{r_i} \times \left(\underline{e}_{\theta_i}\cos\varphi_p + \underline{e}_{\varphi_i}\sin\varphi_p\right) = H_0\left(\underline{e}_{\varphi_i}\cos\varphi_p - \underline{e}_{\theta_i}\sin\varphi_p\right)$$

$$= H_0\left[\underline{e}_x\left(-\cos\theta_i\cos\varphi_i\sin\varphi_p - \sin\varphi_i\cos\varphi_p\right) + \underline{e}_y\left(-\cos\theta_i\sin\varphi_i\sin\varphi_p + \cos\varphi_i\cos\varphi_p\right) + \underline{e}_z\sin\theta_i\sin\varphi_p\right]$$

where

$$\frac{\omega\mu_0}{k} = \eta_0$$

$$E_0 = \eta_0 H_0$$

and we can then write

$$\underline{H}^{inc} = H_0\underline{e}_{r_i} \times \left(\underline{e}_{\theta_i}\cos\varphi_p + \underline{e}_{\varphi_i}\sin\varphi_p\right)e^{i\underline{k}\cdot\underline{r}} = H_0\left(\underline{e}_{\varphi_i}\cos\varphi_p - \underline{e}_{\theta_i}\sin\varphi_p\right)e^{i\underline{k}\cdot\underline{r}}$$

$$= H_0\left(-\underline{e}_{\rho_i}\cos\theta_i\sin\varphi_p + \underline{e}_{\varphi_i}\cos\varphi_p + \underline{e}_z\sin\theta_i\sin\varphi_p\right)e^{ik_{\rho_i}\rho\cos(\varphi-\varphi_i)+ik_{z_i}z}$$

2.2 Cylindrical TM-TE Decomposition & Short Circuit Current Density

Let us consider this wave to be made up of a TM part and a TE part with respect to the z axis. Beginning with the TM part we write

$$E_z^{inc} = -E_0\sin\theta_i\cos\varphi_p e^{ik_{\rho_i}\rho\cos(\varphi-\varphi_i)+ik_{z_i}z}$$

We now use [8], [13]

$$e^{ik_\rho\rho\cos(\varphi-\varphi_i)} = \sum_{n=-\infty}^{\infty} i^n J_n\left(k_\rho\rho\right)e^{in(\varphi-\varphi_i)} = \sum_{m=0}^{\infty}\varepsilon_m i^m J_m\left(k_\rho\rho\right)\cos m\left(\varphi-\varphi_i\right)$$

where the Neumann numbers are defined as

$$\varepsilon_m = 2\ ,\ m \geq 1$$

$$= 1\ ,\ m = 0$$

to write

$$E_z^{inc} = -E_0\sin\theta_i\cos\varphi_p e^{ik_{z_i}z}\sum_{m=0}^{\infty}\varepsilon_m i^m J_m\left(k_{\rho_i}\rho\right)\cos m\left(\varphi-\varphi_i\right)$$

Noting that

$$E_z\left(\rho=b\right)=0$$

we find the total field

$$E_z=-E_0\sin\theta_i\cos\varphi_p e^{ik_{z_i}z}\sum_{m=0}^{\infty}\varepsilon_m i^m\left[J_m\left(k_{\rho_i}\rho\right)-\frac{J_m\left(k_{\rho_i}b\right)}{H_m^{(1)}\left(k_{\rho_i}b\right)}H_m^{(1)}\left(k_{\rho_i}\rho\right)\right]\cos m\left(\varphi-\varphi_i\right)$$

Then we can find the other field components by the procedure of separating out the axial and transverse parts

$$\nabla=\nabla_t+\underline{e}_z\frac{\partial}{\partial z}$$

$$\underline{E}=\underline{E}_t+E_z\underline{e}_z$$

and for the TM field from Maxwell's equations

$$\nabla_t\times\underline{E}_t=0$$

$$\left(\underline{e}_z\frac{\partial}{\partial z}\right)\times\underline{E}_t+\nabla_t\times\left(\underline{e}_zE_z\right)=i\omega\mu_0\underline{H}_t$$

$$\nabla_t\times\underline{H}_t=-i\omega\varepsilon_0\underline{e}_zE_z$$

$$\left(\underline{e}_z\frac{\partial}{\partial z}\right)\times\underline{H}_t=-i\omega\varepsilon_0\underline{E}_t$$

Using the fourth equation in the second

$$\left(\underline{e}_z\frac{\partial}{\partial z}\right)\times\left(\underline{e}_z\frac{\partial}{\partial z}\right)\times\underline{H}_t-i\omega\varepsilon_0\nabla_t\times\left(\underline{e}_zE_z\right)=k^2\underline{H}_t$$

or

$$-i\omega\varepsilon_0\nabla_t\times\left(\underline{e}_zE_z\right)=i\omega\varepsilon_0\left(\underline{e}_z\times\nabla_tE_z\right)=\left(\frac{\partial^2}{\partial z^2}+k^2\right)\underline{H}_t=\left(k^2-k_{z_i}^2\right)\underline{H}_t$$

$$\underline{H}_t=\frac{i\omega\varepsilon_0}{k^2-k_{z_i}^2}\left(\underline{e}_z\times\nabla_tE_z\right)$$

From the fourth equation

$$\left(\underline{e}_z\frac{\partial}{\partial z}\right)\times\underline{H}_t=\frac{-i\omega\varepsilon_0}{k^2-k_{z_i}^2}\frac{\partial}{\partial z}\nabla_tE_z=\frac{\omega\varepsilon_0k_{z_i}}{k^2-k_{z_i}^2}\nabla_tE_z=-i\omega\varepsilon_0\underline{E}_t$$

$$\frac{ik_{z_i}}{k^2-k_{z_i}^2}\nabla_tE_z=\underline{E}_t$$

from which we see that if $E_z\left(\rho=b\right)=0$ then so does this contribution to $E_\varphi\left(\rho=b\right)$ since it is a tangential derivative of $E_z\left(\rho=b\right)$, where b is the exterior radius of the cylinder.

Next for the TE case

$$H_z^{inc} = H_0 \sin\theta_i \sin\varphi_p e^{ik_{\rho_i}\rho\cos(\varphi-\varphi_i)+ik_{z_i}z}$$

$$= H_0 \sin\theta_i \sin\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \varepsilon_m i^m J_m(k_\rho \rho) \cos m(\varphi - \varphi_i)$$

and noting (from the preceding results as well as the relation below for E_φ) that

$$E_\varphi(\rho = b) = 0 \rightarrow \frac{\partial H_z}{\partial \rho}(\rho = b) = 0$$

we find the total field

$$H_z = H_0 \sin\theta_i \sin\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \varepsilon_m i^m \left[J_m\left(k_{\rho_i}\rho\right) - \frac{J_m'\left(k_{\rho_i}b\right)}{H_m^{(1)\prime}\left(k_{\rho_i}b\right)} H_m^{(1)}\left(k_{\rho_i}\rho\right) \right] \cos m(\varphi - \varphi_i)$$

Separating the axial and transverse components

$$\underline{H} = \underline{H}_t + H_z \underline{e}_z$$

from Maxwell's equations

$$\nabla_t \times \underline{H}_t = 0$$

$$\left(\underline{e}_z \frac{\partial}{\partial z}\right) \times \underline{H}_t + \nabla_t \times (\underline{e}_z H_z) = -i\omega\varepsilon_0 \underline{E}_t$$

$$\nabla_t \times \underline{E}_t = i\omega\mu_0 \underline{e}_z H_z$$

$$\left(\underline{e}_z \frac{\partial}{\partial z}\right) \times \underline{E}_t = i\omega\mu_0 \underline{H}_t$$

From the second and fourth equations

$$\left(\underline{e}_z \frac{\partial}{\partial z}\right) \times \left(\underline{e}_z \frac{\partial}{\partial z}\right) \times \underline{E}_t + i\omega\mu_0 \nabla_t \times (\underline{e}_z H_z) = -\frac{\partial^2}{\partial z^2}\underline{E}_t - i\omega\mu_0 \underline{e}_z \times \nabla_t H_z = k^2 \underline{E}_t$$

or

$$-i\omega\mu_0 \underline{e}_z \times \nabla_t H_z = \left(\frac{\partial^2}{\partial z^2} + k^2\right) \underline{E}_t = \left(k^2 - k_{z_i}^2\right) \underline{E}_t$$

$$\underline{E}_t = \frac{-i\omega\mu_0}{k^2 - k_{z_i}^2} \underline{e}_z \times \nabla_t H_z$$

Then substituting into the fourth equation

$$\frac{i\omega\mu_0}{k^2 - k_{z_i}^2} \frac{\partial}{\partial z} \nabla_t H_z = \frac{-\omega\mu_0 k_{z_i}}{k^2 - k_{z_i}^2} \nabla_t H_z = i\omega\mu_0 \underline{H}_t$$

or

$$\underline{H}_t = \frac{ik_{z_i}}{k^2 - k_{z_i}^2} \nabla_t H_z$$

Finally, the total azimuthal field (sum of TM and TE parts) is

$$H_\varphi = \frac{i\omega\varepsilon_0}{k^2 - k_{z_i}^2} \frac{\partial}{\partial\rho} E_z + \frac{ik_{z_i}}{k^2 - k_{z_i}^2} \frac{1}{\rho} \frac{\partial}{\partial\varphi} H_z$$

$$= -\frac{i\omega\varepsilon_0 k_{\rho_i}}{k^2 - k_{z_i}^2} E_0 \sin\theta_i \cos\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \varepsilon_m i^m \left[J_m'\left(k_{\rho_i}\rho\right) - \frac{J_m\left(k_{\rho_i}b\right)}{H_m^{(1)}\left(k_{\rho_i}b\right)} H_m^{(1)\prime}\left(k_{\rho_i}\rho\right) \right] \cos m\left(\varphi - \varphi_i\right)$$

$$-\frac{ik_{z_i}}{k^2 - k_{z_i}^2} \frac{1}{\rho} H_0 \sin\theta_i \sin\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \varepsilon_m i^m \left[J_m\left(k_{\rho_i}\rho\right) - \frac{J_m'\left(k_{\rho_i}b\right)}{H_m^{(1)\prime}\left(k_{\rho_i}b\right)} H_m^{(1)}\left(k_{\rho_i}\rho\right) \right] m \sin m\left(\varphi - \varphi_i\right)$$

Taking $\rho = b$

$$H_\varphi\left(b, \varphi, z\right) = \frac{\omega\varepsilon_0 k_{\rho_i}}{k^2 - k_{z_i}^2} E_0 \sin\theta_i \cos\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{H_m^{(1)}\left(k_{\rho_i}b\right)} \left[J_m'\left(k_{\rho_i}b\right) Y_m\left(k_{\rho_i}b\right) - J_m\left(k_{\rho_i}b\right) Y_m'\left(k_{\rho_i}b\right) \right] \cos m\left(\varphi - \varphi_i\right)$$

$$+\frac{k_{z_i}}{k^2 - k_{z_i}^2} \frac{1}{b} H_0 \sin\theta_i \sin\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{H_m^{(1)\prime}\left(k_{\rho_i}b\right)} \left[J_m\left(k_{\rho_i}b\right) Y_m'\left(k_{\rho_i}b\right) - J_m'\left(k_{\rho_i}b\right) Y_m\left(k_{\rho_i}b\right) \right] m \sin m\left(\varphi - \varphi_i\right)$$

Using the Wronskian

$$J_m\left(k_{\rho_i}b\right) Y_m'\left(k_{\rho_i}b\right) - J_m'\left(k_{\rho_i}b\right) Y_m\left(k_{\rho_i}b\right) = 2/\left(\pi k_{\rho_i}b\right)$$

gives

$$H_\varphi\left(b, \varphi, z\right) = -\frac{\omega\varepsilon_0 k_{\rho_i}}{k^2 - k_{z_i}^2} E_0 \sin\theta_i \cos\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{2\varepsilon_m i^m}{\pi k_{\rho_i} b H_m^{(1)}\left(k_{\rho_i}b\right)} \cos m\left(\varphi - \varphi_i\right)$$

$$+\frac{k_{z_i} k_{\rho_i}}{k^2 - k_{z_i}^2} H_0 \sin\theta_i \sin\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{2\varepsilon_m i^m}{\pi k_{\rho_i}^2 b^2 H_m^{(1)\prime}\left(k_{\rho_i}b\right)} m \sin m\left(\varphi - \varphi_i\right)$$

Using

$$k_{\rho_i} = k \sin\theta_i$$

$$k_{z_i} = k \cos\theta_i$$

$$k^2 - k_{z_i}^2 = k_{\rho_i}^2 = k_{\rho_i} k \sin\theta_i$$

$$\frac{k}{\omega\varepsilon_0} = \eta_0$$

$$E_0 = \eta_0 H_0$$

gives

$$H_\varphi(b,\varphi,z) = -2H_0 \cos\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b H_m^{(1)}\left(k_{\rho_i} b\right)} \cos m(\varphi - \varphi_i)$$

$$+2H_0 \cos\theta_i \sin\varphi_p e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i}^2 b^2 H_m^{(1)\prime}\left(k_{\rho_i} b\right)} m \sin m(\varphi - \varphi_i)$$

or the short circuit exterior current density

$$K_z^{sc}(\varphi,z) = H_\varphi^{sc}(b,\varphi,z)$$

$$= 2H_0 e^{ik_{z_i}z} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[-\frac{\cos m(\varphi - \varphi_i) \cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m \sin m(\varphi - \varphi_i) \cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right]$$

We can now apply this field (or current density) to the slot excitation.

3 INTERIOR MODAL POTENTIALS & FIELDS

Now let us look at the fields and potentials for the various cavity modes. We examine the potentials and fields for the TM and TE modes in the cylindrical cavity of height h_c with $0 < z < h_c$ and radius a with $0 < \rho < a$.

3.1 Interior TM Modal Potentials And Fields

The TM modes (driven by a slot at axial location z_0 of the cylindrical cavity) have axial field

$$E_z = A_{p,m,n} J_m \left(j_{m,p} \rho / a \right) \cos \left[m \left(\varphi - \varphi_0 \right) \right] \cos \left(n\pi z / h_c \right)$$

where the roots of the Bessel function $j_{m,p}$ satisfy

$$J_m \left(j_{m,p} \right) = 0$$

The total electric field is

$$\underline{E} = \underline{E}_t + E_z \underline{e}_z$$

with transverse components

$$\underline{E}_t = \frac{1}{k_{p,m,n}^2 - \left(n\pi / h_c \right)^2} \frac{\partial}{\partial z} \nabla_t E_z$$

and magnetic field components

$$\underline{H}_t = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - \left(n\pi / h_c \right)^2} \left(\underline{e}_z \times \nabla_t E_z \right)$$

or

$$E_\rho = \frac{-1}{k_{p,m,n}^2 - \left(n\pi / h_c \right)^2} A_{p,m,n} \frac{j_{m,p}}{a} J'_m \left(j_{m,p} \rho / a \right) \cos \left[m \left(\varphi - \varphi_0 \right) \right] \frac{n\pi}{h_c} \sin \left(n\pi z / h_c \right)$$

$$E_\varphi = \frac{1}{k_{p,m,n}^2 - \left(n\pi / h_c \right)^2} \frac{1}{\rho} A_{p,m,n} J_m \left(j_{m,p} \rho / a \right) m \sin \left[m \left(\varphi - \varphi_0 \right) \right] \frac{n\pi}{h_c} \sin \left(n\pi z / h_c \right)$$

$$H_\rho = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - \left(n\pi / h_c \right)^2} \frac{1}{\rho} A_{p,m,n} J_m \left(j_{m,p} \rho / a \right) m \sin \left[m \left(\varphi - \varphi_0 \right) \right] \cos \left(n\pi z / h_c \right)$$

$$H_\varphi = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - \left(n\pi / h_c \right)^2} A_{p,m,n} \frac{j_{m,p}}{a} J'_m \left(j_{m,p} \rho / a \right) \cos \left[m \left(\varphi - \varphi_0 \right) \right] \cos \left(n\pi z / h_c \right)$$

Noting that

$$\begin{aligned} \left(\nabla^2 + k^2 \right) E_z &= \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) E_z = 0 \\ &= \left[k^2 - \left(j_{m,p} / a \right)^2 - \left(n\pi / h_c \right)^2 \right] E_z = 0 \end{aligned}$$

where we have used the Bessel equation

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m}{\rho^2}+(j_{m,p}/a)^2\right]J_m\left(j_{m,p}\rho/a\right)=0$$

gives the TM modal wavenumbers

$$k_{p,m,n}^2=\left(\frac{j_{m,p}}{a}\right)^2+(n\pi/h_c)^2$$

Using the electric vector potential and scalar magnetic potential for the fields

$$\underline{D}=\varepsilon_0\underline{E}=-\nabla\times\underline{A}_e$$

$$\underline{H}=i\omega\underline{A}_e-\nabla\phi_m$$

we take modal potentials satisfying the vector Helmholtz equation

$$\left(\nabla^2+k_{p,m,n}^2\right)\underline{A}_e^{(p,m,n)}=0$$

$$i\omega\underline{A}_e^{(p,m,n)}=\underline{H}^{(p,m,n)}$$

with ρ and φ components

$$A_{e\rho}^{(p,m,n)}=\frac{\varepsilon_0}{k_{p,m,n}^2-(n\pi/h_c)^2}\frac{1}{\rho}A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)}=\frac{\varepsilon_0}{k_{p,m,n}^2-(n\pi/h_c)^2}A_{p,m,n}\frac{j_{m,p}}{a}J'_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

The divergence (because of the construction of the potential from the magnetic field it is the Coulomb gauge) is

$$\nabla\cdot\underline{A}_e^{(p,m,n)}=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_{e\rho}\right)+\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\varphi}$$

$$=\frac{\varepsilon_0\left(j_{m,p}/a\right)}{k_{p,m,n}^2-(n\pi/h_c)^2}\frac{1}{\rho}A_{p,m,n}J'_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$-\frac{\varepsilon_0}{k_{p,m,n}^2-(n\pi/h_c)^2}\frac{m}{\rho}A_{p,m,n}\frac{j_{m,p}}{a}J'_m\left(j_{m,p}\rho/a\right)\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)=0$$

Note in this case that the axial electric field does not vanish

$$-\varepsilon_0E_z=\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)A_{e\varphi}-\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\rho}\propto\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)\left[\frac{j_{m,p}}{a}J'_m\left(j_{m,p}\rho/a\right)\right]-\frac{m^2}{\rho^2}J_m\left(j_{m,p}\rho/a\right)$$

$$=\left(\frac{j_{m,p}}{a}\right)^2J''_m\left(j_{m,p}\rho/a\right)+\frac{j_{m,p}}{a\rho}J'_m\left(j_{m,p}\rho/a\right)-\frac{m^2}{\rho^2}J_m\left(j_{m,p}\rho/a\right)$$

$$=\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\right]J_m\left(j_{m,p}\rho/a\right)=-\left(\frac{j_{m,p}}{a}\right)^2J_m\left(j_{m,p}\rho/a\right)$$

Note also that the normal component vanishes on the wall

$$i\omega A_{e\rho}^{(p,m,n)}(\rho=a) = i\omega A_{en}^{(p,m,n)}(\rho=a) = 0$$

with ρ component of the vector Helmholtz equation

$$0 = \left(\nabla^2 - \frac{1}{\rho^2} + k_{p,m,n}^2\right) A_{e\rho}^{(p,m,n)} - \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A_{e\varphi}^{(p,m,n)} = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2} + k_{p,m,n}^2\right) A_{e\rho}^{(p,m,n)} - \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A_{e\varphi}^{(p,m,n)}$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n}$$

$$\left[\left(\frac{j_{m,p}}{a}\right)^2 \frac{1}{\rho} J_m''(j_{m,p}\rho/a) - \left(\frac{j_{m,p}}{a}\right)\frac{2}{\rho^2}J_m'(j_{m,p}\rho/a) + \left(\frac{j_{m,p}}{a}\right)\frac{1}{\rho^2}J_m'(j_{m,p}\rho/a) + \frac{2}{\rho^3}J_m(j_{m,p}\rho/a)\right.$$

$$\left. - \frac{2}{\rho^3}J_m(j_{m,p}\rho/a) + \left(k_{p,m,n}^2 - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{h_c^2}\right)\frac{1}{\rho}J_m(j_{m,p}\rho/a) + \frac{2}{\rho^2}\frac{j_{m,p}}{a}J_m'(j_{m,p}\rho/a)\right] m\sin[m(\varphi-\varphi_0)]\cos(n\pi z/h_c)$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n}\left[\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + k_{p,m,n}^2 - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{h_c^2}\right) J_m(j_{m,p}\rho/a)\right]\frac{m}{\rho}\sin[m(\varphi-\varphi_0)]\cos(n\pi z/h_c)$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n}\left(k_{p,m,n}^2 - \frac{j_{m,p}^2}{a^2} - \frac{n^2\pi^2}{h_c^2}\right) J_m(j_{m,p}\rho/a)\frac{m}{\rho}\sin[m(\varphi-\varphi_0)]\cos(n\pi z/h_c) = 0$$

and φ component of the vector Helmholtz equation

$$0 = \left(\nabla^2 - \frac{1}{\rho^2} + k_{p,m,n}^2\right) A_{e\varphi}^{(p,m,n)} + \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A_{e\rho} = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\rho^2} + k_{p,m,n}^2\right) A_{e\varphi}^{(p,m,n)} + \frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A_{e\rho}^{(p,m,n)}$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n}$$

$$\left[\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2} - \frac{n^2\pi^2}{h_c^2} - \frac{1}{\rho^2} + k_{p,m,n}^2\right)\frac{\partial}{\partial\rho}J_m(j_{m,p}\rho/a) + \frac{1}{\rho}J_m(j_{m,p}\rho/a)\frac{2m^2}{\rho^2}\right]$$

$$\cos[m(\varphi-\varphi_0)]\cos(n\pi z/h_c)$$

Applying a derivative to the Bessel equation gives

$$\left(\frac{\partial^3}{\partial\rho^3} + \frac{1}{\rho}\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho^2}\frac{\partial}{\partial\rho} + \frac{j_{m,p}^2}{a^2}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\frac{\partial}{\partial\rho} + \frac{2m^2}{\rho^3}\right) J_m(j_{m,p}\rho/a) = 0$$

or

$$\left(\frac{\partial^3}{\partial\rho^3}+\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\frac{\partial}{\partial\rho}\right)J_m\left(j_{m,p}\rho/a\right)=-\frac{j_{m,p}^2}{a^2}\frac{\partial}{\partial\rho}J_m\left(j_{m,p}\rho/a\right)-\frac{2m^2}{\rho^3}J_m\left(j_{m,p}\rho/a\right)$$

so that we can verify the φ component of the vector Helmholtz equation vanishes

$$\left(\nabla^2-\frac{1}{\rho^2}+k^2\right)A_{e\varphi}^{(p,m,n)}+\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A_{e\rho}^{(p,m,n)}$$

$$=\frac{\varepsilon_0}{k_{p,m,n}^2-\left(n\pi/h_c\right)^2}A_{p,m,n}\left[\left(k_{p,m,n}^2-\frac{j_{m,p}^2}{a^2}-\frac{n^2\pi^2}{h_c^2}\right)\frac{\partial}{\partial\rho}J_m\left(j_{m,p}\rho/a\right)\right]\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)=0$$

The axial electric field for this potential mode is

$$E_z=A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

which follows from

$$\varepsilon_0E_z=\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\rho}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_{e\varphi}\right)$$

$$=\frac{\varepsilon_0}{k_{p,m,n}^2-\left(n\pi/h_c\right)^2}A_{p,m,n}\left[\left(\frac{m}{u}\right)^2J_m\left(u\right)-\frac{1}{u}\frac{\partial}{\partial u}\left\{uJ_m'\left(u\right)\right\}\right]_{u=j_{m,p}\rho/a}\left(\frac{j_{m,p}}{a}\right)^2\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$=\frac{\varepsilon_0}{k_{p,m,n}^2-\left(n\pi/h_c\right)^2}A_{p,m,n}\left(\frac{j_{m,p}}{a}\right)^2J_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$=\varepsilon_0A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

and we have used

$$J_m''\left(u\right)+\frac{1}{u}J_m'\left(u\right)+\left(1-\frac{m^2}{u^2}\right)J_m\left(u\right)=0\rightarrow\frac{1}{u}\frac{\partial}{\partial u}\left[uJ_m'\left(u\right)\right]+\left(1-\frac{m^2}{u^2}\right)J_m\left(u\right)=0$$

$$\frac{1}{u}\frac{\partial}{\partial u}\left[uJ_m'\left(u\right)\right]=-\left(1-\frac{m^2}{u^2}\right)J_m\left(u\right)$$

$$k_{p,m,n}^2-\left(n\pi/h_c\right)^2=\left(\frac{j_{m,p}}{a}\right)^2$$

The radial electric field

$$E_\rho=-\frac{1}{k_{p,m,n}^2-\left(n\pi/h_c\right)^2}A_{p,m,n}\frac{j_{m,p}}{a}J_m'\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

follows from

$$\varepsilon_0E_\rho=\frac{\partial}{\partial z}A_{e\varphi}-\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{ez}=-\frac{\varepsilon_0}{k_{p,m,n}^2-\left(n\pi/h_c\right)^2}A_{p,m,n}\frac{j_{m,p}}{a}J_m'\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

The azimuthal field

$$E_\varphi = \frac{1}{k_{p,m,n}^2 - (n\pi/h_c)^2}\frac{1}{\rho}A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

we get from

$$\varepsilon_0 E_\varphi = \frac{\partial}{\partial\rho}A_{ez} - \frac{\partial}{\partial z}A_{e\rho} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}\frac{1}{\rho}A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

The radial magnetic field

$$H_\rho = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}\frac{1}{\rho}A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

we obtain from the solenoidal result

$$H_\rho = i\omega A_{e\rho} = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}\frac{1}{\rho}A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

The azimuthal magnetic field

$$H_\varphi = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}A_{p,m,n}\frac{j_{m,p}}{a}J_m'\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

we also find from the solenoidal result

$$H_\varphi = i\omega A_{e\varphi} = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}A_{p,m,n}\frac{j_{m,p}}{a}J_m'\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

The solenoidal axial magnetic field is found as

$$H_z = i\omega A_{ez} = 0$$

Note that modes which have E_z odd with respect to the slot locations could be excited if the drive field had an odd component in z. For a narrow slot aperture with width w much less than the slot length ℓ, $w << \ell$, we expect the interior drive to be significantly reduced. Furthermore, if the slot depth d is also much larger than the slot width $d >> w$ we expect it to be reduced even further. It is worth noting that radiation damping out of the slot is also reduced for these modes and therefore the cavity quality factor can be increased.

3.1.1 TM Modes Lorentz Gauge

The TM potential in the Lorentz gauge can be taken as the axial component of the magnetic vector potential

$$A_z = -i\omega\mu_0\varepsilon_0\frac{A_{p,m,n}^{TM}}{k^2 - (n\pi/h_c)^2}J_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

The electric field is then

$$\underline{E} = i\omega\underline{A} - \nabla\phi = \frac{i}{\omega\mu_0\varepsilon_0}\left[\nabla\left(\nabla\cdot\underline{A}\right) + k^2\underline{A}\right]$$

$$\nabla \cdot \underline{A} = i\omega\mu_0\varepsilon_0\phi$$

$$E_z = A^{TM}_{p,m,n} J_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi - \varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$E_\varphi = \frac{i}{\omega\mu_0\varepsilon_0}\frac{1}{\rho}\frac{\partial^2}{\partial z\partial\varphi}A_z = \frac{A^{TM}_{p,m,n}\left(n\pi/h_c\right)}{k^2 - \left(n\pi/h_c\right)^2}\frac{m}{\rho}J_m\left(j_{m,p}\rho/a\right)\sin\left[m\left(\varphi - \varphi_0\right)\right]\sin\left(n\pi z/h_c\right)$$

The magnetic field is

$$\mu_0\underline{H} = \nabla \times \underline{A}$$

$$H_\varphi = -\frac{1}{\mu_0}\frac{\partial}{\partial\rho}A_z = i\omega\varepsilon_0\frac{A^{TM}_{p,m,n}\left(j_{m,p}/a\right)}{k^2 - \left(n\pi/h_c\right)^2}J'_m\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi - \varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

3.2 Magnetic Scalar Potential

There is of course a contribution to H_z from the magnetic scalar potential $-\nabla\phi_m$

$$\underline{H} = i\omega\underline{A}_e - \nabla\phi_m$$

which in the Coulomb gauge satisfies the Poisson equation

$$\nabla^2\phi^{(p,m,n)}_m = -\rho_m/\mu_0$$

with boundary condition on the cavity walls (at the aperture location the volume magnetic charge density ρ_m, or surface magnetic charge density σ_m, accounts for the normal magnetic field)

$$-\mu_0\frac{\partial\phi^{(p,m,n)}_m}{\partial n} = \sigma_m$$

Because this potential satisfies Poisson's equation it typically diminishes in size away from the aperture. These magnetic charge sources are later determined from the slot voltage derivative.

3.3 Interior TE Modal Potentials & Fields

The TE modes of interest (driven by a slot at the center of the cylindrical cavity) are odd in z with axial field

$$H_z = A_{p,m,n} J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi - \varphi_0\right)\right]\sin\left(n\pi z/h_c\right)$$

where the roots of the derivative of the Bessel function $j'_{m,p}$ satisfy

$$J'_m\left(j'_{m,p}\right) = 0$$

The arbitrary angle φ_0 describes the angle of the modal field distribution in this symmetric cavity geometry (without the slot). The total magnetic field is

$$\underline{H} = \underline{H}_t + H_z\underline{e}_z$$

and the transverse magnetic field components are

$$\underline{H}_t = \frac{1}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{\partial}{\partial z} \nabla_t H_z$$

with transverse electric field components

$$\underline{E}_t = \frac{-i\omega\mu_0}{k'^2_{p,m,n} - (n\pi/h_c)^2} \underline{e}_z \times \nabla_t H_z$$

or

$$H_\rho = \frac{1}{k'^2_{p,m,n} - (n\pi/h_c)^2} A_{p,m,n} \frac{j'_{m,p}}{a} J'_m \left(j'_{m,p}\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$H_\varphi = \frac{-1}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{m}{\rho} A_{p,m,n} J_m \left(j'_{m,p}\rho/a\right) \sin\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$E_\rho = \frac{-i\omega\mu_0}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{m}{\rho} A_{p,m,n} J_m \left(j'_{m,p}\rho/a\right) \sin\left[m\left(\varphi - \varphi_0\right)\right] \sin\left(n\pi z/h_c\right)$$

$$E_\varphi = \frac{-i\omega\mu_0}{k'^2_{p,m,n} - (n\pi/h_c)^2} A_{p,m,n} \frac{j'_{m,p}}{a} J'_m \left(j'_{m,p}\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \sin\left(n\pi z/h_c\right)$$

Noting that

$$\left(\nabla^2 + k^2\right) H_z = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_z = 0$$

$$= \left[k^2 - \left(j'_{m,p}/a\right)^2 - (n\pi/h_c)^2\right] H_z = 0$$

where we have used the Bessel equation

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m}{\rho^2} + \left(j'_{m,p}/a\right)^2\right] J_m \left(j'_{m,p}\rho/a\right) = 0$$

gives the TE modal wavenumbers

$$k'^2_{p,m,n} = \left(\frac{j'_{m,p}}{a}\right)^2 + (n\pi/h_c)^2$$

Using the electric vector potential and scalar magnetic potential for the fields

$$\underline{D} = \varepsilon_0 \underline{E} = -\nabla \times \underline{A}_e$$

$$\underline{H} = i\omega \underline{A}_e - \nabla\phi_m$$

we take modal potentials satisfying the vector Helmholtz equation

$$\left(\nabla^2 + k'^2_{p,m,n}\right) \underline{A}_e^{(p,m,n)} = 0$$

$$i\omega \underline{A}_e^{(p,m,n)} = \underline{H}^{(p,m,n)}$$

with components

$$A_{ez}^{(p,m,n)} = \frac{A_{p,m,n}}{i\omega} J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\sin\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{-1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{m}{\rho}A_{p,m,n}J_m\left(j'_{m,p}\rho/a\right)\sin\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\cos\left(n\pi z/h_c\right)$$

$$A_{e\rho}^{(p,m,n)} = \frac{1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}\frac{j'_{m,p}}{a}J'_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\cos\left(n\pi z/h_c\right)$$

The divergence is

$$\nabla\cdot\underline{A}_e^{(p,m,n)} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_{e\rho}\right)+\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\varphi}+\frac{\partial}{\partial z}A_{ez}$$

$$= \frac{1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}\left(\frac{j'_{m,p}}{a}\right)^2\frac{1}{\left(j'_{m,p}\rho/a\right)}\frac{\partial}{\partial\left(j'_{m,p}\rho/a\right)}\left[\left(j'_{m,p}\rho/a\right)J'_m\left(j'_{m,p}\rho/a\right)\right]\cos\left[m\left(\varphi-\varphi_0\right)\right]\left(\frac{n\pi}{h_c}\right)\cos\left(n\pi z/h_c\right)$$
$$+\frac{-1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{m^2}{\rho^2}A_{p,m,n}J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\left(\frac{n\pi}{h_c}\right)\cos\left(n\pi z/h_c\right)$$
$$+\frac{A_{p,m,n}}{i\omega}\left(n\pi/h_c\right)J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$= \frac{-1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}\left[\left(\frac{j'_{m,p}}{a}\right)^2-\frac{m^2}{\rho^2}\right]J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\left(\frac{n\pi}{h_c}\right)\cos\left(n\pi z/h_c\right)$$
$$+\frac{-1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{m^2}{\rho^2}A_{p,m,n}J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\left(\frac{n\pi}{h_c}\right)\cos\left(n\pi z/h_c\right)$$
$$+\frac{A_{p,m,n}}{i\omega}\left(n\pi/h_c\right)J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)=0$$

Note in this case that the axial electric field vanishes

$$-\varepsilon_0 E_z = \left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)A_{e\varphi}-\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\rho}\propto\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)\left[\frac{m}{\rho}J_m\left(j'_{m,p}\rho/a\right)\right]-\frac{m}{\rho}\frac{j'_{m,p}}{a}J'_m\left(j'_{m,p}\rho/a\right)=0$$

and

$$i\omega A_{e\rho}^{(p,m,n)}\left(\rho=a\right)=i\omega A_{en}^{(p,m,n)}\left(\rho=a\right)=0$$

Note also that

$$0=\left(\nabla^2+k'^2_{p,m,n}\right)A_{ez}^{(p,m,n)}=\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k'^2_{p,m,n}\right)A_{ez}^{(p,m,n)}$$

$$=\left[k'^2_{p,m,n}-\left(j'_{m,p}/a\right)^2-\left(n\pi/h_c\right)^2\right]\frac{A_{p,m,n}}{i\omega}J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

with ρ component

$$0=\left(\nabla^2-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\rho}-\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\varphi}=\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\rho}-\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\varphi}$$

$$=\frac{-1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}\left[\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}-\left(n\pi z/h_c\right)^2-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)\right.$$

$$\left.+2\frac{m^2}{\rho^3}J_m\left(j'_{m,p}\rho/a\right)\right]\cos\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

$$=\frac{-1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}\left[\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}-\frac{1}{\rho^2}+\left(\frac{j'_{m,p}}{a}\right)^2\right)\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)\right.$$

$$\left.+2\frac{m^2}{\rho^3}J_m\left(j'_{m,p}\rho/a\right)\right]\cos\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)=0$$

and φ component

$$0=\left(\nabla^2-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\varphi}+\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\rho}=\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\varphi}+\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\rho}$$

$$=\frac{1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}$$

$$\left[\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}-\frac{n^2\pi^2}{h_c^2}-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)\frac{m}{\rho}J_m\left(j'_{m,p}\rho/a\right)+\frac{2m}{\rho^2}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)\right]$$

$$\sin\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

$$=\frac{m/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}$$

$$\left[\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}-\frac{1}{\rho^2}+\left(\frac{j'_{m,p}}{a}\right)^2\right)\frac{1}{\rho}J_m\left(j'_{m,p}\rho/a\right)+\frac{2}{\rho^2}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)\right]$$

$$\sin\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

Noting that

$$\frac{\partial^2}{\partial\rho^2}\frac{1}{\rho}J_m\left(j'_{m,p}\rho/a\right)=\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)\right]-\frac{\partial}{\partial\rho}\left[\frac{1}{\rho^2}J_m\left(j'_{m,p}\rho/a\right)\right]=\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}J_m\left(j'_{m,p}\rho/a\right)-\frac{2}{\rho^2}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)$$

$$+\frac{2}{\rho^3}J_m\left(j'_{m,p}\rho/a\right)$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}J_m\left(j'_{m,p}\rho/a\right)\right]=\frac{1}{\rho^2}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)-\frac{1}{\rho^3}J_m\left(j'_{m,p}\rho/a\right)$$

and

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\frac{1}{\rho}J_m\left(j'_{m,p}\rho/a\right)=\frac{1}{\rho}\left[\frac{\partial^2}{\partial\rho^2}J_m\left(j'_{m,p}\rho/a\right)-\frac{1}{\rho}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)+\frac{1}{\rho^2}J_m\left(j'_{m,p}\rho/a\right)\right]$$

so that

$$\left(\nabla^2-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\varphi}+\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\rho}=\frac{m/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}A_{p,m,n}$$

$$\frac{1}{\rho}\left[\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}-\frac{m^2}{\rho^2}-\frac{1}{\rho^2}+\left(\frac{j'_{m,p}}{a}\right)^2\right]J_m\left(j'_{m,p}\rho/a\right)\sin\left[m\left(\varphi-\varphi_0\right)\right]\frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)=0$$

Applying a derivative to the Bessel equation gives

$$\left(\frac{\partial^3}{\partial\rho^3}+\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial}{\partial\rho}+\frac{j'^2_{m,p}}{a^2}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\frac{\partial}{\partial\rho}+\frac{2m^2}{\rho^3}\right)J_m\left(j'_{m,p}\rho/a\right)=0$$

or

$$\left(\frac{\partial^3}{\partial\rho^3}+\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}\frac{\partial}{\partial\rho}\right)J_m\left(j'_{m,p}\rho/a\right)=-\frac{j'^2_{m,p}}{a^2}\frac{\partial}{\partial\rho}J_m\left(j'_{m,p}\rho/a\right)-\frac{2m^2}{\rho^3}J_m\left(j'_{m,p}\rho/a\right)$$

so that

$$0=\left(\nabla^2-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\rho}-\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\varphi}$$

and from the preceding

$$\left(\nabla^2-\frac{1}{\rho^2}+k'^2_{p,m,n}\right)A^{(p,m,n)}_{e\varphi}+\frac{2}{\rho^2}\frac{\partial}{\partial\varphi}A^{(p,m,n)}_{e\rho}=0$$

The electric field components are

$$-\varepsilon_0E_\rho=\frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{ez}-\frac{\partial}{\partial z}A_{e\varphi}=-\frac{k'^2_{p,m,n}}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{m}{\rho}\frac{A_{p,m,n}}{i\omega}J_m\left(j'_{m,p}\rho/a\right)\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$=-\frac{-i\omega\mu_0\varepsilon_0}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{m}{\rho}A_{p,m,n}J_m\left(j'_{m,p}\rho/a\right)\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$-\varepsilon_0E_\varphi=\frac{\partial}{\partial z}A_{e\rho}-\frac{\partial}{\partial\rho}A_{ez}=-\frac{k'^2_{p,m,n}}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{A_{p,m,n}}{i\omega}\frac{j'_{m,p}}{a}J'_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$= -\frac{-i\omega\mu_0\varepsilon_0}{k'^2_{p,m,n} - (n\pi/h_c)^2} A_{p,m,n} \frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$-\varepsilon_0 E_z = \left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right) A_{e\varphi} - \frac{1}{\rho}\frac{\partial}{\partial\varphi} A_{e\rho}$$

$$= \frac{m/(i\omega)}{k'^2_{p,m,n} - (n\pi/h_c)^2} A_{p,m,n} \left\{\left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right)\left[\frac{1}{\rho} J_m\left(j'_{m,p}\rho/a\right)\right] - \frac{1}{\rho}\frac{\partial}{\partial\rho} J_m\left(j'_{m,p}\rho/a\right)\right\} \sin\left[m\left(\varphi-\varphi_0\right)\right] \frac{n\pi}{h_c}\sin\left(n\pi z/h_c\right)$$

$$= 0$$

where we used

$$\frac{\partial}{\partial\rho}\left[\frac{1}{\rho} J_m\left(j'_{m,p}\rho/a\right)\right] = \frac{1}{\rho}\frac{\partial}{\partial\rho} J_m\left(j'_{m,p}\rho/a\right) - \frac{1}{\rho^2} J_m\left(j'_{m,p}\rho/a\right)$$

Note that modes which have H_z even (and H_φ odd) with respect to the slot locations could be excited if the drive field had an odd component in z. For a narrow slot aperture with width w much less than the slot length ℓ, $w << \ell$, we expect the interior drive to be significantly reduced. Furthermore, if the slot depth d is also much larger than the slot width $d >> w$ we expect it to be reduced even further. It is worth noting that radiation damping out of the slot is also reduced for these modes and therefore the cavity quality factor can be increased.

3.3.1 TE Modes Lorentz Gauge

The TE potential in the Lorentz gauge can be taken as the axial component of the electric vector potential

$$A_{ez} = -i\omega\mu_0\varepsilon_0 \frac{A^{TE}_{p,m,n}}{k^2 - (n\pi/h_c)^2} J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

The magnetic field is then

$$\underline{H} = i\omega\underline{A}_e - \nabla\phi_m = \frac{i}{\omega\mu_0\varepsilon_0}\left[\nabla\left(\nabla\cdot\underline{A}_e\right) + k^2\underline{A}_e\right]$$

$$\nabla\cdot\underline{A}_e = i\omega\mu_0\varepsilon_0\phi_m$$

$$H_z = A^{TE}_{p,m,n} J_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$H_\varphi = \frac{i}{\omega\mu_0\varepsilon_0}\frac{1}{\rho}\frac{\partial^2}{\partial z\partial\varphi} A_{ez} = \frac{A^{TE}_{p,m,n}\left(n\pi/h_c\right)}{k^2 - (n\pi/h_c)^2}\frac{m}{\rho} J_m\left(j'_{m,p}\rho/a\right)\sin\left[m\left(\varphi-\varphi_0\right)\right]\sin\left(n\pi z/h_c\right)$$

$$H_\rho = -\frac{A^{TE}_{p,m,n}\left(n\pi/h_c\right)\left(j'_{m,p}/a\right)}{k^2 - (n\pi/h_c)^2} J'_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\sin\left(n\pi z/h_c\right)$$

The electric field is

$$\varepsilon_0\underline{E} = -\nabla\times\underline{A}_e$$

$$E_\varphi = \frac{1}{\varepsilon_0}\frac{\partial}{\partial\rho}A_{ez} = -i\omega\mu_0 \frac{A_{p,m,n}^{TE}\left(j'_{m,p}/a\right)}{k^2 - \left(n\pi/h_c\right)^2} J'_m\left(j'_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

4 PROJECTIONS ON MODES

The coupling problem is carried out by finding the projections on the interior cavity modes. The modal amplitudes are then determined in terms of integrations of the slot voltage distribution, which later are used to set up the integro-differential equation for the slot voltage.

4.1 Interior Coupling Formulation

Maxwell's equations driven by a magnetic current are

$$\nabla \times \underline{E} = -\underline{J}_m + i\omega \underline{B}$$

$$\nabla \times \underline{H} = -i\omega \underline{D}$$

with the constitutive relations

$$\underline{B} = \mu_0 \underline{H}$$

$$\underline{D} = \varepsilon_0 \underline{E}$$

Addition of the continuity equation

$$\nabla \cdot \underline{J}_m = i\omega \rho_m$$

gives the Gauss law

$$\nabla \cdot \underline{B} = \rho_m$$

Taking the electric vector potential to be defined by

$$\underline{D} = -\nabla \times \underline{A}_e$$

and the scalar magnetic potential

$$\underline{H} = i\omega \underline{A}_e - \nabla \phi_m$$

we find from the first equation

$$\nabla \times (\nabla \times \underline{A}_e) = \nabla (\nabla \cdot \underline{A}_e) - \nabla^2 \underline{A}_e = \varepsilon_0 \underline{J}_m + k^2 \underline{A}_e + i\omega \mu_0 \varepsilon_0 \nabla \phi_m$$

and from Gauss's law

$$i\omega \nabla \cdot \underline{A}_e - \nabla^2 \phi_m = \rho_m / \mu_0$$

where

$$k^2 = \omega^2 \mu_0 \varepsilon_0$$

If we choose the Lorentz gauge

$$\nabla \cdot \underline{A}_e = i\omega \mu_0 \varepsilon_0 \phi_m$$

$$\left(\nabla^2 + k^2\right) \underline{A}_e = -\varepsilon_0 \underline{J}_m$$

$$\left(\nabla^2 + k^2\right) \phi_m = -\rho_m/\mu_0$$

This approach usually proceeds using the axial components of the electric and magnetic vector potentials or the Hertz potentials [7]. Instead we choose the Coulomb gauge

$$\nabla \cdot \underline{A}_e = 0$$

we find the Helmholtz equation for the electric vector potential

$$\left(\nabla^2 + k^2\right) \underline{A}_e = -\varepsilon_0 \underline{J}_m - i\omega\mu_0\varepsilon_0 \nabla\phi_m$$

and the Poisson equation for the magnetic scalar potential

$$\nabla^2 \phi_m = -\rho_m/\mu_0$$

The magnetic scalar potential will be generated by the magnetic charge $i\omega q_m = \partial I_m/\partial s$ existing at the slot; because in the Coulomb gauge it obeys Poisson's equation generating a magnetic dipole, its effect shrinks in strength as one moves away from the slot. The local magnetic fields from this scalar potential will also include the axial component H_z; because the slot can be made to resonate in combination with the cavity loading of the slot, these scalar potential generated local fields will show resonant enhancements due to the resonant enhancements of the slot magnetic current. If we break up the electric vector potential into modes (in general, the modal representation will need to include both TM and TE modes) then

$$\underline{A}_e = \sum_{p,m,n} \underline{A}_e^{(p,m,n)}$$

where the modal potentials satisfy

$$\left(\nabla^2 + k^2_{p,m,n}\right) \underline{A}_e^{(p,m,n)} = 0$$

Taking these to be orthogonal over the cavity volume (again we need to include both TM and TE modes)

$$\int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{\left(p',m',n'\right)} dV = \delta_{pp'}\delta_{mm'}\delta_{nn'} \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV$$

We also note that the TM and TE modes are orthogonal over the volume

$$\int_V \underline{A}_e^{TE(p,m,n)} \cdot \underline{A}_e^{TM(p,m,n)} dV = \int_V \left[A_{e\rho}^{TM(p,m,n)} A_{e\rho}^{TE(p,m,n)} + A_{e\varphi}^{TM(p,m,n)} A_{e\varphi}^{TE(p,m,n)} \right] dV$$

$$= \frac{\varepsilon_0 \left(n\pi/h_c\right) m/\left(i\omega\right)}{\left\{k^2_{p,m,n} - \left(n\pi/h_c\right)^2\right\} \left\{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2\right\}}$$

$$\int_0^a \int_0^{2\pi} \int_0^{h_c} \Bigg[J_m\left(j_{m,p}\rho/a\right) \left\{ A^{TM}_{p,m,n} B^{TE}_{p,m,n} \sin^2\left(m\varphi\right) - B^{TM}_{p,m,n} A^{TE}_{p,m,n} \cos^2\left(m\varphi\right) \right\} \frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right)$$

$$+ \frac{j_{m,p}}{a} J'_m\left(j_{m,p}\rho/a\right) \left\{ A^{TM}_{p,m,n} B^{TE}_{p,m,n} \cos^2\left(m\varphi\right) - B^{TM}_{p,m,n} A^{TE}_{p,m,n} \sin^2\left(m\varphi\right) \right\} J_m\left(j'_{m,p}\rho/a\right) \Bigg] \cos^2\left(n\pi z/h_c\right) dz d\varphi d\rho$$

$$= \frac{\varepsilon_0 (n\pi/h_c)(\pi h_c/\varepsilon_n) m/(i\omega)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}\left\{k_{p,m,n}'^2 - (n\pi/h_c)^2\right\}} \left\{A_{p,m,n}^{TM} B_{p,m,n}^{TE} - B_{p,m,n}^{TM} A_{p,m,n}^{TE}\right\}$$

$$\int_0^a \left[\frac{j_{m,p}'}{a} J_m (j_{m,p}\rho/a) J_m' \left(j_{m,p}'\rho/a\right) + \frac{j_{m,p}}{a} J_m' (j_{m,p}\rho/a) J_m \left(j_{m,p}'\rho/a\right)\right] d\rho$$

$$= \frac{\varepsilon_0 (n\pi/h_c)(\pi h_c/\varepsilon_n) m/(i\omega)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}\left\{k_{p,m,n}'^2 - (n\pi/h_c)^2\right\}} \left\{A_{p,m,n}^{TM} B_{p,m,n}^{TE} - B_{p,m,n}^{TM} A_{p,m,n}^{TE}\right\}$$

$$\int_0^a \frac{d}{d\rho} \left\{J_m (j_{m,p}\rho/a) J_m \left(j_{m,p}'\rho/a\right)\right\} d\rho$$

$$= \frac{\varepsilon_0 (n\pi/h_c)(\pi h_c/\varepsilon_n) m/(i\omega)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}\left\{k_{p,m,n}'^2 - (n\pi/h_c)^2\right\}} \left\{A_{p,m,n}^{TM} B_{p,m,n}^{TE} - B_{p,m,n}^{TM} A_{p,m,n}^{TE}\right\}$$

$$\left\{J_m (j_{m,p}) J_m \left(j_{m,p}'\right) - J_m (j_{m,p}0) J_m \left(j_{m,p}'0\right)\right\} = 0$$

where the multiplier by m guarantees this vanishes for the symmetric case, and we used the TM and TE potentials with $\cos(m\varphi)$ and $\sin(m\varphi)$ dependences in the next subsections. We also note that if we take either sine or cosine dependences these are also orthogonal over the volume.

Then

$$\left(\nabla^2 + k^2\right) \underline{A}_e = \sum_{p,m,n} \left(k^2 - k_{p,m,n}^2\right) \underline{A}_e^{(p,m,n)} = -\varepsilon_0 \underline{J}_m - i\omega\mu_0\varepsilon_0 \nabla\phi_m$$

and

$$\left(k^2 - k_{p,m,n}^2\right) \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV = -\varepsilon_0 \int_V \underline{A}_e^{(p,m,n)} \cdot \left(\underline{J}_m + i\omega\mu_0 \nabla\phi_m\right) dV$$

We note that

$$\underline{J}_m + i\omega\mu_0 \nabla\phi_m = \underline{J}_{ms}$$

where $\underline{J}_{ms}$ is the solenoidal part of the magnetic current. However, since the vector potential is orthogonal to the gradient of the scalar potential over the cavity volume with PEC walls, we can find the vector potential by taking projections of the total magnetic current on the vector potential eigenmodes [14]

$$\left(k^2 - k_{p,m,n}^2\right) \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV = -\varepsilon_0 \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{J}_m dV$$

4.1.1 Azimuthal Single Slot Coupling

The azimuthal magnetic current in an azimuthal slot, projected on the cavity modes, gives

$$\left(k^2 - k_{p,m,n}^2\right) \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV = -\varepsilon_0 \int_V A_{e\varphi}^{(p,m,n)} J_{m\varphi} dV$$

where the magnetic current on the interior side (plus sign) is (the slot arc length is $s = a\varphi$)

$$J_{m\varphi} = \frac{1}{2} I_m^+ (s)\, \delta (z - z_0)\, \delta (\rho - a) = V_+ (s)\, \delta (z - z_0)\, \delta (\rho - a) \ , \ \ - h < s < h$$

The factor of one half is present in this equation because: 1) in prior work with slot apertures on plane conductors we defined the magnetic current $I_m^+(s)$ by imaging the slot magnetic current in the cavity wall so that $I_m^+(s) = 2V_+(s)$, but, 2) when we expand the electric vector potential in cylindrical cavity modes, regarding the magnetic current source as interior to the cavity, and hence the source is effectively imaged in the conductive boundary wall (we are therefore allowing the modal solution to accomplish the image). Note that the voltage $V_+(s)$ is taken to be positive on the positive z side of the narrow slot with I_m^+ being φ directed. Then with the slot at $z = z_0$

$$\left(k^2 - k_{p,m,n}^2\right) \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV = -\varepsilon_0 \int_{-h}^{h} A_{e\varphi}^{(p,m,n)}(a, \varphi, z_0) V_+(s)\, ds$$

We evaluate these integrals for TM and TE modes below.

4.1.2 Azimuthal Slot Array Coupling

The azimuthal magnetic current from an array of slots around the circumference, projected on the cavity modes, gives

$$\left(k^2 - k_{p,m,n}^2\right) \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV = -\varepsilon_0 \int_V A_{e\varphi}^{(p,m,n)} J_{m\varphi} dV$$

$$= -\varepsilon_0 a \int_0^{2\pi} A_{e\varphi}^{(p,m,n)}(a, \varphi, z_0) V_+(s)\, d\varphi$$

where the magnetic current is

$$J_{m\varphi} = \frac{1}{2} I_m^+(s)\, \delta(z - z_0)\, \delta(\rho - a) = V_+(s)\, \delta(z - z_0)\, \delta(\rho - a) \ , \ \ -\pi a < s < \pi a$$

$$= \sum_{n'=0}^{N-1} \frac{1}{2} I_{mn'}^+(s - s_{n'})\, \delta(z - z_0)\, \delta(\rho - a) = \sum_{n'=0}^{N-1} V_{n'}(s - s_{n'})\, \delta(z - z_0)\, \delta(\rho - a)$$

$$\left(k^2 - k_{p,m,n}^2\right) \int_V \underline{A}_e^{(p,m,n)} \cdot \underline{A}_e^{(p,m,n)} dV = -\varepsilon_0 \sum_{n'=0}^{N-1} \int_{-h}^{h} A_{e\varphi}^{(p,m,n)}(a, \varphi + \varphi_{n'}, z_0) V_{n'}(s)\, ds$$

where arc length is $a\varphi = s$ and $a\varphi_{n'} = s_{n'}$.

4.1.3 TM Modes

For the TM modes the potential components are

$$A_{e\rho}^{(p,m,n)} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{1}{\rho} A_{p,m,n} J_m(j_{m,p}\rho/a)\, m \sin[m(\varphi - \varphi_0)] \cos(n\pi z/h_c)$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{1}{\rho} J_m(j_{m,p}\rho/a)\, m \left[A_{p,m,n} \sin(m\varphi) - B_{p,m,n} \cos(m\varphi)\right] \cos(n\pi z/h_c)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n} \frac{j_{m,p}}{a} J_m'(j_{m,p}\rho/a) \cos[m(\varphi - \varphi_0)] \cos(n\pi z/h_c)$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{j_{m,p}}{a} J_m'(j_{m,p}\rho/a) \left[A_{p,m,n} \cos(m\varphi) + B_{p,m,n} \sin(m\varphi)\right] \cos(n\pi z/h_c)$$

Note that we can take $m\varphi_0 = 0$ to generate the $\cos(m\varphi)$ behavior in $A_{e\varphi}$ or take $m\varphi_0 = \pi/2$ to generate $\sin(m\varphi)$ behavior in $A_{e\varphi}$. The volume integral, needed in the preceding orthogonality expression for the modal amplitudes, is

$$\int_V \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2}\right] dV = \int_0^{h_c}\int_0^{2\pi}\int_0^a \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2}\right]\rho d\rho d\varphi dz = \frac{\varepsilon_0^2 A_{p,m,n}^2}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2}$$

$$\int_0^a \left[\left(\frac{m}{\rho}\right)^2 J_m^2(j_{m,p}\rho/a)\int_0^{2\pi}\sin^2 m(\varphi-\varphi_0)\, d\varphi + \left(\frac{j_{m,p}}{a}\right)^2 J_m'^2(j_{m,p}\rho/a)\int_0^{2\pi}\cos^2 m(\varphi-\varphi_0)\, d\varphi\right]\rho d\rho$$

$$\int_0^{h_c}\cos^2(n\pi z/h_c)\, dz = \frac{\varepsilon_0^2 A_{p,m,n}^2 2\pi h_c j_{m,p}^2/(\varepsilon_n\varepsilon_m)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2}\int_0^1\left[\left(\frac{m}{j_{m,p}u}\right)^2 J_m^2(j_{m,p}u) + J_m'^2(j_{m,p}u)\right]udu$$

$$= \frac{\varepsilon_0^2 A_{p,m,n}^2 \pi h_c j_{m,p}^2 J_{m-1}^2(j_{m,p})/(\varepsilon_n\varepsilon_m)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2}$$

$$= \varepsilon_0^2 A_{p,m,n}^2\left[\pi a^4 h_c/(\varepsilon_n\varepsilon_m)\right]\frac{J_{m-1}^2(j_{m,p})}{j_{m,p}^2}$$

$$= \varepsilon_0^2 A_{p,m,n}^2\left(\frac{1}{\varepsilon_n}\pi a^4 h_c/\varepsilon_m\right)\frac{J_{m-1}^2(j_{m,p})}{j_{m,p}^2}$$

where we used the identity

$$\int_0^1\left[J_m'^2(j_{m,p}u) + \frac{m^2}{j_{m,p}^2u^2}J_m^2(j_{m,p}u)\right]udu = \frac{1}{2}J_{m-1}^2(j_{m,p})$$

which is derived below.

It ends up convenient to use the $A_{p,m,n}\cos(m\varphi)$ and $B_{p,m,n}\sin(m\varphi)$ form of the solution with the two unknown coefficients in the projection, even though it consolidated the modal setup and integration to set up the modes with the phase shifted form $\cos[m(\varphi-\varphi_0)]$ or $\sin[m(\varphi-\varphi_0)]$. Because $\sin(m\varphi)$ and $\cos(m\varphi)$ are orthogonal over 2π we can consider we will see that they can be considered separately in the projection. The calculation of the integral on the left hand side of the projection involves the evaluation of the volume integral

$$\int_V \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2}\right] dV = \int_0^{h_c}\int_0^{2\pi}\int_0^a \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2}\right]\rho d\rho d\varphi dz = \frac{\varepsilon_0^2}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2}$$

$$\int_0^a\left[\left(\frac{m}{\rho}\right)^2 J_m^2(j_{m,p}\rho/a)\int_0^{2\pi}\left\{A_{p,m,n}\sin(m\varphi) - B_{p,m,n}\cos(m\varphi)\right\}^2 d\varphi\right.$$

$$\left. + \left(\frac{j_{m,p}}{a}\right)^2 J_m'^2(j_{m,p}\rho/a)\int_0^{2\pi}\left\{A_{p,m,n}\cos(m\varphi) + B_{p,m,n}\sin(m\varphi)\right\}^2 d\varphi\right]\rho d\rho$$

$$\int_0^{h_c} \cos^2(n\pi z/h_c)\,dz = \frac{\varepsilon_0^2\left(A_{p,m,n}^2 + B_{p,m,n}^2\right) 2\pi h_c j_{m,p}^2/(\varepsilon_n\varepsilon_m)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2} \int_0^1 \left[\left(\frac{m}{j_{m,p}u}\right)^2 J_m^2(j_{m,p}u) + J_m'^2(j_{m,p}u)\right] udu$$

$$= \frac{\varepsilon_0^2\left(A_{p,m,n}^2 + B_{p,m,n}^2\right)\pi h_c j_{m,p}^2 J_{m-1}^2(j_{m,p})/(\varepsilon_n\varepsilon_m)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2}$$

$$= \varepsilon_0^2\left(A_{p,m,n}^2 + B_{p,m,n}^2\right)\left[\pi a^4 h_c/(\varepsilon_n\varepsilon_m)\right] \frac{J_{m-1}^2(j_{m,p})}{j_{m,p}^2}$$

$$= \varepsilon_0^2\left(A_{p,m,n}^2 + B_{p,m,n}^2\right)\left(\frac{1}{\varepsilon_n}\pi a^4 h_c/\varepsilon_m\right) \frac{J_{m-1}^2(j_{m,p})}{j_{m,p}^2}$$

where we have used

$$k_{p,m,n}'^2 = \left(j_{m,p}'/a\right)^2 + (n\pi/h_c)^2$$

and again

$$\int_0^1 \left[J_m'^2(j_{m,p}u) + \frac{m^2}{j_{m,p}^2 u^2} J_m^2(j_{m,p}u)\right] udu = \frac{1}{2} J_{m-1}^2(j_{m,p})$$

This identity follows

$$\int_0^1 \left[J_m'^2(j_{m,p}u) + \frac{m^2}{j_{m,p}^2 u^2} J_m^2(j_{m,p}u)\right] udu = \int_0^1 \left[\left\{J_{m-1}(j_{m,p}u) - \frac{m}{j_{m,p}u} J_m(j_{m,p}u)\right\}^2 + \frac{m^2}{j_{m,p}^2 u^2} J_m^2(j_{m,p}u)\right] udu$$

$$= \int_0^1 \left[J_{m-1}^2(j_{m,p}u)\,u - \frac{2m}{j_{m,p}} J_m(j_{m,p}u) J_{m-1}(j_{m,p}u) + \frac{2m^2}{j_{m,p}^2 u} J_m^2(j_{m,p}u)\right] du$$

$$= \int_0^1 \left[J_{m-1}^2(j_{m,p}u)\,u - \frac{2m}{j_{m,p}} \left\{J_{m-1}(j_{m,p}u) - \frac{m}{j_{m,p}u} J_m(j_{m,p}u)\right\} J_m(j_{m,p}u)\right] du$$

$$= \int_0^1 \left[J_{m-1}^2(j_{m,p}u)\,u - \frac{2m}{j_{m,p}} J_m'(j_{m,p}u) J_m(j_{m,p}u)\right] du$$

$$= \int_0^1 J_{m-1}^2(j_{m,p}u)\,udu - \frac{m}{j_{m,p}^2}\left[J_m^2(j_{m,p}u)\right]_0^1$$

$$= \int_0^1 J_{m-1}^2(j_{m,p}u)\,udu - \frac{m}{j_{m,p}^2} J_m^2(j_{m,p}) = \int_0^1 J_{m-1}^2(j_{m,p}u)\,udu$$

$$\int_0^1 J_{m-1}^2(j_{m,p}u)\,udu = \frac{1}{2j_{m,p}^2}\left[\frac{a^2}{b^2} + j_{m,p}^2 - (m-1)^2\right] J_{m-1}^2(j_{m,p})$$

$$= \frac{1}{2j_{m,p}^2}\left[(m-1)^2 + j_{m,p}^2 - (m-1)^2\right] J_{m-1}^2(j_{m,p}) = \frac{1}{2} J_{m-1}^2(j_{m,p}) \ , \ m \geq 0$$

where

$$aJ_{m-1}(j_{m,p}) + bj_{m,p}J'_{m-1}(j_{m,p}) = aJ_{m-1}(j_{m,p}) + bj_{m,p}\left\{-J_m(j_{m,p}) + \frac{m-1}{j_{m,p}}J_{m-1}(j_{m,p})\right\}$$

$$= \{a + (m-1)b\}J_{m-1}(j_{m,p}) - bj_{m,p}J_m(j_{m,p}) = 0 \rightarrow a = -(m-1)b$$

and thus

$$\int_0^1 \left[J'^2_m(j_{m,p}u) + \frac{m^2}{j^2_{m,p}u^2}J^2_m(j_{m,p}u)\right]udu = \int_0^1 J^2_{m-1}(j_{m,p}u)\,udu = \frac{1}{2}J^2_{m-1}(j_{m,p}) = \frac{1}{2}J'^2_m(j_{m,p})$$

We used the indefinite integral [16]

$$\int J^2_\nu(ku)\,udu = \frac{1}{2}u^2\left[\left(1 - \frac{\nu^2}{k^2u^2}\right)J^2_\nu(ku) + J'^2_\nu(ku)\right]$$

and the recurrence relation

$$J'_\nu(ku) = J_{\nu-1}(ku) - \frac{\nu}{ku}J_\nu(ku)$$

Then for the single slot

$$\left(k^2 - k^2_{p,m,n}\right)A_{p,m,n} = \frac{\varepsilon_m\varepsilon_n}{k^2_{p,m,n} - (n\pi/h_c)^2}\frac{j^3_{m,p}\cos(n\pi z_0/h_c)}{\pi h_c a^5 J_{m-1}(j_{m,p})}\int_{-h}^{h}\cos[m(s'/a - \varphi_0)]V_+(s')\,ds'$$

We can again use results from the phase shifted form involving $\cos[m(\varphi-\varphi_0)]$ or $\sin[m(\varphi-\varphi_0)]$ to arrive at results for the form $A_{p,m,n}\cos(m\varphi)$ and $B_{p,m,n}\sin(m\varphi)$ by taking $m\varphi_0 = 0$ to generate the even behavior $\cos(m\varphi) = \cos(ms/a)$ or take $m\varphi_0 = \pi/2$ to generate the odd behavior $\sin(m\varphi) = \sin(ms/a)$, where $s = a\varphi$ is azimuth arc length. The result is

$$\left(k^2 - k^2_{p,m,n}\right)A_{p,m,n} = \frac{\varepsilon_m\varepsilon_n}{k^2_{p,m,n} - (n\pi/h_c)^2}\frac{j^3_{m,p}\cos(n\pi z_0/h_c)}{\pi h_c a^5 J_{m-1}(j_{m,p})}\int_{-h}^{h}\cos(ms'/a)V_+(s')\,ds'$$

$$\left(k^2 - k^2_{p,m,n}\right)B_{p,m,n} = \frac{\varepsilon_m\varepsilon_n}{k^2_{p,m,n} - (n\pi/h_c)^2}\frac{j^3_{m,p}\cos(n\pi z_0/h_c)}{\pi h_c a^5 J_{m-1}(j_{m,p})}\int_{-h}^{h}\sin(ms'/a)V_+(s')\,ds'$$

$$A^{(p,m,n)}_{e\rho} = \frac{\varepsilon_0}{k^2_{p,m,n} - (n\pi/h_c)^2}\frac{1}{\rho}J_m(j_{m,p}\rho/a)\,m\left[A_{p,m,n}\sin(m\varphi) - B_{p,m,n}\cos(m\varphi)\right]\cos(n\pi z/h_c)$$

$$A^{(p,m,n)}_{e\varphi} = \frac{\varepsilon_0}{k^2_{p,m,n} - (n\pi/h_c)^2}\frac{j_{m,p}}{a}J'_m(j_{m,p}\rho/a)\left[A_{p,m,n}\cos(m\varphi) + B_{p,m,n}\sin(m\varphi)\right]\cos(n\pi z/h_c)$$

For the array

$$\left(k^2 - k_{p,m,n}^2\right) A_{p,m,n} = \frac{\varepsilon_m \varepsilon_n \cos\left(n\pi z_0/h_c\right)}{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2} \frac{j_{m,p}^3}{\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} \sum_{n'=0}^{N-1} \int_{-h}^{h} \cos\left[m\left(s'/a + s_{n'}/a - \varphi_0\right)\right] V_{n'}\left(s'\right) ds'$$

or

$$\left(k^2 - k_{p,m,n}^2\right) A_{p,m,n} = \frac{\varepsilon_m \varepsilon_n \cos\left(n\pi z_0/h_c\right)}{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2} \frac{j_{m,p}^3}{\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} \sum_{n'=0}^{N-1} \int_{-h}^{h} \cos\left[m\left(s'/a + s_{n'}/a\right)\right] V_{n'}\left(s'\right) ds'$$

$$\left(k^2 - k_{p,m,n}^2\right) B_{p,m,n} = \frac{\varepsilon_m \varepsilon_n \cos\left(n\pi z_0/h_c\right)}{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2} \frac{j_{m,p}^3}{\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} \sum_{n'=0}^{N-1} \int_{-h}^{h} \sin\left[m\left(s'/a + s_{n'}/a\right)\right] V_{n'}\left(s'\right) ds'$$

4.1.4 TE Modes

For the TE modes the potential components are

$$A_{e\rho}^{(p,m,n)} = \frac{1/\left(i\omega\right)}{k_{p,m,n}'^2 - \left(n\pi/h_c\right)^2} A_{p,m,n} \frac{j_{m,p}'}{a} J_m'\left(j_{m,p}'\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$= \frac{1/\left(i\omega\right)}{k_{p,m,n}'^2 - \left(n\pi/h_c\right)^2} \frac{j_{m,p}'}{a} J_m'\left(j_{m,p}'\rho/a\right) \left[A_{p,m,n}\cos\left(m\varphi\right) - B_{p,m,n}\sin\left(m\varphi\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{-1/\left(i\omega\right)}{k_{p,m,n}'^2 - \left(n\pi/h_c\right)^2} \frac{m}{\rho} A_{p,m,n} J_m\left(j_{m,p}'\rho/a\right) \sin\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$= \frac{-1/\left(i\omega\right)}{k_{p,m,n}'^2 - \left(n\pi/h_c\right)^2} \frac{m}{\rho} J_m\left(j_{m,p}'\rho/a\right) \left[A_{p,m,n}\sin\left(m\varphi\right) + B_{p,m,n}\cos\left(m\varphi\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$A_{ez}^{(p,m,n)} = \frac{A_{p,m,n}}{i\omega} J_m\left(j_{m,p}'\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \sin\left(n\pi z/h_c\right)$$

$$= \frac{1}{i\omega} J_m\left(j_{m,p}'\rho/a\right) \left[A_{p,m,n}\cos\left(m\varphi\right) + B_{p,m,n}\sin\left(m\varphi\right)\right] \sin\left(n\pi z/h_c\right)$$

Note that we can take $m\varphi_0 = 0$ to generate the $\sin\left(m\varphi\right)$ behavior in $A_{e\varphi}$ or take $m\varphi_0 = -\pi/2$ to generate $\cos\left(m\varphi\right)$ behavior in $A_{e\varphi}$. The integral on the left hand side of the projection involves the evaluation of the volume integral (note here that taking the conjugate of the projection will require a conjugate on the right side as well, which will change the sign since there is an imaginary unit involved in the modal amplitude)

$$\int_V \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2} + A_{ez}^{(p,m,n)2}\right] dV = \int_0^{h_c} \int_0^{2\pi} \int_0^{a} \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2} + A_{ez}^{(p,m,n)2}\right] \rho d\rho d\varphi dz$$

$$= -\frac{A_{p,m,n}^2 h_c n^2 \pi^2 / \left(h_c^2 \omega^2\right)}{2\left\{k_{p,m,n}'^2 - \left(n\pi/h_c\right)^2\right\}^2}$$

$$\int_0^a \left[\left(\frac{j'_{m,p}}{a}\right)^2 J_m'^2\left(j'_{m,p}\rho/a\right)\int_0^{2\pi}\cos^2 m\left(\varphi-\varphi_0\right)d\varphi+\left(\frac{m}{\rho}\right)^2 J_m^2\left(j'_{m,p}\rho/a\right)\int_0^{2\pi}\sin^2 m\left(\varphi-\varphi_0\right)d\varphi\right]\rho d\rho$$

$$-\left(1-\delta_{n0}\right)\frac{h_c}{2\omega^2}A^2_{p,m,n}\int_0^a J_m^2\left(j'_{m,p}\rho/a\right)\int_0^{2\pi}\cos^2 m\left(\varphi-\varphi_0\right)d\varphi\rho d\rho$$

$$=-\frac{A^2_{p,m,n}\pi h_c\left(n\pi/h_c\right)^2/\left(\varepsilon_m\omega^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}$$

$$\int_0^a \left[\left(\frac{j'_{m,p}}{a}\right)^2 J_m'^2\left(j'_{m,p}\rho/a\right)+\left(\frac{m}{\rho}\right)^2 J_m^2\left(j'_{m,p}\rho/a\right)\right]\rho d\rho-\left(1-\delta_{n0}\right)A^2_{p,m,n}\left[\pi h_c/\left(\varepsilon_m\omega^2\right)\right]\int_0^a J_m^2\left(j'_{m,p}\rho/a\right)\rho d\rho$$

$$=-\frac{A^2_{p,m,n}\pi h_c j'^2_{m,p}\left(n\pi/h_c\right)^2/\left(\varepsilon_m\omega^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}\int_0^1\left[J_m'^2\left(j'_{m,p}u\right)+\left(\frac{m}{j'_{m,p}u}\right)^2 J_m^2\left(j'_{m,p}u\right)\right]udu$$

$$-\frac{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2\left(1-\delta_{n0}\right)A^2_{p,m,n}\pi h_c j'^2_{m,p}/\left(\varepsilon_m\omega^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}\left(\frac{a}{j'_{m,p}}\right)^2\int_0^1 J_m^2\left(j'_{m,p}u\right)udu$$

$$=-\left[\frac{A^2_{p,m,n}\pi h_c j'^2_{m,p}\left(n\pi/h_c\right)^2/\left(\varepsilon_m\omega^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}+\frac{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2A^2_{p,m,n}\pi h_c j'^2_{m,p}/\left(\varepsilon_m\omega^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}\left(\frac{a}{j'_{m,p}}\right)^2\right]$$

$$\frac{1}{2}\left(1-\frac{m^2}{j'^2_{m,p}}\right)J_m^2\left(j'_{m,p}\right)\left(1-\delta_{n0}\right)$$

$$=-\frac{A^2_{p,m,n}\pi h_c j'^2_{m,p}/\left(\varepsilon_m\omega^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}\left[\left(n\pi/h_c\right)^2+k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right]\frac{1}{2}\left(1-\frac{m^2}{j'^2_{m,p}}\right)J_m^2\left(j'_{m,p}\right)\left(1-\delta_{n0}\right)$$

$$=-A^2_{p,m,n}\left[\pi h_c/\left(2\varepsilon_m\omega^2\right)\right]\frac{k'^2_{p,m,n}\left(j'^2_{m,p}-m^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}J_m^2\left(j'_{m,p}\right)\left(1-\delta_{n0}\right)$$

$$=\left|A_{p,m,n}\right|^2\left[\pi h_c/\left(2\varepsilon_m\right)\right]\frac{\mu_0\varepsilon_0\left(k'_{p,m,n}/k\right)^2\left(j'^2_{m,p}-m^2\right)}{\left\{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2\right\}^2}J_m^2\left(j'_{m,p}\right)\left(1-\delta_{n0}\right)$$

$$=-A^2_{p,m,n}\left(\frac{\pi}{2}/\varepsilon_m\right)\left(j'^2_{m,p}-m^2\right)a^4h_c\mu_0\varepsilon_0\left(k'_{p,m,n}/k\right)^2\frac{J_m^2\left(j'_{m,p}\right)}{j'^4_{m,p}}\left(1-\delta_{n0}\right)$$

$$=-\mu_0\varepsilon_0A^2_{p,m,n}\left(\pi a^4h_c/\varepsilon_m\right)\left(1-m^2/j'^2_{m,p}\right)\left(k'_{p,m,n}/k\right)^2\frac{J_m^2\left(j'_{m,p}\right)}{j'^2_{m,p}}\left(1-\delta_{n0}\right)$$

Using

$$0 = J'_m\left(j'_{m,p}\right) = J_{m-1}\left(j'_{m,p}\right) - \frac{m}{j'_{m,p}} J_m\left(j'_{m,p}\right)$$

this can also be written as

$$\int_V \left[A_{e\rho}^{(p,m,n)2} + A_{e\varphi}^{(p,m,n)2} + A_{ez}^{(p,m,n)2}\right] dV = -\mu_0\varepsilon_0 A_{p,m,n}^2 \left(\frac{\pi}{2} a^4 h_c/\varepsilon_m\right)\left(j_{m,p}'^2/m^2 - 1\right)\left(k'_{p,m,n}/k\right)^2 \frac{J_{m-1}^2\left(j'_{m,p}\right)}{j_{m,p}'^2}\left(1-\delta_{n0}\right)$$

where we have used

$$k_{p,m,n}'^2 = \left(j'_{m,p}/a\right)^2 + \left(n\pi/h_c\right)^2$$

and

$$\int_0^1 \left[J_m'^2\left(j'_{m,p}u\right) + \frac{m^2}{j_{m,p}'^2 u^2} J_m^2\left(j'_{m,p}u\right)\right] udu = \int_0^1 \left[\left\{J_{m-1}\left(j'_{m,p}u\right) - \frac{m}{j'_{m,p}u} J_m\left(j'_{m,p}u\right)\right\}^2 + \frac{m^2}{j_{m,p}'^2 u^2} J_m^2\left(j'_{m,p}u\right)\right] udu$$

$$= \int_0^1 \left[J_{m-1}^2\left(j'_{m,p}u\right) u - 2\left\{J_{m-1}\left(j'_{m,p}u\right) - \frac{m}{j'_{m,p}u} J_m\left(j'_{m,p}u\right)\right\} \frac{m}{j'_{m,p}} J_m\left(j'_{m,p}u\right)\right] du$$

$$= \int_0^1 \left[J_{m-1}^2\left(j'_{m,p}u\right) u - 2\frac{m}{j'_{m,p}} J_m\left(j'_{m,p}u\right) J'_m\left(j'_{m,p}u\right)\right] du$$

$$= \left[\frac{1}{2}u^2\left\{\left(1 - \frac{(m-1)^2}{j_{m,p}'^2 u^2}\right) J_{m-1}^2\left(j'_{m,p}u\right) + J_{m-1}'^2\left(j'_{m,p}u\right)\right\} - \frac{m}{j_{m,p}'^2} J_m^2\left(j'_{m,p}u\right)\right]_0^1$$

$$= \frac{1}{2}\left\{\left(1 - \frac{(m-1)^2}{j_{m,p}'^2}\right) J_{m-1}^2\left(j'_{m,p}\right) + J_{m-1}'^2\left(j'_{m,p}\right)\right\} - \frac{m}{j_{m,p}'^2} J_m^2\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{\left(1 - \frac{(m-1)^2}{j_{m,p}'^2}\right) J_{m-1}^2\left(j'_{m,p}\right) + \left[-J_m\left(j'_{m,p}\right) + \frac{m-1}{j'_{m,p}} J_{m-1}\left(j'_{m,p}\right)\right]^2\right\} - \frac{m}{j_{m,p}'^2} J_m^2\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{J_{m-1}^2\left(j'_{m,p}\right) + J_m^2\left(j'_{m,p}\right) - 2J_m\left(j'_{m,p}\right)\frac{(m-1)}{j'_{m,p}} J_{m-1}\left(j'_{m,p}\right)\right\} - \frac{m}{j_{m,p}'^2} J_m^2\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{J_{m-1}^2\left(j'_{m,p}\right) + J_m^2\left(j'_{m,p}\right)\right\} - \frac{1}{j'_{m,p}}\left[(m-1) J_{m-1}\left(j'_{m,p}\right) + \frac{m}{j'_{m,p}} J_m\left(j'_{m,p}\right)\right] J_m\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{J_{m-1}^2\left(j'_{m,p}\right) + J_m^2\left(j'_{m,p}\right)\right\} - \frac{1}{j'_{m,p}}\left[mJ_{m-1}\left(j'_{m,p}\right) - J'_m\left(j'_{m,p}\right)\right] J_m\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{J_{m-1}^2\left(j'_{m,p}\right) + J_m^2\left(j'_{m,p}\right)\right\} - \frac{m}{j'_{m,p}} J_{m-1}\left(j'_{m,p}\right) J_m\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{-J_{m-1}^2\left(j'_{m,p}\right) + J_m^2\left(j'_{m,p}\right)\right\} + \left[J_{m-1}\left(j'_{m,p}\right) - \frac{m}{j'_{m,p}} J_m\left(j'_{m,p}\right)\right] J_{m-1}\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{-J_{m-1}^2\left(j'_{m,p}\right)+J_m^2\left(j'_{m,p}\right)\right\}+J'_m\left(j'_{m,p}\right)J_{m-1}\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left\{J_m\left(j'_{m,p}\right)-J_{m-1}\left(j'_{m,p}\right)\right\}\left\{J_m\left(j'_{m,p}\right)+J_{m-1}\left(j'_{m,p}\right)\right\}$$

$$= \frac{1}{2}\left\{J_m\left(j'_{m,p}\right)-J'_m\left(j'_{m,p}\right)-\frac{m}{j'_{m,p}}J_m\left(j'_{m,p}\right)\right\}\left\{J_m\left(j'_{m,p}\right)+J'_m\left(j'_{m,p}\right)+\frac{m}{j'_{m,p}}J_m\left(j'_{m,p}\right)\right\}$$

$$= \frac{1}{2}\left(1-\frac{m}{j'_{m,p}}\right)\left(1+\frac{m}{j'_{m,p}}\right)J_m^2\left(j'_{m,p}\right)$$

$$= \frac{1}{2}\left(1-\frac{m^2}{j'^2_{m,p}}\right)J_m^2\left(j'_{m,p}\right)$$

as well as

$$\int_0^1 J_m^2\left(j'_{m,p}u\right)udu = \frac{1}{2}\left[\left(1-\frac{m^2}{j'^2_{m,p}}\right)J_m^2\left(j'_{m,p}\right)+J'^2_m\left(j'_{m,p}\right)\right] = \frac{1}{2}\left(1-\frac{m^2}{j'^2_{m,p}}\right)J_m^2\left(j'_{m,p}\right)$$

which used the indefinite integral [16]

$$\int J_\nu^2(ku)\,udu = \frac{1}{2}u^2\left[\left(1-\frac{\nu^2}{k^2u^2}\right)J_\nu^2(ku)+J'^2_\nu(ku)\right]$$

and the recurrence relation

$$J'_\nu(ku) = J_{\nu-1}(ku)-\frac{\nu}{ku}J_\nu(ku)$$

Then for the single slot

$$\left(k^2-k'^2_{p,m,n}\right)A_{p,m,n} = \frac{\varepsilon_m/\left(-i\omega\mu_0\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{2m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{\pi h_c^2 a^5\left(k'_{p,m,n}/k\right)^2\left(1-m^2/j'^2_{m,p}\right)J_m\left(j'_{m,p}\right)}\int_{-h}^{h}\sin\left[m\left(s'/a-\varphi_0\right)\right]V_+\left(s'\right)ds'$$

Note that we can take $m\varphi_0 = 0$ to generate the $\sin(ms/a)$ behavior in $A_{e\varphi}$ or take $m\varphi_0 = -\pi/2$ to generate $\cos(ms/a)$ behavior in $A_{e\varphi}$. The result is

$$\left(k^2-k'^2_{p,m,n}\right)A_{p,m,n} = \frac{\varepsilon_m/\left(-i\omega\mu_0\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{2m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{\pi h_c^2 a^5\left(k'_{p,m,n}/k\right)^2\left(1-m^2/j'^2_{m,p}\right)J_m\left(j'_{m,p}\right)}\int_{-h}^{h}\sin\left(ms'/a\right)V_+\left(s'\right)ds'$$

$$\left(k^2-k'^2_{p,m,n}\right)B_{p,m,n} = \frac{\varepsilon_m/\left(-i\omega\mu_0\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{2m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{\pi h_c^2 a^5\left(k'_{p,m,n}/k\right)^2\left(1-m^2/j'^2_{m,p}\right)J_m\left(j'_{m,p}\right)}\int_{-h}^{h}\cos\left(ms'/a\right)V_+\left(s'\right)ds'$$

$$A_{e\rho}^{(p,m,n)} = \frac{1/\left(i\omega\right)}{k'^2_{p,m,n}-\left(n\pi/h_c\right)^2}\frac{j'_{m,p}}{a}J'_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}\cos\left(m\varphi\right)-B_{p,m,n}\sin\left(m\varphi\right)\right]\frac{n\pi}{h_c}\cos\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{-1/(i\omega)}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{m}{\rho} J_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}\sin(m\varphi) + B_{p,m,n}\cos(m\varphi)\right]\frac{n\pi}{h_c}\cos(n\pi z/h_c)$$

$$A_{ez}^{(p,m,n)} = \frac{1}{i\omega} J_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}\cos(m\varphi) + B_{p,m,n}\sin(m\varphi)\right]\sin(n\pi z/h_c)$$

For the array

$$\left(k^2 - k'^2_{p,m,n}\right) A_{p,m,n} = \frac{\varepsilon_m/(-i\omega\mu_0)}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{2m\pi n j'^2_{m,p}\cos(n\pi z_0/h_c)}{\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)}$$
$$\sum_{n'=0}^{N-1} \int_{-h}^{h} \sin\left[m\left(s'/a + s_{n'}/a - \varphi_0\right)\right] V_{n'}(s')\, ds'$$

or

$$\left(k^2 - k'^2_{p,m,n}\right) A_{p,m,n} = \frac{\varepsilon_m/(-i\omega\mu_0)}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{2m\pi n j'^2_{m,p}\cos(n\pi z_0/h_c)}{\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)}$$
$$\sum_{n'=0}^{N-1} \int_{-h}^{h} \sin\left[m\left(s'/a + s_{n'}/a\right)\right] V_{n'}(s')\, ds'$$

$$\left(k^2 - k'^2_{p,m,n}\right) B_{p,m,n} = \frac{\varepsilon_m/(-i\omega\mu_0)}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{2m\pi n j'^2_{m,p}\cos(n\pi z_0/h_c)}{\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)}$$
$$\sum_{n'=0}^{N-1} \int_{-h}^{h} \cos\left[m\left(s'/a + s_{n'}/a\right)\right] V_{n'}(s')\, ds'$$

4.2 Coulomb Gauge Potential Representation

In the Coulomb gauge

$$\nabla \cdot \underline{A}_e = 0$$

with the Helmholtz equation for the electric vector potential

$$\left(\nabla^2 + k^2\right)\underline{A}_e = -\varepsilon_0\left(\underline{J}_m + i\omega\mu_0\nabla\phi_m\right) = -\varepsilon_0\underline{J}_{ms}$$

where $\underline{J}_{ms}$ is the solenoidal part of the magnetic current density, and the Poisson equation for the magnetic scalar potential

$$\nabla^2\phi_m = -\rho_m/\mu_0$$

where for the slot we take the volume magnetic charge density to be an azimuthal filament, at locations $\rho = \rho_0 \to a$ and $z = z_0$, with magnetic charge per unit length $q_m^+(\varphi)$ (the plus sign is used to denote the slot magnetic charge on the cavity side of the slot)

$$\rho_m = \frac{1}{2} q_m^+(\varphi) \frac{1}{\rho} \delta(\rho - \rho_0)\, \delta(z - z_0)$$

and the volume magnetic current to be an azimuthal filament with current $I_m^+(\varphi)$ (the plus sign is used to denote the slot magnetic current on the cavity side of the slot)

$$\underline{J}_m = \underline{e}_\varphi \frac{1}{2} I_m^+(\varphi) \frac{1}{\rho} \delta(\rho - \rho_0) \delta(z - z_0)$$

where the use of these concentrated filament sources then requires the use of an equivalent (nonzero) equivalent radius for field matching to include local reactive slot properties. The factor of one half in these equations is used for the same reason as discussed above, because we are regarding the sources of magnetic current and magnetic charge as interior to the cylindrical cavity, later taking $\rho_0 \to a$. The fields are found from the electric vector potential and magnetic scalar potential by means of

$$\underline{H} = i\omega \underline{A}_e - \nabla \phi_m$$

$$\underline{D} = \varepsilon_0 \underline{E} = -\nabla \times \underline{A}_e$$

In the cavity we expand in terms of modes

$$\underline{A}_e = \sum_{njm} \underline{A}_e^{(njm)}$$

where the modal potentials satisfy

$$\left(\nabla^2 + k_{njm}^2\right) \underline{A}_e^{(njm)} = 0$$

In the Coulomb gauge all three components of the electric vector potential are present from the azimuthal slot excitation (three components of the electric vector potential are present for the TE modes and two components of the electric vector potential are present for the TM modes).

4.2.1 Magnetic Scalar Potential

The magnetic scalar potential can be expanded in quasistatic modes

$$\phi_m(\rho, \varphi, z) = \sum_{pmn} \left[C_{pmn} \cos(m\varphi) + D_{pmn} \sin(m\varphi)\right] J_m\left(j'_{m,p}\rho/a\right) \cos(n\pi z/h_c)$$

$$\nabla^2 \phi_m = \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right] \phi_m$$

$$= -\sum_{pmn} \left(j'^2_{m,p}/a^2 + n^2\pi^2/h_c^2\right) \left[C_{pmn} \cos(m\varphi) + D_{pmn} \sin(m\varphi)\right] \cos(n\pi z/h_c) J_m\left(j'_{m,p}\rho/a\right)$$

$$= -\sum_{pmn} k'^2_{pmn} \left[C_{pmn} \cos(m\varphi) + D_{pmn} \sin(m\varphi)\right] J_m\left(j'_{m,p}\rho/a\right) \cos(n\pi z/h_c) = -\frac{1}{2\mu_0} q_m^+(\varphi) \frac{1}{\rho} \delta(\rho - \rho_0) \delta(z - z_0)$$

where the Bessel function satisfies

$$\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + j'^2_{m,p}/a^2 - m^2/\rho^2\right) J_m\left(j'_{m,p}\rho/a\right) = 0$$

and taking

$$\underline{n} \cdot \nabla \phi_m = \frac{\partial}{\partial n} \phi_m = 0$$

we choose the roots to satisfy

$$J'_m \left(j'_{m,p} \right) = 0$$

where

$$k'^2_{pmn} = j'^2_{m,p}/a^2 + (n\pi/h_c)^2$$

To illustrate the general behavior of the roots, an asymptotic formula for the roots is

$$j'_{m,p} \sim \beta' - \frac{4m^2 + 3}{8\beta'} - \cdots$$

where $(p >> m)$

$$\beta' = (p + m/2 - 3/4)\,\pi$$

Multiplying by the trigonometric functions and rho times the Bessel function, followed by integration over the volume of the cavity, gives

$$\sum_{pmn} k'^2_{pmn} \int_{-\pi}^{\pi} \left[C_{pmn} \cos(m\varphi) + D_{pmn} \sin(m\varphi) \right] \left\{ \begin{array}{c} \cos(m'\varphi) \\ \sin(m'\varphi) \end{array} \right\} d\varphi$$

$$\int_0^a J_{m'} \left(j'_{m',p'} \rho/a \right) J_m \left(j'_{m,p} \rho/a \right) \rho d\rho \int_0^{h_c} \cos(n'\pi z/h_c) \cos(n\pi z/h_c)\, dz$$

$$= \frac{1}{2\mu_0} \int_{-h}^{h} q_m^+(\varphi) \left\{ \begin{array}{c} \cos(m'\varphi) \\ \sin(m'\varphi) \end{array} \right\} d\varphi \int_0^a J_{m'} \left(j'_{m',p'} \rho/a \right) \delta(\rho - \rho_0)\, d\rho \int_0^{h_c} \delta(z - z_0) \cos(n'\pi z/h_c)\, dz$$

Using the orthogonality conditions

$$\int_0^{h_c} \cos(n'\pi z/h_c) \cos(n\pi z/h_c)\, dz = \frac{1}{2} \int_0^{h_c} \left\{ \cos((n'-n)\,\pi z/h_c) + \cos((n'+n)\,\pi z/h_c) \right\} dz$$

$$= \frac{1}{2} \left\{ \frac{\sin((n'-n)\,\pi z/h_c)}{(n'-n)\,\pi/h_c} + \frac{\sin((n'+n)\,\pi z/h_c)}{(n'+n)\,\pi/h_c} \right\}_0^{h_c} = \frac{h_c}{2} \left\{ \frac{\sin((n'-n)\,\pi)}{(n'-n)\,\pi} + \frac{\sin((n'+n)\,\pi)}{(n'+n)\,\pi} \right\} = \frac{h_c}{\varepsilon_n} \delta_{nn'}$$

$$\int_0^a J_{m'} \left(j'_{m',p'} \rho/a \right) J_m \left(j'_{m',p} \rho/a \right) \rho d\rho = a^2 \int_0^1 J_{m'} \left(j'_{m',p'} u \right) J_{m'} \left(j'_{m',p} u \right) u du = \delta_{pp'} \frac{a^2}{2} \left(1 - m'^2/j'^2_{m',p} \right) J^2_{m'} \left(j'_{m',p} \right)$$

$$\int_{-\pi}^{\pi} \cos(m\varphi) \sin(m'\varphi)\, d\varphi = 0 = \int_{-\pi}^{\pi} \sin(m\varphi) \cos(m'\varphi)\, d\varphi$$

$$\int_{-\pi}^{\pi} \cos(m\varphi) \cos(m'\varphi)\, d\varphi = \int_0^{\pi} \left[\cos((m-m')\,\varphi) + \cos((m+m')\,\varphi) \right] d\varphi$$

$$= \left[\frac{\sin\left(\left(m-m'\right)\varphi\right)}{\left(m-m'\right)} + \frac{\sin\left(\left(m+m'\right)\varphi\right)}{\left(m+m'\right)} \right]_0^\pi = \frac{2\pi}{\varepsilon_m}\delta_{mm'}$$

$$\int_{-\pi}^{\pi} \sin\left(m\varphi\right)\sin\left(m'\varphi\right)d\varphi = \int_0^\pi \left[\cos\left(\left(m-m'\right)\varphi\right) - \cos\left(\left(m+m'\right)\varphi\right)\right]d\varphi$$

$$= \left[\frac{\sin\left(\left(m-m'\right)\varphi\right)}{\left(m-m'\right)} - \frac{\sin\left(\left(m+m'\right)\varphi\right)}{\left(m+m'\right)} \right]_0^\pi = \pi\delta_{mm'} \ , \ m \neq 0$$

then gives

$$k_{pmn}'^2 \frac{h_c}{\varepsilon_n} \left\{ \begin{array}{c} C_{pmn} 2\pi/\varepsilon_m \\ D_{pmn}\pi \end{array} \right\} \frac{a^2}{2} \left(1 - m^2/j_{m,p}'^2\right) J_m^2\left(j_{m,p}'\right)$$

$$= \frac{1}{2\mu_0} J_m\left(j_{m,p}'\rho_0/a\right)\cos\left(n\pi z_0/h_c\right) \int_{-h}^{h} q_m^+\left(\varphi\right) \left\{ \begin{array}{c} \cos\left(m\varphi\right) \\ \sin\left(m\varphi\right) \end{array} \right\} d\varphi$$

and thus

$$\phi_m\left(\rho,\varphi,z\right) = \frac{1}{V_{cav}} \sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\varepsilon_n\varepsilon_m}{k_{pmn}'^2\left(1 - m^2/j_{m,p}'^2\right)J_m^2\left(j_{m,p}'\right)} J_m\left(j_{m,p}'\rho_0/a\right) J_m\left(j_{m,p}'\rho/a\right)\cos\left(nz_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

$$\frac{1}{2\mu_0}\left[\cos\left(m\varphi\right)\int_{-h}^{h} q_m^+\left(\varphi'\right)\cos\left(m\varphi'\right)d\varphi' + \sin\left(m\varphi\right)\int_{-h}^{h} q_m^+\left(\varphi'\right)\sin\left(m\varphi'\right)d\varphi'\right]$$

$$= \frac{1}{V_{cav}} \sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\varepsilon_n\varepsilon_m}{k_{pmn}'^2\left(1 - m^2/j_{m,p}'^2\right)J_m^2\left(j_{m,p}'\right)} J_m\left(j_{m,p}'\rho_0/a\right) J_m\left(j_{m,p}'\rho/a\right)\cos\left(nz_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

$$\frac{1}{2\mu_0}\cos\left(m\varphi\right)\int_{-h}^{h} q_m^+\left(\varphi'\right)\cos\left(m\left(\varphi-\varphi'\right)\right)d\varphi'$$

$$V_{cav} = \pi a^2 h_c$$

Here we take the magnetic charge to approach the wall with $\rho_0 \to a$ and set it equal to twice the magnetic flux per unit length from the slot aperture (the magnetic charge per unit length is defined by imaging it in the conductive wall consistent with our definition of the slot magnetic current)

$$q_m^+\left(s = a\varphi\right) = 2\Phi_+\left(s\right)$$

where again $s = a\varphi$ is azimuth arc length. From the continuity equation $\nabla \cdot \underline{J}_m = i\omega\rho_m$ we can write the magnetic charge per unit length as

$$\frac{\partial}{\partial s} I_m^+ = i\omega q_m^+$$

where consistent with including the image we take

$$I_m^+\left(s\right) = 2V_+\left(s\right)$$

Then inserting this

$$\phi_m(\rho,\varphi,z)=\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)J_m\left(j_{m,p}'\right)}$$

$$J_m\left(j_{m,p}'\rho/a\right)\cos(n\pi z/h_c)\frac{1/2}{i\omega\mu_0}\int_{-h}^{h}\frac{\partial I_m^+}{\partial s'}\cos\left(m\left(\varphi-s'/a\right)\right)ds'$$

and integrating by parts (using $I_m^+(\pm h)=0$ to eliminate the boundary terms)

$$\phi_m(\rho,\varphi,z)=\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)J_m\left(j_{m,p}'\right)}$$

$$J_m\left(j_{m,p}'\rho/a\right)\cos(n\pi z/h_c)\frac{1/2}{-i\omega\mu_0}\frac{m}{a}\int_{-h}^{h}I_m^+(s')\sin\left(m\left(\varphi-s'/a\right)\right)ds'$$

$$=\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)J_m\left(j_{m,p}'\right)}$$

$$J_m\left(j_{m,p}'\rho/a\right)\cos(n\pi z/h_c)\frac{1/2}{i\omega\mu_0}\frac{1}{a}\frac{\partial}{\partial\varphi}\int_{-h}^{h}I_m^+(s')\cos\left(m\left(\varphi-s'/a\right)\right)ds'$$

The contribution to the azimuthal magnetic field is then

$$-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m=\frac{1}{a\rho}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)J_m\left(j_{m,p}'\right)}$$

$$J_m\left(j_{m,p}'\rho/a\right)\cos(n\pi z/h_c)\frac{1/2}{-i\omega\mu_0}\int_{-h}^{h}I_m^+(s')\cos\left(m\left(\varphi-s'/a\right)\right)ds'$$

or

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z)=\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1/2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}\cos(n\pi z/h_c)$$

$$\frac{i\omega\varepsilon_0}{k^2}\int_{-h}^{h}I_m^+(s')\cos\left(m\left(\varphi-s'/a\right)\right)ds'$$

or

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z)=\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}\cos(n\pi z/h_c)$$

$$\frac{i\omega\varepsilon_0}{2k^2}\left[\cos(m\varphi)\int_{-h}^{h}I_m^+(s')\cos(ms'/a)\,ds'+\sin(m\varphi)\int_{-h}^{h}I_m^+(s')\sin(ms'/a)\,ds'\right]$$

or

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z)=\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}\cos(n\pi z/h_c)$$

$$\frac{i\omega\varepsilon_0}{k^2}\left[\cos(m\varphi)\int_{-h}^{h}V_+(s')\cos(ms'/a)\,ds' + \sin(m\varphi)\int_{-h}^{h}V_+(s')\sin(ms'/a)\,ds'\right]$$

or

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z) = \frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{2V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{k'^2_{pmn}\left(1-m^2/j'^2_{m,p}\right)}\cos(n\pi z/h_c)\left[\cos(m\varphi)\,m^+_{es} + \sin(m\varphi)\,m^+_{os}\right]$$

$$m^+_{es} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos(ms'/a)\,V_+(s')\,ds'$$

$$m^+_{os} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin(ms'/a)\,V_+(s')\,ds'$$

4.2.2 Electric Vector Potential

The TM mode potentials are

$$i\omega A^{TM}_{e\rho} = \frac{-i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{\left(k^2-k^2_{p,m,n}\right)}\frac{1}{\left(j'_{m,p}/a\right)}\frac{m}{\rho}\frac{J_m(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

$$\left[\sin(m\varphi)\int_{-h}^{h}\cos(ms'/a)\,V_+(s')\,ds' - \cos(m\varphi)\int_{-h}^{h}\sin(ms'/a)\,V_+(s')\,ds'\right]$$

$$i\omega A^{TM}_{e\varphi} = \frac{-i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{\left(k^2-k^2_{p,m,n}\right)}\frac{J'_m(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

$$\left[\cos(m\varphi)\int_{-h}^{h}\cos(ms'/a)\,V_+(s')\,ds' + \sin(m\varphi)\int_{-h}^{h}\sin(ms'/a)\,V_+(s')\,ds'\right]$$

where

$$J_m(j_{m,p}) = 0$$

with asymptotic form

$$j_{m,p} \sim \beta - \frac{4m^2-1}{8\beta} - \cdots$$

$$\beta = (p + m/2 - 1/4)\,\pi$$

The TE mode potentials are

$$i\omega A^{TE}_{e\rho} = \frac{i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{\left(k^2-k'^2_{p,m,n}\right)k'^2_{p,m,n}}\frac{2m/a}{\left(j'_{m,p}/a\right)\left(1-m^2/j'^2_{m,p}\right)}\frac{J'_m\left(j'_{m,p}\rho/a\right)}{J_m\left(j'_{m,p}\right)}\left(\frac{n\pi}{h_c}\right)^2\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

$$\left[\cos\left(m\varphi\right)\int_{-h}^{h}\sin\left(ms'/a\right)V_+\left(s'\right)ds' - \sin\left(m\varphi\right)\int_{-h}^{h}\cos\left(ms'/a\right)V_+\left(s'\right)ds'\right]$$

$$i\omega A_{e\varphi}^{TE} = \frac{-i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{\left(k^2 - k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{2m/a}{\left(j_{m,p}'/a\right)^2\left(1 - m^2/j_{m,p}'^2\right)}\frac{m}{\rho}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n\pi}{h_c}\right)^2$$

$$\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\left[\sin\left(m\varphi\right)\int_{-h}^{h}\sin\left(ms'/a\right)V_+\left(s'\right)ds' + \cos\left(m\varphi\right)\int_{-h}^{h}\cos\left(ms'/a\right)V_+\left(s'\right)ds'\right]$$

$$i\omega A_{ez}^{TE} = \frac{i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{\left(k^2 - k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{2m/a}{\left(1 - m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n\pi}{h_c}\right)\cos\left(n\pi z_0/h_c\right)\sin\left(n\pi z/h_c\right)$$

$$\left[\cos\left(m\varphi\right)\int_{-h}^{h}\sin\left(ms'/a\right)V_+\left(s'\right)ds' + \sin\left(m\varphi\right)\int_{-h}^{h}\cos\left(ms'/a\right)V_+\left(s'\right)ds'\right]$$

The total azimuthal potential is then the sum of TM and TE parts

$$i\omega A_{e\varphi} = \frac{-i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\varepsilon_m\left[\frac{\varepsilon_n}{\left(k^2 - k_{p,m,n}^2\right)}\frac{J_m'\left(j_{m,p}\rho/a\right)}{J_{m-1}\left(j_{m,p}\right)}\right.$$

$$\left. + \frac{1}{\left(k^2 - k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{2m/a}{\left(j_{m,p}'/a\right)^2\left(1 - m^2/j_{m,p}'^2\right)}\frac{m}{\rho}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n\pi}{h_c}\right)^2\right]$$

$$\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\left[\cos\left(m\varphi\right)\int_{-h}^{h}\cos\left(ms'/a\right)V_+\left(s'\right)ds' + \sin\left(m\varphi\right)\int_{-h}^{h}\sin\left(ms'/a\right)V_+\left(s'\right)ds'\right]$$

or

$$i\omega A_{e\varphi} = \frac{-k^2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\frac{\varepsilon_n}{\left(k^2 - k_{p,m,n}^2\right)}\frac{J_m'\left(j_{m,p}\rho/a\right)}{J_m'\left(j_{m,p}\right)}\right.$$

$$\left. + \frac{1}{\left(k^2 - k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{2m/a}{\left(j_{m,p}'/a\right)^2\left(1 - m^2/j_{m,p}'^2\right)}\frac{m}{\rho}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n\pi}{h_c}\right)^2\right]$$

$$\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]$$

and setting $\rho = a$

$$i\omega A_{e\varphi}\left(a,\varphi,z\right) = \frac{-k^2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{2}\left[\frac{1}{\left(k^2 - k_{p,m,n}^2\right)} + \frac{1}{\left(k^2 - k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1 - m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right]$$

$$\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]$$

4.2.3 Extraction Of Dominant Operator

It is convenient to extract the dominant operator at the slot to improve convergence of the modal summations (particularly important at higher frequencies where many simultaneous resonant modes are retained); this identification of the differential operator at the slot in the integro-differential equation for the slot voltage will also allow generalizations of the slot properties to include depth, wall losses, and possible gasket materials (as discussed below). Noting that the roots for large p are

$$j'_{m,p} \sim (p + m/2 - 3/4)\,\pi$$

and thus

$$k'^2_{pmn}\left(1 - m^2/j'^2_{m,p}\right) \sim k'^2_{pmn}$$

We can thus consider the summation over p by means of

$$\sum_p \frac{1}{k'^2_{pmn}\left(1 - m^2/j'^2_{m,p}\right)} \to \sum_p \frac{1}{k'^2_{pmn}} \to \sum_p \frac{1}{(p + m/2 - 3/4)^2\,\pi^2/a^2 + n^2\pi^2/h_c^2}$$

Then using

$$\begin{aligned}
\frac{1}{(p + m/2 - 3/4)^2 + n^2a^2/h_c^2} &= \frac{1}{(p + m/2 - 3/4 + ina/h_c)\,(p + m/2 - 3/4 - ina/h_c)} \\
&= \frac{1}{-i2na/h_c}\left[\frac{1}{(p + m/2 - 3/4 + ina/h_c)} - \frac{1}{(p + m/2 - 3/4 - ina/h_c)}\right] \\
&= \frac{1}{-i2na/h_c}\left[\frac{1}{(p + m/2 - 3/4 + ina/h_c)} - \frac{1}{p} + \frac{1}{p} - \frac{1}{(p + m/2 - 3/4 - ina/h_c)}\right] \\
&= \frac{1}{-i2na/h_c}\left[-\frac{m/2 - 3/4 + ina/h_c}{p\,(p + m/2 - 3/4 + ina/h_c)} + \frac{m/2 - 3/4 - ina/h_c}{p\,(p + m/2 - 3/4 - ina/h_c)}\right]
\end{aligned}$$

we can use the summation (where $\psi(z)$ is the digamma function or logarithmic derivative of the gamma function $\Gamma(z)$)

$$\sum_{p=1}^{\infty} \frac{z}{p\,(p + z)} = \psi(z) + \gamma' + \frac{1}{z}$$

to find

$$\sum_p \frac{1}{(p + m/2 - 3/4)^2 + n^2a^2/h_c^2}$$

$$= \frac{1}{-i2na/h_c}\left[\psi(m/2 - 3/4 - ina/h_c) - \psi(m/2 - 3/4 + ina/h_c) + \frac{1}{m/2 - 3/4 - ina/h_c} - \frac{1}{m/2 - 3/4 + ina/h_c}\right]$$

$$= \frac{1}{-i2na/h_c}\left[\psi(m/2 - 3/4 - ina/h_c) - \psi(m/2 - 3/4 + ina/h_c) + \frac{i2na/h_c}{(m/2 - 3/4)^2 + (na/h_c)^2}\right]$$

Then using the first two terms of the asymptotic form

$$\psi(z) \sim \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + \cdots$$

gives

$$\sum_{p=1}^{\infty} \frac{1}{(p+m/2-3/4)^2 + n^2a^2/h_c^2}$$

$$\sim \frac{1}{-i2na/h_c}\left[\ln(m/2-3/4-ina/h_c) - \ln(m/2-3/4+ina/h_c) + \frac{1/2}{m/2-3/4-ina/h_c} - \frac{1/2}{m/2-3/4+ina/h_c}\right]$$

or for large n

$$\sum_{p=1}^{\infty} \frac{1}{(p+m/2-3/4)^2 + n^2a^2/h_c^2} \sim \frac{1}{-i2na/h_c}\left[-i\pi + \frac{m-1/2}{-ina/h_c}\right] = \frac{1}{2na/h_c}\left[\pi - \frac{m-1/2}{na/h_c}\right] \sim \frac{\pi}{2na/h_c}$$

Then we can write

$$\sum_{p} \frac{1}{k'^2_{pmn}\left(1 - m^2/j'^2_{m,p}\right)} \to \frac{a}{2n\pi/h_c}$$

Using

$$\sum_{n=1}^{\infty} \frac{2}{n}\cos(n\pi z_0/h_c)\cos(n\pi z/h_c) = \sum_{n=1}^{\infty} \frac{1}{n}\cos(n\pi(z-z_0)/h_c) + \sum_{n=1}^{\infty} \frac{1}{n}\cos(n\pi(z+z_0)/h_c)$$

$$= \frac{1}{2}\ln\left\{\frac{1/2}{1-\cos(\pi(z-z_0)/h_c)}\right\} + \frac{1}{2}\ln\left\{\frac{1/2}{1-\cos(\pi(z+z_0)/h_c)}\right\}$$

$$= -\frac{1}{2}\ln\left\{\sin^2(\pi(z-z_0)/(2h_c))\right\} - \frac{1}{2}\ln\left\{\sin^2(\pi(z+z_0)/(2h_c))\right\}$$

$$= -\ln|\sin(\pi(z-z_0)/(2h_c))| - \ln|\sin(\pi(z+z_0)/(2h_c))|$$

$$\sim -\ln|(\pi(z-z_0)/(2h_c))| - \ln|\sin(\pi z_0/h_c)| \ \ , \ \ z - z_0 << h_c$$

we can write

$$\frac{a}{2\pi/h_c}\sum_{n=1}^{\infty} \frac{2}{n}\cos(n\pi z_0/h_c)\cos(n\pi z/h_c) = \frac{a}{2\pi/h_c}\left[-\ln|\sin(\pi(z-z_0)/(2h_c))| - \ln|\sin(\pi(z+z_0)/(2h_c))|\right]$$

$$\sim \frac{a}{2\pi/h_c}\left[-\ln|(\pi(z-z_0)/(2h_c))| - \ln|\sin(\pi z_0/h_c)|\right]$$

and then

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z) = \frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^{+} + \sin(m\varphi)\,m_{os}^{+}\right]$$

$$\left[\sum_{n=0}^{\infty}\varepsilon_n\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\right.$$

$$\left.\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{1-\delta_{n0}}{2n\pi/\left(ah_c\right)}\right\}+\frac{ah_c}{2\pi}\left\{-\ln\left|\left(\pi\left(z-z_0\right)/\left(2h_c\right)\right)\right|-\ln\left|\sin\left(\pi z_0/h_c\right)\right|\right\}\right]$$

Now we note that

$$i\omega\varepsilon_0 V_+\left(s\right)=\frac{k^2}{2\pi a}\sum_m\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^+ +\sin\left(m\varphi\right)m_{os}^+\right]$$

and thus

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m\left(a,\varphi,z\right)=\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^+ +\sin\left(m\varphi\right)m_{os}^+\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{1-\delta_{n0}}{2n\pi/\left(ah_c\right)}\right\}$$

$$+i\omega\varepsilon_0\frac{1}{k^2a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{\pi}V_+\left(s\right)\left\{-\ln\left|\left(\pi\left(z-z_0\right)/\left(2h_c\right)\right)\right|-\ln\left|\sin\left(\pi z_0/h_c\right)\right|\right\}$$

Similarly, using

$$j_{m,p}\sim\left(p+m/2-1/4\right)\pi$$

$$\sum_p\frac{1}{k_{pmn}^2}\to\sum_p\frac{1}{\left(p+m/2-1/4\right)^2\pi^2/a^2+n^2\pi^2/h_c^2}$$

$$\frac{1}{\left(p+m/2-1/4\right)^2+n^2a^2/h_c^2}=\frac{1}{\left(p+m/2-1/4+ina/h_c\right)\left(p+m/2-1/4-ina/h_c\right)}$$

$$=\frac{1}{-i2na/h_c}\left[\frac{1}{\left(p+m/2-1/4+ina/h_c\right)}-\frac{1}{\left(p+m/2-1/4-ina/h_c\right)}\right]$$

$$=\frac{1}{-i2na/h_c}\left[\frac{1}{\left(p+m/2-1/4+ina/h_c\right)}-\frac{1}{p}+\frac{1}{p}-\frac{1}{\left(p+m/2-1/4-ina/h_c\right)}\right]$$

$$=\frac{1}{-i2na/h_c}\left[-\frac{m/2-1/4+ina/h_c}{p\left(p+m/2-1/4+ina/h_c\right)}+\frac{m/2-1/4-ina/h_c}{p\left(p+m/2-1/4-ina/h_c\right)}\right]$$

we can use the summation (where $\psi\left(z\right)$ is the digamma function or logarithmic derivative of the gamma function $\Gamma\left(z\right)$) to find

$$\sum_p\frac{1}{\left(p+m/2-1/4\right)^2+n^2a^2/h_c^2}$$

$$= \frac{1}{-i2na/h_c} \left[\psi\left(m/2 - 1/4 - ina/h_c\right) - \psi\left(m/2 - 1/4 + ina/h_c\right) + \frac{1}{m/2 - 1/4 - ina/h_c} - \frac{1}{m/2 - 1/4 + ina/h_c} \right]$$

$$= \frac{1}{-i2na/h_c} \left[\psi\left(m/2 - 1/4 - ina/h_c\right) - \psi\left(m/2 - 1/4 + ina/h_c\right) + \frac{i2na/h_c}{\left(m/2 - 1/4\right)^2 + \left(na/h_c\right)^2} \right]$$

Then using the first two terms of the asymptotic form of the digamma function gives

$$\sum_{p=1}^{\infty} \frac{1}{\left(p + m/2 - 1/4\right)^2 + n^2 a^2 / h_c^2}$$

$$\sim \frac{1}{-i2na/h_c} \left[\ln\left(m/2 - 1/4 - ina/h_c\right) - \ln\left(m/2 - 1/4 + ina/h_c\right) + \frac{1/2}{m/2 - 1/4 - ina/h_c} - \frac{1/2}{m/2 - 1/4 + ina/h_c} \right]$$

or for large n

$$\sum_{p=1}^{\infty} \frac{1}{\left(p + m/2 - 1/4\right)^2 + n^2 a^2 / h_c^2} \sim \frac{1}{-i2na/h_c} \left[-i\pi + \frac{m + 1/2}{-ina/h_c} \right] = \frac{1}{2na/h_c} \left[\pi - \frac{m + 1/2}{na/h_c} \right] \sim \frac{\pi}{2na/h_c}$$

Then we can write

$$\sum_{p} \frac{1}{k_{pmn}^2} \to \frac{a}{2n\pi/h_c}$$

Then we can write the electric vector potential as

$$i\omega A_{e\varphi}\left(a, \varphi, z\right) = \frac{-k^2}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2} \sum_{n=0}^{\infty} \varepsilon_n \left[\sum_{p=1}^{\infty} \left\{ \frac{1}{\left(k^2 - k_{p,m,n}^2\right)} + \frac{1}{\left(k^2 - k_{p,m,n}'^2\right) k_{p,m,n}'^2} \frac{m^2 / j_{m,p}'^2}{\left(1 - m^2 / j_{m,p}'^2\right)} \left(\frac{n\pi}{h_c}\right)^2 \right\} + \frac{a\left(1 - \delta_{n0}\right)}{2n\pi/h_c} \right]$$

$$\cos\left(n\pi z_0/h_c\right) \cos\left(n\pi z/h_c\right) \left[\cos\left(m\varphi\right) m_{es}^{+} + \sin\left(m\varphi\right) m_{os}^{+}\right]$$

$$+ \frac{k^2}{2\pi a} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2\pi} \left[\cos\left(m\varphi\right) m_{es}^{+} + \sin\left(m\varphi\right) m_{os}^{+}\right] \sum_{n=1}^{\infty} \frac{2}{n} \cos\left(n\pi z_0/h_c\right) \cos\left(n\pi z/h_c\right)$$

or

$$i\omega A_{e\varphi}\left(a, \varphi, z\right) = \frac{-k^2}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2} \sum_{n=0}^{\infty} \varepsilon_n \left[\sum_{p=1}^{\infty} \left\{ \frac{1}{\left(k^2 - k_{p,m,n}^2\right)} + \frac{1}{\left(k^2 - k_{p,m,n}'^2\right) k_{p,m,n}'^2} \frac{m^2 / j_{m,p}'^2}{\left(1 - m^2 / j_{m,p}'^2\right)} \left(\frac{n\pi}{h_c}\right)^2 \right\} + \frac{a\left(1 - \delta_{n0}\right)}{2n\pi/h_c} \right]$$

$$\cos\left(n\pi z_0/h_c\right) \cos\left(n\pi z/h_c\right) \left[\cos\left(m\varphi\right) m_{es}^{+} + \sin\left(m\varphi\right) m_{os}^{+}\right]$$

$$+ i\omega\varepsilon_0 \frac{1}{\pi} V_{+}\left(s\right) \left[-\ln\left|\left(\pi\left(z - z_0\right) / \left(2h_c\right)\right)\right| - \ln\left|\sin\left(\pi z_0/h_c\right)\right|\right]$$

The total azimuthal field is then

$$H_\varphi(a,\varphi,z) = i\omega A_{e\varphi}(a,\varphi,z) - \frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z)$$

$$= i\omega\varepsilon_0\frac{1}{\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\left\{-\ln\left|\left(\pi\left(z-z_0\right)/\left(2h_c\right)\right)\right| - \ln\left|\sin\left(\pi z_0/h_c\right)\right|\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]\sum_{n=0}^{\infty}\varepsilon_n\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{1-\delta_{n0}}{2n\pi/\left(ah_c\right)}\right\}$$

Finally setting $z - z_0 = \rho_+$

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

where

$$\Omega_{e\rho_+} = \Omega_{\rho_+} + C_e$$

$$\Omega_{\rho_+} = 2\ln\left(2h/\rho_+\right)$$

4.2.4 Approximate Extraction Of Interior Quasistatic Operator

It is sometimes useful to approximately extract the Coulomb gauge scalar potential contribution to the magnetic field at the slot. Our prior interior representation

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m\left(a,\varphi,z\right)=\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{2V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos\left(n\pi z_0/h_c\right)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}\cos\left(n\pi z/h_c\right)\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$m_{es}^{+}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_{+}\left(s'\right)ds'$$

$$m_{os}^{+}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_{+}\left(s'\right)ds'$$

is approximated by the half space form

$$\phi_m\left(\rho_+,s\right)=\frac{1}{\mu_0}\int_{-h}^{h}\frac{q_m^{+}\left(s'\right)}{4\pi\sqrt{\rho_+^2+\left(s-s'\right)^2}}ds'=\frac{1}{i\omega\mu_0}\frac{\partial}{\partial s}\int_{-h}^{h}\frac{V_{+}\left(s'\right)}{2\pi\sqrt{\rho_+^2+\left(s-s'\right)^2}}ds'$$

where we used the continuity equation

$$\frac{\partial}{\partial s}I_m^{+}=2\frac{\partial}{\partial s}V_{+}=i\omega q_m^{+}$$

and the vanishing of the voltage $V\left(\pm h\right)=0$. Hence the approximation to the magnetic field at the slot is

$$-\frac{\partial}{\partial s}\phi_m\left(\rho_+,s\right)=\frac{i}{\omega\mu_0}\frac{\partial^2}{\partial s^2}\int_{-h}^{h}\frac{V_{+}\left(s'\right)}{2\pi\sqrt{\rho_+^2+\left(s-s'\right)^2}}ds'$$

where we choose

$$z-z_0=\rho_+$$

Then using

$$\int_{-h}^{h}V_{+}\left(s'\right)\frac{ds'}{2\pi\sqrt{\rho_+^2+\left(s-s'\right)^2}}ds'\approx\frac{\Omega_{e\rho_+}}{2\pi}V_{+}\left(s\right)$$

$$+\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_{+}\left(s'\right)+\int_{-h}^{h}\frac{V_{+}\left(s'\right)-V_{+}\left(s\right)}{2\pi\left|s-s'\right|}ds'$$

$$\Omega_{e\rho_+}=\Omega_{\rho_+}+C_e$$

$$\Omega_{\rho_+}=2\ln\left(2h/\rho_+\right)$$

gives

$$-\frac{\partial}{\partial s}\phi_m\left(\rho_+,s\right)=\frac{i}{\omega\mu_0}\frac{\Omega_{e\rho_+}}{2\pi}\frac{\partial^2}{\partial s^2}V_{+}\left(s\right)$$

$$+\frac{i}{\omega\mu_0 2\pi}\frac{\partial^2}{\partial s^2}\left[\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_+\left(s'\right)+\int_{-h}^{h}\frac{V_+\left(s'\right)-V_+\left(s\right)}{\left|s-s'\right|}ds'\right]$$

Using this with the electric vector potential part of the representation of the preceding subsection

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega A_{e\varphi}\left(a,\varphi,z_0+\rho_+\right)-\frac{1}{a}\frac{\partial\phi_m}{\partial\varphi}\left(a,\varphi,z_0+\rho_+\right)$$

$$\approx i\omega\varepsilon_0\frac{\Omega_{e\rho_+}}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)+i\omega\varepsilon_0\frac{1}{2\pi}V_+\left(s\right)\left\{-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$+\frac{i}{\omega\mu_0 2\pi}\frac{\partial^2}{\partial s^2}\left[\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_+\left(s'\right)+\int_{-h}^{h}\frac{V_+\left(s'\right)-V_+\left(s\right)}{\left|s-s'\right|}ds'\right]$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

where

$$\Omega_{e\rho_+}=\Omega_{\rho_+}+C_e$$

$$\Omega_{\rho_+}=2\ln\left(2h/\rho_+\right)$$

4.3 Lorentz Gauge Potential Representation

From the Coulomb gauge form of the electric vector potential we can construct the Lorentz gauge form by replacing (where we must determine ψ appropriately)

$$\underline{A}_e=\underline{A}_e^C+\nabla\psi$$

In the Lorentz gauge

$$\nabla\cdot\underline{A}_e=\nabla^2\psi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2}+\frac{\partial^2\psi}{\partial z^2}=i\omega\mu_0\varepsilon_0\phi_m$$

where we need the Helmholtz solution for the magnetic vector potential in the Lorentz gauge

$$\left(\nabla^2+k^2\right)\phi_m=-\rho_m/\mu_0=-\frac{1}{2\mu_0}q_m^{+}\left(s\right)\delta\left(z-z_0\right)$$

From the preceding subsection we can take the Coulomb gauge form and replace $k_{pmn}'^2\rightarrow k_{pmn}'^2-k^2$

$$\phi_m\left(\rho,\varphi,z\right)=\frac{1}{V_{cav}}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\cos\left(nz_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

$$\frac{1}{2\mu_0}\left[\cos(m\varphi)\int_{-h}^{h} q_m^{+}(\varphi')\cos(m\varphi')\,d\varphi' + \sin(m\varphi)\int_{-h}^{h} q_m^{+}(\varphi')\sin(m\varphi')\,d\varphi'\right]$$

$$= \frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

$$\frac{1/2}{i\omega\mu_0}\frac{1}{a}\frac{\partial}{\partial\varphi}\int_{-h}^{h} I_m^{+}(s')\cos(m(\varphi-s'/a))\,ds'$$

and

$$\psi = -\frac{i\omega\mu_0\varepsilon_0}{V_{cav}}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{k_{pmn}'^2\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\cos(nz_0/h_c)\cos(n\pi z/h_c)$$

$$\frac{1/2}{i\omega\mu_0}\frac{1}{a}\frac{\partial}{\partial\varphi}\int_{-h}^{h} I_m^{+}(s')\cos(m(\varphi-s'/a))\,ds'$$

$$= -\frac{i\omega\mu_0\varepsilon_0}{V_{cav}}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{k_{pmn}'^2\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\cos(nz_0/h_c)\cos(n\pi z/h_c)$$

$$\frac{1}{i\omega\mu_0}\frac{1}{a}\frac{\partial}{\partial\varphi}\left[\cos(m\varphi)\int_{-h}^{h} V_{+}(s')\cos(ms'/a)\,ds' + \sin(m\varphi)\int_{-h}^{h} V_{+}(s')\sin(ms'/a)\,ds'\right]$$

The total azimuthal electric vector potential in the Lorentz gauge is then

$$i\omega A_{e\varphi} = i\omega\left(A_{e\varphi}^{C}+\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi}\right) = \frac{-i\omega\varepsilon_0}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\varepsilon_m\varepsilon_n\left[\frac{1}{\left(k^2-k_{p,m,n}^2\right)}\frac{J_m'\left(j_{m,p}\rho/a\right)}{J_{m-1}\left(j_{m,p}\right)}\right.$$

$$\left.+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{1}{\left(j_{m,p}'/a\right)^2\left(1-m^2/j_{m,p}'^2\right)}\frac{1}{a\rho}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left\{m^2\left(\frac{n\pi}{h_c}\right)^2-\left(j_{m,p}'/a\right)^2\frac{\partial^2}{\partial\varphi^2}\right\}\right]$$

$$\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)\left[\cos(m\varphi)\int_{-h}^{h}\cos(ms'/a)\,V_{+}(s')\,ds' + \sin(m\varphi)\int_{-h}^{h}\sin(ms'/a)\,V_{+}(s')\,ds'\right]$$

$$= -\frac{k^2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\varepsilon_n\left[\frac{1}{\left(k^2-k_{p,m,n}^2\right)}\frac{J_m'\left(j_{m,p}\rho/a\right)}{J_m'\left(j_{m,p}\right)} + \frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{1}{\left(j_{m,p}'/a\right)^2\left(1-m^2/j_{m,p}'^2\right)}\frac{m^2}{a\rho}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\right]$$

$$\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)\left[\cos(m\varphi)\,m_{es}^{+}+\sin(m\varphi)\,m_{os}^{+}\right]$$

where

$$m_{es}^{+} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_{+}\left(s'\right)ds'$$

$$m_{os}^{+} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_{+}\left(s'\right)ds'$$

Setting $\rho = a$

$$i\omega A_{e\varphi} = -\frac{k^2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\varepsilon_m\varepsilon_n\left[\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right]$$

$$\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\frac{1}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

The contribution to the azimuthal magnetic field from the scalar potential in the Lorentz gauge is

$$-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m = \frac{1}{a\rho}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

$$\frac{1}{-i\omega\mu_0}\left[\cos\left(m\varphi\right)\int_{-h}^{h}V_{+}\left(s'\right)\cos\left(ms'/a\right)ds'+\sin\left(m\varphi\right)\int_{-h}^{h}V_{+}\left(s'\right)\sin\left(ms'/a\right)ds'\right]$$

or at $\rho = a$

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m\left(a,\varphi,z\right) = \frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}$$

$$\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)\frac{1}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

4.3.1 Extraction Of Dominant Operator

For the same reasons discussed in the Coulomb gauge representation is convenient to extract the dominant differential operator at the slot here in the Lorentz gauge representation. Therefore, extracting the dominant operator from the Lorentz gauge representation using

$$\sum_{p}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)} \to \frac{a}{2n\pi/h_c}$$

$$\sum_{n=1}^{\infty}\frac{2}{n}\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right) \sim -\ln\left|\left(\pi\left(z-z_0\right)/\left(2h_c\right)\right)\right| - \ln\left|\sin\left(\pi z_0/h_c\right)\right| \;,\; z-z_0 << h_c$$

$$i\omega\varepsilon_0 V_{+}\left(s\right) = \frac{k^2}{2\pi a}\sum_{m}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

gives

$$-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z)=\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^{+}+\sin(m\varphi)\,m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\,(1-\delta_{0n})}{2n\pi/h_c}\right\}$$

$$+i\omega\varepsilon_0\frac{1}{k^2}\frac{\partial^2}{\partial s^2}\frac{1}{\pi}V_+(s)\left\{-\ln\left|\left(\pi\,(z-z_0)\,/\,(2h_c)\right)\right|-\ln\left|\sin\left(\pi z_0/h_c\right)\right|\right\}$$

Also using

$$\sum_p\frac{1}{k_{pmn}^2}\to\frac{a}{2n\pi/h_c}$$

we can write

$$i\omega A_{e\varphi}=-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^{+}+\sin(m\varphi)\,m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\,(1-\delta_{n0})}{2n\pi/h_c}\right]$$

$$+i\omega\varepsilon_0\frac{1}{\pi}V_+(s)\left\{-\ln\left|\left(\pi\,(z-z_0)\,/\,(2h_c)\right)\right|-\ln\left|\sin\left(\pi z_0/h_c\right)\right|\right\}$$

The total azimuthal field is then

$$H_\varphi(a,\varphi,z)=i\omega A_{e\varphi}(a,\varphi,z)-\frac{1}{a}\frac{\partial}{\partial\varphi}\phi_m(a,\varphi,z)$$

$$=i\omega\varepsilon_0\frac{1}{\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\left\{-\ln\left|\left(\pi\,(z-z_0)\,/\,(2h_c)\right)\right|-\ln\left|\sin\left(\pi z_0/h_c\right)\right|\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^{+}+\sin(m\varphi)\,m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\,(1-\delta_{n0})}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^{+}+\sin(m\varphi)\,m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

$$\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\,(1-\delta_{0n})}{2n\pi/h_c}\right\}$$

Finally setting $z-z_0=\rho_+$

$$H_\varphi(a,\varphi,z_0+\rho_+)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

where

$$\Omega_{e\rho_+}=\Omega_{\rho_+}+C_e$$

$$\Omega_{\rho_+}=2\ln\left(2h/\rho_+\right)$$

4.3.2 Coulomb Gauge Lorentz Gauge Comparison

The previous Coulomb gauge form is

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

Partial fractions

$$\frac{k^2}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}=\frac{1}{k_{p,m,n}'^2}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}$$

or

$$\frac{k^2}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}+\frac{1}{\left(k_{p,m,n}'^2-k^2\right)}=\frac{1}{k_{p,m,n}'^2}$$

then changes the Coulomb form to

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right) V_+(s)\left\{\Omega_{e\rho_+} - C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{k^2}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}\right\}$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{p,m,n}'^2-k^2\right)}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

or

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right) V_+(s)\left\{\Omega_{e\rho_+} - C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{j_{m,p}'^2/a^2+n^2\pi^2/h_c^2}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{p,m,n}'^2-k^2\right)}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

or

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right) V_+(s)\left\{\Omega_{e\rho_+} - C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{p,m,n}'^2-k^2\right)}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

which is the same as the Lorentz gauge result

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

Noting that near the TE resonance the substitution

$$k^2/j_{m,p}'^2\to k_{p,m,n}'^2/j_{m,p}'^2=\left(j_{m,p}'^2/a^2+n^2\pi^2/h_c^2\right)/j_{m,p}'^2=1/a^2+n^2\pi^2/\left(h_c^2j_{m,p}'^2\right)=1/a^2\ ,\ n=0$$

in the Lorentz form

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{k^2}{\left(k^2-k_{p,m,n}^2\right)}+\frac{k^2}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+k^2\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$-\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{p,m,n}'^2-k^2\right)}\frac{m^2/a^2}{\left(1-m^2/j_{m,p}'^2\right)}-\left(m^2/a^2\right)\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

shows that the TE resonance terms cancel for $n=0$ as was obvious from the beginning in the preceding Coulomb form.

Issue With Quality Factor Perturbation The preceding subsection demonstrated the required equivalence of the representation in the two gauges. However, when a perturbation method is used to introduce cavity wall losses in the representations these are not necessarily equivalent any longer. Again beginning with the Coulomb gauge form, but with the quality factors introduced

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left\{k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}+\frac{1}{\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}^{\prime 2}\right\}k_{p,m,n}^{\prime 2}}\frac{m^2/j_{m,p}^{\prime 2}}{\left(1-m^2/j_{m,p}^{\prime 2}\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}\right.$$

$$\left.+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}^{\prime 2}\left(1-m^2/j_{m,p}^{\prime 2}\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

In the preceding subsection, partial fractions were used

$$\frac{k^2}{\left(k^2-k_{p,m,n}^{\prime 2}\right)k_{p,m,n}^{\prime 2}}=\frac{1}{k_{p,m,n}^{\prime 2}}+\frac{1}{\left(k^2-k_{p,m,n}^{\prime 2}\right)}$$

but here with the quality factor there is a change

$$\frac{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}}{\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}^{\prime 2}\right\}k_{p,m,n}^{\prime 2}}=\frac{1}{k_{p,m,n}^{\prime 2}}+\frac{1}{\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}^{\prime 2}\right\}}$$

or the analog of the perfectly conducting form

$$\frac{k^2}{\left(k^2-k_{p,m,n}^{\prime 2}\right)k_{p,m,n}^{\prime 2}}+\frac{1}{\left(k_{p,m,n}^{\prime 2}-k^2\right)}=\frac{1}{k_{p,m,n}^{\prime 2}}$$

can be written as

$$\frac{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}}{\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}^{\prime 2}\right\}k_{p,m,n}^{\prime 2}}+\frac{1}{\left\{k_{p,m,n}^{\prime 2}-kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k^2\right\}}=\frac{1}{k_{p,m,n}^{\prime 2}}$$

or

$$\frac{k^2}{\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}^{\prime 2}\right\}k_{p,m,n}^{\prime 2}}+\frac{1}{\left\{k_{p,m,n}^{\prime 2}-kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k^2\right\}}$$

$$= \frac{1}{k'^2_{p,m,n}} - \frac{\left(k/k'_{p,m,n}\right)(1+i)/Q^{TE}_{p,m,n}}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\}}$$

Then the Coulomb form becomes

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\left\{\Omega_{e\rho_+} - C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m^+_{es} + \sin(m\varphi)\,m^+_{os}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n} - k^2_{p,m,n}\right\}} + \frac{1}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\}k'^2_{p,m,n}}\frac{m^2/j'^2_{m,p}}{\left(1-m^2/j'^2_{m,p}\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}\right.$$

$$\left.+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m^+_{es} + \sin(m\varphi)\,m^+_{os}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}\frac{k^2 + kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\}k'^2_{p,m,n}}\right\}$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m^+_{es} + \sin(m\varphi)\,m^+_{os}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}\frac{1}{\left\{k'^2_{p,m,n} - kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n} - k^2\right\}} - \frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

or

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\left\{\Omega_{e\rho_+} - C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m^+_{es} + \sin(m\varphi)\,m^+_{os}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n} - k^2_{p,m,n}\right\}} + \frac{j'^2_{m,p}/a^2 + n^2\pi^2/h_c^2}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\}k'^2_{p,m,n}}\frac{m^2/j'^2_{m,p}}{\left(1-m^2/j'^2_{m,p}\right)}\right\}\right.$$

$$\left.+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right] + \frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m^+_{es} + \sin(m\varphi)\,m^+_{os}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}\frac{\left(k/k'_{p,m,n}\right)(1+i)/Q^{TE}_{p,m,n}}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\}}\right\}$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}\frac{1}{\left\{k_{p,m,n}'^2-kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k^2\right\}}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

or

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left\{k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}-\frac{1}{\left\{k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}}\right\}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}\frac{\left(k/k_{p,m,n}'\right)\left(1+i\right)/Q_{p,m,n}^{TE}}{\left\{k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}}\right\}$$

$$+\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j_{m,p}'^2\right)}\frac{1}{\left\{k_{p,m,n}'^2-kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k^2\right\}}-\frac{a\left(1-\delta_{n0}\right)}{n\pi/h_c}\right\}$$

Compared to the Lorentz form

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}-k_{p,m,n}^2}+\frac{1}{k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}-k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k'^2_{p,m,n}-kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n}-k^2}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

or

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m^+_{es}+\sin\left(m\varphi\right)m^+_{os}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\sum_{p=1}^{\infty}\left\{\frac{1}{k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}-k^2_{p,m,n}}-\frac{1}{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}-k'^2_{p,m,n}}\right\}$$

$$+\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m^+_{es}+\sin\left(m\varphi\right)m^+_{os}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k'^2_{p,m,n}-kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n}-k^2}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

we see that the Coulomb form contains the extra term

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m^+_{es}+\sin\left(m\varphi\right)m^+_{os}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}\frac{\left(k/k'_{p,m,n}\right)\left(1+i\right)/Q^{TE}_{p,m,n}}{\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}}\right\}$$

Hence for very large quality factors this term represents a very small difference $O\left(1/Q^{TE}_{p,m,n}\right)$.

4.4 Interior Modal Series Representation And Finite Quality Factor

The interior field will be represented by the electric potential modal series

$$\underline{A}_e\left(\underline{r}\right)=\sum_{p,m,n}\underline{A}_e^{(p,m,n)}\left(\underline{r}\right)$$

where for the TM modes

$$\left(k^2-k^2_{p,m,n}\right)\int_V\underline{A}_e^{(p,m,n)}\cdot\underline{A}_e^{(p,m,n)}dV=-\varepsilon_0\int_V\underline{A}_e^{(p,m,n)}\cdot\underline{J}_m dV$$

and for the TE modes

$$\left(k^2-k'^2_{p,m,n}\right)\int_V\underline{A}_e^{(p,m,n)}\cdot\underline{A}_e^{(p,m,n)}dV=-\varepsilon_0\int_V\underline{A}_e^{(p,m,n)}\cdot\underline{J}_m dV$$

Now to introduce cavity losses we perturb the leading factor multiplying the volume integral [15], which for the TM modes is

$$k^2 - k_{p,m,n}^2 \rightarrow k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n} - k_{p,m,n}^2$$

and for the TE modes is

$$k^2 - k_{p,m,n}'^2 \rightarrow k^2 + kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n} - k_{p,m,n}'^2$$

where the cavity quality factors are given by

$$Q_{p,m,n} = \frac{\omega_{p,m,n}\mu_0 \int_V \underline{H}^{(p,m,n)*} \cdot \underline{H}^{(p,m,n)} dV}{R_s^{(p,m,n)} \oint_S \underline{H}^{(p,m,n)*} \cdot \underline{H}^{(p,m,n)} dS}$$

with surface resistance

$$R_s^{(p,m,n)} = 1/\left(\sigma\delta_{p,m,n}\right)$$

and skin depth

$$\delta_{p,m,n} = \sqrt{2/\left(\omega_{p,m,n}\mu\sigma\right)}$$

4.4.1 TM Modes

For the TM modes the magnetic field components are

$$H_\rho = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2}\frac{1}{\rho}A_{p,m,n}J_m\left(j_{m,p}\rho/a\right)m\sin\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

$$H_\varphi = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2}A_{p,m,n}\frac{j_{m,p}}{a}J_m'\left(j_{m,p}\rho/a\right)\cos\left[m\left(\varphi-\varphi_0\right)\right]\cos\left(n\pi z/h_c\right)$$

The surface integral in the quality factor is

$$\begin{aligned}
\oint_S \underline{H}^{(p,m,n)*} \cdot \underline{H}^{(p,m,n)} dS &= \int_0^{h_c} dz \int_{-\pi}^{\pi} \left|H_\varphi\left(\rho=a\right)\right|^2 a d\varphi + \int_0^a \int_{-\pi}^{\pi} \left[\left|H_\rho\left(z=0\right)\right|^2 + \left|H_\varphi\left(z=0\right)\right|^2\right]\rho d\varphi d\rho \\
&+ \int_0^a \int_{-\pi}^{\pi} \left[\left|H_\rho\left(z=h_c\right)\right|^2 + \left|H_\varphi\left(z=h_c\right)\right|^2\right]\rho d\varphi d\rho \\
&= \int_0^{h_c} dz \int_{-\pi}^{\pi} \left|H_\varphi\left(\rho=a\right)\right|^2 a d\varphi + 2\int_0^a \int_{-\pi}^{\pi} \left[\left|H_\rho\left(z=0\right)\right|^2 + \left|H_\varphi\left(z=0\right)\right|^2\right]\rho d\varphi d\rho \\
&= \frac{\left(\omega\varepsilon_0\right)^2\left|A_{p,m,n}\right|^2 J_m'^2\left(j_{m,p}\right)j_{m,p}^2}{\left\{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2\right\}^2 a}\int_0^{h_c}\cos^2\left(n\pi z/h_c\right)dz\int_{-\pi}^{\pi}\cos^2\left[m\left(\varphi-\varphi_0\right)\right]d\varphi \\
&+2\frac{\left(\omega\varepsilon_0\right)^2\left|A_{p,m,n}\right|^2\left(2\pi/\varepsilon_m\right)}{\left\{k_{p,m,n}^2 - \left(n\pi/h_c\right)^2\right\}^2}\int_0^a\left\{\frac{m^2}{\rho^2}J_m^2\left(j_{m,p}\rho/a\right) + \left(\frac{j_{m,p}}{a}\right)^2 J_m'^2\left(j_{m,p}\rho/a\right)\right\}\rho d\rho
\end{aligned}$$

$$= \frac{(\omega\varepsilon_0)^2 |A_{p,m,n}|^2 J_m'^2(j_{m,p}) j_{m,p}^2}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2} \left(\frac{h_c}{a}/\varepsilon_n\right)(2\pi/\varepsilon_m)$$

$$+2\frac{(\omega\varepsilon_0)^2 |A_{p,m,n}|^2 j_{m,p}^2}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2}(2\pi/\varepsilon_m)\int_0^1 \left\{J_m'^2(j_{m,p}u) + \frac{m^2}{j_{m,p}^2 u^2} J_m^2(j_{m,p}u)\right\} udu$$

$$= |A_{p,m,n}|^2 \frac{2\pi}{\varepsilon_m} \frac{(\omega\varepsilon_0)^2}{\left\{k^2 - (n\pi/h_c)^2\right\}^2} J_{m-1}^2(j_{m,p}) j_{m,p}^2 \left[1 + \frac{h_c/a}{\varepsilon_n}\right]$$

where we used

$$\int_0^1 \left[J_m'^2(j_{m,p}u) + \frac{m^2}{j_{m,p}^2 u^2} J_m^2(j_{m,p}u)\right] udu = \frac{1}{2} J_m'^2(j_{m,p}) = \frac{1}{2} J_{m-1}^2(j_{m,p})$$

The volume integral in the quality factor is

$$\int_V \underline{H}^{(p,m,n)*} \cdot \underline{H}^{(p,m,n)} dV = \int_0^{h_c} \int_0^a \int_{-\pi}^{\pi} \left[|H_\rho|^2 + |H_\varphi|^2\right] \rho d\varphi d\rho dz$$

$$= \frac{(\omega\varepsilon_0)^2 |A_{p,m,n}|^2 (2\pi/\varepsilon_m)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2} \int_0^{h_c} \cos^2(n\pi z/h_c)\, dz \int_0^a \left[\frac{m^2}{\rho^2} J_m^2(j_{m,p}\rho/a) + \left(\frac{j_{m,p}}{a}\right)^2 J_m'^2(j_{m,p}\rho/a)\right] \rho d\rho$$

$$= \frac{(\omega\varepsilon_0)^2 |A_{p,m,n}|^2 (2\pi/\varepsilon_m)}{\left\{k_{p,m,n}^2 - (n\pi/h_c)^2\right\}^2} (h_c/\varepsilon_n)\, j_{m,p}^2 \int_0^1 \left[J_m'^2(j_{m,p}u) + \frac{m^2}{j_{m,p}^2 u^2} J_m^2(j_{m,p}u)\right] udu$$

$$= |A_{p,m,n}|^2 \frac{(\omega\varepsilon_0)^2}{\left\{k^2 - (n\pi/h_c)^2\right\}^2} j_{m,p}^2 \frac{h_c}{\varepsilon_n} \frac{\pi}{\varepsilon_m} J_{m-1}^2(j_{m,p})$$

Hence the TM mode quality factor is

$$Q_{p,m,n} = \frac{k_{p,m,n} a}{\varepsilon_n a/h_c + 1} \frac{\eta_0}{2R_s^{(p,m,n)}}$$

where again the TM modal wavenumbers and frequencies are

$$k_{p,m,n}^2 = \omega_{p,m,n}^2 \mu_0 \varepsilon_0 = \left(\frac{j_{m,p}}{a}\right)^2 + (n\pi/h_c)^2$$

and the surface resistance and skin depth are

$$R_s^{(p,m,n)} = 1/(\sigma\delta_{p,m,n})$$

$$\delta_{p,m,n} = \sqrt{2/(\omega_{p,m,n}\mu\sigma)}$$

4.4.2 TE Modes

For the TE modes the magnetic field components are

$$H_\rho = \frac{1}{k'^2_{p,m,n} - (n\pi/h_c)^2} A_{p,m,n} \frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$H_\varphi = \frac{-1}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{m}{\rho} A_{p,m,n} J_m\left(j'_{m,p}\rho/a\right) \sin\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$H_z = A_{p,m,n} J_m\left(j'_{m,p}\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \sin\left(n\pi z/h_c\right)$$

Note that $n = 0$ has all components vanishing, so this does not exist. The surface integral in the quality factor is

$$\oint_S \underline{H}^{(p,m,n)*} \cdot \underline{H}^{(p,m,n)} dS = \int_0^{h_c} dz \int_{-\pi}^{\pi} \left[\left|H_\varphi\left(\rho = a\right)\right|^2 + \left|H_z\left(\rho = a\right)\right|^2\right] a d\varphi$$

$$+ \int_0^a \int_{-\pi}^{\pi} \left[\left|H_\rho\left(z = 0\right)\right|^2 + \left|H_\varphi\left(z = 0\right)\right|^2\right] \rho d\varphi d\rho + \int_0^a \int_{-\pi}^{\pi} \left[\left|H_\rho\left(z = h_c\right)\right|^2 + \left|H_\varphi\left(z = h_c\right)\right|^2\right] \rho d\varphi d\rho$$

$$= \frac{2\pi}{\varepsilon_m} \frac{h_c}{\varepsilon_n} \left|A_{p,m,n}\right|^2 \left[\frac{(n\pi/h_c)^2 (m/a)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} + 1 - \delta_{n0}\right] a J_m^2\left(j'_{m,p}\right)$$

$$+2 \frac{\pi \left|A_{p,m,n}\right|^2 (n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} \int_0^a \left[(2/\varepsilon_m) \left(\frac{j'_{m,p}}{a}\right)^2 J'^2_m\left(j'_{m,p}\rho/a\right) + \left(\frac{m}{\rho}\right)^2 J_m^2\left(j'_{m,p}\rho/a\right)\right] \rho d\rho$$

$$= \left|A_{p,m,n}\right|^2 h_c \left(\pi/\varepsilon_m\right) \left(1 - \delta_{n0}\right) \left[\frac{(n\pi/h_c)^2 (m/a)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} + 1\right] a J_m^2\left(j'_{m,p}\right)$$

$$+ \frac{2\left|A_{p,m,n}\right|^2 \left(2\pi/\varepsilon_m\right) (n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} \int_0^a \left[\left(\frac{j'_{m,p}}{a}\right)^2 J'^2_m\left(j'_{m,p}\rho/a\right) + \left(\frac{m}{\rho}\right)^2 J_m^2\left(j'_{m,p}\rho/a\right)\right] \rho d\rho$$

$$= \left|A_{p,m,n}\right|^2 h_c \left(\pi/\varepsilon_m\right) \left(1 - \delta_{n0}\right) \left[\frac{(n\pi/h_c)^2 (m/a)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} + 1\right] a J_m^2\left(j'_{m,p}\right)$$

$$+ \frac{2\left|A_{p,m,n}\right|^2 \left(2\pi/\varepsilon_m\right) j'^2_{m,p} (n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} \int_0^1 \left[J'^2_m\left(j'_{m,p} u\right) + \left(\frac{m}{j'_{m,p} u}\right)^2 J_m^2\left(j'_{m,p} u\right)\right] u du$$

$$= \left|A_{p,m,n}\right|^2 \left(\pi/\varepsilon_m\right) \left\{a h_c \left(1 - \delta_{n0}\right) \left[\frac{(n\pi/h_c)^2 (m/a)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} + 1\right] + \frac{2 j'^2_{m,p} (n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} \left(1 - \frac{m^2}{j'^2_{m,p}}\right)\right\} J_m^2\left(j'_{m,p}\right)$$

$$= |A_{p,m,n}|^2 (\pi/\varepsilon_m) \left\{ ah_c (1-\delta_{n0}) \left[\frac{(n\pi/h_c)^2 (m/a)^2}{\left(j'^2_{m,p}/a^2\right)^2} + 1 \right] + \frac{2j'^2_{m,p} (n\pi/h_c)^2}{\left(j'^2_{m,p}/a^2\right)^2} \left(1 - \frac{m^2}{j'^2_{m,p}}\right) \right\} J^2_m \left(j'_{m,p}\right)$$

$$= |A_{p,m,n}|^2 (\pi/\varepsilon_m) (1-\delta_{n0}) \left\{ ah_c \left[\frac{(n\pi/h_c)^2 (ma)^2}{j'^2_{m,p}} + j'^2_{m,p} \right] + 2a^4 (n\pi/h_c)^2 \left(1 - \frac{m^2}{j'^2_{m,p}}\right) \right\} \frac{J^2_m \left(j'_{m,p}\right)}{j'^2_{m,p}}$$

$$= |A_{p,m,n}|^2 (\pi/\varepsilon_m) (1-\delta_{n0}) \left\{ j'^2_{m,p} ah_c + 2a^4 (n\pi/h_c)^2 \frac{m^2}{j'^2_{m,p}} h_c/(2a) + 2a^4 (n\pi/h_c)^2 \left(1 - \frac{m^2}{j'^2_{m,p}}\right) \right\} \frac{J^2_m \left(j'_{m,p}\right)}{j'^2_{m,p}}$$

$$= |A_{p,m,n}|^2 (\pi/\varepsilon_m) (1-\delta_{n0}) \left\{ j'^2_{m,p} ah_c + 2a^4 (n\pi/h_c)^2 \left(1 - \frac{m^2}{j'^2_{m,p}} \left(1 - \frac{h_c}{2a}\right)\right) \right\} \frac{J^2_m \left(j'_{m,p}\right)}{j'^2_{m,p}}$$

The volume integral in the quality factor is

$$\int_V \underline{H}^{(p,m,n)*} \cdot \underline{H}^{(p,m,n)} dV = \int_0^{h_c} \int_0^a \int_{-\pi}^{\pi} \left[|H_\rho|^2 + |H_\varphi|^2 + |H_z|^2 \right] \rho d\varphi d\rho dz = |A_{p,m,n}|^2 (\pi/\varepsilon_m) h_c$$

$$\int_0^a \left[\frac{(n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} \left\{ \left(\frac{j'_{m,p}}{a}\right)^2 J'^2_m \left(j'_{m,p}\rho/a\right) + \left(\frac{m}{\rho}\right)^2 J^2_m \left(j'_{m,p}\rho/a\right) \right\} + (1-\delta_{n0}) J^2_m \left(j'_{m,p}\rho/a\right) \right] \rho d\rho$$

$$= |A_{p,m,n}|^2 (\pi/\varepsilon_m) h_c \frac{j'^2_{m,p} (n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} \int_0^1 \left\{ J'^2_m \left(j'_{m,p} u\right) + \left(\frac{m}{j'_{m,p} u}\right)^2 J^2_m \left(j'_{m,p} u\right) \right\} udu$$

$$+ (1-\delta_{n0}) |A_{p,m,n}|^2 (\pi/\varepsilon_m) h_c j'^2_{m,p} \left(\frac{a}{j'_{m,p}}\right)^2 \int_0^1 J^2_m \left(j'_{m,p} u\right) udu$$

$$= |A_{p,m,n}|^2 \left(\frac{\pi}{2}/\varepsilon_m\right) h_c a^2 \left[\frac{\left(j'_{m,p}/a\right)^2 (n\pi/h_c)^2}{\left\{k'^2_{p,m,n} - (n\pi/h_c)^2\right\}^2} + 1 - \delta_{n0} \right] \left(1 - \frac{m^2}{j'^2_{m,p}}\right) J^2_m \left(j'_{m,p}\right)$$

$$= |A_{p,m,n}|^2 \left(\frac{\pi}{2}/\varepsilon_m\right) h_c a^2 \left[\frac{(n\pi/h_c)^2}{k'^2_{p,m,n} - (n\pi/h_c)^2} + 1 \right] \left(1 - \frac{m^2}{j'^2_{m,p}}\right) J^2_m \left(j'_{m,p}\right) (1-\delta_{n0})$$

$$= |A_{p,m,n}|^2 \left(\frac{\pi}{2}/\varepsilon_m\right) h_c a^2 \frac{k'^2_{p,m,n}}{k'^2_{p,m,n} - (n\pi/h_c)^2} \left(1 - \frac{m^2}{j'^2_{m,p}}\right) J^2_m \left(j'_{m,p}\right) (1-\delta_{n0})$$

$$= |A_{p,m,n}|^2 \left(\frac{\pi}{2}/\varepsilon_m\right) a^4 h_c k'^2_{p,m,n} \left(1 - \frac{m^2}{j'^2_{m,p}}\right) \frac{J^2_m \left(j'_{m,p}\right)}{j'^2_{m,p}} (1-\delta_{n0})$$

where

$$\int_0^1 \left[J_m'^2\left(j'_{m,p}u\right) + \frac{m^2}{j'^2_{m,p}u^2} J_m^2\left(j'_{m,p}u\right)\right] udu = \frac{1}{2}\left(1 - \frac{m^2}{j'^2_{m,p}}\right) J_m^2\left(j'_{m,p}\right) = \int_0^1 J_m^2\left(j'_{m,p}u\right) udu$$

Then the TE mode quality factor is

$$\begin{aligned} Q_{p,m,n} &= \frac{\left(1 - m^2/j'^2_{m,p}\right)}{j'^2_{m,p}/a^2 + 2\left(a/h_c\right)\left(n\pi/h_c\right)^2 \left(1 - \frac{m^2}{j'^2_{m,p}}\left(1 - \frac{h_c}{2a}\right)\right)} k'^3_{p,m,n} a \frac{\eta_0}{2R_s^{(p,m,n)}} \\ &= \frac{\left(1 - m^2/j'^2_{m,p}\right)}{j'^2_{m,p}/a^2 + 2\left(a/h_c\right)\left(n\pi/h_c\right)^2 \left(1 - m^2/j'^2_{m,p}\right) + \left(n\pi/h_c\right)^2 m^2/j'^2_{m,p}} k'^3_{p,m,n} a \frac{\eta_0}{2R_s^{(p,m,n)}} \end{aligned}$$

where again the TE modal wavenumbers are

$$k'^2_{p,m,n} = \omega'^2_{p,m,n}\mu_0\varepsilon_0 = \left(\frac{j'_{m,p}}{a}\right)^2 + \left(n\pi/h_c\right)^2$$

and the surface resistance and skin depth are

$$R_s^{(p,m,n)} = 1/\left(\sigma\delta_{p,m,n}\right)$$

$$\delta_{p,m,n} = \sqrt{2/\left(\omega'_{p,m,n}\mu\sigma\right)}$$

4.4.3 Approximate Quality Factors

The previous "exact" results for the quality factors (found by integrating the square of the perfectly conducting magnetic field distribution) are

$$Q^{TM}_{p,m,n} = \frac{k_{p,m,n}a}{\varepsilon_n a/h_c + 1}\frac{\eta_0}{2R_s^{(p,m,n)}}$$

$$Q^{TE}_{p,m,n} = \frac{\left(1 - m^2/j'^2_{m,p}\right)}{j'^2_{m,p}/a^2 + \left(a/h_c\right)\left(n\pi/h_c\right)^2\left(1 - m^2/j'^2_{m,p}\right) + \left(n\pi/h_c\right)^2 m^2/j'^2_{m,p}} k'^3_{p,m,n} a \frac{\eta_0}{2R_s^{(p,m,n)}}$$

In the general cavity geometry we can write the quality factor as

$$\begin{aligned} Q = \frac{\omega W}{P} &\approx \frac{\omega V\left(\frac{1}{2}\mu_0\left\langle H^2\right\rangle_V + \frac{1}{2}\varepsilon_0\left\langle E^2\right\rangle_V\right)}{R_s S\left\langle H^2\right\rangle_S} \approx \frac{\omega V\mu_0\left\langle H^2\right\rangle_V}{R_s S\left\langle H^2\right\rangle_S} \approx \frac{\omega V_{cav}\mu_0 3\left\langle H_j^2\right\rangle_V}{R_s S_{cav} 2\left\langle H_j^2\right\rangle_S} \\ &\approx \frac{3}{4}\frac{\omega\mu_0 V_{cav}}{R_s S_{cav}} = \frac{3}{4}\frac{\eta_0 k V_{cav}}{R_s S_{cav}} = \frac{3}{2}\frac{V_{cav}}{\delta S_{cav}}\frac{\mu_0}{\mu} \end{aligned}$$

For the cylindrical geometry this becomes

$$Q \approx \frac{3}{2}\frac{ka}{2\left(a/h_c + 1\right)}\frac{\eta_0}{2R_s}$$

Now for the TM field configuration

$$Q^{TM} = \frac{\omega W}{P} \approx \frac{\omega V_{cav}\left(\frac{1}{2}\mu_0\left\langle H^2\right\rangle_V + \frac{1}{2}\varepsilon_0\left\langle E^2\right\rangle_V\right)}{R_s S_{cav}\left\langle H^2\right\rangle_S} \approx \frac{\omega V_{cav}\mu_0\left\langle H^2\right\rangle_V}{R_s S_{cav}\left\langle H^2\right\rangle_S} \approx \frac{\omega\pi a^2 h_c\mu_0 2\left\langle H_j^2\right\rangle_V}{R_s\left(2\pi a^2 2\left\langle H_j^2\right\rangle_S + 2\pi a h_c\left\langle H_j^2\right\rangle_S\right)}$$

$$\approx \frac{\omega\pi a^2 h_c \mu_0 2\left\langle H_j^2\right\rangle_V}{R_s\left(2\pi a^2 4\left\langle H_j^2\right\rangle_V + 2\pi a h_c 2\left\langle H_j^2\right\rangle_V\right)} \approx \frac{\eta_0 k a}{2R_s\left(2a/h_c+1\right)}$$

the same as the preceding exact result when $n \neq 0$. Introducing $n = 0$ and $m = 0$ terms

$$Q^{TM} \approx \frac{\omega\pi a^2 h_c \mu_0 2\left\langle H_j^2\right\rangle_V (2/\varepsilon_m)(2/\varepsilon_n)}{R_s\left[2\pi a^2 4\left\langle H_j^2\right\rangle_V (2/\varepsilon_m) + 2\pi a h_c 2\left\langle H_j^2\right\rangle_V (2/\varepsilon_m)(2/\varepsilon_n)\right]} \approx \frac{\eta_0 k a}{2R_s\left(\varepsilon_n a/h_c+1\right)}$$

which is the exact TM quality factor.

For the TE configuration

$$Q^{TE} = \frac{\omega W}{P} \approx \frac{\omega V_{cav}\left(\frac{1}{2}\mu_0\left\langle H^2\right\rangle_V + \frac{1}{2}\varepsilon_0\left\langle E^2\right\rangle_V\right)}{R_s S_{cav}\left\langle H^2\right\rangle_S} \approx \frac{\omega V_{cav}\mu_0\left\langle H^2\right\rangle_V}{R_s S_{cav}\left\langle H^2\right\rangle_S} \approx \frac{\omega V_{cav}\mu_0 3\left\langle H_j^2\right\rangle_V}{R_s S_{cav} 2\left\langle H_j^2\right\rangle_S}$$

$$\approx \frac{\omega V_{cav}\mu_0 3\left\langle H_j^2\right\rangle_V}{R_s S_{cav} 4\left\langle H_j^2\right\rangle_V} \approx \frac{3}{4}\frac{\omega V_{cav}\mu_0}{R_s S_{cav}} = \frac{3}{4}\frac{\eta_0 k V_{cav}}{R_s S_{cav}} = \frac{3}{2}\frac{V_{cav}}{\delta S_{cav}}\frac{\mu_0}{\mu} = \frac{3}{4}\frac{\eta_0 k a}{2R_s\left(a/h_c+1\right)}$$

Introducing the $m = 0$ (the $n = 0$ modes do not exist for PEC boundary conditions) terms gives the same result

$$Q^{TE} \approx \frac{\omega\left(\pi a^2 h_c\right)\mu_0 3\left\langle H_j^2\right\rangle_V (2/\varepsilon_m)}{R_s\left(2\pi a h_c 2/\varepsilon_n + 2\pi a^2\right) 4\left\langle H_j^2\right\rangle_V (2/\varepsilon_m)} = \frac{3}{4}\frac{\eta_0 k a}{2R_s\left(2/\varepsilon_n + a/h_c\right)} \approx \frac{3}{4}\frac{\omega V_{cav}\mu_0}{R_s S_{cav}} = \frac{3}{4}\frac{\eta_0 k V_{cav}}{R_s S_{cav}} = \frac{3}{4}\frac{\eta_0 k a}{2R_s\left(a/h_c+1\right)}$$

which is somewhat different than the preceding exact result.

This page left blank

5 EXTERIOR MODAL POTENTIALS AND FIELDS

We now turn to the exterior radiation on the cylinder. The incident plane wave and interaction with the cylinder (with slot shorted) is contained in the short circuit field.

5.1 Coulomb Gauge Electric Vector Potential

The Coulomb gauge potentials can be constructed from the form of the interior modal potentials with a continuous spectrum in z and a radiation condition in ρ.

5.1.1 TM Potentials

Based on the interior TM potentials

$$A_{e\rho}^{(p,m,n)} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{1}{\rho} A_{p,m,n} J_m\left(j_{m,p}\rho/a\right) m \sin\left[m\left(\varphi - \varphi_0\right)\right] \cos\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n} \frac{j_{m,p}}{a} J_m'\left(j_{m,p}\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \cos\left(n\pi z/h_c\right)$$

we take the exterior TM representation to be

$$A_{e\rho} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} \frac{m}{\rho} A_m^{TM}(\alpha) H_m^{(1)}(k_t\rho) \sin\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} A_m^{TM}(\alpha) H_m^{(1)\prime}(k_t\rho) \cos\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

where

$$k_t^2 = k^2 - \alpha^2$$

5.1.2 TE Potentials

Similarly, based on the interior TE potentials

$$A_{ez}^{(p,m,n)} = \frac{A_{p,m,n}}{i\omega} J_m\left(j_{m,p}'\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \sin\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{-1/(i\omega)}{k_{p,m,n}'^2 - (n\pi/h_c)^2} \frac{m}{\rho} A_{p,m,n} J_m\left(j_{m,p}'\rho/a\right) \sin\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$A_{e\rho}^{(p,m,n)} = \frac{1/(i\omega)}{k_{p,m,n}'^2 - (n\pi/h_c)^2} A_{p,m,n} \frac{j_{m,p}'}{a} J_m'\left(j_{m,p}'\rho/a\right) \cos\left[m\left(\varphi - \varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

we take the exterior TE representation to be

$$A_{ez} = \sum_m \int_0^\infty A_m^{TE}(\alpha) H_m^{(1)}(k_t\rho) \cos\left[m\left(\varphi - \varphi_m^{TE}\right)\right] \left\{ \begin{array}{c} \sin(\alpha z) \\ -\cos(\alpha z) \end{array} \right\} d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{-1}{k_t^2} \frac{m}{\rho} A_m^{TE}(\alpha) H_m^{(1)}(k_t\rho) \sin\left[m\left(\varphi - \varphi_m^{TE}\right)\right] \alpha \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$A_{e\rho} = \sum_m \int_0^\infty \frac{1}{k_t} A_m^{TE}(\alpha) H_m^{(1)\prime}(k_t\rho) \cos\left[m\left(\varphi - \varphi_m^{TE}\right)\right] \alpha \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

We should be able to use these representations along with orthogonality on the $\rho = b$ surface in z and φ to expand the field. Matching the tangential electric field on the surface (along with the normal magnetic field) can be used to find the coefficients $A_m^{TM}(\alpha)$ and $A_m^{TE}(\alpha)$. We rewrite the TM potentials as (where we take $m\varphi_m^{TM} = 0, \pi/2$) for the even parity about $z = z_0$ as

$$A_{e\rho} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} \frac{m}{\rho} H_m^{(1)}(k_t\rho) \left[A_m^{TM}(\alpha) \sin(m\varphi) - B_m^{TM}(\alpha) \cos(m\varphi)\right] \cos\alpha(z - z_0)\, d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} H_m^{(1)\prime}(k_t\rho) \left[A_m^{TM}(\alpha) \cos(m\varphi) + B_m^{TM}(\alpha) \sin(m\varphi)\right] \cos\alpha(z - z_0)\, d\alpha$$

and the TE potentials as (where we take $m\varphi_m^{TE} = -\pi/2, 0$) for the even parity of $A_{e\varphi}$ about $z = z_0$

$$A_{ez} = \sum_m \int_0^\infty H_m^{(1)}(k_t\rho) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi)\right] \sin\alpha(z - z_0)\, d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{-1}{k_t^2} \frac{m}{\rho} H_m^{(1)}(k_t\rho) \left[A_m^{TE}(\alpha) \cos(m\varphi) + B_m^{TE}(\alpha) \sin(m\varphi)\right] \alpha \cos\alpha(z - z_0)\, d\alpha$$

$$A_{e\rho} = \sum_m \int_0^\infty \frac{1}{k_t} H_m^{(1)\prime}(k_t\rho) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi)\right] \alpha \cos\alpha(z - z_0)\, d\alpha$$

We check the vanishing of divergences of these potentials in the Coulomb gauge. For the TM modes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_{e\rho}) + \frac{1}{\rho}\frac{\partial}{\partial\varphi} A_{e\varphi} =$$

$$\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} \frac{m}{\rho} H_m^{(1)\prime}(k_t\rho) \left[A_m^{TM}(\alpha) \sin(m\varphi) - B_m^{TM}(\alpha) \cos(m\varphi)\right] \cos\alpha(z - z_0)\, d\alpha$$

$$-\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} \frac{m}{\rho} H_m^{(1)\prime}(k_t\rho) \left[A_m^{TM}(\alpha) \sin(m\varphi) - B_m^{TM}(\alpha) \cos(m\varphi)\right] \cos\alpha(z - z_0)\, d\alpha = 0$$

For the TE modes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_{e\rho}) + \frac{1}{\rho}\frac{\partial}{\partial\varphi} A_{e\varphi} + \frac{\partial}{\partial z} A_{ez} =$$

$$\sum_m \int_0^\infty \frac{1}{k_t^2} \frac{1}{\rho} \frac{\partial}{\partial\rho} \left\{\rho \frac{\partial}{\partial\rho} H_m^{(1)}(k_t\rho)\right\} \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi)\right] \alpha \cos\alpha(z - z_0)\, d\alpha$$

$$+\sum_m \int_0^\infty \frac{-1}{k_t^2} \frac{m^2}{\rho^2} H_m^{(1)}(k_t\rho) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi)\right] \alpha \cos\alpha(z - z_0)\, d\alpha$$

$$+\sum_m \int_0^\infty H_m^{(1)}(k_t\rho) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi)\right] \alpha \cos\alpha(z - z_0)\, d\alpha =$$

$$\sum_m \int_0^\infty \frac{1}{k_t^2} \left[\frac{1}{\rho} \frac{\partial}{\partial\rho} \left\{ \rho \frac{\partial}{\partial\rho} H_m^{(1)}(k_t\rho) \right\} + \left(k_t^2 - \frac{m^2}{\rho^2} \right) H_m^{(1)}(k_t\rho) \right]$$

$$\left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi) \right] \alpha \cos\alpha (z - z_0)\, d\alpha = 0$$

5.1.3 Matching Of Surface Electric Fields

For the narrow slot the azimuthal electric field on the surface is taken to be near zero for all φ and z on the cylinder surface $\rho = b$

$$\varepsilon_0 E_\varphi = -\frac{\partial}{\partial z} A_{e\rho} + \frac{\partial}{\partial\rho} A_{ez} \approx 0\ ,\ \ \rho = b$$

$$\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} \frac{m}{b} H_m^{(1)}(k_t b) \left[A_m^{TM}(\alpha) \sin(m\varphi) - B_m^{TM}(\alpha) \cos(m\varphi) \right] \alpha \sin\alpha (z - z_0)\, d\alpha$$

$$+ \sum_m \int_0^\infty \frac{1}{k_t} H_m^{(1)\prime}(k_t b) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi) \right] \alpha^2 \sin\alpha (z - z_0)\, d\alpha$$

$$+ \sum_m \int_0^\infty H_m^{(1)\prime}(k_t b) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi) \right] \sin\alpha (z - z_0)\, k_t d\alpha \to 0$$

or

$$\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} \frac{m}{b} H_m^{(1)}(k_t b) \left[A_m^{TM}(\alpha) \sin(m\varphi) - B_m^{TM}(\alpha) \cos(m\varphi) \right] \alpha \sin\alpha (z - z_0)\, d\alpha$$

$$+ \sum_m \int_0^\infty \frac{k^2}{k_t} H_m^{(1)\prime}(k_t b) \left[-A_m^{TE}(\alpha) \sin(m\varphi) + B_m^{TE}(\alpha) \cos(m\varphi) \right] \sin\alpha (z - z_0)\, d\alpha \to 0$$

which means that we can take

$$\frac{\varepsilon_0}{k_t} \alpha \frac{m}{b} H_m^{(1)}(k_t b)\, A_m^{TM}(\alpha) = k^2 H_m^{(1)\prime}(k_t b)\, A_m^{TE}(\alpha)$$

$$\frac{\varepsilon_0}{k_t} \alpha \frac{m}{b} H_m^{(1)}(k_t b)\, B_m^{TM}(\alpha) = k^2 H_m^{(1)\prime}(k_t b)\, B_m^{TE}(\alpha)$$

For a φ directed magnetic current (and consistency of this sign with previous slot analysis on a plane) with $I_m^- = -2V_-$ (because later we use the continuity equation $dI_m^-/ds = i\omega q_m^-$ with $s = b\varphi$, where the minus superscript denotes the incident side of the slot) the electric field around the magnetic current is consistent with the axial field being $-z$ directed for a positive voltage (with positive reference on the plus z side of the slot). Hence we must introduce a minus sign in front of the delta function in z (the delta function is used for the very narrow slot limit)

$$\varepsilon_0 E_z = -\frac{1}{\rho} \frac{\partial}{\partial\rho} (\rho A_{e\varphi}) + \frac{1}{\rho} \frac{\partial}{\partial\varphi} A_{e\rho} \approx -\varepsilon_0 V_-(s)\, \delta(z - z_0)$$

$$\sum_m \int_0^\infty \varepsilon_0 \left(1 - \frac{m^2}{k_t^2 b^2} \right) H_m^{(1)}(k_t b) \left[A_m^{TM}(\alpha) \cos(m\varphi) + B_m^{TM}(\alpha) \sin(m\varphi) \right] \cos\alpha (z - z_0)\, d\alpha$$

$$+\sum_m \int_0^\infty \frac{1}{k_t}\frac{m}{b} H_m^{(1)\prime}(k_t b)\left[A_m^{TE}(\alpha)\cos(m\varphi)+B_m^{TE}(\alpha)\sin(m\varphi)\right]\alpha\cos\alpha(z-z_0)\,d\alpha$$

$$+\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2}\frac{m^2}{b^2} H_m^{(1)}(k_t b)\left[A_m^{TM}(\alpha)\cos(m\varphi)+B_m^{TM}(\alpha)\sin(m\varphi)\right]\cos\alpha(z-z_0)\,d\alpha$$

$$-\sum_m \int_0^\infty \frac{1}{k_t}\frac{m}{b} H_m^{(1)\prime}(k_t b)\left[A_m^{TE}(\alpha)\cos(m\varphi)+B_m^{TE}(\alpha)\sin(m\varphi)\right]\alpha\cos\alpha(z-z_0)\,d\alpha \approx -\varepsilon_0 V_-(s)\,\delta(z-z_0)$$

or

$$\sum_m \int_0^\infty \varepsilon_0 H_m^{(1)}(k_t b)\left[A_m^{TM}(\alpha)\cos(m\varphi)+B_m^{TM}(\alpha)\sin(m\varphi)\right]\cos\alpha(z-z_0)\,d\alpha \approx -\varepsilon_0 V_-(s)\,\delta(z-z_0)$$

where we used

$$\left(\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{dz}+1-m^2/z^2\right)H_m^{(1)}(z)=0=\left[\frac{1}{z}\frac{d}{dz}\left(z\frac{d}{dz}\right)+1-m^2/z^2\right]H_m^{(1)}(z)$$

$$\frac{1}{k_t\rho}\frac{\partial}{\partial k_t\rho}\left(k_t\rho H_m^{(1)\prime}(k_t\rho)\right)=-\left(1-\frac{m^2}{k_t^2\rho^2}\right)H_m^{(1)}(k_t\rho)$$

The odd parity representation about $z=z_0$ can be written as

$$A_{e\rho}=\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2}\frac{m}{\rho} H_m^{(1)}(k_t\rho)\left[A_m^{TMo}(\alpha)\sin(m\varphi)-B_m^{TMo}(\alpha)\cos(m\varphi)\right]\sin\alpha(z-z_0)\,d\alpha$$

$$A_{e\varphi}=\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} H_m^{(1)\prime}(k_t\rho)\left[A_m^{TMo}(\alpha)\cos(m\varphi)+B_m^{TMo}(\alpha)\sin(m\varphi)\right]\sin\alpha(z-z_0)\,d\alpha$$

$$A_{ez}=\sum_m \int_0^\infty H_m^{(1)}(k_t\rho)\left[A_m^{TEo}(\alpha)\sin(m\varphi)-B_m^{TEo}(\alpha)\cos(m\varphi)\right]\cos\alpha(z-z_0)\,d\alpha$$

$$A_{e\varphi}=\sum_m \int_0^\infty \frac{-1}{k_t^2}\frac{m}{\rho} H_m^{(1)}(k_t\rho)\left[A_m^{TEo}(\alpha)\cos(m\varphi)+B_m^{TEo}(\alpha)\sin(m\varphi)\right]\alpha\sin\alpha(z-z_0)\,d\alpha$$

$$A_{e\rho}=\sum_m \int_0^\infty \frac{1}{k_t} H_m^{(1)\prime}(k_t\rho)\left[-A_m^{TEo}(\alpha)\sin(m\varphi)+B_m^{TEo}(\alpha)\cos(m\varphi)\right]\alpha\sin\alpha(z-z_0)\,d\alpha$$

and for all φ and z

$$\varepsilon_0 E_\varphi=-\frac{\partial}{\partial z}A_{e\rho}+\frac{\partial}{\partial\rho}A_{ez}\approx 0\ ,\ \ \rho=b$$

$$-\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2}\frac{m}{b} H_m^{(1)}(k_t b)\left[A_m^{TMo}(\alpha)\sin(m\varphi)-B_m^{TMo}(\alpha)\cos(m\varphi)\right]\alpha\cos\alpha(z-z_0)\,d\alpha$$

$$+\sum_m \int_0^\infty \frac{\alpha^2}{k_t} H_m^{(1)\prime}(k_t b)\left[A_m^{TEo}(\alpha)\sin(m\varphi)-B_m^{TEo}(\alpha)\cos(m\varphi)\right]\cos\alpha(z-z_0)\,d\alpha$$

$$+\sum_m \int_0^\infty k_t H_m^{(1)\prime}(k_t b)\left[A_m^{TEo}(\alpha)\sin(m\varphi) - B_m^{TEo}(\alpha)\cos(m\varphi)\right]\cos\alpha(z-z_0)\,d\alpha \to 0$$

or

$$\varepsilon_0 \sum_m \int_0^\infty \frac{\alpha}{k_t^2}\frac{m}{b} H_m^{(1)}(k_t b)\left[A_m^{TMo}(\alpha)\sin(m\varphi) - B_m^{TMo}(\alpha)\cos(m\varphi)\right]d\alpha$$
$$= \sum_m \int_0^\infty \frac{k^2}{k_t} H_m^{(1)\prime}(k_t b)\left[A_m^{TEo}(\alpha)\sin(m\varphi) - B_m^{TEo}(\alpha)\cos(m\varphi)\right]d\alpha$$

We can thus take

$$\varepsilon_0 \frac{\alpha}{k_t}\frac{m}{b} H_m^{(1)}(k_t b)\, A_m^{TMo}(\alpha) = k^2 H_m^{(1)\prime}(k_t b)\, A_m^{TEo}(\alpha)$$
$$\varepsilon_0 \frac{\alpha}{k_t}\frac{m}{b} H_m^{(1)}(k_t b)\, B_m^{TMo}(\alpha) = k^2 H_m^{(1)\prime}(k_t b)\, B_m^{TEo}(\alpha)$$

Also, in the narrow slot limit this odd slot voltage (in the width direction) is driven to zero

$$\varepsilon_0 E_z = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_{e\varphi}) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\rho} \approx -\varepsilon_0 V_-^o(s)\,\delta(z-z_0) \to 0$$

and thus

$$\sum_m \int_0^\infty \varepsilon_0\left(1-\frac{m^2}{k_t^2 b^2}\right) H_m^{(1)}(k_t b)\left[A_m^{TMo}(\alpha)\cos(m\varphi) + B_m^{TMo}(\alpha)\sin(m\varphi)\right]\sin\alpha(z-z_0)\,d\alpha$$
$$+\sum_m \int_0^\infty \frac{1}{k_t}\frac{m}{b} H_m^{(1)\prime}(k_t b)\left[A_m^{TEo}(\alpha)\cos(m\varphi) + B_m^{TEo}(\alpha)\sin(m\varphi)\right]\alpha\sin\alpha(z-z_0)\,d\alpha$$
$$+\sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2}\frac{m^2}{b^2} H_m^{(1)}(k_t b)\left[A_m^{TMo}(\alpha)\cos(m\varphi) + B_m^{TMo}(\alpha)\sin(m\varphi)\right]\sin\alpha(z-z_0)\,d\alpha$$

$$-\sum_m \int_0^\infty \frac{1}{k_t}\frac{m}{b} H_m^{(1)\prime}(k_t b)\left[A_m^{TEo}(\alpha)\cos(m\varphi) + B_m^{TEo}(\alpha)\sin(m\varphi)\right]\alpha\sin\alpha(z-z_0)\,d\alpha \to -\varepsilon_0 V_-^o(s)\,\delta(z-z_0) \to 0$$

or

$$\sum_m \int_0^\infty \varepsilon_0 H_m^{(1)}(k_t b)\left[A_m^{TMo}(\alpha)\cos(m\varphi) + B_m^{TMo}(\alpha)\sin(m\varphi)\right]\sin\alpha(z-z_0)\,d\alpha \to -\varepsilon_0 V_-^o(s)\,\delta(z-z_0) \to 0$$

We can therefore take

$$A_m^{TMo}(\alpha) = 0 = B_m^{TMo}(\alpha)$$

and thus

$$A_m^{TEo}(\alpha) = 0 = B_m^{TEo}(\alpha)$$

The Fourier representation for the delta function

$$\int_0^{\infty} A(\alpha)\cos\alpha(z-z_0)\,d\alpha = \delta(z-z_0)$$

can be inverted by noting that

$$\int_{-\infty}^{\infty}\cos(\alpha' u)\cos(\alpha u)\,du = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\left[\cos(\alpha-\alpha')u+\cos(\alpha+\alpha')u\right]du$$

$$=\lim_{R\to\infty}\left[\frac{\sin(\alpha-\alpha')R}{(\alpha-\alpha')}+\frac{\sin(\alpha+\alpha')R}{(\alpha+\alpha')}\right]=\pi\lim_{R/\pi\to\infty}R/\pi\left[\frac{\sin(\alpha-\alpha')\pi R/\pi}{(\alpha-\alpha')\pi R/\pi}+\frac{\sin(\alpha+\alpha')\pi R/\pi}{(\alpha+\alpha')\pi R/\pi}\right]$$

$$=\pi\left[\delta(\alpha-\alpha')+\delta(\alpha+\alpha')\right]$$

and hence

$$\int_0^{\infty}A(\alpha)\int_{-\infty}^{\infty}\cos\alpha'(z-z_0)\cos\alpha(z-z_0)\,d(z-z_0)\,d\alpha=\int_0^{\infty}A(\alpha)\pi\left[\delta(\alpha-\alpha')+\delta(\alpha+\alpha')\right]d\alpha=1$$

and therefore

$$A(\alpha')=1/\pi$$

Another check on this representation is

$$\frac{1}{\pi}\lim_{R\to\infty}\int_0^{R}\cos\alpha(z-z_0)\,d\alpha=\lim_{R\to\infty}\frac{\sin R(z-z_0)}{\pi(z-z_0)}=\lim_{R/\pi\to\infty}\frac{\sin(\pi(z-z_0)R/\pi)}{\pi(z-z_0)}=\delta(z-z_0)$$

Thus, because the unknowns are not really functions of α (by virtue of the preceding delta function representation), the preceding equation

$$\sum_m\int_0^{\infty}\varepsilon_0 H_m^{(1)}(k_t b)\left[A_m^{TM}(\alpha)\cos(m\varphi)+B_m^{TM}(\alpha)\sin(m\varphi)\right]\cos\alpha(z-z_0)\,d\alpha\approx-\varepsilon_0 V_-(s)\,\delta(z-z_0)$$

can be reduced to

$$\sum_m H_m^{(1)}(k_t b)\left[A_m^{TM}\cos(m\varphi)+B_m^{TM}\sin(m\varphi)\right]\approx-\frac{1}{\pi}V_-(s)$$

Then using the integrals

$$\int_{-\pi}^{\pi}\cos(m'\varphi)\cos(m\varphi)\,d\varphi=\frac{2\pi}{\varepsilon_m}\delta_{mm'}$$

$$\int_{-\pi}^{\pi}\sin(m'\varphi)\cos(m\varphi)\,d\varphi=0$$

$$\int_{-\pi}^{\pi}\sin(m'\varphi)\sin(m\varphi)\,d\varphi=\pi\delta_{mm'}$$

the Fourier series can be inverted to give

$$H_m^{(1)}(k_t b)\, A_m^{TM} \approx -\frac{\varepsilon_m}{2\pi^2}\int_{-\pi}^{\pi} V_-(s)\cos(m\varphi)\, d\varphi$$

$$H_m^{(1)}(k_t b)\, B_m^{TM} \approx -\frac{1}{\pi^2}\int_{-\pi}^{\pi} V_-(s)\sin(m\varphi)\, d\varphi$$

Therefore the coefficients are found as

$$-\varepsilon_0\frac{\alpha}{k_t}\frac{m}{b}\frac{\varepsilon_m}{2\pi^2}\int_{-\pi}^{\pi} V_-(s)\cos(m\varphi)\, d\varphi = \frac{\varepsilon_0}{k_t}\alpha\frac{m}{b}H_m^{(1)}(k_t b)\, A_m^{TM}(\alpha) = k^2 H_m^{(1)\prime}(k_t b)\, A_m^{TE}(\alpha)$$

$$-\varepsilon_0\frac{\alpha}{k_t}\frac{m}{b}\frac{1}{\pi^2}\int_{-\pi}^{\pi} V_-(s)\sin(m\varphi)\, d\varphi = \frac{\varepsilon_0}{k_t}\alpha\frac{m}{b}H_m^{(1)}(k_t b)\, B_m^{TM}(\alpha) = k^2 H_m^{(1)\prime}(k_t b)\, B_m^{TE}(\alpha)$$

The azimuthal potential is

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} H_m^{(1)\prime}(k_t\rho)\left[A_m^{TM}(\alpha)\cos(m\varphi) + B_m^{TM}(\alpha)\sin(m\varphi)\right]\cos\alpha(z-z_0)\, d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{-1}{k_t^2}\frac{m}{\rho} H_m^{(1)}(k_t\rho)\left[A_m^{TE}(\alpha)\cos(m\varphi) + B_m^{TE}(\alpha)\sin(m\varphi)\right]\alpha\cos\alpha(z-z_0)\, d\alpha$$

and radial potential is

$$A_{e\rho} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2}\frac{m}{\rho} H_m^{(1)}(k_t\rho)\left[A_m^{TM}(\alpha)\sin(m\varphi) - B_m^{TM}(\alpha)\cos(m\varphi)\right]\cos\alpha(z-z_0)\, d\alpha$$

$$A_{e\rho} = \sum_m \int_0^\infty \frac{1}{k_t} H_m^{(1)\prime}(k_t\rho)\left[-A_m^{TE}(\alpha)\sin(m\varphi) + B_m^{TE}(\alpha)\cos(m\varphi)\right]\alpha\cos\alpha(z-z_0)\, d\alpha$$

The total is then

$$A_{e\varphi} = -\frac{\varepsilon_0}{\pi^2}\sum_m \int_0^\infty \left[\frac{H_m^{(1)\prime}(k_t\rho)}{k_t H_m^{(1)}(k_t b)} - \frac{m^2\alpha^2/k^2}{k_t^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\cos(m\varphi)\frac{\varepsilon_m}{2}\int_{-\pi}^{\pi} V_-(s')\cos(m\varphi')\, d\varphi' + \sin(m\varphi)\int_{-\pi}^{\pi} V_-(s')\sin(m\varphi')\, d\varphi'\right]\cos\alpha(z-z_0)\, d\alpha$$

and

$$A_{e\rho} = -\frac{\varepsilon_0}{\pi^2}\sum_m \int_0^\infty \frac{m}{k_t^2}\left[\frac{H_m^{(1)}(k_t\rho)}{\rho H_m^{(1)}(k_t b)} - \frac{\alpha^2}{k^2 b}\frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\sin(m\varphi)\frac{\varepsilon_m}{2}\int_{-\pi}^{\pi} V_-(s')\cos(m\varphi')\, d\varphi' - \cos(m\varphi)\int_{-\pi}^{\pi} V_-(s')\sin(m\varphi')\, d\varphi'\right]\cos\alpha(z-z_0)\, d\alpha$$

Evaluation at $\rho = b$ and near the line source $z = z_0 + \Delta$ (with Δ a small quantity) and exterior arc length $s = b\varphi$, the azimuthal component is

$$i\omega A_{e\varphi}\left(b,\varphi,z_0+\Delta\right)=-\frac{i\omega\varepsilon_0}{\pi^2}\sum_m\int_0^\infty\left[\frac{H_m^{(1)\prime}\left(k_tb\right)}{k_tbH_m^{(1)}\left(k_tb\right)}-\frac{m^2\left(k^2-k_t^2\right)}{k_t^2k^2b^2}\frac{H_m^{(1)}\left(k_tb\right)}{k_tbH_m^{(1)\prime}\left(k_tb\right)}\right]$$

$$\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}\int_{-h}^{h}V_-\left(s'\right)\cos\left(ms'/b\right)ds'+\sin\left(m\varphi\right)\int_{-h}^{h}V_-\left(s'\right)\sin\left(ms'/b\right)ds'\right]\cos\left(\alpha\Delta\right)d\alpha$$

$$i\omega A_{e\rho}\left(b,\varphi,z_0+\Delta\right)=-\frac{i\omega\varepsilon_0}{\pi^2b^2}\sum_m\int_0^\infty\frac{m}{k_t^2}\left(1-\frac{\alpha^2}{k^2}\right)$$

$$\left[\sin\left(m\varphi\right)\frac{\varepsilon_m}{2}\int_{-h}^{h}V_-\left(s'\right)\cos\left(ms'/b\right)ds'-\cos\left(m\varphi\right)\int_{-h}^{h}V_-\left(s'\right)\sin\left(ms'/b\right)ds'\right]\cos\left(\alpha\Delta\right)d\alpha$$

or

$$i\omega A_{e\varphi}\left(b,\varphi,z_0+\Delta\right)=\frac{\omega^2\mu_0\varepsilon_0}{2\pi^2}\sum_m\int_0^\infty\left[\frac{H_m^{(1)\prime}\left(k_tb\right)}{k_tbH_m^{(1)}\left(k_tb\right)}-\frac{m^2\left(k^2-k_t^2\right)}{k_t^2k^2b^2}\frac{H_m^{(1)}\left(k_tb\right)}{k_tbH_m^{(1)\prime}\left(k_tb\right)}\right]$$

$$\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]\cos\left(\alpha\Delta\right)d\alpha$$

and the radial component is

$$i\omega A_{e\rho}\left(b,\varphi,z_0+\Delta\right)=\frac{\omega^2\mu_0\varepsilon_0}{2\pi^2k^2b^2}\sum_m m\left[\sin\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^--\cos\left(m\varphi\right)m_{os}^-\right]\int_0^\infty\cos\left(\alpha\Delta\right)d\alpha$$

$$=-\frac{1}{2\pi^2b^2}\frac{\partial}{\partial\varphi}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]\int_0^\infty\cos\left(\alpha\Delta\right)d\alpha$$

where

$$m_{es}^-=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/b\right)V_-\left(s'\right)ds'$$

$$m_{os}^-=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/b\right)V_-\left(s'\right)ds'$$

and we note

$$\lim_{R\to\infty}\int_0^R\cos\left(\alpha\Delta\right)d\alpha=\lim_{R\to\infty}\frac{\sin\left(R\Delta\right)}{\Delta}=\pi\lim_{R/\pi\to\infty}\left(R/\pi\right)\frac{\sin\left(\pi\Delta R/\pi\right)}{\pi\Delta R/\pi}=\pi\lim_{R/\pi\to\infty}\left(R/\pi\right)\operatorname{sinc}\left(\Delta R/\pi\right)=\pi\delta\left(\Delta\right)$$

so that

$$i\omega A_{e\rho}\left(b,\varphi,z_0+\Delta\right)=-\frac{1}{2\pi b^2}\frac{\partial}{\partial\varphi}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]\delta\left(\Delta\right)$$

5.1.4 Quasi-static Limit

The quasi-static limit of the azimuthal vector potential is

$$k_t = i\sqrt{\alpha^2 - k^2} \to i\alpha$$

$$i\omega A_{e\varphi}\left(b,\varphi,z_0+\Delta\right) \to \frac{\omega^2\mu_0\varepsilon_0}{2\pi^2}\sum_m \int_0^\infty \left[\frac{H_m^{(1)\prime}\left(i\alpha b\right)}{i\alpha b H_m^{(1)}\left(i\alpha b\right)} + \frac{m^2}{k^2 b^2}\frac{H_m^{(1)}\left(i\alpha b\right)}{i\alpha b H_m^{(1)\prime}\left(i\alpha b\right)}\right]$$
$$\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^{-} + \sin\left(m\varphi\right)m_{os}^{-}\right]\cos\left(\alpha\Delta\right)d\alpha$$

5.1.5 Local Quasi-Static Planar Contribution

To avoid convergence issues near the line source it is important to remove the local contribution to the potential. One approach is to remove the local planar quasi-static contribution, which can be written as

$$i\omega A_{es}^{s}\left(\rho_-,s\right) = i\omega\varepsilon_0\int_{-h}^{h} I_m^{-}\left(s'\right)\frac{ds'}{4\pi\sqrt{\left(s-s'\right)^2+\rho_-^2}} = -i\omega\varepsilon_0\frac{1}{2\pi}\int_{-h}^{h} V_-\left(s'\right)\frac{ds'}{\sqrt{\left(s-s'\right)^2+\rho_-^2}}$$

$$= -i\omega\varepsilon_0\frac{1}{2\pi}\left[V_-\left(s\right)\int_{-h}^{h}\frac{ds'}{\sqrt{\left(s-s'\right)^2+\rho_-^2}} + \int_{-h}^{h}\left\{V_-\left(s'\right)-V_-\left(s\right)\right\}\frac{ds'}{\sqrt{\left(s-s'\right)^2+\rho_-^2}}\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}\left[V_-\left(s\right)\int_{s-h}^{s+h}\frac{ds'}{\sqrt{\left(s-s'\right)^2+\rho_-^2}} + \int_{-h}^{h}\frac{V_-\left(s'\right)-V_-\left(s\right)}{\left|s-s'\right|}ds'\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}\left[V_-\left(s\right)\int_{s-h}^{s+h}\frac{du}{\sqrt{u^2+\rho_-^2}} + \int_{-h}^{h}\frac{V_-\left(s'\right)-V_-\left(s\right)}{\left|s-s'\right|}ds'\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}\left[V_-\left(s\right)\int_{-(h-s)/\rho_-}^{(h+s)/\rho_-}\frac{du}{\sqrt{u^2+1}} + \int_{-h}^{s}\frac{V_-\left(s'\right)-V_-\left(s\right)}{s-s'}ds' + \int_{s}^{h}\frac{V_-\left(s'\right)-V_-\left(s\right)}{s'-s}ds'\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}\left[V_-\left(s\right)\left\{\mathrm{Arcsinh}\left(\left(h+s\right)/\rho_-\right) + \mathrm{Arcsinh}\left(\left(h-s\right)/\rho_-\right)\right\} + \int_{-h}^{s}\frac{V_-\left(s'\right)-V_-\left(s\right)}{s-s'}ds' + \int_{s}^{h}\frac{V_-\left(s'\right)-V_-\left(s\right)}{s'-s}ds'\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}\left[V_-\left(s\right)\ln\left\{\left(h+s\right)/\rho_- + \sqrt{\left(h+s\right)^2/\rho_-^2+1}\right\} + V_-\left(s\right)\ln\left\{\left(h-s\right)/\rho_- + \sqrt{\left(h-s\right)^2/\rho_-^2+1}\right\}\right.$$
$$\left. + \int_{-h}^{s}\frac{V_-\left(s'\right)-V_-\left(s\right)}{s-s'}ds' + \int_{s}^{h}\frac{V_-\left(s'\right)-V_-\left(s\right)}{s'-s}ds'\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}V_-\left(s\right)\left[2\ln\left(2h/\rho_-\right) + \ln\left\{\left(h+s\right)/h\right\} + \ln\left\{\left(h-s\right)/h\right\}\right]$$

$$-i\omega\varepsilon_0\frac{1}{2\pi}\left[\int_{-h}^{s}\frac{V_-(s')-V_-(s)}{s-s'}ds'+\int_{s}^{h}\frac{V_-(s')-V_-(s)}{s'-s}ds'\right]$$

$$\approx -i\omega\varepsilon_0\frac{1}{2\pi}V_-(s)\left[\Omega_{\rho_-}+\ln\left\{\left(h^2-s^2\right)/h^2\right\}\right]$$

$$-i\omega\varepsilon_0\frac{1}{2\pi}\left[\int_{-h}^{s}\frac{V_-(s')-V_-(s)}{s-s'}ds'+\int_{s}^{h}\frac{V_-(s')-V_-(s)}{s'-s}ds'\right]$$

where ρ_- is a small displacement from the line source and the fatness parameter is

$$\Omega_{\rho_-}=2\ln\left(2h/\rho_-\right)$$

By expanding this approximate quasi-static half space form of the vector potential static axial magnetic field in the Fourier series

$$i\omega A_{es}^{s}\left(\rho_-,s\right)=-i\omega\varepsilon_0\sum_m\left[A_m\cos(m\varphi)+B_m\sin(m\varphi)\right]$$

we can then remove this from the preceding cylindrical expansion.

5.1.6 Local Transmission Line Operator

Because we intend to use the transmission line approximation for the dominant operator, it is simpler to remove only the leading Hallen or transmission line term (rather than the complete half space form)

$$i\omega A_{es}^{s}\left(\rho_-,s\right)\sim -i\omega\varepsilon_0\frac{\Omega_{e\rho_-}}{2\pi}V_-(s)$$

where

$$\Omega_{e\rho_-}=\Omega_{\rho_-}+C_e$$

(Note that we could also take different constants for the even and odd parts C_e^e and C_e^o of the voltage.) Breaking the voltage into even and odd parts

$$V_-(s)=V_e^-(s)+V_o^-(s)$$

we extract the coefficients in the Fourier series

$$A_m\int_{-\pi}^{\pi}\cos^2(m\varphi)\,d\varphi=\frac{2\pi}{\varepsilon_m}A_m\sim\frac{\Omega_{e\rho_-}}{2\pi}\int_{-h/b}^{h/b}V_e^-(s)\cos(m\varphi)\,d\varphi$$

$$\sim\frac{\Omega_{e\rho_-}}{2\pi b}\int_{-h}^{h}V_e^-(s)\cos(ms/b)\,ds=\frac{k^2}{-i\omega\varepsilon_0}\frac{\Omega_{e\rho_-}}{4\pi b}m_{es}^-$$

$$B_m\int_{-\pi}^{\pi}\sin^2(m\varphi)\,d\varphi=\pi B_m\sim\frac{\Omega_{e\rho_-}}{2\pi}\int_{-h/b}^{h/b}V_o^-(s)\sin(m\varphi)\,d\varphi$$

$$\sim\frac{\Omega_{e\rho_-}}{2\pi b}\int_{-h}^{h}V_o^-(s)\sin(ms/b)\,ds=\frac{k^2}{-i\omega\varepsilon_0}\frac{\Omega_{e\rho_-}}{4\pi b}m_{os}^-$$

$$m_{es}^{-}=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/b\right)V_{-}\left(s'\right)ds'$$

$$m_{os}^{-}=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/b\right)V_{-}\left(s'\right)ds'$$

gives the Fourier series representation

$$i\omega A_{es}^{s}\left(\rho_{-},s\right)=-i\omega\varepsilon_0\frac{\Omega_{e\rho_{-}}}{2\pi}V_{-}\left(s\right)=\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^{-}+\sin\left(m\varphi\right)m_{os}^{-}\right]\Omega_{e\rho_{-}}/2$$

We can also operate on this to obtain

$$i\omega\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)A_{es}^{s}\left(\rho_{-},s\right)=\frac{k^2}{2\pi^2 b}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^{-}+\sin\left(m\varphi\right)m_{os}^{-}\right]\Omega_{e\rho_{-}}/2$$

$$=\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^{-}+\sin\left(m\varphi\right)m_{os}^{-}\right]\left(1-\frac{m^2}{k^2b^2}\right)\Omega_{e\rho_{-}}/2$$

Then we can write

$$i\omega\left[A_{e\varphi}\left(b,\varphi,z\right)-A_{es}^{s}\left(\rho_{-},s\right)\right]\approx\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^{-}+\sin\left(m\varphi\right)m_{os}^{-}\right]$$

$$\int_0^\infty\left[\frac{H_m^{(1)\prime}\left(k_tb\right)}{H_m^{(1)}\left(k_tb\right)}-m^2\left(1-k_t^2/k^2\right)\frac{H_m^{(1)}\left(k_tb\right)}{\left(k_tb\right)^2H_m^{(1)\prime}\left(k_tb\right)}\right]\cos\alpha\left(z-z_0\right)\frac{d\alpha}{k_t}$$

$$-\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m_{es}^{-}+\sin\left(m\varphi\right)m_{os}^{-}\right]\Omega_{e\rho_{-}}/2$$

Considering the convergence properties in α: near $\alpha\to 0$ there is no issue since $k_t\to k$, for $\alpha\to\infty$ we have $k_t=i\sqrt{\alpha^2-k^2}\sim i\alpha$ and therefore

$$\frac{H_m^{(1)\prime}\left(k_tb\right)}{H_m^{(1)}\left(k_tb\right)}\sim\frac{K_m'\left(\alpha b\right)}{iK_m\left(\alpha b\right)}\sim i$$

$$\left[\frac{H_m^{(1)\prime}\left(k_tb\right)}{H_m^{(1)}\left(k_tb\right)}-m^2\left(1-k_t^2/k^2\right)\frac{H_m^{(1)}\left(k_tb\right)}{\left(k_tb\right)^2H_m^{(1)\prime}\left(k_tb\right)}\right]\sim i\left\{1-m^2/\left(kb\right)^2\right\}$$

where we used

$$\frac{K_m\left(\alpha b\right)}{\alpha K_m'\left(\alpha b\right)}=\frac{H_m^{(1)}\left(i\alpha b\right)}{i\alpha H_m^{(1)\prime}\left(i\alpha b\right)}$$

and the integrand behaves as $O\left(1/\alpha\right)$. We used

$$k_t=\sqrt{k^2-\alpha^2}=i\sqrt{\alpha^2-k^2}$$

with the original branch cut topology $0<\arg\left(k^2-\alpha^2\right)<2\pi$, where the cut from the branch point at $\alpha=k$ proceeds from the branch point to the origin and up the positive imaginary axis, the cut from the branch point at $\alpha = -k$ proceeds from the branch point to the origin and down the negative imaginary axis. The integration contour from $\alpha = 0$ to $\alpha = \infty$ is along the bottom of the cut on the positive real α axis (up to $\alpha = k$). Then $\arg(\alpha + k) = 0$ on the contour, $\arg(k - \alpha) = 0$ for $0 < \alpha < k$, and $\arg(k - \alpha) = \pi$ with $\arg(\alpha - k) = 0$ on the remainder of the contour $k < \alpha < \infty$. Then the integral is

$$\begin{aligned}
\int_0^\infty \cos(\alpha\Delta)\frac{d\alpha}{k_t} &= \int_0^k \cos(\alpha\Delta)\frac{d\alpha}{\sqrt{k^2-\alpha^2}} + \int_k^\infty \cos(\alpha\Delta)\frac{d\alpha}{i\sqrt{\alpha^2-k^2}} \\
&= \int_0^1 \cos(\alpha k\Delta)\frac{d\alpha}{\sqrt{1-\alpha^2}} - i\int_1^\infty \cos(\alpha k\Delta)\frac{d\alpha}{\sqrt{\alpha^2-1}} \\
&= \frac{\pi}{2}J_0(k\Delta) + i\frac{\pi}{2}Y_0(k\Delta) = \frac{\pi}{2}H_0^{(1)}(k\Delta)
\end{aligned}$$

Note also that the asymptotic form is

$$\frac{\pi}{2}H_0^{(1)}(k\Delta) \sim \frac{\pi}{2} + i\left\{\ln(k\Delta/2) + \gamma'\right\}\ ,\ k\Delta << 1$$

We can then write

$$\begin{aligned}
i\int_0^\infty \cos(\alpha\Delta)\frac{d\alpha}{k_t} - \Omega_{\rho_-}/2 &= K_0(-ik\Delta) - \Omega_{\rho_-}/2 = i\frac{\pi}{2}H_0^{(1)}(k\Delta) - \Omega_{\rho_-}/2 \\
&\sim i\frac{\pi}{2} - \left\{\ln(k\Delta/2) + \gamma'\right\} - \ln\left(2h/\rho_-\right) = i\frac{\pi}{2} - \left\{\ln\left(kh\Delta/\rho_-\right) + \gamma'\right\}
\end{aligned}$$

and

$$i\omega\left[A_{e\varphi}(b,\varphi,z_0+\Delta) - A_{es}^s\left(\rho_-, s\right)\right] \approx \frac{k^2}{2\pi^2 b}\sum_m\left[\cos(m\varphi)\frac{\varepsilon_m}{2}m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^\infty\left\{\frac{H_m^{(1)\prime}(k_t b)}{H_m^{(1)}(k_t b)} - m^2\left(1 - k_t^2/k^2\right)\frac{H_m^{(1)}(k_t b)}{(k_t b)^2 H_m^{(1)\prime}(k_t b)}\right\}\cos(\alpha\Delta)\frac{d\alpha}{k_t} - \Omega_{e\rho_-}/2\right]$$

$$\approx \frac{k^2}{2\pi^2 b}\sum_m\left[\cos(m\varphi)\frac{\varepsilon_m}{2}m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^\infty\left\{\frac{H_m^{(1)\prime}(k_t b)}{H_m^{(1)}(k_t b)} - i - m^2\left(1 - k_t^2/k^2\right)\frac{H_m^{(1)}(k_t b)}{(k_t b)^2 H_m^{(1)\prime}(k_t b)}\right\}\cos(\alpha\Delta)\frac{d\alpha}{k_t} + i\frac{\pi}{2}H_0^{(1)}(k\Delta) - \Omega_{e\rho_-}/2\right]$$

$$\approx -\frac{k^2}{2\pi^2 b}\sum_m\left[\cos(m\varphi)\frac{\varepsilon_m}{2}m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^\infty\left\{\frac{H_m^{(1)\prime}(k_t b)}{H_m^{(1)}(k_t b)} - i - m^2\left(1 - k_t^2/k^2\right)\frac{H_m^{(1)}(k_t b)}{(k_t b)^2 H_m^{(1)\prime}(k_t b)}\right\}\cos(\alpha\Delta)\frac{d\alpha}{k_t} + i\frac{\pi}{2} - \gamma' - \ln\left(kh\Delta/\rho_-\right)\right]$$

Setting the displacement from the line source in the modal sum Δ to the same value as the displacement from the line source in the extracted transmission line term ρ_- or $\Delta = \rho_-$, we see that ρ_- cancels out in the logarithm of this potential difference (at this point we cannot drop the $\cos(\alpha\Delta) = \cos\left(\alpha\rho_-\right)$ factor due to the slow decay of the final term of the integrand at infinity).

If we use the form of the local potential with the operator applied

$$i\omega\left[A_{e\varphi}\left(b,\varphi,z_0+\rho_-\right)-\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)A^s_{es}\left(\rho_-,s\right)\right]\approx-\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m^-_{es}+\sin\left(m\varphi\right)m^-_{os}\right]$$

$$\int_0^\infty\left[\frac{H^{(1)\prime}_m\left(k_t b\right)}{H^{(1)}_m\left(k_t b\right)}-m^2\left(1-k_t^2/k^2\right)\frac{H^{(1)}_m\left(k_t b\right)}{\left(k_t b\right)^2 H^{(1)\prime}_m\left(k_t b\right)}\right]\cos\left(\alpha\rho_-\right)\frac{d\alpha}{k_t}$$

$$-\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m^-_{es}+\sin\left(m\varphi\right)m^-_{os}\right]\left(1-\frac{m^2}{k^2b^2}\right)\Omega_{e\rho_-}/2$$

$$\approx-\frac{k^2}{2\pi^2 b}\sum_m\left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2}m^-_{es}+\sin\left(m\varphi\right)m^-_{os}\right]$$

$$\left[\int_0^\infty\left\{\frac{H^{(1)\prime}_m\left(k_t b\right)}{H^{(1)}_m\left(k_t b\right)}-i\left(1-\frac{m^2}{k^2b^2}\right)-m^2\left(1-k_t^2/k^2\right)\frac{H^{(1)}_m\left(k_t b\right)}{\left(k_t b\right)^2 H^{(1)\prime}_m\left(k_t b\right)}\right\}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{k_t}\right.$$

$$\left.+\left(1-\frac{m^2}{k^2b^2}\right)i\frac{\pi}{2}H^{(1)}_0\left(k\rho_-\right)-\left(1-\frac{m^2}{k^2b^2}\right)\Omega_{e\rho_-}/2+\left(1-\frac{m^2}{k^2b^2}\right)\Omega_{e\rho_-}\right]$$

$$\frac{H^{(1)\prime}_m\left(k_t b\right)}{H^{(1)}_m\left(k_t b\right)}\sim\frac{K'_m\left(\alpha b\right)}{iK_m\left(\alpha b\right)}\sim i$$

$$\left\{\frac{H^{(1)\prime}_m\left(k_t b\right)}{H^{(1)}_m\left(k_t b\right)}-i\left(1-\frac{m^2}{k^2b^2}\right)-m^2\left(1-k_t^2/k^2\right)\frac{H^{(1)}_m\left(k_t b\right)}{\left(k_t b\right)^2 H^{(1)\prime}_m\left(k_t b\right)}\right\}$$

$$=\left\{\frac{H^{(1)\prime}_m\left(k_t b\right)}{H^{(1)}_m\left(k_t b\right)}-i\left(1-\frac{m^2}{k^2b^2}\right)+\frac{m^2}{k^2b^2}\left(1-k^2/k_t^2\right)\frac{H^{(1)}_m\left(k_t b\right)}{H^{(1)\prime}_m\left(k_t b\right)}\right\}$$

$$\sim\left\{i-i\left(1-\frac{m^2}{k^2b^2}\right)-\frac{m^2}{k^2b^2}\left(1-k^2/k_t^2\right)i\right\}=O\left(1/k_t^2\right)=O\left(1/\alpha^2\right)$$

Thus with this form we remove the slow convergence at infinity. Hence with this we can write the electric vector potential part as

$$i\omega A_{e\varphi}\left(b,\varphi,z_0+\rho_-\right)\approx-\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m^-_{es}+\sin\left(m\varphi\right)m^-_{os}\right]$$

$$\left[\int_0^\infty\left\{\frac{H^{(1)\prime}_m\left(k_t b\right)}{H^{(1)}_m\left(k_t b\right)}-i\left(1-\frac{m^2}{k^2b^2}\right)-m^2\left(1-k_t^2/k^2\right)\frac{H^{(1)}_m\left(k_t b\right)}{\left(k_t b\right)^2 H^{(1)\prime}_m\left(k_t b\right)}\right\}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{k_t}\right.$$

$$\left.+\left(1-\frac{m^2}{k^2b^2}\right)i\frac{\pi}{2}H^{(1)}_0\left(k\rho_-\right)-\left(1-\frac{m^2}{k^2b^2}\right)\Omega_{e\rho_-}/2+\left(1-\frac{m^2}{k^2b^2}\right)\Omega_{e\rho_-}\right]$$

$$+i\omega\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)A^s_{es}\left(\rho_-,s\right)$$

where

$$i\omega A^s_{es}\left(\rho_-, s\right) \sim -i\omega\varepsilon_0 \frac{\Omega_{e\rho_-}}{2\pi} V_-\left(s\right)$$

Because the integrand converges sufficiently rapidly at infinity we could set $\cos\left(\alpha\rho_-\right) \to 1$. If we expand for $\rho_- \to 0$

$$i\frac{\pi}{2} H_0^{(1)}\left(k\rho_-\right) \sim i\frac{\pi}{2} - \left\{\ln\left(k\rho_-/2\right) + \gamma'\right\}$$

then

$$i\omega A_{e\varphi}\left(b, \varphi, z_0 + \rho_-\right) \approx -\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right) m^-_{es} + \sin\left(m\varphi\right) m^-_{os}\right]$$

$$\left[\int_0^\infty \left\{\frac{H_m^{(1)\prime}\left(k_t b\right)}{H_m^{(1)}\left(k_t b\right)} - i\left(1 - \frac{m^2}{k^2 b^2}\right) - m^2\left(1 - k_t^2/k^2\right)\frac{H_m^{(1)}\left(k_t b\right)}{\left(k_t b\right)^2 H_m^{(1)\prime}\left(k_t b\right)}\right\}\frac{d\alpha}{k_t}\right.$$

$$\left. + \left(1 - \frac{m^2}{k^2 b^2}\right)\left\{i\frac{\pi}{2} - \ln\left(kh\right) - \gamma' + \Omega_{e\rho_-}\right\}\right]$$

$$+ i\omega\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right) A^s_{es}\left(\rho_-, s\right)$$

Using the identity from the preceding

$$-i\omega\varepsilon_0 V_-\left(s\right) = \frac{k^2}{2\pi b}\sum_m \left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2} m^-_{es} + \sin\left(m\varphi\right) m^-_{os}\right]$$

we can write the total radial potential as

$$i\omega A_{e\rho}\left(b, \varphi, z_0 + \Delta\right) = \frac{1}{2\pi b}\frac{\partial}{\partial s}\sum_m \left[\cos\left(m\varphi\right)\frac{\varepsilon_m}{2} m^-_{es} + \sin\left(m\varphi\right) m^-_{os}\right]\delta\left(\Delta\right)$$

$$= \frac{-i\omega\varepsilon_0}{k^2}\frac{\partial}{\partial s} V_-\left(s\right)\delta\left(\Delta\right) = \frac{1}{i\omega\mu_0}\frac{\partial}{\partial s} V_-\left(s\right)\delta\left(\Delta\right)$$

Because this is the radial magnetic field contribution from the electric vector potential we can write the magnetic charge contribution from this as

$$\Phi^e_-\left(s\right) = \frac{1}{2} q^-_{me} = \mu_0 i\omega A_{e\rho}\left(b, \varphi, z_0 + \Delta\right)/\delta\left(\Delta\right) = \frac{1}{i\omega}\frac{\partial}{\partial s} V_-\left(s\right)$$

5.1.7 Inner Integration

Breaking up the inner integral into two ranges

$$\int_0^\infty \left\{\frac{H_m^{(1)\prime}\left(k_t b\right)}{k_t H_m^{(1)}\left(k_t b\right)} - \frac{i}{k_t} - \frac{m^2}{k^2 b^2}\frac{\left(k^2 - k_t^2\right)}{k_t^2}\frac{H_m^{(1)}\left(k_t b\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)} - \frac{m^2}{k^2 b^2}\frac{H_m^{(1)}\left(i\alpha b\right)}{i\alpha H_m^{(1)\prime}\left(i\alpha b\right)}\right\} d\alpha$$

$$= \int_0^{k-\delta} \left\{\frac{H_m^{(1)\prime}\left(k_t b\right)}{k_t H_m^{(1)}\left(k_t b\right)} - \frac{i}{k_t} - \frac{m^2}{k^2 b^2}\frac{\left(k^2 - k_t^2\right)}{k_t^2}\frac{H_m^{(1)}\left(k_t b\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)} - \frac{m^2}{k^2 b^2}\frac{H_m^{(1)}\left(i\alpha b\right)}{i\alpha H_m^{(1)\prime}\left(i\alpha b\right)}\right\} d\alpha$$

$$+\frac{i}{2kb}\int_0^{\pi}\left\{\frac{\sqrt{2k\delta e^{i\theta}}bH_m^{(1)\prime}\left(\sqrt{2k\delta e^{i\theta}}b\right)}{H_m^{(1)}\left(\sqrt{2k\delta e^{i\theta}}b\right)}-m^2\frac{H_m^{(1)}\left(\sqrt{2k\delta e^{i\theta}}b\right)}{\sqrt{2k\delta e^{i\theta}}bH_m^{(1)\prime}\left(\sqrt{2k\delta e^{i\theta}}b\right)}\right\}d\theta$$

$$+\int_{k+\delta}^{\infty}\left\{\frac{H_m^{(1)\prime}(iub)}{iuH_m^{(1)}(iub)}-\frac{1}{u}+\frac{m^2}{k^2b^2}\frac{\left(k^2+u^2\right)}{u^2}\frac{H_m^{(1)}(iub)}{iuH_m^{(1)\prime}(iub)}-\frac{m^2}{k^2b^2}\frac{H_m^{(1)}(i\alpha b)}{i\alpha H_m^{(1)\prime}(i\alpha b)}\right\}d\alpha$$

where

$$u=\sqrt{\alpha^2-k^2}\ ,\ k_t=\sqrt{k^2-\alpha^2}$$

$$u\sim\sqrt{2k\delta}\ ,\ k_t\sim\sqrt{2k\delta}$$

Changing variables

$$\int_0^{\infty}\left\{\frac{H_m^{(1)\prime}(k_tb)}{k_tH_m^{(1)}(k_tb)}-\frac{i}{k_t}-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}(k_tb)}{k_tH_m^{(1)\prime}(k_tb)}-\frac{m^2}{k^2b^2}\frac{H_m^{(1)}(i\alpha b)}{i\alpha H_m^{(1)\prime}(i\alpha b)}\right\}d\alpha$$

$$=\int_{\sqrt{2k\delta}}^{k}\left\{\frac{H_m^{(1)\prime}(k_tb)}{k_tH_m^{(1)}(k_tb)}-\frac{i}{k_t}-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}(k_tb)}{k_tH_m^{(1)\prime}(k_tb)}-\frac{m^2}{k^2b^2}\frac{H_m^{(1)}(i\alpha b)}{i\alpha H_m^{(1)\prime}(i\alpha b)}\right\}\frac{k_t}{\alpha}dk_t$$

$$+\frac{i}{2kb}\int_0^{\pi}\left\{\frac{\sqrt{2k\delta e^{i\theta}}bH_m^{(1)\prime}\left(\sqrt{2k\delta e^{i\theta}}b\right)}{H_m^{(1)}\left(\sqrt{2k\delta e^{i\theta}}b\right)}-m^2\frac{H_m^{(1)}\left(\sqrt{2k\delta e^{i\theta}}b\right)}{\sqrt{2k\delta e^{i\theta}}bH_m^{(1)\prime}\left(\sqrt{2k\delta e^{i\theta}}b\right)}\right\}d\theta$$

$$+\int_{\sqrt{2k\delta}}^{\infty}\left\{\frac{H_m^{(1)\prime}(iub)}{iuH_m^{(1)}(iub)}-\frac{1}{u}+\frac{m^2}{k^2b^2}\frac{\left(k^2+u^2\right)}{u^2}\frac{H_m^{(1)}(iub)}{iuH_m^{(1)\prime}(iub)}-\frac{m^2}{k^2b^2}\frac{H_m^{(1)}(i\alpha b)}{i\alpha H_m^{(1)\prime}(i\alpha b)}\right\}\frac{u}{\alpha}du$$

$$=\int_{\sqrt{2k\delta}}^{k}\left\{\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)}-i-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}-\frac{m^2}{k^2b^2}\frac{k_t}{\alpha}\frac{K_m(\alpha b)}{K_m'(\alpha b)}\right\}\frac{1}{\alpha}dk_t$$

$$+\frac{i}{2kb}\int_0^{\pi}\left\{\frac{\sqrt{2k\delta e^{i\theta}}bH_m^{(1)\prime}\left(\sqrt{2k\delta e^{i\theta}}b\right)}{H_m^{(1)}\left(\sqrt{2k\delta e^{i\theta}}b\right)}-m^2\frac{H_m^{(1)}\left(\sqrt{2k\delta e^{i\theta}}b\right)}{\sqrt{2k\delta e^{i\theta}}bH_m^{(1)\prime}\left(\sqrt{2k\delta e^{i\theta}}b\right)}\right\}d\theta$$

$$+\int_{\sqrt{2k\delta}}^{\infty}\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{k^2b^2}\frac{\left(k^2+u^2\right)}{u^2}\frac{K_m(ub)}{K_m'(ub)}-\frac{m^2}{k^2b^2}\frac{u}{\alpha}\frac{K_m(\alpha b)}{K_m'(\alpha b)}\right\}\frac{1}{\alpha}du$$

where we used

$$\frac{K_m(\alpha b)}{K_m'(\alpha b)}=\frac{H_m^{(1)}(i\alpha b)}{iH_m^{(1)\prime}(i\alpha b)}$$

For $m=0$

$$\int_0^{\infty}\left\{\frac{H_m^{(1)\prime}(k_tb)}{k_tH_m^{(1)}(k_tb)}-\frac{i}{k_t}\right\}d\alpha$$

$$= -\int_{\sqrt{2k\delta}}^{k}\left\{\frac{H_1^{(1)}(k_t b)}{H_0^{(1)}(k_t b)}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_{\sqrt{2k\delta}}^{\infty}\left\{\frac{K_1(ub)}{K_0(ub)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$-\frac{1}{\pi kb}\int_0^{\pi}\frac{d\theta}{1+i(2/\pi)\ln\left(e^{\gamma'}\sqrt{k\delta e^{i\theta}/2b}\right)}$$

Near the lower limits

$$-\int_{\sqrt{2k\delta}}^{k}\left\{\frac{H_1^{(1)}(k_t b)}{H_0^{(1)}(k_t b)}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_{\sqrt{2k\delta}}^{\infty}\left\{\frac{K_1(ub)}{K_0(ub)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$\to -\frac{1}{k}\int_{\sqrt{2k\delta}}^{K}\left\{\frac{-i2/(\pi k_t b)}{1+i(2/\pi)\ln(k_t b e^{\gamma'}/2)}+i\right\}dk_t-\frac{1}{k}\int_{\sqrt{2k\delta}}^{U}\left\{\frac{1}{(ub)\ln(ube^{\gamma'}/2)}+1\right\}du$$

$$\to \frac{1}{kb}\int_{\sqrt{2k\delta}}^{K}\left\{\frac{1}{(k_t b)\ln\left(k_t b e^{\gamma'}e^{-i\pi/2}/2\right)}+i\right\}bdk_t-\frac{1}{kb}\int_{\sqrt{2k\delta}}^{U}\left\{\frac{1}{(ub)\ln(ube^{\gamma'}/2)}+1\right\}bdu$$

$$\to \frac{1}{kb}\int_{\sqrt{2k\delta}b}^{Kb}\left\{\frac{1}{\zeta\ln\left(\zeta e^{\gamma'}e^{-i\pi/2}/2\right)}+i\right\}d\zeta-\frac{1}{kb}\int_{\sqrt{2k\delta}b}^{Ub}\left\{\frac{1}{\zeta\ln(\zeta e^{\gamma'}/2)}+1\right\}d\zeta$$

$$\to \frac{1}{kb}\left\{\ln\left(\ln\left(\zeta e^{\gamma'}e^{-i\pi/2}/2\right)\right)+i\zeta\right\}_{\sqrt{2k\delta}b}^{Kb}-\frac{1}{kb}\left\{\ln\left(\ln\left(\zeta e^{\gamma'}/2\right)\right)+\zeta\right\}_{\sqrt{2k\delta}b}^{Ub}$$

$$\to \frac{1}{kb}\left\{\ln\left(\ln\left(Kbe^{\gamma'}e^{-i\pi/2}/2\right)\right)-\ln\left(\ln\left(\sqrt{2k\delta}be^{\gamma'}e^{-i\pi/2}/2\right)\right)+iKb-i\sqrt{2k\delta}b\right\}$$

$$-\frac{1}{kb}\left\{\ln\left(\ln\left(Ube^{\gamma'}/2\right)\right)-\ln\left(\ln\left(\sqrt{2k\delta}be^{\gamma'}/2\right)\right)+Ub-\sqrt{2k\delta}b\right\}$$

$$\to \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}e^{-i\pi/2}/2\right)}{\ln(Ube^{\gamma'}/2)}\right)-\ln\left(\frac{\ln\left(\sqrt{2k\delta}be^{\gamma'}e^{-i\pi/2}/2\right)}{\ln\left(\sqrt{2k\delta}be^{\gamma'}/2\right)}\right)-(U-iK)b+(1-i)\sqrt{2k\delta}b\right\}$$

$$\to \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}/2\right)-i\pi/2}{\ln(Ube^{\gamma'}/2)}\right)-\ln\left(\frac{\ln\left(\sqrt{2k\delta}be^{\gamma'}/2\right)-i\pi/2}{\ln\left(\sqrt{2k\delta}be^{\gamma'}/2\right)}\right)-(U-iK)b+(1-i)\sqrt{2k\delta}b\right\}$$

$$\to \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}/2\right)-i\pi/2}{\ln(Ube^{\gamma'}/2)}\right)-(U-iK)b\right\}$$

so that

$$-\int_{\sqrt{2k\delta}}^{k}\left\{\frac{H_1^{(1)}(k_t b)}{H_0^{(1)}(k_t b)}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_{\sqrt{2k\delta}}^{\infty}\left\{\frac{K_1(ub)}{K_0(ub)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$\approx \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}/2\right)-i\pi/2}{\ln\left(Ube^{\gamma'}/2\right)}\right)-\left(U-iK\right)b\right\}$$

$$-\int_K^k\left\{\frac{H_1^{(1)}\left(k_tb\right)}{H_0^{(1)}\left(k_tb\right)}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_U^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$\approx \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}/2\right)-i\pi/2}{\ln\left(Ube^{\gamma'}/2\right)}\right)-\left(U-iK\right)b\right\}$$

$$-\int_K^k\left\{\frac{H_1^{(1)}\left(k_tb\right)H_0^{(2)}\left(k_tb\right)}{\left|H_0^{(1)}\left(k_tb\right)\right|^2}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_U^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$\approx \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}/2\right)-i\pi/2}{\ln\left(Ube^{\gamma'}/2\right)}\right)-\left(U-iK\right)b\right\}$$

$$-\int_K^k\left\{\frac{\left\{J_0\left(k_tb\right)J_1\left(k_tb\right)+Y_0\left(k_tb\right)Y_1\left(k_tb\right)\right\}+i\left\{J_0\left(k_tb\right)Y_1\left(k_tb\right)-J_1\left(k_tb\right)Y_0\left(k_tb\right)\right\}}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+\int_U^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\approx \frac{1}{kb}\left\{\ln\left(\frac{\ln\left(Kbe^{\gamma'}/2\right)-i\pi/2}{\ln\left(Ube^{\gamma'}/2\right)}\right)-\left(U-iK\right)b\right\}$$

$$-\int_K^k\left\{\frac{-\frac{1}{2k_t}\frac{\partial}{\partial b}\left\{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)\right\}-i2/\left(\pi k_tb\right)}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}+i\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_U^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

where $0<K,U<<1/b$. Because of the very slow convergence at the branch point ($k_t\to 0$ and $u\to 0$) it is useful to choose the parameters K and U away from zero.

Now for the $m\geq 1$ terms

$$\int_0^\infty\left\{\frac{H_m^{(1)\prime}\left(k_tb\right)}{k_tH_m^{(1)}\left(k_tb\right)}-\frac{i}{k_t}-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}\left(k_tb\right)}{k_tH_m^{(1)\prime}\left(k_tb\right)}-\frac{m^2}{k^2b^2}\frac{H_m^{(1)}\left(i\alpha b\right)}{i\alpha H_m^{(1)\prime}\left(i\alpha b\right)}\right\}d\alpha$$

$$=\int_{\sqrt{2k\delta}}^k\left\{\frac{H_m^{(1)\prime}\left(k_tb\right)}{H_m^{(1)}\left(k_tb\right)}-i-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}\left(k_tb\right)}{H_m^{(1)\prime}\left(k_tb\right)}-\frac{m^2}{k^2b^2}\frac{k_t}{\sqrt{k^2-k_t^2}}\frac{K_m\left(\sqrt{k^2-k_t^2}b\right)}{K_m'\left(\sqrt{k^2-k_t^2}b\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+\frac{i}{2kb}\int_0^\pi\left\{\frac{\sqrt{2k\delta e^{i\theta}}bY_m'\left(\sqrt{2k\delta e^{i\theta}}b\right)}{Y_m\left(\sqrt{2k\delta e^{i\theta}}b\right)}-m^2\frac{Y_m\left(\sqrt{2k\delta e^{i\theta}}b\right)}{\sqrt{2k\delta e^{i\theta}}bY_m'\left(\sqrt{2k\delta e^{i\theta}}b\right)}\right\}d\theta$$

$$+\int_{\sqrt{2k\delta}}^{\infty}\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{k^2b^2}\frac{\left(k^2+u^2\right)}{u^2}\frac{K_m(ub)}{K_m'(ub)}-\frac{m^2}{k^2b^2}\frac{u}{\sqrt{u^2+k^2}}\frac{K_m\left(\sqrt{u^2+k^2}b\right)}{K_m'\left(\sqrt{u^2+k^2}b\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$\frac{zY_m'(z)}{Y_m(z)}\sim\frac{m\Gamma(m)(z/2)^{-m}/\pi}{-\Gamma(m)(z/2)^{-m}/\pi}=-m$$

The branch point integral can thus be dropped and near the lower limits

$$\int_{\sqrt{2k\delta}}^{k}\left\{\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)}-i-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}-\frac{m^2}{k^2b^2}\frac{k_t}{\sqrt{k^2-k_t^2}}\frac{K_m\left(\sqrt{k^2-k_t^2}b\right)}{K_m'\left(\sqrt{k^2-k_t^2}b\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+\int_{\sqrt{2k\delta}}^{\infty}\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{k^2b^2}\frac{\left(k^2+u^2\right)}{u^2}\frac{K_m(ub)}{K_m'(ub)}-\frac{m^2}{k^2b^2}\frac{u}{\sqrt{u^2+k^2}}\frac{K_m\left(\sqrt{u^2+k^2}b\right)}{K_m'\left(\sqrt{u^2+k^2}b\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$\rightarrow\frac{1}{k}\int_{\sqrt{2k\delta}}^{K}\left\{\frac{Y_m'(k_tb)}{Y_m(k_tb)}-i-\frac{m}{k_tb}\frac{mY_m(k_tb)}{k_tbY_m'(k_tb)}-\frac{m^2}{k^2b^2}\frac{k_t}{k}\frac{K_m(kb)}{K_m'(kb)}\right\}dk_t$$

$$+\frac{1}{k}\int_{\sqrt{2k\delta}}^{U}\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m}{ub}\frac{mK_m(ub)}{ubK_m'(ub)}-\frac{m^2}{k^2b^2}\frac{u}{k}\frac{K_m(kb)}{K_m'(kb)}\right\}du$$

$$\frac{-mY_m(k_tb)}{k_tbY_m'(k_tb)}\sim 1$$

Thus for $m\geq 1$ we can also drop the branch point contribution and write

$$\int_0^{\infty}\left\{\frac{H_m^{(1)\prime}(k_tb)}{k_tH_m^{(1)}(k_tb)}-\frac{i}{k_t}-\frac{m^2}{k^2b^2}\frac{\left(k^2-k_t^2\right)}{k_t^2}\frac{H_m^{(1)}(k_tb)}{k_tH_m^{(1)\prime}(k_tb)}-\frac{m^2}{k^2b^2}\frac{H_m^{(1)}(i\alpha b)}{i\alpha H_m^{(1)\prime}(i\alpha b)}\right\}d\alpha$$

$$=\int_0^{k}\left\{\frac{k_tH_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)}-ik_t-\frac{m^2}{k^2b^2}\left(k^2-k_t^2\right)\frac{H_m^{(1)}(k_tb)}{k_tH_m^{(1)\prime}(k_tb)}-\frac{m^2}{k^2b^2}\frac{k_t^2}{\sqrt{k^2-k_t^2}}\frac{K_m\left(\sqrt{k^2-k_t^2}b\right)}{K_m'\left(\sqrt{k^2-k_t^2}b\right)}\right\}\frac{dk_t/k_t}{\sqrt{k^2-k_t^2}}$$

$$+\int_0^{\infty}\left\{-\frac{u\,K_m'(ub)}{K_m(ub)}-u+\frac{m^2}{k^2b^2}\left(k^2+u^2\right)\frac{K_m(ub)}{u\,K_m'(ub)}-\frac{m^2}{k^2b^2}\frac{u^2}{\sqrt{u^2+k^2}}\frac{K_m\left(\sqrt{u^2+k^2}b\right)}{K_m'\left(\sqrt{u^2+k^2}b\right)}\right\}\frac{du/u}{\sqrt{u^2+k^2}}$$

5.2 Scalar Potential Contribution

The magnetic scalar potential

$$\underline{H}=-\nabla\phi_m$$

from Gauss's law

$$\nabla \cdot \underline{B} = \nabla \cdot (\mu_0 \underline{H}) = \rho_m$$

satisfies the Poisson equation

$$\nabla^2 \phi_m = -\rho_m/\mu_0$$

Then

$$\nabla^2 \phi_m = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\phi_m\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}\phi_m + \frac{\partial^2}{\partial z^2}\phi_m \rightarrow \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \alpha^2 - \frac{m^2}{\rho^2}\right)\phi_m$$

where

$$\phi_m = \sum_m \int_0^\infty \frac{K_m(\alpha\rho)}{\alpha K'_m(\alpha b)}\left[A_m(\alpha)\cos(m\varphi) + B_m(\alpha)\sin(m\varphi)\right]\cos(\alpha(z - z_0))\, d\alpha$$

Applying the boundary condition

$$-\mu_0 \frac{\partial \phi_m}{\partial \rho}(b) = \sigma_m^- = \Phi_-(s)\,\delta(z - z_0) = \frac{1}{2}q_m^-(s)\,\delta(z - z_0)$$

and multiplying by the trigonometric and Bessel functions, followed by integration over the volume of the cavity, gives

$$\sum_m \int_0^\infty \left[A_m(\alpha)\cos(m\varphi) + B_m(\alpha)\sin(m\varphi)\right]\int_{-\infty}^\infty \cos(\alpha'(z - z_0))\cos(\alpha(z - z_0))\, d\alpha\, d(z - z_0)$$

$$= -\frac{1}{2\mu_0}q_m^-(s)\int_{-\infty}^\infty \delta(z - z_0)\cos\alpha'(z - z_0)\, d(z - z_0)$$

Using

$$\int_{-\infty}^\infty \cos(\alpha' u)\cos(\alpha u)\, du = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \left[\cos(\alpha - \alpha')u + \cos(\alpha + \alpha')u\right] du$$

$$= \lim_{R\to\infty}\left[\frac{\sin(\alpha - \alpha')R}{(\alpha - \alpha')} + \frac{\sin(\alpha + \alpha')R}{(\alpha + \alpha')}\right] = \pi \lim_{R/\pi\to\infty}(R/\pi)\left[\frac{\sin(\alpha - \alpha')\pi R/\pi}{(\alpha - \alpha')\pi R/\pi} + \frac{\sin(\alpha + \alpha')\pi R/\pi}{(\alpha + \alpha')\pi R/\pi}\right]$$

$$= \pi\left[\delta(\alpha - \alpha') + \delta(\alpha + \alpha')\right]$$

gives

$$\sum_m \left[A_m(\alpha)\cos(m\varphi) + B_m(\alpha)\sin(m\varphi)\right] = -\frac{1}{2\pi\mu_0}q_m^-(s)$$

Then using

$$\int_{-\pi}^\pi \cos(m'\varphi)\cos(m\varphi)\, d\varphi = \frac{1}{2}\int_{-\pi}^\pi \left[\cos(m - m')\varphi + \cos(m + m')\varphi\right] d\varphi$$

$$= \frac{\sin(m - m')\pi}{(m - m')} + \frac{\sin(m + m')\pi}{(m + m')} = \frac{2\pi}{\varepsilon_m}\delta_{mm'}$$

$$\int_{-\pi}^{\pi} \sin(m'\varphi)\sin(m\varphi)\,d\varphi = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(m-m')\varphi - \cos(m+m')\varphi\right]d\varphi$$

$$= \frac{\sin(m-m')\pi}{(m-m')} - \frac{\sin(m+m')\pi}{(m+m')} = \pi\delta_{mm'}$$

$$\int_{-\pi}^{\pi} \sin(m'\varphi)\cos(m\varphi)\,d\varphi = \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(m'-m)\varphi + \sin(m'+m)\varphi\right]d\varphi = 0$$

gives

$$\frac{2\pi}{\varepsilon_m}A_m(\alpha) = -\frac{1}{2\pi\mu_0}\int_{-\pi}^{\pi} q_m^-(s)\cos(m\varphi)\,d\varphi = -\frac{1}{2\pi b\mu_0}\int_{-h}^{h} q_m^-(s)\cos(ms/b)\,ds$$

$$\pi B_m(\alpha) = -\frac{1}{2\pi\mu_0}\int_{-\pi}^{\pi} q_m^-(s)\sin(m\varphi)\,d\varphi = -\frac{1}{2\pi b\mu_0}\int_{-h}^{h} q_m^-(s)\sin(ms/b)\,ds$$

Then we have

$$\phi_m = -\frac{1}{2\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{K_m(\alpha\rho)}{\alpha K_m'(\alpha b)}\left[\cos(m\varphi)\int_{-h}^{h} q_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} q_m^-(s')\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

The total magnetic charge can be found from the continuity equation as

$$\frac{\partial}{\partial s}I_m^- = -2\frac{\partial}{\partial s}V_-(s) = i\omega\left[q_m^-(s) + q_{me}^-\right] = i\omega q_m^-(s) + 2\frac{\partial}{\partial s}V_-(s)$$

so that

$$\phi_m = \frac{1}{i\omega\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{K_m(\alpha\rho)}{\alpha K_m'(\alpha b)}\left[\cos(m\varphi)\int_{-h}^{h} \frac{\partial}{\partial s'}V_-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} \frac{\partial}{\partial s'}V_-(s')\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

Integration by parts gives

$$\phi_m = \frac{1}{-i\omega\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{K_m(\alpha\rho)}{\alpha K_m'(\alpha b)}\left[\cos(m\varphi)\int_{-h}^{h} V_-(s)\frac{\partial}{\partial s}\cos(ms/b)\,ds + \sin(m\varphi)\int_{-h}^{h} V_-(s)\frac{\partial}{\partial s}\sin(ms/b)\,ds\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

Then

$$\frac{\partial}{\partial\varphi}\phi_m = \frac{1}{-i\omega\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{K_m(\alpha\rho)}{\alpha K_m'(\alpha b)}$$

$$\frac{\partial}{\partial\varphi}\left[\cos\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\frac{\partial}{\partial s'}\cos\left(ms'/b\right)ds'+\sin\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\frac{\partial}{\partial s'}\sin\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$=\frac{1}{-i\omega\pi\mu_0}\sum_m\frac{\varepsilon_m}{2\pi}\int_0^{\infty}\frac{K_m\left(\alpha\rho\right)}{\alpha K_m'\left(\alpha b\right)}\left[\frac{m^2}{b^2}\sin\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\sin\left(ms'/b\right)ds'+\frac{m^2}{b^2}\cos\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\cos\left(ms'/b\right)ds'\right]$$

$$\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$=\frac{1}{i\omega\mu_0\pi b^2}\frac{\partial^2}{\partial\varphi^2}\sum_m\frac{\varepsilon_m}{2\pi}\int_0^{\infty}\frac{K_m\left(\alpha\rho\right)}{\alpha K_m'\left(\alpha b\right)}\left[\sin\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\sin\left(ms'/b\right)ds'+\cos\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\cos\left(ms'/b\right)ds'\right]$$

$$\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

Taking $\rho=b$

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m\left(b,\varphi,z\right)=\frac{1}{-i\omega\mu_0\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\int_0^{\infty}\frac{K_m\left(\alpha b\right)}{\alpha K_m'\left(\alpha b\right)}$$

$$\left[\sin\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\sin\left(ms'/b\right)ds'+\cos\left(m\varphi\right)\int_{-h}^{h}V_{-}\left(s'\right)\cos\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

Then near the slot

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m\left(b,\varphi,z_0+\Delta\right)=-\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\int_0^{\infty}\frac{K_m\left(\alpha b\right)}{\alpha K_m'\left(\alpha b\right)}\left[\sin\left(m\varphi\right)m_{os}^{-}+\cos\left(m\varphi\right)m_{es}^{-}\right]\cos\left(\alpha\Delta\right)d\alpha$$

$$=-\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\int_0^{\infty}\frac{H_m^{(1)}\left(i\alpha b\right)}{i\alpha H_m^{(1)\prime}\left(i\alpha b\right)}\left[\sin\left(ms/b\right)m_{os}^{-}+\cos\left(ms/b\right)m_{es}^{-}\right]\cos\left(\alpha\Delta\right)d\alpha$$

where

$$m_{es}^{-}=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_{-}\left(s'\right)ds'$$

$$m_{os}^{-}=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_{-}\left(s'\right)ds'$$

$$i\frac{\pi}{2}e^{im\pi/2}H_m^{(1)}\left(iz\right)=K_m\left(z\right)$$

Replacing $\Delta=\rho_{-}$

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m\left(b,\varphi,z_0+\rho_{-}\right)=-\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\sin\left(m\varphi\right)m_{os}^{-}+\cos\left(m\varphi\right)m_{es}^{-}\right]\int_0^{\infty}\frac{K_m\left(\alpha b\right)}{K_m'\left(\alpha b\right)}\cos\left(\alpha\rho_{-}\right)\frac{d\alpha}{\alpha}$$

$$= -\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\sin(m\varphi)\, m_{os}^- + \cos(m\varphi)\, m_{es}^-\right]\int_0^\infty \frac{K_m\left(ub/\rho_-\right)}{K_m'\left(ub/\rho_-\right)}\cos(u)\,\frac{du}{u}$$

$$= -\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\sin(m\varphi)\, m_{os}^- + \cos(m\varphi)\, m_{es}^-\right]\int_0^\infty \frac{K_m(u)}{K_m'(u)}\cos\left(u\rho_-/b\right)\frac{du}{u}$$

$$= -\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\sin(ms/b)\, m_{os}^- + \cos(ms/b)\, m_{es}^-\right]\int_0^\infty \frac{H_m^{(1)}(i\alpha b)}{iH_m^{(1)\prime}(i\alpha b)}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{\alpha}$$

Using

$$\frac{K_m(\alpha b)}{K_m'(\alpha b)} = \frac{H_m^{(1)}(i\alpha b)}{iH_m^{(1)\prime}(i\alpha b)} \sim -1\ ,\ \alpha b >> 1$$

we can write

$$\int_0^\infty \frac{K_m(\alpha b)}{K_m'(\alpha b)}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{\alpha} = \int_0^A \frac{K_m(\alpha b)}{K_m'(\alpha b)}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{\alpha} + \int_A^\infty \left\{\frac{K_m(\alpha b)}{K_m'(\alpha b)} + 1\right\}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{\alpha} + \mathrm{Ci}\left(A\rho_-\right)$$

$$\mathrm{Ci}(z) = -\int_z^\infty \cos(t)\,\frac{dt}{t} = \gamma' + \ln(z) + \sum_{n=1}^\infty \frac{(-1)^n z^{2n}}{2n\,(2n)!}$$

Next assuming that $A\rho_- << 1$ we can approximate as

$$\int_0^\infty \frac{K_m(\alpha b)}{K_m'(\alpha b)}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{\alpha} \approx \int_0^A \frac{K_m(\alpha b)}{K_m'(\alpha b)}\frac{d\alpha}{\alpha} + \int_A^\infty \left\{\frac{K_m(\alpha b)}{K_m'(\alpha b)} + 1\right\}\frac{d\alpha}{\alpha} + \gamma' + \ln\left(A\rho_-\right)$$

If we set $A = 1/b$

$$\int_0^\infty \frac{K_m(\alpha b)}{K_m'(\alpha b)}\cos\left(\alpha\rho_-\right)\frac{d\alpha}{\alpha} \approx \int_0^1 \frac{K_m(u)}{K_m'(u)}\frac{du}{u} + \int_1^\infty \left\{\frac{K_m(u)}{K_m'(u)} + 1\right\}\frac{du}{u} + \gamma' + \ln\left(\rho_-/b\right)$$

$$\approx \int_0^U \frac{K_m(u)}{K_m'(u)}\frac{du}{u} + \ln(U) + \int_U^\infty \left\{-\frac{1 + \left(4m^2 - 1\right)/(8u)}{1 - \left(4m^2 + 2\right)/(8u)} + 1\right\}\frac{du}{u} + \gamma' + \ln\left(\rho_-/b\right)$$

$$\approx \int_0^U \frac{K_m(u)}{K_m'(u)}\frac{du}{u} + \ln(U) - \left(m^2 + 1/8\right)\int_U^\infty \frac{du}{u^2} + \gamma' + \ln\left(\rho_-/b\right)$$

$$\approx \int_0^U \frac{K_m(u)}{K_m'(u)}\frac{du}{u} + \ln(U) - \left(m^2 + 1/8\right)/U + \gamma' + \ln\left(\rho_-/b\right)$$

$$\approx C_m + \gamma' + \ln\left(\rho_-/b\right)$$

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m\left(b,\varphi,z_0+\rho_-\right) = -\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\sin(m\varphi)\, m_{os}^- + \cos(m\varphi)\, m_{es}^-\right]\int_0^\infty \frac{K_m(u)}{K_m'(u)}\cos\left(u\rho_-/b\right)\frac{du}{u}$$

$$\approx -\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\sin(m\varphi)\,m_{os}^- + \cos(m\varphi)\,m_{es}^-\right]\left\{C_m + \gamma' + \ln\left(\rho_-/b\right)\right\}$$

$$C_m \sim C_{m-1} + \ln\left(\frac{m+0.4}{m-1+0.4}\right)$$

m	C_m
0	-1.0331453
1	-0.31851065
2	0.19293576
3	0.53504092
4	0.79063387
5	0.99441230
6	1.1637820
7	1.3086626
8	1.4352316
9	1.5475940

5.2.1 Local Quasi-Static Planar Contribution

Because the scalar potential satisfies Poisson's equation in the Coulomb gauge, it decreases away from the aperture. We might alternatively approximate the magnetic scalar potential when the aperture is small by means of a half space approximation

$$\phi_m(\underline{r}) = \frac{1}{4\pi\mu_0}\int_V \frac{\rho_m(\underline{r}')}{|\underline{r}-\underline{r}'|}dV'$$

$$|\underline{r}-\underline{r}'| = \sqrt{\{\rho\cos(\varphi)-\rho'\cos(\varphi')\}^2 + \{\rho\sin(\varphi)-\rho'\sin(\varphi')\}^2 + (z-z')^2}$$

With φ and φ' small we can expand as

$$|\underline{r}-\underline{r}'| \approx \sqrt{(\rho-\rho')^2 + (\rho\varphi-\rho'\varphi')^2 + (z-z')^2} = \sqrt{(\rho-\rho')^2 + (s-s')^2 + (z-z')^2}$$

$$s = \rho\varphi\ ,\ s' = \rho'\varphi'$$

and therefore

$$\phi_m(\rho,\varphi,z) \approx \frac{1}{4\pi\mu_0}\int_{-h}^{h}\frac{q_m^+(s')}{\sqrt{\rho_0^2+(s-s')^2}}ds'\ ,\ \rho_0^2 = (\rho-b)^2 + (z-z_0)^2\ \ ,\ s=\rho\varphi\ ,\ s' = b\varphi'$$

where here we have included the image in the definition of the magnetic charge per unit length

$$\rho_m = q_m^+\delta(\rho-b)\,\delta(z-z')$$

$$q_m^+(s) = 2\Phi_+(s)$$

Using

$$\frac{\partial}{\partial s} I_m^+ = i\omega q_m^+$$

$$\phi_m(\rho,\varphi,z) \approx \frac{1}{4\pi i\omega\mu_0}\int_{-h}^{h}\frac{\partial}{\partial s'}I_m^+(s')\frac{1}{\sqrt{\rho_0^2+(s-s')^2}}ds' \ , \ \rho_0^2 = (\rho-b)^2+(z-z_0)^2 \ , \ s=\rho\varphi \ , \ s' = b\varphi'$$

and integrating by parts

$$\phi_m(\rho,\varphi,z) \approx \frac{1}{4\pi i\omega\mu_0}\frac{\partial}{\partial s}\int_{-h}^{h}\frac{I_m^+(s')}{\sqrt{\rho_0^2+(s-s')^2}}ds' \ , \ \rho_0^2 = (\rho-b)^2+(z-z_0)^2 \ , \ s=\rho\varphi \ , \ s' = b\varphi'$$

The contribution to the azimuthal field is then

$$-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m(\rho,\varphi,z) \approx \frac{1}{4\pi(-i\omega\mu_0)}\frac{1}{\rho}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial s}\int_{-h}^{h}\frac{I_m^+(s')}{\sqrt{\rho_0^2+(s-s')^2}}ds' \ , \ \rho_0^2 = (\rho-b)^2+(z-z_0)^2 \ , \ s=\rho\varphi \ , \ s' = b\varphi'$$

or near $\rho = b$

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m(b,\varphi,z) \approx \frac{i\omega\varepsilon_0}{4\pi k^2}\frac{\partial^2}{\partial s^2}\int_{-h}^{h}\frac{I_m^+(s')}{\sqrt{\rho_0^2+(s-s')^2}}ds'$$

with

$$I_m^+(s) = 2V_+(s)$$

and

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m(b,\varphi,z) \approx \frac{i\omega\varepsilon_0}{2\pi k^2}\frac{\partial^2}{\partial s^2}\int_{-h}^{h}\frac{V_+(s')}{\sqrt{\rho_0^2+(s-s')^2}}ds'$$

5.3 Lorentz Gauge Electric Vector Potential

From the Coulomb gauge form of the electric vector potential we should be able to construct the Lorentz gauge form by replacing

$$\underline{A}_e = \underline{A}_e^C + \nabla\psi$$

In the Lorentz gauge

$$\nabla\cdot\underline{A}_e = \nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\psi\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2} = i\omega\mu_0\varepsilon_0\phi_m$$

We must use the Helmholtz solution for the scalar potential (with outer radiation condition)

$$\left(\nabla^2+k^2\right)\phi_m = -\rho_m/\mu_0$$

$$k_t = \sqrt{k^2-\alpha^2} = i\sqrt{\alpha^2-k^2}$$

$$\nabla^2\phi_m = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\phi_m\right) + k^2\phi_m + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\phi_m + \frac{\partial^2}{\partial z^2}\phi_m = \left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + k_t^2 - \frac{m^2}{\rho^2}\right)\phi_m$$

$$\phi_m = \sum_m \int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\left[A_m(\alpha)\cos(m\varphi) + B_m(\alpha)\sin(m\varphi)\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$-\mu_0\frac{\partial\phi_m}{\partial\rho}(b) = \sigma_m^- = \Phi_-(s)\,\delta(z-z_0) = \frac{1}{2}q_m^-(s)\,\delta(z-z_0)$$

and multiplying by the trigonometric and Bessel functions, followed by integration over the volume of the cavity, gives

$$\sum_m \int_0^\infty \left[A_m(\alpha)\cos(m\varphi) + B_m(\alpha)\sin(m\varphi)\right]\int_{-\infty}^\infty \cos(\alpha'(z-z_0))\cos(\alpha(z-z_0))\,d\alpha d(z-z_0)$$

$$= -\frac{1}{2\mu_0}q_m^-(s)\int_{-\infty}^\infty \delta(z-z_0)\cos\alpha'(z-z_0)\,d(z-z_0)$$

$$\int_{-\infty}^\infty \cos(\alpha' u)\cos(\alpha u)\,du = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^R \left[\cos(\alpha-\alpha')u + \cos(\alpha+\alpha')u\right]du$$

$$= \lim_{R\to\infty}\left[\frac{\sin(\alpha-\alpha')R}{(\alpha-\alpha')} + \frac{\sin(\alpha+\alpha')R}{(\alpha+\alpha')}\right] = \pi\lim_{R/\pi\to\infty}(R/\pi)\left[\frac{\sin(\alpha-\alpha')\pi R/\pi}{(\alpha-\alpha')\pi R/\pi} + \frac{\sin(\alpha+\alpha')\pi R/\pi}{(\alpha+\alpha')\pi R/\pi}\right]$$

$$= \pi\left[\delta(\alpha-\alpha') + \delta(\alpha+\alpha')\right]$$

$$\sum_m \left[A_m(\alpha)\cos(m\varphi) + B_m(\alpha)\sin(m\varphi)\right] = -\frac{1}{2\pi\mu_0}q_m^-(s)$$

$$\int_{-\pi}^\pi \cos(m'\varphi)\cos(m\varphi)\,d\varphi = \frac{1}{2}\int_{-\pi}^\pi \left[\cos(m-m')\varphi + \cos(m+m')\varphi\right]d\varphi$$

$$= \frac{\sin(m-m')\pi}{(m-m')} + \frac{\sin(m+m')\pi}{(m+m')} = \frac{2\pi}{\varepsilon_m}\delta_{mm'}$$

$$\int_{-\pi}^\pi \sin(m'\varphi)\sin(m\varphi)\,d\varphi = \frac{1}{2}\int_{-\pi}^\pi \left[\cos(m-m')\varphi - \cos(m+m')\varphi\right]d\varphi$$

$$= \frac{\sin(m-m')\pi}{(m-m')} - \frac{\sin(m+m')\pi}{(m+m')} = \pi\delta_{mm'}$$

$$\int_{-\pi}^\pi \sin(m'\varphi)\cos(m\varphi)\,d\varphi = \frac{1}{2}\int_{-\pi}^\pi \left[\sin(m'-m)\varphi + \sin(m'+m)\varphi\right]d\varphi = 0$$

$$\frac{2\pi}{\varepsilon_m}A_m(\alpha) = -\frac{1}{2\pi\mu_0}\int_{-\pi}^\pi q_m^-(s)\cos(m\varphi)\,d\varphi = -\frac{1}{2\pi b\mu_0}\int_{-h}^h q_m^-(s)\cos(ms/b)\,ds$$

$$\pi B_m(\alpha) = -\frac{1}{2\pi\mu_0}\int_{-\pi}^{\pi} q_m^-(s)\sin(m\varphi)\,d\varphi = -\frac{1}{2\pi b\mu_0}\int_{-h}^{h} q_m^-(s)\sin(ms/b)\,ds$$

Then we have

$$\phi_m = -\frac{1}{2\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\left[\cos(m\varphi)\int_{-h}^{h} q_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} q_m^-(s')\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

where we take the magnetic charge to be equal to twice the magnetic flux

$$q_m^-(s) = 2\Phi_-(s)$$

Then

$$\nabla^2\psi = i\omega\mu_0\varepsilon_0\phi_m$$

$$\psi = \frac{i\omega\mu_0\varepsilon_0/k^2}{2\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\left[\cos(m\varphi)\int_{-h}^{h} q_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} q_m^-(s')\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

and

$$\nabla^2\psi = \frac{i\omega\mu_0\varepsilon_0/k^2}{2\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{1}{k_t H_m^{(1)\prime}(k_t b)}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \alpha^2 - \frac{m^2}{\rho^2}\right) H_m^{(1)}(k_t\rho)$$

$$\left[\cos(m\varphi)\int_{-h}^{h} q_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} q_m^-(s')\sin(ms'/b)\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + k_t^2 - \frac{m^2}{\rho^2}\right) H_m^{(1)}(k_t\rho) = 0$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \alpha^2 - \frac{m^2}{\rho^2}\right) H_m^{(1)}(k_t\rho) = -\left(k_t^2 + \alpha^2\right) H_m^{(1)}(k_t\rho) = -k^2 H_m^{(1)}(k_t\rho)$$

$$\nabla^2\psi = -\frac{i\omega\mu_0\varepsilon_0}{2\pi b\mu_0}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\left[\cos(m\varphi)\int_{-h}^{h} q_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} q_m^-(s')\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

Now the magnetic field is

$$\underline{H} = -\nabla\phi_m + i\omega\underline{A}_e = -\nabla(\phi_m - \psi) + i\omega\underline{A}_e^C$$

$$\phi_m - \psi = -\frac{1 - 1/(i\omega)}{2\pi b\mu_0} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}$$

$$\left[\cos(m\varphi) \int_{-h}^{h} q_m^-(s') \cos(ms'/b)\, ds' + \sin(m\varphi) \int_{-h}^{h} q_m^-(s') \sin(ms'/b)\, ds'\right] \cos(\alpha(z - z_0))\, d\alpha$$

$$A_{e\varphi}^C = \frac{\varepsilon_0}{2\pi} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \left[\frac{H_m^{(1)\prime}(k_t\rho)}{k_t H_m^{(1)}(k_t b)} - \frac{m^2\alpha^2/k^2}{k_t^2 \rho b} \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\cos(m\varphi) \int_{-\pi}^{\pi} I_m^-(s') \cos(m\varphi')\, d\varphi' + \sin(m\varphi) \int_{-\pi}^{\pi} I_m^-(s') \sin(m\varphi')\, d\varphi'\right] \cos(\alpha(z - z_0))\, d\alpha$$

and

$$A_{e\rho}^C = \frac{\varepsilon_0}{2\pi} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{m}{k_t^2} \left[\frac{H_m^{(1)}(k_t\rho)}{\rho H_m^{(1)}(k_t b)} - \frac{\alpha^2}{k^2 b} \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\sin(m\varphi) \int_{-\pi}^{\pi} I_m^-(s') \cos(m\varphi')\, d\varphi' - \cos(m\varphi) \int_{-\pi}^{\pi} I_m^-(s') \sin(m\varphi')\, d\varphi'\right] \cos(\alpha(z - z_0))\, d\alpha$$

$$I_m^- = -2V_-$$

$$\frac{\partial}{\partial s} I_m^- = i\omega q_m^-$$

$$\psi = \frac{1}{2\pi b\omega^2\mu_0} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}$$

$$\left[\cos(m\varphi) \int_{-h}^{h} \frac{\partial}{\partial s'} I_m^-(s') \cos(ms'/b)\, ds' + \sin(m\varphi) \int_{-h}^{h} \frac{\partial}{\partial s'} I_m^-(s') \sin(ms'/b)\, ds'\right] \cos(\alpha(z - z_0))\, d\alpha$$

$$\int_{-h}^{h} \frac{\partial}{\partial s'} I_m^-(s') \cos(ms'/b)\, ds' = -\int_{-h}^{h} I_m^-(s') \frac{\partial}{\partial s'} \cos(ms'/b)\, ds'$$

$$\int_{-h}^{h} \frac{\partial}{\partial s'} I_m^-(s') \sin(ms'/b)\, ds' = -\int_{-h}^{h} I_m^-(s') \frac{\partial}{\partial s'} \sin(ms'/b)\, ds'$$

$$\psi = -\frac{1}{2\pi b\omega^2\mu_0} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}$$

$$\left[\cos(m\varphi) \int_{-h}^{h} I_m^-(s') \frac{\partial}{\partial s'} \cos(ms'/b)\, ds' + \sin(m\varphi) \int_{-h}^{h} I_m^-(s') \frac{\partial}{\partial s'} \sin(ms'/b)\, ds\right] \cos(\alpha(z - z_0))\, d\alpha$$

$$= \frac{1}{2\pi b^2\omega^2\mu_0} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)} m \left[\cos(m\varphi) \int_{-h}^{h} I_m^-(s') \sin(ms'/b)\, ds' - \sin(m\varphi) \int_{-h}^{h} I_m^-(s') \cos(ms'/b)\, ds\right]$$

$$\cos(\alpha(z - z_0))\, d\alpha$$

$$= \frac{1}{2\pi b^2\omega^2\mu_0} \frac{\partial}{\partial\varphi} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)} \left[\sin(m\varphi) \int_{-h}^{h} I_m^-(s') \sin(ms'/b)\, ds' + \cos(m\varphi) \int_{-h}^{h} I_m^-(s') \cos(ms'/b)\, ds\right]$$

$$\cos(\alpha(z - z_0))\, d\alpha$$

The total ρ component is

$$\frac{\partial\psi}{\partial\rho} = -\frac{\varepsilon_0}{2\pi b} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_t b)} \frac{m}{k^2 b} \left[\sin(m\varphi) \int_{-h}^{h} I_m^-(s') \cos(ms'/b)\, ds - \cos(m\varphi) \int_{-h}^{h} I_m^-(s') \sin(ms'/b)\, ds'\right]$$

$$\cos(\alpha(z - z_0))\, d\alpha$$

and then the Lorentz gauge electric vector potential is

$$A_{e\rho} = A_{e\rho}^C + \frac{\partial\psi}{\partial\rho} = \frac{\varepsilon_0}{2\pi b} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{m}{k_t^2} \left[\frac{H_m^{(1)}(k_t\rho)}{\rho H_m^{(1)}(k_t b)} - \frac{\alpha^2}{k^2 b} \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\sin(m\varphi) \int_{-h}^{h} I_m^-(s') \cos(m\varphi')\, ds' - \cos(m\varphi) \int_{-h}^{h} I_m^-(s') \sin(m\varphi')\, ds'\right] \cos(\alpha(z - z_0))\, d\alpha$$

$$-\frac{\varepsilon_0}{2\pi b} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_t b)} \frac{m}{k^2 b} \left[\sin(m\varphi) \int_{-h}^{h} I_m^-(s') \cos(ms'/b)\, ds - \cos(m\varphi) \int_{-h}^{h} I_m^-(s') \sin(ms'/b)\, ds'\right]$$

$$\cos(\alpha(z - z_0))\, d\alpha$$

$$= \frac{\varepsilon_0}{2\pi b} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{m}{k_t^2} \left[\frac{H_m^{(1)}(k_t\rho)}{\rho H_m^{(1)}(k_t b)} - \frac{H_m^{(1)\prime}(k_t\rho)}{b H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\sin(m\varphi) \int_{-h}^{h} I_m^-(s') \cos(m\varphi')\, ds' - \cos(m\varphi) \int_{-h}^{h} I_m^-(s') \sin(m\varphi')\, ds'\right] \cos(\alpha(z - z_0))\, d\alpha$$

which we see vanishes at $\rho = b$. The scalar potential ρ derivative is

$$-\frac{\partial}{\partial\rho}\phi_m = -\frac{i\omega\varepsilon_0}{2\pi b k^2} \sum_m \frac{\varepsilon_m}{2\pi} \int_0^\infty \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_t b)}$$

$$\left[\cos(m\varphi)\int_{-h}^{h} i\omega q_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} i\omega q_m^-(s')\sin(ms'/b)\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$= -\frac{i\omega\varepsilon_0}{2\pi bk^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_tb)}\left[\cos(m\varphi)\int_{-h}^{h}\frac{\partial}{\partial s'}I_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h}\frac{\partial}{\partial s'}I_m^-(s')\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

$$= \frac{i\omega\varepsilon_0}{2\pi bk^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_tb)}\left[\cos(m\varphi)\int_{-h}^{h}I_m^-(s')\frac{\partial}{\partial s'}\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h}I_m^-(s')\frac{\partial}{\partial s'}\sin(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

$$= \frac{i\omega\varepsilon_0}{2\pi bk^2}\sum_m \frac{\varepsilon_m}{2\pi}\frac{m}{b}\int_0^\infty \frac{H_m^{(1)\prime}(k_t\rho)}{H_m^{(1)\prime}(k_tb)}\left[-\cos(m\varphi)\int_{-h}^{h}I_m^-(s')\sin(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h}I_m^-(s')\cos(ms'/b)\,ds'\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

and the total ρ component of the magnetic field is then

$$H_\rho = -\frac{\partial}{\partial\rho}\phi_m + i\omega A_{e\rho}$$

$$= \frac{i\omega\varepsilon_0}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{m}{k_t^2}\left[\frac{H_m^{(1)}(k_t\rho)}{\rho H_m^{(1)}(k_tb)} - \left(1-k_t^2/k^2\right)\frac{H_m^{(1)\prime}(k_t\rho)}{bH_m^{(1)\prime}(k_tb)}\right]$$

$$\left[\sin(m\varphi)\int_{-h}^{h}I_m^-(s')\cos(m\varphi')\,ds' - \cos(m\varphi)\int_{-h}^{h}I_m^-(s')\sin(m\varphi')\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$= -\frac{i\omega\varepsilon_0}{2\pi b}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{1}{k_t^2}\left[\frac{H_m^{(1)}(k_t\rho)}{\rho H_m^{(1)}(k_tb)} - \left(1-k_t^2/k^2\right)\frac{H_m^{(1)\prime}(k_t\rho)}{bH_m^{(1)\prime}(k_tb)}\right]$$

$$\left[\cos(m\varphi)\int_{-h}^{h}I_m^-(s')\cos(m\varphi')\,ds' + \sin(m\varphi)\int_{-h}^{h}I_m^-(s')\sin(m\varphi')\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

At $\rho = b$

$$H_\rho(b,\varphi,z) = -\frac{i\omega\varepsilon_0}{2\pi k^2b^2}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\int_{-h}^{h}I_m^-(s')\cos(m\varphi')\,ds' + \sin(m\varphi)\int_{-h}^{h}I_m^-(s')\sin(m\varphi')\,ds'\right]$$

$$\int_0^\infty \cos(\alpha(z-z_0))\,d\alpha$$

$$= -\frac{i\omega\varepsilon_0}{2k^2b^2}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\int_{-h}^{h} I_m^-(s')\cos(m\varphi')\,ds' + \sin(m\varphi)\int_{-h}^{h} I_m^-(s')\sin(m\varphi')\,ds'\right]\delta(z-z_0)$$

$$= \frac{i\omega\varepsilon_0}{k^2b^2}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\int_{-h}^{h} V_-(s')\cos(m\varphi')\,ds' + \sin(m\varphi)\int_{-h}^{h} V_-(s')\sin(m\varphi')\,ds'\right]\delta(z-z_0)$$

$$= -\frac{1}{2b}\frac{\partial}{\partial s}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\, m_{es}^- + \sin(m\varphi)\, m_{os}^-\right]\delta(z-z_0) = \frac{i\omega\varepsilon_0}{k^2}\frac{\partial}{\partial s}V_-(s)\,\delta(z-z_0) = \frac{1}{-i\omega\mu_0}\frac{\partial}{\partial s}V_-(s)\,\delta(z-z_0)$$

$$= \frac{1/2}{i\omega\mu_0}\frac{\partial}{\partial s}I_m^-(s)\,\delta(z-z_0) = \frac{1}{2}q_m^-(s)\,\delta(z-z_0) = \Phi_-(s)\,\delta(z-z_0)$$

where we used

$$m_{es}^- = \frac{-i2}{\omega\mu_0}\int_{-h}^{h}\cos(ms'/b)\,V_-(s')\,ds'$$

$$m_{os}^- = \frac{-i2}{\omega\mu_0}\int_{-h}^{h}\sin(ms'/b)\,V_-(s')\,ds'$$

and the Fourier series result

$$-i\omega\varepsilon_0\frac{1}{\pi}V_-(s) = \frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\, m_{es}^- + \sin(m\varphi)\, m_{os}^-\right]$$

The φ component is then

$$\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} = \frac{\varepsilon_0}{2\pi bk^2}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}(k_t\rho)}{k_t b\rho H_m^{(1)\prime}(k_t b)}$$

$$\left[\sin(m\varphi)\int_{-h}^{h} I_m^-(s')\sin(ms'/b)\,ds' + \cos(m\varphi)\int_{-h}^{h} I_m^-(s')\cos(ms'/b)\,ds\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$= -\frac{\varepsilon_0}{2\pi bk^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{m^2 H_m^{(1)}(k_t\rho)}{k_t b\rho H_m^{(1)\prime}(k_t b)}\left[\sin(m\varphi)\int_{-h}^{h} I_m^-(s')\sin(ms'/b)\,ds' + \cos(m\varphi)\int_{-h}^{h} I_m^-(s')\cos(ms'/b)\,ds\right]$$

$$\cos(\alpha(z-z_0))\,d\alpha$$

and the Lorentz gauge electric vector potential

$$A_{e\varphi}^C + \frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} = \frac{\varepsilon_0}{2\pi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \left[\frac{H_m^{(1)\prime}(k_t\rho)}{k_t H_m^{(1)}(k_t b)} - \frac{m^2\left(\alpha^2+k_t^2\right)/k^2}{k_t^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\cos\left(m\varphi\right)\int_{-\pi}^{\pi} I_m^-\left(s'\right)\cos\left(m\varphi'\right)d\varphi' + \sin\left(m\varphi\right)\int_{-\pi}^{\pi} I_m^-\left(s'\right)\sin\left(m\varphi'\right)d\varphi'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$A_{e\varphi} = \frac{\varepsilon_0}{2\pi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \left[\frac{H_m^{(1)\prime}\left(k_t\rho\right)}{k_t H_m^{(1)}\left(k_t b\right)} - \frac{m^2}{k_t^2 \rho b}\frac{H_m^{(1)}\left(k_t\rho\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)}\right]$$

$$\left[\cos\left(m\varphi\right)\int_{-\pi}^{\pi} I_m^-\left(s'\right)\cos\left(m\varphi'\right)d\varphi' + \sin\left(m\varphi\right)\int_{-\pi}^{\pi} I_m^-\left(s'\right)\sin\left(m\varphi'\right)d\varphi'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m = \frac{1/\rho}{2\pi b\mu_0}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}\left(k_t\rho\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)}$$

$$\left[\cos\left(m\varphi\right)\int_{-h}^{h} q_m^-\left(s'\right)\cos\left(ms'/b\right)ds' + \sin\left(m\varphi\right)\int_{-h}^{h} q_m^-\left(s'\right)\sin\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$= \frac{1/\rho}{2\pi b i\omega\mu_0}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}\left(k_t\rho\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)}$$

$$\left[\cos\left(m\varphi\right)\int_{-h}^{h} \frac{\partial}{\partial s'} I_m^-\left(s'\right)\cos\left(ms'/b\right)ds' + \sin\left(m\varphi\right)\int_{-h}^{h} \frac{\partial}{\partial s'} I_m^-\left(s'\right)\sin\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$= -\frac{1/\rho}{2\pi b i\omega\mu_0}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}\left(k_t\rho\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)}$$

$$\left[\cos\left(m\varphi\right)\int_{-h}^{h} I_m^-\left(s'\right)\frac{\partial}{\partial s'}\cos\left(ms'/b\right)ds' + \sin\left(m\varphi\right)\int_{-h}^{h} I_m^-\left(s'\right)\frac{\partial}{\partial s'}\sin\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$= -\frac{1/\rho}{2\pi b^2 i\omega\mu_0}\frac{\partial}{\partial\varphi}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{mH_m^{(1)}\left(k_t\rho\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)}$$

$$\left[-\cos\left(m\varphi\right)\int_{-h}^{h} I_m^-\left(s'\right)\sin\left(ms'/b\right)ds' + \sin\left(m\varphi\right)\int_{-h}^{h} I_m^-\left(s'\right)\cos\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

$$= \frac{1/\rho}{2\pi b^2 i\omega\mu_0}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{H_m^{(1)}\left(k_t\rho\right)}{k_t H_m^{(1)\prime}\left(k_t b\right)}$$

$$\left[\sin\left(m\varphi\right)\int_{-h}^{h} I_m^-\left(s'\right)\sin\left(ms'/b\right)ds' + \cos\left(m\varphi\right)\int_{-h}^{h} I_m^-\left(s'\right)\cos\left(ms'/b\right)ds'\right]\cos\left(\alpha\left(z-z_0\right)\right)d\alpha$$

The total magnetic field φ component is then

$$H_\varphi = -\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m + i\omega A_{e\varphi}$$

$$= -\frac{i\omega\varepsilon_0}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{1}{k^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}$$

$$\left[\cos(m\varphi)\int_{-h}^{h} I_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} I_m^-(s')\sin(ms'/b)\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$+\frac{i\omega\varepsilon_0}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \left[\frac{H_m^{(1)\prime}(k_t\rho)}{k_t H_m^{(1)}(k_t b)} - \frac{m^2}{k_t^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\cos(m\varphi)\int_{-h}^{h} I_m^-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} I_m^-(s')\sin(ms'/b)\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

or

$$H_\varphi = -\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m + i\omega A_{e\varphi}$$

$$= \frac{i\omega\varepsilon_0}{\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{1}{k^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}$$

$$\left[\cos(m\varphi)\int_{-h}^{h} V_-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} V_-(s')\sin(ms'/b)\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$-\frac{i\omega\varepsilon_0}{\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \left[\frac{H_m^{(1)\prime}(k_t\rho)}{k_t H_m^{(1)}(k_t b)} - \frac{m^2}{k_t^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\right]$$

$$\left[\cos(m\varphi)\int_{-h}^{h} V_-(s')\cos(ms'/b)\,ds' + \sin(m\varphi)\int_{-h}^{h} V_-(s')\sin(ms'/b)\,ds'\right]\cos(\alpha(z-z_0))\,d\alpha$$

or

$$H_\varphi = -\frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \frac{1}{k^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]\cos(\alpha(z-z_0))\,d\alpha$$

$$+\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\int_0^\infty \left[\frac{H_m^{(1)\prime}(k_t\rho)}{k_t H_m^{(1)}(k_t b)} - \frac{m^2}{k_t^2\rho b}\frac{H_m^{(1)}(k_t\rho)}{k_t H_m^{(1)\prime}(k_t b)}\right]\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]\cos(\alpha(z-z_0))\,d\alpha$$

Now taking $\rho = b$

$$H_{\varphi}(b,\varphi,z) = -\frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]\frac{1}{k^2b^2}\int_0^\infty \frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\cos(\alpha(z-z_0))\frac{d\alpha}{k_t}$$

$$+\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]\int_0^\infty \left[\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)} - \frac{m^2}{k_t^2b^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\right]\cos(\alpha(z-z_0))\frac{d\alpha}{k_t}$$

where

$$k_t = \sqrt{k^2-\alpha^2} = i\sqrt{\alpha^2-k^2} = iu \sim i\alpha\ ,\ \ \alpha >> 1$$

$$\frac{H_m^{(1)}(iub)}{iH_m^{(1)\prime}(iub)} = \frac{K_m(ub)}{K_m'(ub)} \sim -1\ ,\ \ ub >> 1$$

Using the integral

$$\int_0^\infty \cos(\alpha\Delta)\frac{d\alpha}{k_t} = \int_0^k \cos(\alpha\Delta)\frac{d\alpha}{\sqrt{k^2-\alpha^2}} + \int_k^\infty \cos(\alpha\Delta)\frac{d\alpha}{i\sqrt{\alpha^2-k^2}}$$

$$= \int_0^1 \cos(\alpha k\Delta)\frac{d\alpha}{\sqrt{1-\alpha^2}} - i\int_1^\infty \cos(\alpha k\Delta)\frac{d\alpha}{\sqrt{\alpha^2-1}}$$

$$= \frac{\pi}{2}J_0(k\Delta) + i\frac{\pi}{2}Y_0(k\Delta) = \frac{\pi}{2}H_0^{(1)}(k\Delta)$$

to extract the large α behavior of the integrand we rewrite this as

$$H_{\varphi}(b,\varphi,z) = \frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\frac{1}{k^2b^2}\left[\int_0^\infty \left\{-\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)} - i\right\}\cos(\alpha(z-z_0))\frac{d\alpha}{k_t} + i\frac{\pi}{2}H_0^{(1)}(k(z-z_0))\right]$$

$$+\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^\infty \left\{\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)} - i - \frac{m^2}{k_t^2b^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\right\}\cos(\alpha(z-z_0))\frac{d\alpha}{k_t} + i\frac{\pi}{2}H_0^{(1)}(k(z-z_0))\right]$$

Because we now have absolute convergence at infinity for $k(z-z_0) = k\Delta << 1$

$$i\frac{\pi}{2}H_0^{(1)}(k\Delta) \sim i\frac{\pi}{2} - \{\ln(k\Delta/2) + \gamma'\}\ ,$$

and

$$H_{\varphi}(b,\varphi,z_0+\Delta) \sim \frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\frac{1}{k^2b^2}\left[\int_0^\infty\left\{-\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}-i\right\}\frac{d\alpha}{k_t}+i\frac{\pi}{2}-\{\ln(k\Delta/2)+\gamma'\}\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^-+\sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^\infty\left\{\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)}-i-\frac{m^2}{k_t^2b^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\right\}\frac{d\alpha}{k_t}+i\frac{\pi}{2}-\{\ln(k\Delta/2)+\gamma'\}\right]$$

Breaking up the integrals into two ranges

$$H_\varphi(b,\varphi,z)=\frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^-+\sin(m\varphi)\,m_{os}^-\right]$$

$$\frac{1}{k^2b^2}\left[\int_0^k\left\{-\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}-i\right\}\frac{d\alpha}{k_t}+\int_k^\infty\left\{-\frac{K_m(ub)}{K_m'(ub)}-1\right\}\frac{d\alpha}{u}+i\frac{\pi}{2}-\{\ln(k\Delta/2)+\gamma'\}\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^-+\sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)}-i-\frac{m^2}{k_t^2b^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\right\}\frac{d\alpha}{k_t}+\int_k^\infty\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{u^2b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{d\alpha}{u}+i\frac{\pi}{2}-\{\ln(k\Delta/2)+\gamma'\}\right]$$

Noting that

$$\int_0^k\cos(\alpha\Delta)\frac{d\alpha}{\sqrt{k^2-\alpha^2}}=\int_0^1\cos(\alpha k\Delta)\frac{d\alpha}{\sqrt{1-\alpha^2}}=\frac{\pi}{2}J_0(k\Delta)\sim\frac{\pi}{2}\ ,\ k\Delta<<1$$

we can drop the subtraction in the initial range, and also using the result

$$-i\omega\varepsilon_0\frac{1}{\pi}V_-(s)=\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^-+\sin(m\varphi)\,m_{os}^-\right]$$

we can sum the new terms outside the integrals, to find

$$H_\varphi(b,\varphi,z_0+\Delta)=\frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^-+\sin(m\varphi)\,m_{os}^-\right]$$

$$\frac{1}{k^2b^2}\left[\int_0^k\left\{-\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\right\}\frac{d\alpha}{k_t}+\int_k^\infty\left\{-\frac{K_m(ub)}{K_m'(ub)}-1\right\}\frac{d\alpha}{u}+C_0\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^-+\sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{H_m^{(1)\prime}(k_tb)}{H_m^{(1)}(k_tb)}-\frac{m^2}{k_t^2b^2}\frac{H_m^{(1)}(k_tb)}{H_m^{(1)\prime}(k_tb)}\right\}\frac{d\alpha}{k_t}+\int_k^\infty\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{u^2b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{d\alpha}{u}+C_0\right]$$

$$+i\omega\varepsilon_0\frac{1}{\pi}\left\{\ln\left(k\Delta/2\right)+\gamma'+C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)$$

where we are free to choose the value of C_0. Changing variables

$$\frac{dk_t}{d\alpha}=-\alpha/k_t$$

$$\frac{du}{d\alpha}=\alpha/u$$

gives

$$H_\varphi\left(b,\varphi,z_0+\Delta\right)=\frac{k^2}{2\pi b}\frac{\partial^2}{\partial\varphi^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\frac{1}{k^2b^2}\left[\int_0^k\left\{-\frac{H_m^{(1)}\left(k_tb\right)}{H_m^{(1)\prime}\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_0^\infty\left\{-\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}+C_0\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{H_m^{(1)\prime}\left(k_tb\right)}{H_m^{(1)}\left(k_tb\right)}-\frac{m^2}{k_t^2b^2}\frac{H_m^{(1)}\left(k_tb\right)}{H_m^{(1)\prime}\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+\int_0^\infty\left\{-\frac{K_m'\left(ub\right)}{K_m\left(ub\right)}-1+\frac{m^2}{u^2b^2}\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}+C_0\right]$$

$$+i\omega\varepsilon_0\frac{1}{\pi}\left\{\ln\left(k\Delta/2\right)+\gamma'+C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)$$

Convergence near $k_t,u\to 0$ in the second sum (for $m\geq 1$)

$$[]\to\frac{1}{k}\left[\int_0\left\{\frac{-\left(1/2\right)\left(-m\right)\left(1/\pi\right)\left(m-1\right)!\left(\frac{2}{k_tb}\right)^{m+1}}{-\left(1/\pi\right)\left(m-1\right)!\left(\frac{2}{k_tb}\right)^{m}}-\frac{m^2}{k_t^2b^2}\frac{-\left(1/\pi\right)\left(m-1\right)!\left(\frac{2}{k_tb}\right)^{m}}{-\left(1/2\right)\left(-m\right)\left(1/\pi\right)\left(m-1\right)!\left(\frac{2}{k_tb}\right)^{m+1}}\right\}dk_t\right.$$

$$\left.+\int_0\left\{-\frac{\frac{1}{4}\left(-m\right)\left(m-1\right)!\left(\frac{2}{ub}\right)^{m+1}}{\frac{1}{2}\left(m-1\right)!\left(\frac{2}{ub}\right)^{m}}-1+\frac{m^2}{u^2b^2}\frac{\frac{1}{2}\left(m-1\right)!\left(\frac{2}{ub}\right)^{m}}{\frac{1}{4}\left(-m\right)\left(m-1\right)!\left(\frac{2}{ub}\right)^{m+1}}\right\}du\right]$$

$$\{\}\to\left\{-\frac{m}{k_tb}+\frac{m}{k_tb}\right\}$$

$$\{\}=\left\{\frac{m}{ub}-1-\frac{m}{ub}\right\}$$

and for $m=0$

$$[]=\frac{1}{k}\left[\int_0^k\left\{-\frac{H_1^{(1)}\left(k_tb\right)}{H_0^{(1)}\left(k_tb\right)}\right\}dk_t+\int_0^\infty\left\{\frac{K_1'\left(ub\right)}{K_0\left(ub\right)}-1\right\}du\right]$$

$$\to \frac{1}{k}\left[\int_0^\delta \left\{-\frac{(-i2/\pi)/(k_t b)}{1+i(2/\pi)\{\ln(k_t b/2)+\gamma'\}}\right\} dk_t + \int_0^\delta \left\{\frac{1/(ub)}{-\{\ln(ub/2)+\gamma'\}} - 1\right\} du\right]$$

$$\to \frac{1}{kb}\left[-\int_0^\delta \left\{-\frac{i(2/\pi)}{1+i(2/\pi)\{\ln(ub/2)+\gamma'\}} + \frac{1}{\{\ln(ub/2)+\gamma'\}}\right\}\frac{du}{u} - b\int_0 du\right]$$

$$\to -\frac{1}{kb}\int_0^\delta \left[-\frac{1}{-i\pi/2+\{\ln(ub/2)+\gamma'\}} + \frac{1}{\{\ln(ub/2)+\gamma'\}}\right]\frac{du}{u}$$

$$\to -\frac{1}{kb}\int_0^\delta \frac{-\{\ln(ub/2)+\gamma'\} + [-i\pi/2+\{\ln(ub/2)+\gamma'\}]}{[-i\pi/2+\{\ln(ub/2)+\gamma'\}]\{\ln(ub/2)+\gamma'\}}\frac{du}{u}$$

$$\to \frac{i\pi/2}{kb}\int_0^\delta \frac{1}{[-i\pi/2+\{\ln(ub/2)+\gamma'\}]\{\ln(ub/2)+\gamma'\}}\frac{du}{u}$$

$$\to \frac{i\pi/2}{kb}\int_0^\delta \frac{1}{[-i\pi/2+\ln(ube^{\gamma'}/2)]\ln(ube^{\gamma'}/2)}\frac{du}{u} = \frac{i\pi/2}{kb}\int_0^{\delta be^{\gamma'}/2} \frac{1}{[-i\pi/2+\ln(u)]\ln(u)}\frac{du}{u}$$

$$= \frac{1}{kb}\int_{-\ln\left(\delta be^{\gamma'}/2\right)}^\infty \frac{i\pi/2}{i\pi/2+v}\frac{dv}{v}$$

$$= \frac{1}{kb}\int_{-\ln\left(\delta be^{\gamma'}/2\right)}^\infty \left(\frac{1}{v} - \frac{1}{i\pi/2+v}\right) dv = \frac{1}{kb}\left[\ln\left(\frac{i\pi/2+v}{v}\right)\right]_{-\ln\left(\delta be^{\gamma'}/2\right)}$$

$$= \frac{1}{kb}\ln\left[\frac{i\pi/2-\ln\left(\delta be^{\gamma'}/2\right)}{-\ln(\delta be^{\gamma'}/2)}\right] \sim \frac{1}{kb}\frac{i\pi/2}{-\ln(\delta be^{\gamma'}/2)}$$

$$-\ln(u) = v \to dv = -du/u$$

$$\frac{i\pi/2}{i\pi/2+v}\frac{1}{v} = \frac{1}{v} - \frac{1}{i\pi/2+v}$$

Thus the combination of the two integrals is convergent for $m = 0$ (barely). Then rationalizing the first integrals

$$\int_0^k \left\{-\frac{H_m^{(1)}(k_t b)}{H_m^{(1)\prime}(k_t b)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}} = \int_0^k \left\{-\frac{H_m^{(1)}(k_t b)\,H_m^{(2)\prime}(k_t b)}{H_m^{(1)\prime}(k_t b)\,H_m^{(2)\prime}(k_t b)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$= \int_0^k \left[-\frac{\{J_m(k_t b)+iY_m(k_t b)\}\{J'_m(k_t b)-iY'_m(k_t b)\}}{J'^2_m(k_t b)+Y'^2_m(k_t b)}\right]\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$= \int_0^k \left[-\frac{J_m(k_t b)\,J'_m(k_t b)+Y_m(k_t b)\,Y'_m(k_t b)}{J'^2_m(k_t b)+Y'^2_m(k_t b)}\right]\frac{dk_t}{\sqrt{k^2-k_t^2}} + i\int_0^k \left[-\frac{J'_m(k_t b)\,Y_m(k_t b)-J_m(k_t b)\,Y'_m(k_t b)}{J'^2_m(k_t b)+Y'^2_m(k_t b)}\right]\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$\int_0^k \left\{ \frac{H_m^{(1)\prime}(k_t b)}{H_m^{(1)}(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{H_m^{(1)}(k_t b)}{H_m^{(1)\prime}(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} = \int_0^k \left\{ \frac{H_m^{(1)\prime}(k_t b)\, H_m^{(2)}(k_t b)}{H_m^{(1)}(k_t b)\, H_m^{(2)}(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{H_m^{(1)}(k_t b)\, H_m^{(2)\prime}(k_t b)}{H_m^{(1)\prime}(k_t b)\, H_m^{(2)\prime}(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

$$= \int_0^k \left\{ \frac{\{J_m'(k_t b) + iY_m'(k_t b)\}\{J_m(k_t b) - iY_m(k_t b)\}}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{\{J_m(k_t b) + iY_m(k_t b)\}\{J_m'(k_t b) - iY_m'(k_t b)\}}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

$$= \int_0^k \left[\frac{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right] \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

$$+ i \int_0^k \left\{ \frac{J_m(k_t b)\, Y_m'(k_t b) - J_m'(k_t b)\, Y_m(k_t b)}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{J_m'(k_t b)\, Y_m(k_t b) - J_m(k_t b)\, Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

Using the Wronskian

$$J_m(k_t b)\, Y_m'(k_t b) - J_m'(k_t b)\, Y_m(k_t b) = J_{m+1}(k_t b)\, Y_m(k_t b) - J_m(k_t b)\, Y_{m+1}(k_t b) = \frac{2}{\pi k_t b}$$

$$\int_0^k \left\{ -\frac{H_m^{(1)}(k_t b)}{H_m^{(1)\prime}(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

$$= \int_0^k \left\{ -\frac{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} + i \frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}}$$

$$\int_0^k \left\{ \frac{H_m^{(1)\prime}(k_t b)}{H_m^{(1)}(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{H_m^{(1)}(k_t b)}{H_m^{(1)\prime}(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

$$= \int_0^k \left\{ \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)\} \frac{dk_t}{\sqrt{k^2 - k_t^2}}$$

$$+ i \frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} + \frac{m^2}{k_t^2 b^2} \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}}$$

Then we can write this letting $\Delta = \rho_-$

$$H_\varphi(b, \varphi, z_0 + \rho_-) = \frac{1}{2\pi b} \frac{\partial^2}{\partial s^2} \sum_m \frac{\varepsilon_m}{2\pi} \left[\cos(m\varphi)\, m_{es}^- + \sin(m\varphi)\, m_{os}^-\right]$$

$$\left[\int_0^k \left\{ -\frac{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} + i \frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}} \right.$$

$$\left. + \int_0^\infty \left\{ -\frac{K_m(ub)}{K_m'(ub)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2}} + C_0 \right]$$

$$+\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\, m_{es}^- + \sin(m\varphi)\, m_{os}^-\right]$$

$$\left[\int_0^k \left\{\frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\}\left\{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k \left\{\frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} + \frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left.+\int_0^\infty \left\{-\frac{K_m'(ub)}{K_m(ub)} - 1 + \frac{m^2}{u^2 b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{du}{\sqrt{u^2+k^2}} + C_0\right]$$

$$+i\omega\varepsilon_0\frac{1}{\pi}\left\{\ln\left(k\rho_-/2\right) + \gamma' + C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right)V_-(s)$$

This final term can be written as

$$i\omega\varepsilon_0\frac{1}{\pi}\left\{\ln\left(k\rho_-/2\right) + \gamma' + C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right)V_-(s)$$

$$= -i\omega\varepsilon_0\frac{1}{\pi}\left\{-\ln(kh) + \ln\left(2h/\rho_-\right) - \gamma' - C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right)V_-(s)$$

$$= -i\omega\varepsilon_0\frac{1}{\pi}\left\{-\ln(kh) + \Omega_{\rho_-}/2 - \gamma' - C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right)V_-(s)$$

$$= -i\omega\varepsilon_0\frac{1}{\pi}\left\{-\ln(kh) + \Omega_{e\rho_-}/2 - C_e/2 - \gamma' - C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right)V_-(s)$$

and with $\rho_- = a_e^0$

$$-i\omega\varepsilon_0\frac{1}{2\pi}\left\{-2\ln(kh) + \Omega_e^0 - C_e - 2\gamma' - 2C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right)V_-(s)$$

Obviously we can choose

$$2C_0 = -C_e - 2\gamma' - 2\ln(kh)$$

to reduce this to the transmission line term Ω_e^0 only, but we should note that this is a free parameter. As an additional check for $m \geq 1$ the second bracketed term is

$$[] \sim$$

$$\int_\delta^k \left\{\frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\}\left\{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_\delta^k \left\{\frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} + \frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$+\int_{\delta}^{\infty}\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{u^2b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{k}\int_0^{\delta}\{1-1\}\left\{-\frac{m}{k_tb}\right\}dk_t$$

$$+i\frac{\pi b^{2m-1}}{2^{2m-1}k\,(m-1)!^2}\int_0^{\delta}\{1+1\}\,k_t^{2m-1}dk_t$$

$$+\frac{1}{k}\int_0^{\delta}\left\{\frac{m}{ub}-1-\frac{m}{ub}\right\}du+C_0$$

or

$$[]\sim$$

$$\int_{\delta}^{k}\left\{\frac{1}{J_m^2(k_tb)+Y_m^2(k_tb)}-\frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2(k_tb)+Y_m'^2(k_tb)}\right\}\{J_m(k_tb)\,J_m'(k_tb)+Y_m(k_tb)\,Y_m'(k_tb)\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_{\delta}^{k}\left\{\frac{1}{J_m^2(k_tb)+Y_m^2(k_tb)}+\frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2(k_tb)+Y_m'^2(k_tb)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$+\int_{\delta}^{\infty}\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{u^2b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{du}{\sqrt{u^2+k^2}}-\delta/k+C_0$$

As an additional check for $m=0$ the second bracketed term is

$$[]=\int_{\delta}^{k}\left\{\frac{1}{J_0^2(k_tb)+Y_0^2(k_tb)}\right\}\{J_0(k_tb)\,J_0'(k_tb)+Y_0(k_tb)\,Y_0'(k_tb)\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_{\delta}^{k}\left\{\frac{1}{J_0^2(k_tb)+Y_0^2(k_tb)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}+\int_{\delta}^{\infty}\left\{\frac{K_1(ub)}{K_0(ub)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{2b}\int_0^{\delta}\frac{d}{dk_t}\ln\left\{J_0^2(k_tb)+Y_0^2(k_tb)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_0^{\delta}\left\{\frac{1}{J_0^2(k_tb)+Y_0^2(k_tb)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}+\int_0^{\delta}\left\{-\frac{1}{b}\frac{d}{du}\ln\left(K_0(ub)\right)-1\right\}\frac{du}{\sqrt{u^2+k^2}}+C_0$$

$$\sim\int_{\delta}^{k}\left\{\frac{1}{J_0^2(k_tb)+Y_0^2(k_tb)}\right\}\{J_0(k_tb)\,J_0'(k_tb)+Y_0(k_tb)\,Y_0'(k_tb)\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_{\delta}^{k}\left\{\frac{1}{J_0^2(k_tb)+Y_0^2(k_tb)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}+\int_{\delta}^{\infty}\left\{\frac{K_1(ub)}{K_0(ub)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{kb}\int_0^{\delta}\frac{d}{dk_t}\ln\sqrt{1+(2/\pi)^2\left(\ln(k_tb/2)+\gamma'\right)^2}dk_t$$

$$+i\frac{2}{\pi kb}\int_0^\delta\left\{\frac{1}{1+(2/\pi)^2\left(\ln\left(k_tb/2\right)+\gamma'\right)^2}\right\}\frac{dk_t}{k_t}-\frac{1}{kb}\int_0^\delta\frac{d}{du}\ln\left\{-\ln\left(ub/2\right)-\gamma'\right\}du-\delta/k+C_0$$

$$\sim\int_\delta^k\left\{\frac{1}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}\right\}\left\{J_0\left(k_tb\right)J_0'\left(k_tb\right)+Y_0\left(k_tb\right)Y_0'\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_\delta^k\left\{\frac{1}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}+\int_\delta^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{kb}\int_0^\delta\frac{d}{du}\left\{\ln\sqrt{1+(2/\pi)^2\ln^2\left(ube^{\gamma'}/2\right)}-\ln\left\{-\ln\left(ube^{\gamma'}/2\right)\right\}\right\}du$$

$$+i\frac{2}{\pi kb}\int_0^\delta\left\{\frac{1}{1+(2/\pi)^2\ln^2\left(k_tbe^{\gamma'}/2\right)}\right\}\frac{dk_t}{k_t}-\delta/k+C_0$$

$$\sim\int_\delta^k\left\{\frac{1}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}\right\}\left\{J_0\left(k_tb\right)J_0'\left(k_tb\right)+Y_0\left(k_tb\right)Y_0'\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_\delta^k\left\{\frac{1}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}+\int_\delta^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{kb}\left\{\ln\sqrt{1+(2/\pi)^2\ln^2\left(ube^{\gamma'}/2\right)}-\ln\left\{-\ln\left(ube^{\gamma'}/2\right)\right\}\right\}_0^\delta$$

$$+i\frac{2}{\pi kb}\int_{-(2/\pi)\ln\left(\delta be^{\gamma'}/2\right)}^\infty\frac{dv}{1+v^2}-\delta/k+C_0$$

$$v=-\left(2/\pi\right)\ln\left(k_tbe^{\gamma'}/2\right)\to dv=-dk_t/k_t$$

or

$$[]=\int_\delta^k\left\{\frac{1}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}\right\}\left\{J_0\left(k_tb\right)J_0'\left(k_tb\right)+Y_0\left(k_tb\right)Y_0'\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_\delta^k\left\{\frac{1}{J_0^2\left(k_tb\right)+Y_0^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}+\int_\delta^\infty\left\{\frac{K_1\left(ub\right)}{K_0\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{kb}\ln\left\{\frac{\sqrt{1+(2/\pi)^2\ln^2\left(\delta be^{\gamma'}/2\right)}}{-\left(2/\pi\right)\ln\left(\delta be^{\gamma'}/2\right)}\right\}+i\frac{1}{kb}\left\{1-\frac{2}{\pi}\arctan\left(-\left(2/\pi\right)\ln\left(\delta be^{\gamma'}/2\right)\right)\right\}-\delta/k+C_0$$

6 MAGNETIC FIELD REPRESENTATION SUMMARY & INTEGRO-DIFFERENTIAL EQUATION

We now summarize the preceding exterior and interior azimuthal magnetic field representations (in forms we derived from both gauges). We then enforce continuity of the interior azimuthal magnetic field (exterior representation plus short circuit field) and interior azimuthal magnetic field (interior representation) across the slot to set up the integro-differential equation for the slot voltage.

6.1 Exterior Coulomb Gauge Representation

The magnetic scalar potential contribution is

$$-\frac{1}{b}\frac{\partial}{\partial\varphi}\phi_m\left(b,\varphi,z_0+a_e^0\right)=-\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\sin\left(m\varphi\right)m_{os}^-+\cos\left(m\varphi\right)m_{es}^-\right]\int_0^\infty\frac{K_m\left(u\right)}{K_m'\left(u\right)}\cos\left(ua_e^0/b\right)\frac{du}{u}$$

$$\approx-\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\sin\left(m\varphi\right)m_{os}^-+\cos\left(m\varphi\right)m_{es}^-\right]\left\{C_m+\gamma'+\ln\left(a_e^0/b\right)\right\}$$

The azimuthal electric vector potential contribution is

$$i\omega A_{e\varphi}\left(b,\varphi,z_0+a_e^0\right)\approx-\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^\infty\left\{\frac{H_m^{(1)\prime}\left(k_tb\right)}{H_m^{(1)}\left(k_tb\right)}-i\left(1-\frac{m^2}{k^2b^2}\right)-m^2\left(1-k_t^2/k^2\right)\frac{H_m^{(1)}\left(k_tb\right)}{\left(k_tb\right)^2H_m^{(1)\prime}\left(k_tb\right)}\right\}\frac{d\alpha}{k_t}\right.$$

$$\left.+\left(1-\frac{m^2}{k^2b^2}\right)\left\{i\frac{\pi}{2}-\ln\left(kh\right)-\gamma'+\Omega_e^0\right\}\right]$$

$$-i\omega\varepsilon_0\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\frac{\Omega_e^0}{2\pi}V_-\left(s\right)$$

$$\Omega_e^0=\Omega_{a_e^0}+C_e$$

$$\Omega_{a_e^0}=2\ln\left(2h/a_e^0\right)$$

6.2 Exterior Lorentz Gauge Representation

The exterior representation using the Lorentz gauge is

$$H_\varphi\left(b,\varphi,z_0+a_e^0\right)=-i\omega\varepsilon_0\frac{\Omega_e^0}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)-i\omega\varepsilon_0\left\{-2\ln\left(kh\right)-C_e-2\gamma'-2C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\frac{1}{2\pi}V_-\left(s\right)$$

$$+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k \left\{-\frac{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\} \frac{dk_t}{\sqrt{k^2-k_t^2}} + i\frac{2}{\pi b}\int_0^k \left\{\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\} \frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left. + \int_0^\infty \left\{-\frac{K_m(ub)}{K_m'(ub)} - 1\right\} \frac{du}{\sqrt{u^2+k^2}} + C_0\right]$$

$$+\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\, m_{es}^- + \sin(m\varphi)\, m_{os}^-\right]$$

$$\left[\int_0^k \left\{\frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\}\left\{J_m(k_t b)\, J_m'(k_t b) + Y_m(k_t b)\, Y_m'(k_t b)\right\} \frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k \left\{\frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} + \frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)}\right\} \frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left. + \int_0^\infty \left\{-\frac{K_m'(ub)}{K_m(ub)} - 1 + \frac{m^2}{u^2 b^2}\frac{K_m(ub)}{K_m'(ub)}\right\} \frac{du}{\sqrt{u^2+k^2}} + C_0\right]$$

where C_0 is arbitrary and

$$\Omega_e^0 = \Omega_{a_e^0} + C_e$$

$$\Omega_{a_e^0} = 2\ln\left(2h/a_e^0\right)$$

$$a_e^0 \sim \frac{2w}{\pi e}$$

$$m_{es}^- = \frac{-i2}{\omega\mu_0}\int_{-h}^{h} \cos(ms'/b)\, V_-(s')\, ds'$$

$$m_{os}^- = \frac{-i2}{\omega\mu_0}\int_{-h}^{h} \sin(ms'/b)\, V_-(s')\, ds'$$

$$-i\omega\varepsilon_0\frac{\Omega_e^0}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-(s) \to -i\omega\varepsilon_0\frac{\Omega_e}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-(s) - \frac{1}{2}\left(\Delta Y_L\frac{\partial^2}{\partial s^2} - \Delta Y_C\right)V_-(s)$$

For $m=0$ the second bracketed term is broken up as (where $\delta b << 1$)

$$[] = \int_\delta^k \left\{\frac{1}{J_0^2(k_t b) + Y_0^2(k_t b)}\right\}\left\{J_0(k_t b)\, J_0'(k_t b) + Y_0(k_t b)\, Y_0'(k_t b)\right\} \frac{dk_t}{\sqrt{k^2-k_t^2}}$$

$$+i\frac{2}{\pi b}\int_\delta^k \left\{\frac{1}{J_0^2(k_t b) + Y_0^2(k_t b)}\right\} \frac{dk_t}{k_t\sqrt{k^2-k_t^2}} + \int_\delta^\infty \left\{\frac{K_1(ub)}{K_0(ub)} - 1\right\} \frac{du}{\sqrt{u^2+k^2}}$$

$$+\frac{1}{kb}\ln\left\{\frac{\sqrt{1+(2/\pi)^2\ln^2\left(\delta b e^{\gamma'}/2\right)}}{-(2/\pi)\ln\left(\delta b e^{\gamma'}/2\right)}\right\} + i\frac{1}{kb}\left\{1-\frac{2}{\pi}\arctan\left(-(2/\pi)\ln\left(\delta b e^{\gamma'}/2\right)\right)\right\} - \delta/k + C_0$$

6.3 Half Space Exterior Approximation

The exterior half space representation to the magnetic field along the slot is [18], [1], [9] (an accurate approximation the slot length is small compared to the cylinder half perimeter $\ell << \pi a$)

$$H_s^{>}\left(a_{eq}^{-}, s\right) = \frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2} + k^2\right)\int_{-h}^{h} I_m^{-}\left(s'\right)\frac{e^{ik\sqrt{a_{eq}^2+(s-s')^2}}}{4\pi\sqrt{a_{eq}^2+\left(s-s'\right)^2}}ds'$$

$$= -\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2} + k^2\right)\int_{-h}^{h} V_{-}\left(s'\right)\frac{e^{ik\sqrt{a_{eq}^2+(s-s')^2}}}{2\pi\sqrt{a_{eq}^2+\left(s-s'\right)^2}}ds'$$

where

$$I_m^{\pm}\left(s\right) = \pm 2V_{\pm}\left(s\right)$$

with equivalent slot radius [9]

$$a_{eq} \sim \frac{2w}{\pi e}e^{-\pi d/(2w)}\ ,\ d > w/3$$

or if we do not include the slot interior

$$a_{eq}^0 \sim \frac{2w}{\pi e}\ ,\ d > w/3$$

6.3.1 Transmission Line Approximation

If we use a first order Hallen-type approximate form for the local integrals

$$\int_{-h}^{h} I_m^{\pm}\left(s'\right)\frac{e^{ik\sqrt{a_{eq}^2+(s-s')^2}}}{4\pi\sqrt{a_{eq}^2+\left(s-s'\right)^2}}ds' \approx \frac{\Omega_e}{4\pi}I_m^{\pm}\left(s\right)$$

$$+\frac{1}{4\pi}\left\{-C_e + \ln\left(1 - s^2/h^2\right)\right\}I_m^{\pm}\left(s\right) + \int_{-h}^{h}\frac{I_m^{\pm}\left(s'\right)e^{ik\left|s-s'\right|} - I_m^{\pm}\left(s\right)}{4\pi\left|s-s'\right|}ds'$$

where

$$\Omega_e = 2\ln\left(2h/a_{eq}\right) + C_e$$

and if we choose to preserve the low frequency dipole moment of the slot we take

$$C_e = 2\left(\ln 2 - 7/3\right)$$

If we leave out the slot interior

$$\Omega_e^0 = 2\ln\left(2h/a_{eq}^0\right) + C_e$$

Therefore the half space transmission line approximation becomes

$$H_s^{>}\left(a_{eq}^{-}, s\right) \approx \frac{1}{2}\left(\frac{1}{-i\omega L}\frac{\partial^2}{\partial s^2} + i\omega C\right)\frac{1}{2}I_m^{-}\left(s\right)$$

$$+\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{4\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}I_m^-(s)+\int_{-h}^{h}\frac{I_m^-(s')\,e^{ik|s-s'|}-I_m^-(s)}{4\pi|s-s'|}ds'\right]$$

$$\approx -\frac{1}{2}\left(\frac{1}{-i\omega L}\frac{\partial^2}{\partial s^2}+i\omega C\right)V_-(s)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_-(s)+\int_{-h}^{h}\frac{V_-(s')\,e^{ik|s-s'|}-V_-(s)}{2\pi|s-s'|}ds'\right]$$

where

$$L=\mu_0\pi/\Omega_e$$

$$C=\varepsilon_0\Omega_e/\pi$$

If we leave out the slot interior

$$L^0=\mu_0\pi/\Omega_e^0$$

$$C^0=\varepsilon_0\Omega_e^0/\pi$$

6.4 Short Circuit Current Drive

The short circuit magnetic field drive on the exterior is

$$K_z^{sc}(\varphi,z_0)=H_\varphi^{sc}(b,\varphi,z_0)$$

$$=2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\cos m(\varphi-\varphi_i)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin m(\varphi-\varphi_i)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

$$k_{\rho_i}=k\sin\theta_i$$

$$k_{z_i}=k\cos\theta_i$$

6.5 Interior Coulomb Gauge Representation

In the Coulomb gauge

$$-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m(\rho,\varphi,z)=\frac{1}{a\rho}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos\left(n\pi z_0/h_c\right)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}\cos\left(n\pi z/h_c\right)\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}$$

$$\frac{1}{2}\left[\cos(m\varphi)\,m_{es}^{+}+\sin(m\varphi)\,m_{os}^{+}\right]$$

where

$$m_{es}^{+} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_{+}\left(s'\right)ds'$$

$$m_{os}^{+} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_{+}\left(s'\right)ds'$$

$$k_{pmn}'^{2} = \left(j_{m,p}'/a\right)^2 + \left(n\pi/h_c\right)^2$$

$$k_{pmn}^{2} = \left(j_{m,p}/a\right)^2 + \left(n\pi/h_c\right)^2$$

and

$$i\omega A_{e\varphi} = -\frac{k^2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}}\frac{J_m'\left(j_{m,p}\rho/a\right)}{J_{m-1}\left(j_{m,p}\right)}$$
$$\frac{1}{2}\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

$$-\frac{k^2}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_m\varepsilon_n}{\left\{k^2 + kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right\}}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\frac{\left(n\pi/h_c\right)^2}{k_{p,m,n}'^2}\frac{a}{\rho}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}$$
$$\frac{1}{2}\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]\cos\left(n\pi z_0/h_c\right)\cos\left(n\pi z/h_c\right)$$

We note that

$$i\omega A_{es}^{s}\left(\rho_{+},s\right) \sim i\omega\varepsilon_0\frac{\Omega_{e\rho_{+}}}{2\pi}V_{+}\left(s\right)$$

$$i\omega A_{es}^{s}\left(\rho_{+},s\right) = \frac{k^2}{2\pi a}\sum_{m}\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]\Omega_{e\rho_{+}}/2$$

and thus

$$i\omega\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)A_{es}^{s}\left(\rho_{+},s\right) = \frac{k^2}{2\pi a}\sum_{m}\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]\left(1-\frac{m^2}{k^2a^2}\right)\Omega_{e\rho_{+}}/2$$

The total interior field representation is then

$$H_{\varphi}\left(a,\varphi,z_0+a_e^0\right) \approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_{+}\left(s\right)\left\{\Omega_e^0 - C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$
$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+} + \sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$
$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)} + \frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\} + \frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2(n\pi z_0/h_c)\left\{\sum_{p=1}^{\infty}\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

where

$$\Omega_e^0=\Omega_{a_e^0}+C_e$$

$$\Omega_{a_e^0}=2\ln\left(2h/a_e^0\right)$$

$$i\omega\varepsilon_0\frac{\Omega_e^0}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)\to i\omega\varepsilon_0\frac{\Omega_e}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+(s)+\frac{1}{2}\left(\Delta Y_L\frac{\partial^2}{\partial s^2}-\Delta Y_C\right)V_+(s)$$

6.6 Interior Lorentz Gauge Representation

In the Lorentz gauge

$$-\frac{1}{\rho}\frac{\partial}{\partial\varphi}\phi_m(\rho,\varphi,z_0)$$

$$=-\frac{1}{a\rho}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos(n\pi z_0/h_c)}{\left\{k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}\left(1-m^2/j_{m,p}'^2\right)}\cos(n\pi z/h_c)\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}$$

$$\frac{1}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]$$

where

$$m_{es}^+=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos(ms'/a)\,V_+(s')\,ds'$$

$$m_{os}^+=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin(ms'/a)\,V_+(s')\,ds'$$

and we used

$$j_{m,p}^2/a^2=k_{p,m,n}^2-(n\pi/h_c)^2$$

$$j_{m,p}'^2/a^2=k_{p,m,n}'^2-(n\pi/h_c)^2$$

Therefore going to the Lorentz gauge changes the TE mode factor

$$i\omega A_{e\varphi}=i\omega\left(A_{e\varphi}^C+\frac{1}{\rho}\frac{\partial}{\partial\varphi}\psi\right)$$

$$= -\frac{k^2}{V_{cav}} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{m=0}^{\infty} \frac{\varepsilon_m \varepsilon_n}{\left\{k^2 + k k_{p,m,n} (1+i) / Q^{TM}_{p,m,n} - k^2_{p,m,n}\right\}} \frac{J'_m (j_{m,p} \rho / a)}{J_{m-1}(j_{m,p})}$$

$$\frac{1}{2} \left[\cos(m\varphi) m^+_{es} + \sin(m\varphi) m^+_{os}\right] \cos(n\pi z_0 / h_c) \cos(n\pi z / h_c)$$

$$-\frac{k^2}{V_{cav}} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \sum_{m=0}^{\infty} \frac{\varepsilon_m \varepsilon_n}{\left\{k^2 + k k'_{p,m,n} (1+i) / Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\}} \frac{m^2 / j'^2_{m,p}}{\left(1 - m^2 / j'^2_{m,p}\right)} \frac{a}{\rho} \frac{J_m \left(j'_{m,p} \rho / a\right)}{J_m \left(j'_{m,p}\right)}$$

$$\frac{1}{2} \left[\cos(m\varphi) m^+_{es} + \sin(m\varphi) m^+_{os}\right] \cos(n\pi z_0 / h_c) \cos(n\pi z / h_c)$$

The total field representation is then

$$H_\varphi \left(a, \varphi, z_0 + a^0_e\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi} \left(\frac{1}{k^2} \frac{\partial^2}{\partial s^2} + 1\right) V_+(s) \left\{\Omega^0_e - C_e + 2\ln\left(\frac{h_c}{h\pi} \csc(\pi z_0 / h_c)\right)\right\}$$

$$-\frac{k^2}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2} \left[\cos(m\varphi) m^+_{es} + \sin(m\varphi) m^+_{os}\right] \sum_{n=0}^{\infty} \varepsilon_n \cos^2(n\pi z_0 / h_c)$$

$$\left[\sum_{p=1}^{\infty} \left\{\frac{1}{\left(k^2 - k^2_{p,m,n}\right)} + \frac{1}{\left(k^2 - k'^2_{p,m,n}\right)} \frac{m^2 / j'^2_{m,p}}{\left(1 - m^2 / j'^2_{m,p}\right)}\right\} + \frac{a(1 - \delta_{n0})}{2n\pi / h_c}\right]$$

$$+\frac{1}{a^2} \frac{\partial^2}{\partial \varphi^2} \frac{1}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2} \left[\cos(m\varphi) m^+_{es} + \sin(m\varphi) m^+_{os}\right]$$

$$\sum_{n=0}^{\infty} \varepsilon_n \cos^2(n\pi z_0 / h_c) \left\{\sum_{p=1}^{\infty} \frac{1}{\left(k'^2_{pmn} - k^2\right)\left(1 - m^2 / j'^2_{m,p}\right)} - \frac{a(1 - \delta_{n0})}{2n\pi / h_c}\right\}$$

$$\Omega^0_e = \Omega_{a^0_e} + C_e$$

$$\Omega_{a^0_e} = 2\ln\left(2h / a^0_e\right)$$

$$i\omega\varepsilon_0 \frac{\Omega^0_e}{2\pi} \left(\frac{1}{k^2} \frac{\partial^2}{\partial s^2} + 1\right) V_+(s) \to i\omega\varepsilon_0 \frac{\Omega_e}{2\pi} \left(\frac{1}{k^2} \frac{\partial^2}{\partial s^2} + 1\right) V_+(s) + \frac{1}{2} \left(\Delta Y_L \frac{\partial^2}{\partial s^2} - \Delta Y_C\right) V_+(s)$$

Noting that

$$\frac{1}{\left(k^2 - k^2_{p,m,n}\right)} + \frac{1}{\left(k^2 - k'^2_{p,m,n}\right)} \frac{m^2 / j'^2_{m,p}}{\left(1 - m^2 / j'^2_{m,p}\right)} = \frac{1}{\left(k^2 - k^2_{p,m,n}\right)} + \frac{1}{\left(k^2 - k'^2_{p,m,n}\right)} \left\{\frac{1}{\left(1 - m^2 / j'^2_{m,p}\right)} - 1\right\}$$

we can rewrite the field as

$$H_\varphi \left(a, \varphi, z_0 + a^0_e\right) \approx i\omega\varepsilon_0 \frac{1}{2\pi} \left(\frac{1}{k^2} \frac{\partial^2}{\partial s^2} + 1\right) V_+(s) \left\{\Omega^0_e - C_e + 2\ln\left(\frac{h_c}{h\pi} \csc(\pi z_0 / h_c)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$+\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

where for finite quality factors we replace

$$k^2-k_{p,m,n}^2\rightarrow k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2$$

$$k^2-k_{p,m,n}'^2\rightarrow k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2$$

6.6.1 Check Returning To Coulomb Form

Taking the k^2 term from the second summation term back

$$H_\varphi\left(a,\varphi,z_0+a_e^0\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_e^0-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$+\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

gives

$$H_\varphi\left(a,\varphi,z_0+a_e^0\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_e^0-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\frac{m^2/j_{m,p}'^2}{1-m^2/j_{m,p}'^2}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$-\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k^2-k'^2_{pmn}\right)\left(1-m^2/j'^2_{m,p}\right)}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

Now using

$$\frac{k^2}{\left(k^2-k'^2_{p,m,n}\right)k'^2_{p,m,n}}+\frac{1}{\left(k'^2_{p,m,n}-k^2\right)}=\frac{1}{k'^2_{p,m,n}}$$

gives

$$H_\varphi\left(a,\varphi,z_0+a_e^0\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_e^0-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k^2_{p,m,n}\right)}+\frac{1}{\left(k^2-k'^2_{p,m,n}\right)}\frac{m^2/j'^2_{m,p}}{1-m^2/j'^2_{m,p}}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$-\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\frac{1}{\left(1-m^2/j'^2_{m,p}\right)}\left\{\frac{k^2}{\left(k^2-k'^2_{p,m,n}\right)k'^2_{p,m,n}}-\frac{1}{k'^2_{p,m,n}}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

or combining this term with the first sum and using $k'^2_{p,m,n}-j'^2_{m,p}/a^2=n^2\pi^2/h_c^2$

$$H_\varphi\left(a,\varphi,z_0+a_e^0\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_e^0-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k^2_{p,m,n}\right)}+\frac{\left(n\pi/h_c\right)^2}{k'^2_{p,m,n}\left(k^2-k'^2_{p,m,n}\right)}\frac{m^2/j'^2_{m,p}}{1-m^2/j'^2_{m,p}}\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$+\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left\{\sum_{p=1}^{\infty}\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

6.7 Integro-Differential Equation

Now, in summary we note that we have exterior azimuthal magnetic field representations, for a φ directed exterior magnetic current $I_m^- = -2V_-$ (with continuity equation $dI_m^-/ds = i\omega q_m^-$ and exterior arc length $s = b\varphi$, the electric field around the magnetic current is consistent with the axial field being $-z$ directed for a positive exterior voltage V_- on the positive z side of the slot), generated by the exterior surface electric field matching

$$\varepsilon_0 E_z = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_{e\varphi}\right) + \frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\rho} \approx -\varepsilon_0 V_-\left(s\right)\delta\left(z-z_0\right)\ \text{, exterior}$$

and interior azimuthal magnetic field representations (with interior voltage $V_+\left(s\right)$ taken to be positive on the positive z side of the narrow slot with $I_m^+ = 2V_+$ being φ directed), generated by the interior magnetic current source (where the factor of one half is present because the image in the conductive wall is accomplished through the boundary value problem solution)

$$J_{m\varphi} = \frac{1}{2}I_m^+\left(s\right)\delta\left(z-z_0\right)\delta\left(\rho-a\right) = V_+\left(s\right)\delta\left(z-z_0\right)\delta\left(\rho-a\right)\ ,\ \ -h<s<h\ ,\ \text{interior}$$

in addition to a short circuit exterior azimuthal magnetic field representation. In the Thick slot case the two voltages $V_\pm\left(s\right)$ are taken to be approximately the same, and then the matching of the interior and exterior azimuthal magnetic fields results in the required integro-differential equation.

6.7.1 Integro-Differential Equation Using Interior Lorentz Gauge

The matching of the magnetic field results in the equation (here b is the exterior radius and a is the interior radius, and we choose the previously arbitrary constant $C_0 = 0$, which defines the first-order term in the transmission line operator)

$$H_\varphi\left(b,\varphi,z_0+a_e^0\right) - H_\varphi\left(a,\varphi,z_0+a_e^0\right) = -i\omega\varepsilon_0\frac{\Omega_e^0}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\left\{V_-\left(s\right)+V_+\left(s\right)\right\}$$

$$-i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\} - i\omega\varepsilon_0\left\{-2\ln\left(kh\right)-C_e-2\gamma'\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\frac{1}{2\pi}V_-\left(s\right)$$

$$+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^- + \sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{-\frac{J_m\left(k_tb\right)J_m'\left(k_tb\right)+Y_m\left(k_tb\right)Y_m'\left(k_tb\right)}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}} + i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left.+\int_0^\infty\left\{-\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^- + \sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{1}{J_m^2\left(k_tb\right)+Y_m^2\left(k_tb\right)} - \frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\left\{J_m\left(k_tb\right)J_m'\left(k_tb\right)+Y_m\left(k_tb\right)Y_m'\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k \left\{\frac{1}{J_m^2\left(k_t b\right)+Y_m^2\left(k_t b\right)}+\frac{m^2}{k_t^2 b^2}\frac{1}{J_m'^2\left(k_t b\right)+Y_m'^2\left(k_t b\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$+\int_0^\infty \left\{-\frac{K_m'\left(ub\right)}{K_m\left(ub\right)}-1+\frac{m^2}{u^2b^2}\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}\Bigg]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=-H_\varphi^{sc}\left(b,\varphi,z_0\right)=-2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\cos m\left(\varphi-\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin m\left(\varphi-\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

Now ignoring the difference between the two voltages for the Thick case with

$$-i\omega\varepsilon_0\frac{\Omega_e}{\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V\left(s\right)-\left(\Delta Y_L\frac{\partial^2}{\partial s^2}-\Delta Y_C\right)V\left(s\right)=-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V\left(s\right)$$

$$1/Z=\Delta Y_L+\frac{1}{-i\omega\mu_0\pi/\Omega_e}$$

$$Y=\Delta Y_C-i\omega\varepsilon_0\Omega_e/\pi$$

gives

$$-i\omega\varepsilon_0\frac{\Omega_e^0}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)\to-i\omega\varepsilon_0\frac{\Omega_e}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)-\frac{1}{2}\left(\Delta Y_L\frac{\partial^2}{\partial s^2}-\Delta Y_C\right)V_-\left(s\right)$$

Note that the dominant operator contributions to the exterior and interior magnetic fields are

$$H_\varphi\left(b,\varphi,z_0+\rho_-\right)\approx-i\omega\varepsilon_0\frac{1}{2\pi}\left\{-2\ln\left(kh\right)+\Omega_{e\rho_-}-C_e-2\gamma'-2C_0\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)$$

$$H_\varphi\left(a,\varphi,z_0+\rho_+\right)\approx i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{\Omega_{e\rho_+}-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

so that

$$H_\varphi\left(b,\varphi,z_0+\rho_-\right)-H_\varphi\left(a,\varphi,z_0+\rho_+\right)$$

$$\approx-i\omega\varepsilon_0\frac{1}{2\pi}\left\{-2\ln\left(kh\right)+\Omega_{e\rho_-}+\Omega_{e\rho_+}-2C_e-2\gamma'-2C_0+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-\left(s\right)$$

$$\approx -i\omega\varepsilon_0 \frac{1}{\pi}\left\{-\ln(kh)+\Omega_e^0 - C_e - \gamma' - C_0 + \ln\left(\frac{h_c}{h\pi}\csc(\pi z_0/h_c)\right)\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_-(s)$$

and for convenience we set $C_0 = 0$, so we have

$$H_\varphi\left(b,\varphi,z_0+a_e^0\right) - H_\varphi\left(a,\varphi,z_0+a_e^0\right) = -\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2} - Y\right)V(s)$$

$$-i\omega\varepsilon_0\frac{1}{\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V(s)\left\{-C_e-\gamma'-\ln(kh)+\ln\left(\frac{h_c}{h\pi}\csc(\pi z_0/h_c)\right)\right\}$$

$$+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^k\left\{-\frac{J_m(k_tb)\,J_m'(k_tb)+Y_m(k_tb)\,Y_m'(k_tb)}{J_m'^2(k_tb)+Y_m'^2(k_tb)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}} + i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m'^2(k_tb)+Y_m'^2(k_tb)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left.+\int_0^\infty\left\{-\frac{K_m(ub)}{K_m'(ub)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{2\pi b}\sum_m \frac{\varepsilon_m}{2\pi}\left[\cos(m\varphi)\,m_{es}^- + \sin(m\varphi)\,m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{1}{J_m^2(k_tb)+Y_m^2(k_tb)} - \frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2(k_tb)+Y_m'^2(k_tb)}\right\}\left\{J_m(k_tb)\,J_m'(k_tb)+Y_m(k_tb)\,Y_m'(k_tb)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m^2(k_tb)+Y_m^2(k_tb)} + \frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2(k_tb)+Y_m'^2(k_tb)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left.+\int_0^\infty\left\{-\frac{K_m'(ub)}{K_m(ub)}-1+\frac{m^2}{u^2b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^\infty\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]\sum_{n=0}^\infty \varepsilon_n\cos^2(n\pi z_0/h_c)\sum_{p=1}^\infty\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)} - \frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^\infty\frac{\varepsilon_m}{2}\left[\cos(m\varphi)\,m_{es}^+ + \sin(m\varphi)\,m_{os}^+\right]$$

$$\sum_{n=0}^\infty\varepsilon_n\cos^2(n\pi z_0/h_c)\left\{\sum_{p=1}^\infty\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)} - \frac{a\,(1-\delta_{n0})}{2n\pi/h_c}\right\}$$

$$= -H_\varphi^{sc}(b,\varphi,z_0) = -2H_0e^{ik_{z_i}z_0}\sum_{m=0}^\infty\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\cos m(\varphi-\varphi_i)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)} + \frac{m\sin m(\varphi-\varphi_i)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

where for finite quality factors we replace

$$k^2 - k_{p,m,n}^2 \to k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2$$

$$k^2 - k_{p,m,n}^{\prime 2} \to k^2 + kk_{p,m,n}^{\prime}\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}^{\prime 2}$$

Note that we can take the limit

$$\lim_{k\to k_{p,m,n}^{\prime}}\left[-\frac{k^2}{\left(k^2-k_{p,m,n}^{\prime 2}\right)} - \frac{k^2-m^2/a^2}{\left(k_{pmn}^{\prime 2}-k^2\right)\left(1-m^2/j_{m,p}^{\prime 2}\right)}\right]$$

$$= \lim_{k\to k_{p,m,n}^{\prime}}\left[-\frac{k^2}{\left(k^2-k_{p,m,n}^{\prime 2}\right)} - \frac{k^2\left(1-m^2/j_{m,p}^{\prime 2}\right)+k^2m^2/j_{m,p}^{\prime 2}-m^2/a^2}{\left(k_{pmn}^{\prime 2}-k^2\right)\left(1-m^2/j_{m,p}^{\prime 2}\right)}\right]$$

$$= \lim_{k\to k_{p,m,n}^{\prime}}\left[-\frac{k^2-j_{m,p}^{\prime 2}/a^2}{\left(k_{pmn}^{\prime 2}-k^2\right)}\right]\frac{\left(m^2/j_{m,p}^{\prime 2}\right)}{\left(1-m^2/j_{m,p}^{\prime 2}\right)} = \lim_{k\to k_{p,m,n}^{\prime}}\left[-\frac{k^2-k_{p,m,n}^{\prime 2}+k_{p,m,n}^{\prime 2}-j_{m,p}^{\prime 2}/a^2}{\left(k_{pmn}^{\prime 2}-k^2\right)}\right]\frac{\left(m^2/j_{m,p}^{\prime 2}\right)}{\left(1-m^2/j_{m,p}^{\prime 2}\right)}$$

$$= \lim_{k\to k_{p,m,n}^{\prime}}\left[1-\frac{n^2\pi^2/h_c^2}{\left(k_{pmn}^{\prime 2}-k^2\right)}\right]\frac{\left(m^2/j_{m,p}^{\prime 2}\right)}{\left(1-m^2/j_{m,p}^{\prime 2}\right)}$$

where we used

$$k_{p,m,n}^{\prime 2} - n^2\pi^2/h_c^2 = j_{m,p}^{\prime 2}/a^2$$

This shows that the singularity at $k \to k_{p,m,n}$ is absent for $n = 0$ (as clear in the Coulomb representation).

We can also redefine

$$-\left(\frac{1}{\widetilde{Z}}\frac{\partial^2}{\partial s^2} - \widetilde{Y}\right)V\left(s\right) = -\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2} - Y\right)V\left(s\right)$$

$$-i\omega\varepsilon_0\frac{1}{\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V\left(s\right)\left\{-C_e-\gamma'-\ln\left(kh\right)+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

or

$$1/\widetilde{Z} = 1/Z + \frac{1}{-i\omega\mu_0}\frac{1}{\pi}\left\{-C_e-\gamma'-\ln\left(kh\right)+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$\widetilde{Y} = Y - i\omega\varepsilon_0\frac{1}{\pi}\left\{-C_e-\gamma'-\ln\left(kh\right)+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

6.7.2 Integro-Differential Equation Using Interior Coulomb Gauge

The matching of the magnetic field results in the equation

$$H_\varphi\left(b,\varphi,z_0+a_e^0\right)-H_\varphi\left(a,\varphi,z_0+a_e^0\right)=-i\omega\varepsilon_0\frac{\Omega_e^0}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\left\{V_-\left(s\right)+V_+\left(s\right)\right\}$$

$$-i\omega\varepsilon_0\frac{1}{2\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V_+\left(s\right)\left\{-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}+i\omega\varepsilon_0\left\{C_e+2\gamma'+2\ln\left(kh\right)\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\frac{1}{2\pi}V_-\left(s\right)$$

$$+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{-\frac{J_m\left(k_tb\right)J_m'\left(k_tb\right)+Y_m\left(k_tb\right)Y_m'\left(k_tb\right)}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left.+\int_0^\infty\left\{-\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{1}{J_m^2\left(k_tb\right)+Y_m^2\left(k_tb\right)}-\frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\left\{J_m\left(k_tb\right)J_m'\left(k_tb\right)+Y_m\left(k_tb\right)Y_m'\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m^2\left(k_tb\right)+Y_m^2\left(k_tb\right)}+\frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left.+\int_0^\infty\left\{-\frac{K_m'\left(ub\right)}{K_m\left(ub\right)}-1+\frac{m^2}{u^2b^2}\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^\infty\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^++\sin\left(m\varphi\right)m_{os}^+\right]\sum_{n=0}^\infty\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^\infty\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$-\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^\infty\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^++\sin\left(m\varphi\right)m_{os}^+\right]$$

$$\sum_{n=0}^\infty\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^\infty\frac{1}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=-H_\varphi^{sc}\left(b,\varphi,z_0\right)=-2H_0e^{ik_{z_i}z_0}\sum_{m=0}^\infty\frac{\varepsilon_mi^m}{\pi k_{\rho_i}b}\left[-\frac{\cos m\left(\varphi-\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin m\left(\varphi-\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

or using

$$-\left(\frac{1}{\widetilde{Z}}\frac{\partial^2}{\partial s^2}-\widetilde{Y}\right)V\left(s\right)=-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V\left(s\right)$$

$$-i\omega\varepsilon_0\frac{1}{\pi}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V\left(s\right)\left\{-C_e-\gamma'-\ln\left(kh\right)+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$=-i\omega\varepsilon_0\frac{1}{\pi}\left\{\Omega_e-C_e-\gamma'-\ln\left(kh\right)+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)V\left(s\right)$$

we find

$$-\left(\frac{1}{\widetilde{Z}}\frac{\partial^2}{\partial s^2}-\widetilde{Y}\right)V\left(s\right)+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{-\frac{J_m\left(k_tb\right)J'_m\left(k_tb\right)+Y_m\left(k_tb\right)Y'_m\left(k_tb\right)}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left.+\int_0^\infty\left\{-\frac{K_m\left(ub\right)}{K'_m\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{2\pi b}\sum_m\frac{\varepsilon_m}{2\pi}\left[\cos\left(m\varphi\right)m_{es}^-+\sin\left(m\varphi\right)m_{os}^-\right]$$

$$\left[\int_0^k\left\{\frac{1}{J^2_m\left(k_tb\right)+Y^2_m\left(k_tb\right)}-\frac{m^2}{k_t^2b^2}\frac{1}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\left\{J_m\left(k_tb\right)J'_m\left(k_tb\right)+Y_m\left(k_tb\right)Y'_m\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J^2_m\left(k_tb\right)+Y^2_m\left(k_tb\right)}+\frac{m^2}{k_t^2b^2}\frac{1}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left.+\int_0^\infty\left\{-\frac{K'_m\left(ub\right)}{K_m\left(ub\right)}-1+\frac{m^2}{u^2b^2}\frac{K_m\left(ub\right)}{K'_m\left(ub\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^++\sin\left(m\varphi\right)m_{os}^+\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k'^2_{p,m,n}\right)k'^2_{p,m,n}}\frac{m^2/j'^2_{m,p}}{\left(1-m^2/j'^2_{m,p}\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$-\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^++\sin\left(m\varphi\right)m_{os}^+\right]$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{k'^2_{pmn}\left(1-m^2/j'^2_{m,p}\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$= -2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[-\frac{\cos m\left(\varphi - \varphi_i\right) \cos \varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m \sin m\left(\varphi - \varphi_i\right) \cos \theta_i \sin \varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right]$$

where for finite quality factors we replace

$$k^2 - k_{p,m,n}^2 \to k^2 + k k_{p,m,n} \left(1 + i\right) / Q_{p,m,n}^{TM} - k_{p,m,n}^2$$

$$k^2 - k_{p,m,n}^{\prime 2} \to k^2 + k k_{p,m,n}^{\prime} \left(1 + i\right) / Q_{p,m,n}^{TE} - k_{p,m,n}^{\prime 2}$$

6.7.3 Integro-Differential Equation With Half Space Exterior

The matching of the magnetic field results in the equation

$$H_\varphi\left(b, \varphi, z_0 + a_e^0\right) - H_\varphi\left(a, \varphi, z_0 + a_e^0\right) = -\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2} - Y\right) V\left(s\right)$$

$$-i\omega\varepsilon_0 \left\{ -C_e + 2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0 / h_c\right)\right) \right\} \left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2} + 1\right) \frac{1}{2\pi} V\left(s\right)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2} + k^2\right)\left[\frac{1}{2\pi}\left\{-C_e + \ln\left(1 - s^2/h^2\right)\right\} V\left(s\right) + \int_{-h}^{h} \frac{V\left(s'\right) e^{ik\left|s - s'\right|} - V\left(s\right)}{2\pi\left|s - s'\right|} ds'\right]$$

$$+\frac{k^2}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2} \left[\cos\left(m\varphi\right) m_{es}^{+} + \sin\left(m\varphi\right) m_{os}^{+}\right] \sum_{n=0}^{\infty} \varepsilon_n \cos^2\left(n\pi z_0 / h_c\right) \sum_{p=1}^{\infty} \left\{ \frac{1}{\left(k^2 - k_{p,m,n}^2\right)} - \frac{1}{\left(k^2 - k_{p,m,n}^{\prime 2}\right)} \right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2} + k^2\right)\frac{1}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right) m_{es}^{+} + \sin\left(m\varphi\right) m_{os}^{+}\right]$$

$$\sum_{n=0}^{\infty} \varepsilon_n \cos^2\left(n\pi z_0 / h_c\right) \left\{ \sum_{p=1}^{\infty} \frac{1}{\left(k_{pmn}^{\prime 2} - k^2\right)\left(1 - m^2 / j_{m,p}^{\prime 2}\right)} - \frac{a\left(1 - \delta_{n0}\right)}{2n\pi / h_c} \right\}$$

$$= -H_\varphi^{sc}\left(b, \varphi, z_0\right) = -2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[-\frac{\cos m\left(\varphi - \varphi_i\right) \cos \varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m \sin m\left(\varphi - \varphi_i\right) \cos \theta_i \sin \varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right]$$

where for finite quality factors we replace

$$k^2 - k_{p,m,n}^2 \to k^2 + k k_{p,m,n} \left(1 + i\right) / Q_{p,m,n}^{TM} - k_{p,m,n}^2$$

$$k^2 - k_{p,m,n}^{\prime 2} \to k^2 + k k_{p,m,n}^{\prime} \left(1 + i\right) / Q_{p,m,n}^{TE} - k_{p,m,n}^{\prime 2}$$

6.7.4 Integro-Differential Equation With Half Space Exterior & Approximate Quasi-static Planar Interior Component

The matching of the magnetic field results in the equation

$$H_\varphi\left(b,\varphi,z_0+a_e^0\right)-H_\varphi\left(a,\varphi,z_0+a_e^0\right)=-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V\left(s\right)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V\left(s\right)+\int_{-h}^{h}\frac{V\left(s'\right)e^{ik\left|s-s'\right|}-V\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$-\frac{i}{\omega\mu_0}\frac{k^2}{2\pi}V\left(s\right)\left\{-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{i}{\omega\mu_0}\frac{\partial^2}{\partial s^2}\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V\left(s\right)+\int_{-h}^{h}\frac{V\left(s'\right)-V\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\left[\cos\left(m\varphi\right)m_{es}^{+}+\sin\left(m\varphi\right)m_{os}^{+}\right]\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$=-H_\varphi^{sc}\left(b,\varphi,z_0\right)=-2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\cos m\left(\varphi-\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin m\left(\varphi-\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

where for finite quality factors we replace

$$k^2-k_{p,m,n}^2\rightarrow k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2$$

$$k^2-k_{p,m,n}'^2\rightarrow k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2$$

and

$$1/Z=\Delta Y_L+\frac{1}{-i\omega\mu_0\pi/\Omega_e}$$

$$Y=\Delta Y_C-i\omega\varepsilon_0\Omega_e/\pi$$

$$C=\varepsilon_0\Omega_e/\pi\ ,\ 1/L=1/\left(\mu_0\pi/\Omega_e\right)$$

6.8 Even-Odd Equations

We can also break up the integro-differential equation into an even equation

$$-\left(\frac{1}{\widetilde{Z}}\frac{\partial^2}{\partial s^2}-\widetilde{Y}\right)V_e\left(s\right)$$

$$+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2\pi}\cos\left(m\varphi\right)m_{es}^{-}$$

$$\left[\int_0^k\left\{-\frac{J_m\left(k_tb\right)J_m'\left(k_tb\right)+Y_m\left(k_tb\right)Y_m'\left(k_tb\right)}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left.+\int_0^{\infty}\left\{-\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{2\pi b}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2\pi}\cos\left(m\varphi\right)m_{es}^{-}$$

$$\left[\int_0^k\left\{\frac{1}{J_m^2\left(k_tb\right)+Y_m^2\left(k_tb\right)}-\frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\left\{J_m\left(k_tb\right)J_m'\left(k_tb\right)+Y_m\left(k_tb\right)Y_m'\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J_m^2\left(k_tb\right)+Y_m^2\left(k_tb\right)}+\frac{m^2}{k_t^2b^2}\frac{1}{J_m'^2\left(k_tb\right)+Y_m'^2\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left.+\int_0^{\infty}\left\{-\frac{K_m'\left(ub\right)}{K_m\left(ub\right)}-1+\frac{m^2}{u^2b^2}\frac{K_m\left(ub\right)}{K_m'\left(ub\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\cos\left(m\varphi\right)m_{es}^{+}\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\cos\left(m\varphi\right)m_{es}^{+}$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[\frac{\cos\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\cos\left(m\varphi\right)$$

$$m_{es}^{+}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_{+}\left(s'\right)ds'=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_e\left(s'\right)ds'$$

$$m_{es}^{-}=-\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/b\right)V_{-}\left(s'\right)ds'=-\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/b\right)V_e\left(s'\right)ds'$$

and an odd equation

$$-\left(\frac{1}{\widetilde{\widetilde{Z}}}\frac{\partial^2}{\partial s^2}-\widetilde{Y}\right)V_o\left(s\right)$$

$$+\frac{1}{2\pi b}\frac{\partial^2}{\partial s^2}\sum_{m=1}^{\infty}\frac{1}{\pi}\sin\left(m\varphi\right)m_{os}^{-}$$

$$\left[\int_0^k\left\{-\frac{J_m\left(k_tb\right)J'_m\left(k_tb\right)+Y_m\left(k_tb\right)Y'_m\left(k_tb\right)}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}\right.$$

$$\left.+\int_0^{\infty}\left\{-\frac{K_m\left(ub\right)}{K'_m\left(ub\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{2\pi b}\sum_{m=1}^{\infty}\frac{1}{\pi}\sin\left(m\varphi\right)m_{os}^{-}$$

$$\left[\int_0^k\left\{\frac{1}{J^2_m\left(k_tb\right)+Y^2_m\left(k_tb\right)}-\frac{m^2}{k_t^2b^2}\frac{1}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\left\{J_m\left(k_tb\right)J'_m\left(k_tb\right)+Y_m\left(k_tb\right)Y'_m\left(k_tb\right)\right\}\frac{dk_t}{\sqrt{k^2-k_t^2}}\right.$$

$$+i\frac{2}{\pi b}\int_0^k\left\{\frac{1}{J^2_m\left(k_tb\right)+Y^2_m\left(k_tb\right)}+\frac{m^2}{k_t^2b^2}\frac{1}{J'^2_m\left(k_tb\right)+Y'^2_m\left(k_tb\right)}\right\}\frac{dk_t}{k_t\sqrt{k^2-k_t^2}}$$

$$\left.+\int_0^{\infty}\left\{-\frac{K'_m\left(ub\right)}{K_m\left(ub\right)}-1+\frac{m^2}{u^2b^2}\frac{K_m\left(ub\right)}{K'_m\left(ub\right)}\right\}\frac{du}{\sqrt{u^2+k^2}}\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=1}^{\infty}\sin\left(m\varphi\right)m_{os}^{+}\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k^2_{p,m,n}\right)}-\frac{1}{\left(k^2-k'^2_{p,m,n}\right)}\right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=1}^{\infty}\sin\left(m\varphi\right)m_{os}^{+}$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k'^2_{pmn}-k^2\right)\left(1-m^2/j'^2_{m,p}\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=-2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_mi^m}{\pi k_{\rho_i}b}\left[-\frac{\sin\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\cos\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\sin\left(m\varphi\right)$$

where for finite quality factors we replace

$$k^2-k^2_{p,m,n}\rightarrow k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}$$

$$k^2-k'^2_{p,m,n}\rightarrow k^2+kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}$$

and

$$m_{os}^{+}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_{+}\left(s'\right)ds'=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_{o}\left(s'\right)ds'$$

$$m_{os}^{-}=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/b\right)V_{+}\left(s'\right)ds'=\frac{-i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/b\right)V_{o}\left(s'\right)ds'$$

In order to implement this efficiently, we need to tabulate the double series interior terms and integral exterior terms, independent of the Galerkin moments (with indices m', m''), since these multipliers are common). These series multipliers will be dependent on m and will vary with k (so we only tabulate them for m at each frequency k).

6.8.1 Even-Odd Equations With Half Space Exterior

The even equation with half space exterior is

$$-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V_e\left(s\right)$$

$$-i\omega\varepsilon_0\left\{-C_e/2+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\frac{1}{\pi}V_e\left(s\right)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_e\left(s\right)+\int_{-h}^{h}\frac{V_e\left(s'\right)e^{ik\left|s-s'\right|}-V_e\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\cos\left(m\varphi\right)m_{es}^{+}\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}m_{es}^{+}\cos\left(m\varphi\right)$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[\frac{\cos\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\cos\left(m\varphi\right)$$

and odd equation

$$-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V_o\left(s\right)$$

$$-i\omega\varepsilon_0\left\{-C_e/2+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\frac{1}{\pi}V_o\left(s\right)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_o\left(s\right)+\int_{-h}^{h}\frac{V_o\left(s'\right)e^{ik\left|s-s'\right|}-V_o\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\sin\left(m\varphi\right)m_{os}^{+}\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\left(\frac{1}{a^2}\frac{\partial^2}{\partial\varphi^2}+k^2\right)\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}m_{os}^{+}\sin\left(m\varphi\right)$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=-2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\sin\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\cos\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\sin\left(m\varphi\right)$$

where for finite quality factors we replace

$$k^2-k_{p,m,n}^2\rightarrow k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2$$

$$k^2-k_{p,m,n}'^2\rightarrow k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2$$

6.8.2 Even-Odd Equations With Half Space Exterior & Approximate Quasi-static Interior Component

The even equation is

$$-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V_e\left(s\right)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_e\left(s\right)+\int_{-h}^{h}\frac{V_e\left(s'\right)e^{ik\left|s-s'\right|}-V_e\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$-\frac{i}{\omega\mu_0}\frac{k^2}{2\pi}V_e\left(s\right)\left\{-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{i}{\omega\mu_0}\frac{\partial^2}{\partial s^2}\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_e\left(s\right)+\int_{-h}^{h}\frac{V_e\left(s'\right)-V_e\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\cos\left(m\varphi\right)m_{es}^{+}\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$=2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[\frac{\cos\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\cos\left(m\varphi\right)$$

The odd equation is

$$-\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)V_o\left(s\right)$$

$$-\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_o\left(s\right)+\int_{-h}^{h}\frac{V_o\left(s'\right)e^{ik\left|s-s'\right|}-V_o\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$-\frac{i}{\omega\mu_0}\frac{k^2}{2\pi}V_o\left(s\right)\left\{-C_e+2\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}$$

$$-\frac{i}{\omega\mu_0}\frac{\partial^2}{\partial s^2}\left[\frac{1}{2\pi}\left\{-C_e+\ln\left(1-s^2/h^2\right)\right\}V_o\left(s\right)+\int_{-h}^{h}\frac{V_o\left(s'\right)-V_o\left(s\right)}{2\pi\left|s-s'\right|}ds'\right]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}\sin\left(m\varphi\right)m_{os}^{+}\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)$$

$$\left[\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}+\frac{1}{\left(k^2-k_{p,m,n}'^2\right)k_{p,m,n}'^2}\frac{m^2/j_{m,p}'^2}{\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{n\pi}{h_c}\right)^2\right\}+\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right]$$

$$=-2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\sin\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\cos\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\sin\left(m\varphi\right)$$

where for finite quality factors we replace

$$k^2-k_{p,m,n}^2\rightarrow k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2$$

$$k^2-k_{p,m,n}'^2\rightarrow k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2$$

7 FOURIER GALERKIN IMPLEMENTATION HIGHER FREQUENCIES

We now set up the linear systems for the Fourier coefficients from the integro-differential equations. Our purpose in this section is to calculate the slot voltage and interior fields to fairly high frequencies where modal overlap is expected (overmoded region) and compare with some bounding results. In this section we also compare the rigorous broadband solutions for the exterior cylinder with the exterior half space approximation, using the preceding rigorous formulation of the interior and exterior field representations and resulting integro-differential equations (with the dominant transmission line operator rigorously extracted).

7.1 Fourier Basis And Projections

The Fourier projections of interest here are now discussed. These Fourier representations for the voltage distribution enable the distribution to be general and responsive to interior cavity loading as well as slot wall loss or gasket effects. In the even case we take the solution to be expanded in the basis

$$V_e(s) = \sum_{m',odd} V_{em'} \cos(m'\pi s/\ell)$$

and apply the operator

$$\int_{-h}^{h} \cos(m''\pi s/\ell)\, dz$$

In the odd case we take the solution to be expanded in the basis

$$V_o(s) = \sum_{m',even} V_{om'} \sin(m'\pi s/\ell)$$

and apply the operator

$$\int_{-h}^{h} \sin(m''\pi s/\ell)\, dz$$

Then for the transmission line terms we have

$$\int_{-h}^{h} \cos(m''\pi s/\ell)\cos(m'\pi s/\ell)\, ds = \int_{0}^{h} \left[\cos((m''-m')\pi s/\ell) + \cos((m''+m')\pi s/\ell)\right] ds$$

$$= \left[\frac{\sin((m''-m')\pi s/\ell)}{(m''-m')\pi/\ell} + \frac{\sin((m''+m')\pi s/\ell)}{(m''+m')\pi/\ell}\right]_0^h = \left[\frac{\sin((m''-m')\pi/2)}{(m''-m')\pi/\ell} + \frac{\sin((m''+m')\pi/2)}{(m''+m')\pi/\ell}\right]$$

$$= \frac{\ell}{\varepsilon_m}\delta_{mm'}\ ,\ m', m''\ \text{odd}$$

$$\int_{-h}^{h} \sin(m''\pi s/\ell)\sin(m'\pi s/\ell)\, ds = \int_{0}^{h} \left[\cos((m''-m')\pi s/\ell) - \cos((m''+m')\pi s/\ell)\right] ds$$

$$= \left[\frac{\sin((m''-m')\pi s/\ell)}{(m''-m')\pi/\ell} - \frac{\sin((m''+m')\pi s/\ell)}{(m''+m')\pi/\ell}\right]_0^h = \left[\frac{\sin((m''-m')\pi/2)}{(m''-m')\pi/\ell} - \frac{\sin((m''+m')\pi/2)}{(m''+m')\pi/\ell}\right]$$

$$= \frac{\ell}{2}\delta_{mm'}\ ,\ m', m'' \text{ even not zero}$$

$$\int_{-h}^{h} \sin\left(m''\pi s/\ell\right)\cos\left(m'\pi s/\ell\right) ds = 0 = \int_{-h}^{h} \sin\left(m'\pi s/\ell\right)\cos\left(m''\pi s/\ell\right) ds$$

$$\int_{-h}^{h} \cos\left(m''\pi s/\ell\right)\frac{\partial^2}{\partial s^2}\cos\left(m'\pi s/\ell\right) ds = -\left(m'\pi/\ell\right)^2 \frac{\ell}{2}\delta_{m''m'}\ ,\ m', m'' \text{ odd not zero}$$

$$\int_{-h}^{h} \sin\left(m''\pi s/\ell\right)\frac{\partial^2}{\partial s^2}\sin\left(m'\pi s/\ell\right) ds = -\left(m'\pi/\ell\right)^2 \frac{\ell}{2}\delta_{m''m'}\ ,\ m', m'' \text{ even not zero}$$

For the Galerkin implementation we also need to evaluate and store the interior moments

$$\int_{-h}^{h} \cos\left(m'\pi s/\ell\right)\cos\left(ms/a\right) ds = \frac{1}{2}\int_{-h}^{h}\left[\cos\left(\left(m'\pi/\ell - m/a\right)s\right) + \cos\left(\left(m'\pi/\ell + m/a\right)s\right)\right] ds$$

$$= \frac{\sin\left(\left(m'\pi/\ell - m/a\right)h\right)}{\left(m'\pi/\ell - m/a\right)} + \frac{\sin\left(\left(m'\pi/\ell + m/a\right)h\right)}{\left(m'\pi/\ell + m/a\right)}$$

$$\int_{-h}^{h} \sin\left(m'\pi s/\ell\right)\sin\left(ms/a\right) ds = \frac{1}{2}\int_{-h}^{h}\left[\cos\left(\left(m'\pi/\ell - m/a\right)s\right) - \cos\left(\left(m'\pi/\ell + m/a\right)s\right)\right] ds$$

$$= \frac{\sin\left(\left(m'\pi/\ell - m/a\right)h\right)}{\left(m'\pi/\ell - m/a\right)} - \frac{\sin\left(\left(m'\pi/\ell + m/a\right)h\right)}{\left(m'\pi/\ell + m/a\right)}$$

Then the dipole-like terms are

$$m_{es}^{+} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_e\left(s'\right) ds' = \frac{i2}{\omega\mu_0}\sum_{m',odd} V_{em'}\left[\frac{\sin\left(\left(m'\pi/\ell - m/a\right)h\right)}{\left(m'\pi/\ell - m/a\right)} + \frac{\sin\left(\left(m'\pi/\ell + m/a\right)h\right)}{\left(m'\pi/\ell + m/a\right)}\right]$$

$$m_{os}^{+} = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_o\left(s'\right) ds' = \frac{i2}{\omega\mu_0}\sum_{m',even} V_{om'}\left[\frac{\sin\left(\left(m'\pi/\ell - m/a\right)h\right)}{\left(m'\pi/\ell - m/a\right)} - \frac{\sin\left(\left(m'\pi/\ell + m/a\right)h\right)}{\left(m'\pi/\ell + m/a\right)}\right]$$

and with m'' odd

$$\int_{-h}^{h}\cos\left(m''\pi s/\ell\right)\frac{\partial^2}{\partial s^2}\cos\left(ms/a\right) ds = \left[\cos\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\cos\left(ms/a\right)\right]_{-h}^{h} - \int_{-h}^{h}\frac{\partial}{\partial s}\cos\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\cos\left(ms/a\right) ds$$

$$= -\int_{-h}^{h}\frac{\partial}{\partial s}\cos\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\cos\left(ms/a\right) ds = -\left[\frac{\partial}{\partial s}\cos\left(m''\pi s/\ell\right)\cos\left(ms/a\right)\right]_{-h}^{h} + \int_{-h}^{h}\frac{\partial^2}{\partial s^2}\cos\left(m''\pi s/\ell\right)\cos\left(ms/a\right) ds$$

$$= 2\left(m''\pi/\ell\right)\sin\left(m''\pi/2\right)\cos\left(mh/a\right) - \left(m''\pi/\ell\right)^2\int_{-h}^{h}\cos\left(m''\pi s/\ell\right)\cos\left(ms/a\right) ds$$

$$= 2\left(m''\pi/\ell\right)(-1)^{\left(m''-1\right)/2}\cos\left(mh/a\right) - \left(m''\pi/\ell\right)^2\left[\frac{\sin\left(\left(m''\pi/\ell - m/a\right)h\right)}{\left(m''\pi/\ell - m/a\right)} + \frac{\sin\left(\left(m''\pi/\ell + m/a\right)h\right)}{\left(m''\pi/\ell + m/a\right)}\right]$$

$$= -\int_{-h}^{h}\frac{\partial}{\partial s}\cos\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\cos\left(ms/a\right)ds = -\left(m''\pi/\ell\right)\left(m/a\right)\int_{-h}^{h}\sin\left(m''\pi s/\ell\right)\sin\left(ms/a\right)ds$$

$$= -\left(m''\pi/\ell\right)\left(m/a\right)\int_{0}^{h}\left[\cos\left(\left(m''\pi/\ell - m/a\right)s\right) - \cos\left(\left(m''\pi/\ell + m/a\right)s\right)\right]ds$$

$$= -\left(m''\pi/\ell\right)\left(m/a\right)\left[\frac{\sin\left(\left(m''\pi/\ell - m/a\right)h\right)}{\left(m''\pi/\ell - m/a\right)} - \frac{\sin\left(\left(m''\pi/\ell + m/a\right)h\right)}{\left(m''\pi/\ell + m/a\right)}\right]$$

$$= (-1)^{m''/2}\left(m''\pi/\ell\right)\left(m/a\right)\left[\frac{1}{\left(m''\pi/\ell - m/a\right)} + \frac{1}{\left(m''\pi/\ell + m/a\right)}\right]\sin\left(mh/a\right)$$

$$= -\left(m/a\right)^2\left[\frac{\sin\left(\left(m''\pi/\ell - m/a\right)h\right)}{\left(m''\pi/\ell - m/a\right)} + \frac{\sin\left(\left(m''\pi/\ell + m/a\right)h\right)}{\left(m''\pi/\ell + m/a\right)}\right]$$

$$= (-1)^{m''/2}\left(m/a\right)^2\left[\frac{1}{\left(m''\pi/\ell - m/a\right)} - \frac{1}{\left(m''\pi/\ell + m/a\right)}\right]\sin\left(mh/a\right)$$

and with m'' even

$$\int_{-h}^{h}\sin\left(m''\pi s/\ell\right)\frac{\partial^2}{\partial s^2}\sin\left(ms/a\right)ds = \left[\sin\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\sin\left(ms/a\right)\right]_{-h}^{h} - \int_{-h}^{h}\frac{\partial}{\partial s}\sin\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\sin\left(ms/a\right)ds$$

$$= -\int_{-h}^{h}\frac{\partial}{\partial s}\sin\left(m''\pi s/\ell\right)\frac{\partial}{\partial s}\sin\left(ms/a\right)ds = -\left(m''\pi/\ell\right)\left(m/a\right)\int_{-h}^{h}\cos\left(m''\pi s/\ell\right)\cos\left(ms/a\right)ds$$

$$= -\left(m''\pi/\ell\right)\left(m/a\right)\int_{0}^{h}\left[\cos\left(\left(m''\pi/\ell - m/a\right)s\right) + \cos\left(\left(m''\pi/\ell + m/a\right)s\right)\right]ds$$

$$= -\left(m''\pi/\ell\right)\left(m/a\right)\left[\frac{\sin\left(\left(m''\pi/\ell - m/a\right)h\right)}{\left(m''\pi/\ell - m/a\right)} + \frac{\sin\left(\left(m''\pi/\ell + m/a\right)h\right)}{\left(m''\pi/\ell + m/a\right)}\right]$$

$$= (-1)^{m''/2}\left(m''\pi/\ell\right)\left(m/a\right)\left[\frac{1}{\left(m''\pi/\ell - m/a\right)} - \frac{1}{\left(m''\pi/\ell + m/a\right)}\right]\sin\left(mh/a\right)$$

$$= (-1)^{m''/2}\frac{\left(m''\pi/\ell\right)2\left(m/a\right)^2}{\left(m''\pi/\ell\right)^2 - \left(m/a\right)^2}\sin\left(mh/a\right)$$

$$= -\left(m/a\right)^2\left[\frac{\sin\left(\left(m''\pi/\ell - m/a\right)h\right)}{\left(m''\pi/\ell - m/a\right)} - \frac{\sin\left(\left(m''\pi/\ell + m/a\right)h\right)}{\left(m''\pi/\ell + m/a\right)}\right]$$

$$= (-1)^{m''/2}\left(m/a\right)^2\left[\frac{1}{\left(m''\pi/\ell - m/a\right)} + \frac{1}{\left(m''\pi/\ell + m/a\right)}\right]\sin\left(mh/a\right)$$

$$= (-1)^{m''/2} \frac{(m''\pi/\ell)\, 2\, (m/a)^2}{(m''\pi/\ell)^2 - (m/a)^2} \sin(mh/a)$$

We also need to evaluate and store the exterior moments

$$\int_{-h}^{h} \cos(m'\pi s/\ell) \cos(ms/b)\, ds = \frac{1}{2} \int_{-h}^{h} \left[\cos((m'\pi/\ell - m/b)\, s) + \cos((m'\pi/\ell + m/b)\, s)\right] ds$$

$$= \frac{\sin((m'\pi/\ell - m/b)\, h)}{(m'\pi/\ell - m/b)} + \frac{\sin((m'\pi/\ell + m/b)\, h)}{(m'\pi/\ell + m/b)}$$

$$\int_{-h}^{h} \sin(m'\pi s/\ell) \sin(ms/b)\, ds = \frac{1}{2} \int_{-h}^{h} \left[\cos((m'\pi/\ell - m/b)\, s) - \cos((m'\pi/\ell + m/b)\, s)\right] ds$$

$$= \frac{\sin((m'\pi/\ell - m/b)\, h)}{(m'\pi/\ell - m/b)} - \frac{\sin((m'\pi/\ell + m/b)\, h)}{(m'\pi/\ell + m/b)}$$

Then the dipole-like terms are

$$m_{es}^{-} = -\frac{i2}{\omega\mu_0} \int_{-h}^{h} \cos(ms'/b)\, V_e(s')\, ds' = \frac{-i2}{\omega\mu_0} \sum_{m',odd} V_{em'} \left[\frac{\sin((m'\pi/\ell - m/b)\, h)}{(m'\pi/\ell - m/b)} + \frac{\sin((m'\pi/\ell + m/b)\, h)}{(m'\pi/\ell + m/b)}\right]$$

$$m_{os}^{-} = \frac{-i2}{\omega\mu_0} \int_{-h}^{h} \sin(ms'/b)\, V_o(s')\, ds' = \frac{-i2}{\omega\mu_0} \sum_{m',even} V_{om'} \left[\frac{\sin((m'\pi/\ell - m/b)\, h)}{(m'\pi/\ell - m/b)} - \frac{\sin((m'\pi/\ell + m/b)\, h)}{(m'\pi/\ell + m/b)}\right]$$

$$\int_{-h}^{h} \cos(m''\pi s/\ell) \frac{\partial^2}{\partial s^2} \cos(ms/b)\, ds$$

$$= -(m''\pi/\ell)(m/b) \left[\frac{\sin((m''\pi/\ell - m/b)\, h)}{(m''\pi/\ell - m/b)} - \frac{\sin((m''\pi/\ell + m/b)\, h)}{(m''\pi/\ell + m/b)}\right]$$

$$= (-1)^{m''/2} (m''\pi/\ell)(m/b) \left[\frac{1}{(m''\pi/\ell - m/b)} + \frac{1}{(m''\pi/\ell + m/b)}\right] \sin(mh/b)$$

$$= -(m/b)^2 \left[\frac{\sin((m''\pi/\ell - m/b)\, h)}{(m''\pi/\ell - m/b)} + \frac{\sin((m''\pi/\ell + m/b)\, h)}{(m''\pi/\ell + m/b)}\right]$$

$$= (-1)^{m''/2} (m/b)^2 \left[\frac{1}{(m''\pi/\ell - m/b)} - \frac{1}{(m''\pi/\ell + m/b)}\right] \sin(mh/b)$$

$$\int_{-h}^{h} \sin(m''\pi s/\ell) \frac{\partial^2}{\partial s^2} \sin(ms/b)\, ds$$

$$= -(m''\pi/\ell)(m/b) \left[\frac{\sin((m''\pi/\ell - m/b)\, h)}{(m''\pi/\ell - m/b)} + \frac{\sin((m''\pi/\ell + m/b)\, h)}{(m''\pi/\ell + m/b)}\right]$$

$$= (-1)^{m''/2} (m''\pi/\ell)(m/b) \left[\frac{1}{(m''\pi/\ell - m/b)} - \frac{1}{(m''\pi/\ell + m/b)}\right] \sin(mh/b)$$

$$= (-1)^{m''/2} \frac{(m''\pi/\ell)\, 2\, (m/b)^2}{(m''\pi/\ell)^2 - (m/b)^2} \sin(mh/b)$$

$$= -(m/b)^2 \left[\frac{\sin((m''\pi/\ell - m/b)\, h)}{(m''\pi/\ell - m/b)} - \frac{\sin((m''\pi/\ell + m/b)\, h)}{(m''\pi/\ell + m/b)} \right]$$

$$= (-1)^{m''/2} (m/b)^2 \left[\frac{1}{(m''\pi/\ell - m/b)} + \frac{1}{(m''\pi/\ell + m/b)} \right] \sin(mh/b)$$

$$= (-1)^{m''/2} \frac{(m''\pi/\ell)\, 2\, (m/b)^2}{(m''\pi/\ell)^2 - (m/b)^2} \sin(mh/b)$$

7.2 Even System

The even system is

$$- \sum_{m',odd} V_{em'} \int_{-h}^{h} \cos(m''s/\ell) \left(\frac{1}{\widetilde{Z}} \frac{\partial^2}{\partial s^2} - \widetilde{Y} \right) \cos(m'\pi s/\ell)\, ds$$

$$+ \frac{k^2}{2\pi b} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2\pi} m_{es}^{-} \int_{-h}^{h} \cos(m''\pi s/\ell) \frac{1}{k^2} \frac{\partial^2}{\partial s^2} \cos(ms/b)\, ds$$

$$\left[\int_0^k \left\{ - \frac{J_m(k_t b) J_m'(k_t b) + Y_m(k_t b) Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} + i \frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}} \right.$$

$$\left. + \int_0^{\infty} \left\{ - \frac{K_m(ub)}{K_m'(ub)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2}} \right]$$

$$+ \frac{k^2}{2\pi b} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2\pi} m_{es}^{-} \int_{-h}^{h} \cos(m''\pi s/\ell) \cos(ms/b)\, ds$$

$$\left[\int_0^k \left\{ \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \left\{ J_m(k_t b) J_m'(k_t b) + Y_m(k_t b) Y_m'(k_t b) \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} \right.$$

$$+ i \frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} + \frac{m^2}{k_t^2 b^2} \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}}$$

$$\left. + \int_0^{\infty} \left\{ - \frac{K_m'(ub)}{K_m(ub)} - 1 + \frac{m^2}{u^2 b^2} \frac{K_m(ub)}{K_m'(ub)} \right\} \frac{du}{\sqrt{u^2 + k^2}} \right]$$

$$+ \frac{\pi k^2}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2\pi} m_{es}^{+} \int_{-h}^{h} \cos(m''\pi s/\ell) \cos(ms/a)\, ds \sum_{n=0}^{\infty} \varepsilon_n \cos^2(n\pi z_0/h_c) \sum_{p=1}^{\infty} \left\{ \frac{1}{(k^2 - k_{p,m,n}^2)} - \frac{1}{(k^2 - k_{p,m,n}'^2)} \right\}$$

$$- \frac{\pi k^2}{V_{cav}} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{2\pi} m_{es}^{+} \int_{-h}^{h} \cos(m''\pi s/\ell) \left(\frac{1}{k^2} \frac{\partial^2}{\partial s^2} + 1 \right) \cos(ms/a)\, ds$$

$$\sum_{n=0}^{\infty} \varepsilon_n \cos^2(n\pi z_0/h_c) \left\{ \sum_{p=1}^{\infty} \frac{1}{\left(k'^2_{pmn} - k^2\right)\left(1 - m^2/j'^2_{m,p}\right)} - \frac{a(1-\delta_{n0})}{2n\pi/h_c} \right\}$$

$$= -2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[-\frac{\cos(m\varphi_i)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} - \frac{m \sin(m\varphi_i)\cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right] \int_{-h}^{h} \cos(m''\pi s/\ell)\cos(ms/b)\, ds \tag{1}$$

$$m_{es}^{+} = \frac{i2}{\omega\mu_0} \int_{-h}^{h} \cos(ms'/a)\, V_e(s')\, ds' = \frac{i2}{\omega\mu_0} \sum_{m',odd} V_{em'} \int_{-h}^{h} \cos(m'\pi s'/\ell)\cos(ms'/a)\, ds'$$

$$m_{es}^{-} = -\frac{i2}{\omega\mu_0} \int_{-h}^{h} \cos(ms'/b)\, V_e(s')\, ds' = \frac{-i2}{\omega\mu_0} \sum_{m',odd} V_{em'} \int_{-h}^{h} \cos(m'\pi s'/\ell)\cos(ms'/b)\, ds'$$

$$k^2 - k_{p,m,n}^2 \to k^2 + k k_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2$$

$$k^2 - k'^2_{p,m,n} \to k^2 + k k'_{p,m,n}(1+i)/Q_{p,m,n}^{TE} - k'^2_{p,m,n}$$

7.3 Odd System

The odd system

$$-\sum_{m',even} V_{om'} \int_{-h}^{h} \sin(m''s/\ell) \left(\frac{1}{\widetilde{Z}} \frac{\partial^2}{\partial s^2} - \widetilde{Y} \right) \sin(m'\pi s/\ell)\, ds$$

$$+\frac{k^2}{2\pi b} \sum_{m=1}^{\infty} \frac{1}{\pi} m_{os}^{-} \int_{-h}^{h} \sin(m''\pi s/\ell) \frac{1}{k^2} \frac{\partial^2}{\partial s^2} \sin(ms/b)\, ds$$

$$\left[\int_0^k \left\{ -\frac{J_m(k_t b) J'_m(k_t b) + Y_m(k_t b) Y'_m(k_t b)}{J'^2_m(k_t b) + Y'^2_m(k_t b)} \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} + i\frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J'^2_m(k_t b) + Y'^2_m(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}} \right.$$

$$\left. + \int_0^{\infty} \left\{ -\frac{K_m(ub)}{K'_m(ub)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2}} \right]$$

$$+\frac{k^2}{2\pi b} \sum_{m=1}^{\infty} \frac{1}{\pi} m_{os}^{-} \int_{-h}^{h} \sin(m''\pi s/\ell) \sin(ms/b)\, ds$$

$$\left[\int_0^k \left\{ \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} - \frac{m^2}{k_t^2 b^2} \frac{1}{J'^2_m(k_t b) + Y'^2_m(k_t b)} \right\} \left\{ J_m(k_t b) J'_m(k_t b) + Y_m(k_t b) Y'_m(k_t b) \right\} \frac{dk_t}{\sqrt{k^2 - k_t^2}} \right.$$

$$+i\frac{2}{\pi b} \int_0^k \left\{ \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} + \frac{m^2}{k_t^2 b^2} \frac{1}{J'^2_m(k_t b) + Y'^2_m(k_t b)} \right\} \frac{dk_t}{k_t \sqrt{k^2 - k_t^2}}$$

$$+\int_0^\infty \left\{-\frac{K_m'(ub)}{K_m(ub)} - 1 + \frac{m^2}{u^2b^2}\frac{K_m(ub)}{K_m'(ub)}\right\}\frac{du}{\sqrt{u^2+k^2}}\Bigg]$$

$$+\frac{k^2}{V_{cav}}\sum_{m=1}^{\infty} m_{es}^+ \int_{-h}^{h} \sin(m''\pi s/\ell)\sin(ms/a)\,ds \sum_{n=0}^{\infty}\varepsilon_n \cos^2(n\pi z_0/h_c)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)} - \frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}m_{es}^+\int_{-h}^{h}\cos(m''\pi s/\ell)\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\cos(ms/a)\,ds$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2(n\pi z_0/h_c)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)} - \frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$= -2H_0 e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i} b}\left[-\frac{\sin(m\varphi_i)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)} + \frac{m\cos(m\varphi_i)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\int_{-h}^{h}\sin(m''\pi s/\ell)\sin(ms/b)\,ds \tag{2}$$

$$m_{os}^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin(ms'/a)\,V_o(s')\,ds' = \frac{i2}{\omega\mu_0}\sum_{m',even} V_{om'}\int_{-h}^{h}\sin(m'\pi s'/\ell)\sin(ms'/a)\,ds'$$

$$m_{os}^- = -\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin(ms'/b)\,V_o(s')\,ds' = \frac{-i2}{\omega\mu_0}\sum_{m',even} V_{om'}\int_{-h}^{h}\sin(m'\pi s'/\ell)\sin(ms'/b)\,ds'$$

$$k^2 - k_{p,m,n}^2 \to k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2$$

$$k^2 - k_{p,m,n}'^2 \to k^2 + kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2$$

7.4 m th Integrals

First we explore the convergence for $m \neq 0$. The first bracketed term is

$$[] = \int_0^{kb}\left\{-\frac{J_m(u)\,J_m'(u)+Y_m(u)\,Y_m'(u)}{J_m'^2(u)+Y_m'^2(u)}\right\}\frac{du}{\sqrt{k^2b^2-u^2}} + i\frac{2}{\pi}\int_0^{kb}\left\{\frac{1}{J_m'^2(u)+Y_m'^2(u)}\right\}\frac{du}{u\sqrt{k^2b^2-u^2}}$$

$$+\int_0^{\infty}\left\{-\frac{K_m(u)}{K_m'(u)}-1\right\}\frac{du}{\sqrt{u^2+k^2b^2}}$$

Convergence for $u \to 0$ is not an issue in any of the three integrals for $m \neq 0$. For $u >> 1$ we expand the expression in braces of the integrand of the final term

$$\left\{-\frac{K_m(u)}{K_m'(u)}-1\right\} \sim \frac{1+\left(4m^2-1\right)/(8u)+\left(4m^2-1\right)\left(4m^2-9\right)/\left\{2\,(8u)^2\right\}}{1+(4m^2+3)/(8u)+(4m^2-1)(4m^2+15)/\left\{2\,(8u)^2\right\}} - 1$$

$$\sim -1/\left(2u\right)+\left(4m^2-1\right)^2/\left(8u\right)^2+\left(4m^2+3\right)^2/\left(8u\right)^2+\left(4m^2-1\right)\left(4m^2-9\right)/\left\{2\left(8u\right)^2\right\}-\left(4m^2-1\right)\left(4m^2+15\right)/\left\{2\left(8u\right)^2\right\}$$

$$\sim -1/\left(2u\right)+\left[\left(4m^2-1\right)\left(4m^2-1\right)+\left(4m^2+3\right)\left(4m^2+3\right)+\frac{1}{2}\left(4m^2-1\right)\left(4m^2-9\right)-\frac{1}{2}\left(4m^2-1\right)\left(4m^2+15\right)\right]/\left(8u\right)^2$$

$$\sim -1/\left(2u\right)+\left[2\left(4m^2\right)^2+\left(4m^2\right)\left(-2+6-5-7\right)+1+9+9/2+15/2\right]/\left(8u\right)^2$$

$$\sim -1/\left(2u\right)+2\left[\left(4m^2\right)^2-4\left(4m^2\right)+11\right]/\left(8u\right)^2=-1/\left(2u\right)+O\left(m^4/u^2\right)$$

and thus the final term converges at infinity

$$\int_R^\infty\left\{-\frac{K_m\left(u\right)}{K_m'\left(u\right)}-1\right\}\frac{du}{\sqrt{u^2+k^2b^2}}\sim\int_R^\infty\left[-\frac{1}{2u^2}+O\left(m^4/u^3\right)\right]\left\{1+O\left(\frac{k^2b^2}{u^2}\right)\right\}du=-1/\left(2R\right)+O\left(\frac{k^2b^2}{R^3}\right)+O\left(\frac{m^4}{R^2}\right)$$

The second bracketed term is

$$[]=\int_0^{kb}\left\{\frac{1}{J_m^2\left(u\right)+Y_m^2\left(u\right)}-\frac{m^2}{u^2}\frac{1}{J_m'^2\left(u\right)+Y_m'^2\left(u\right)}\right\}\left\{J_m\left(u\right)J_m'\left(u\right)+Y_m\left(u\right)Y_m'\left(u\right)\right\}\frac{du}{\sqrt{k^2b^2-u^2}}$$

$$+i\frac{2}{\pi}\int_0^{kb}\left\{\frac{1}{J_m^2\left(u\right)+Y_m^2\left(u\right)}+\frac{m^2}{u^2}\frac{1}{J_m'^2\left(u\right)+Y_m'^2\left(u\right)}\right\}\frac{du}{u\sqrt{k^2b^2-u^2}}$$

$$+\int_0^\infty\left\{-\frac{K_m'\left(u\right)}{K_m\left(u\right)}-1+\frac{m^2}{u^2}\frac{K_m\left(u\right)}{K_m'\left(u\right)}\right\}\frac{du}{\sqrt{u^2+k^2b^2}}$$

Near $u\to 0$ leading terms in the first set of braces of the first integral cancel and lead to convergence, the second integral converges, and the leading terms in the braces of the third integral cancel leading to convergence. Expanding the braces in the integrand of the final term for $u>>1$

$$\left\{-\frac{K_m'\left(u\right)}{K_m\left(u\right)}-1+\frac{m^2}{u^2}\frac{K_m\left(u\right)}{K_m'\left(u\right)}\right\}\sim\frac{1+\left(4m^2+3\right)/\left(8u\right)+\left(4m^2-1\right)\left(4m^2+15\right)/\left\{2\left(8u\right)^2\right\}}{1+\left(4m^2-1\right)/\left(8u\right)+\left(4m^2-1\right)\left(4m^2-9\right)/\left\{2\left(8u\right)^2\right\}}-1$$

$$+\frac{m^2}{u^2}\frac{1+\left(4m^2-1\right)/\left(8u\right)+\left(4m^2-1\right)\left(4m^2-9\right)/\left\{2\left(8u\right)^2\right\}}{1+\left(4m^2+3\right)/\left(8u\right)+\left(4m^2-1\right)\left(4m^2+15\right)/\left\{2\left(8u\right)^2\right\}}\sim 1/\left(2u\right)+O\left(m^4/u^2\right)$$

$$\int_R^\infty\left\{-\frac{K_m'\left(u\right)}{K_m\left(u\right)}-1+\frac{m^2}{u^2}\frac{K_m\left(u\right)}{K_m'\left(u\right)}\right\}\frac{du}{\sqrt{u^2+k^2b^2}}$$

$$\sim\int_R^\infty\left[\frac{1}{2u^2}+O\left(m^4/u^3\right)\right]\left\{1+O\left(\frac{k^2b^2}{u^2}\right)\right\}du=1/\left(2R\right)+O\left(\frac{k^2b^2}{R^3}\right)+O\left(\frac{m^4}{R^2}\right)$$

7.5 Zero Integrals

Now we explore the convergence for $m = 0$. The first bracketed term for $m = 0$ is

$$[] = \int_0^{kb} \left\{ -\frac{J_0(u) J_0'(u) + Y_0(u) Y_0'(u)}{J_0'^2(u) + Y_0'^2(u)} \right\} \frac{du}{\sqrt{k^2b^2 - u^2}} + i\frac{2}{\pi} \int_0^{kb} \left\{ \frac{1}{J_0'^2(u) + Y_0'^2(u)} \right\} \frac{du}{u\sqrt{k^2b^2 - u^2}}$$

$$+ \int_0^\infty \left\{ \frac{K_0(u)}{K_1(u)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2b^2}}$$

All integrals converge at $u \to 0$. For large u in the final integral

$$\int_R^\infty \left\{ -\frac{K_0(u)}{K_0'(u)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2b^2}} \sim \int_R^\infty \left[-\frac{1}{2u^2} \right] \left\{ 1 + O\left(\frac{k^2b^2}{u^2} \right) \right\} du = -1/(2R) + O\left(\frac{k^2b^2}{R^3} \right)$$

The second bracketed term for $m = 0$ requires cancellation of divergent terms in the several integrals near $u \to 0$, and is therefore written as

$$[] = \int_{\delta b}^{kb} \left\{ \frac{1}{J_0^2(u) + Y_0^2(u)} \right\} \{ J_0(u) J_0'(u) + Y_0(u) Y_0'(u) \} \frac{du}{\sqrt{k^2b^2 - u^2}}$$

$$+ i\frac{2}{\pi} \int_{\delta b}^{kb} \left\{ \frac{1}{J_0^2(u) + Y_0^2(u)} \right\} \frac{du}{u\sqrt{k^2b^2 - u^2}} + \int_{\delta b}^\infty \left\{ \frac{K_1(u)}{K_0(u)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2b^2}}$$

$$+ \frac{1}{kb} \ln \left\{ \frac{\sqrt{1 + (2/\pi)^2 \ln^2(\delta b e^{\gamma'}/2)}}{-(2/\pi) \ln(\delta b e^{\gamma'}/2)} \right\} + i\frac{1}{kb} \left\{ 1 - \frac{2}{\pi} \arctan\left(-(2/\pi) \ln\left(\delta b e^{\gamma'}/2 \right) \right) \right\} - \delta/k$$

where the final integral for $u >> 1$ gives

$$\int_R^\infty \left\{ -\frac{K_0'(u)}{K_0(u)} - 1 \right\} \frac{du}{\sqrt{u^2 + k^2b^2}} \sim \int_R^\infty \frac{1}{2u^2} \left\{ 1 + O\left(\frac{k^2b^2}{u^2} \right) \right\} du = 1/(2R) + O\left(\frac{k^2b^2}{R^3} \right)$$

7.6 Half Space Systems

The Fourier basis half space systems are now given.

7.6.1 Even System

The even system with half-space exterior is

$$-\sum_{m',odd} V_{em'} \int_{-h}^{h} \cos(m''s/\ell) \left(\frac{1}{Z} \frac{\partial^2}{\partial s^2} - Y \right) \cos(m'\pi s/\ell)\, ds$$

$$-i\omega\varepsilon_0 \left\{ -C_e/2 + \ln\left(\frac{h_c}{h\pi} \csc(\pi z_0/h_c) \right) \right\} \frac{1}{\pi} \sum_{m',odd} V_{em'} \int_{-h}^{h} \cos(m''s/\ell) \left(\frac{1}{k^2} \frac{\partial^2}{\partial s^2} + 1 \right) \cos(m'\pi s/\ell)\, ds$$

$$+ \sum_{m',odd} \frac{1}{2} Y_{rad}^{e(m''m')} V_{em'}$$

$$+\frac{k^2}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}m_{es}^{+}\int_{-h}^{h}\cos\left(m''s/\ell\right)\cos\left(ms/a\right)ds\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\frac{1}{V_{cav}}\sum_{m=0}^{\infty}\frac{\varepsilon_m}{2}m_{es}^{+}\int_{-h}^{h}\cos\left(m''\pi s/\ell\right)\left(\frac{\partial^2}{\partial s^2}+k^2\right)\cos\left(ms/a\right)ds$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=2H_0e^{ik_{z_i}z_0}\sum_{m=0}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[\frac{\cos\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\sin\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\int_{-h}^{h}\cos\left(m''\pi s/\ell\right)\cos\left(ms/b\right)ds \quad (3)$$

where

$$m_{es}^{+}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_e\left(s'\right)ds'=\frac{i2}{\omega\mu_0}\sum_{m',odd}V_{em'}\int_{-h}^{h}\cos\left(m'\pi s'/\ell\right)\cos\left(ms'/a\right)ds'$$

$$m_{es}^{-}=-\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/b\right)V_e\left(s'\right)ds'=\frac{-i2}{\omega\mu_0}\sum_{m',odd}V_{em'}\int_{-h}^{h}\cos\left(m'\pi s'/\ell\right)\cos\left(ms'/b\right)ds'$$

and we define

$$Y_{rad}^{e\left(m''m'\right)}=-\frac{1}{-i\omega\mu_0\pi}\int_{-h}^{h}\cos\left(m''s/\ell\right)\left(\frac{\partial^2}{\partial s^2}+k^2\right)$$

$$\left[\cos\left(m's/\ell\right)\left\{-C_e^e+\ln\left(1-s^2/h^2\right)\right\}+\int_{-h}^{h}\frac{\cos\left(m's'/\ell\right)e^{ik\left|s-s'\right|}-\cos\left(m's/\ell\right)}{\left|s-s'\right|}ds'\right]ds$$

The constant for the even case is taken as $C_e^e=C_e=2\left(\ln 2-7/3\right)$. The analytical calculation of these radiation integrals is carried out in [2].

7.6.2 Odd System

The odd system with the half-space exterior is

$$-\sum_{m',even}V_{om'}\int_{-h}^{h}\sin\left(m''s/\ell\right)\left(\frac{1}{Z}\frac{\partial^2}{\partial s^2}-Y\right)\sin\left(m'\pi s/\ell\right)ds$$

$$-i\omega\varepsilon_0\left\{-C_e/2+\ln\left(\frac{h_c}{h\pi}\csc\left(\pi z_0/h_c\right)\right)\right\}\frac{1}{\pi}\sum_{m',even}V_{om'}\int_{-h}^{h}\sin\left(m''s/\ell\right)\left(\frac{1}{k^2}\frac{\partial^2}{\partial s^2}+1\right)\sin\left(m'\pi s/\ell\right)ds$$

$$+\sum_{m',even}\frac{1}{2}Y_{rad}^{o\left(m''m'\right)}V_{om'}$$

$$+\frac{k^2}{V_{cav}}\sum_{m=1}^{\infty}\frac{\varepsilon_m}{2}m_{os}^{+}\int_{-h}^{h}\sin\left(m''\pi s/\ell\right)\sin\left(ms/b\right)ds\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\sum_{p=1}^{\infty}\left\{\frac{1}{\left(k^2-k_{p,m,n}^2\right)}-\frac{1}{\left(k^2-k_{p,m,n}'^2\right)}\right\}$$

$$-\frac{1}{V_{cav}}\sum_{m=1}^{\infty}\frac{\varepsilon_m}{2}m_{os}^{+}\int_{-h}^{h}\sin\left(m''\pi s/\ell\right)\left(\frac{\partial^2}{\partial s^2}+k^2\right)\sin\left(ms/a\right)ds$$

$$\sum_{n=0}^{\infty}\varepsilon_n\cos^2\left(n\pi z_0/h_c\right)\left\{\sum_{p=1}^{\infty}\frac{1}{\left(k_{pmn}'^2-k^2\right)\left(1-m^2/j_{m,p}'^2\right)}-\frac{a\left(1-\delta_{n0}\right)}{2n\pi/h_c}\right\}$$

$$=-2H_0e^{ik_{z_i}z_0}\sum_{m=1}^{\infty}\frac{\varepsilon_m i^m}{\pi k_{\rho_i}b}\left[-\frac{\sin\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i}b\right)}+\frac{m\cos\left(m\varphi_i\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_m^{(1)\prime}\left(k_{\rho_i}b\right)}\right]\int_{-h}^{h}\sin\left(m''\pi s/\ell\right)\sin\left(ms/b\right)ds \tag{4}$$

where

$$m_{os}^{+}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_o\left(s'\right)ds'=\frac{i2}{\omega\mu_0}\sum_{m',even}V_{om'}\int_{-h}^{h}\sin\left(m'\pi s'/\ell\right)\sin\left(ms'/a\right)ds'$$

$$m_{os}^{-}=-\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/b\right)V_o\left(s'\right)ds'=\frac{-i2}{\omega\mu_0}\sum_{m',even}V_{om'}\int_{-h}^{h}\sin\left(m'\pi s'/\ell\right)\sin\left(ms'/b\right)ds'$$

and we define

$$Y_{rad}^{o\left(m''m'\right)}=-\frac{1}{-i\omega\mu_0\pi}\int_{-h}^{h}\sin\left(m''s/\ell\right)\left(\frac{\partial^2}{\partial s^2}+k^2\right)$$

$$\left[\sin\left(m's/\ell\right)\left\{-C_e^o+\ln\left(1-s^2/h^2\right)\right\}+\int_{-h}^{h}\frac{\sin\left(m's'/\ell\right)e^{ik\left|s-s'\right|}-\sin\left(m's/\ell\right)}{\left|s-s'\right|}ds'\right]ds$$

The constant for the odd case C_e^o can be taken as the same as the even case or redefined (see below for a linear drive odd radiation admittance). The analytical calculation of these radiation integrals is carried out in [2].

7.7 Voltage Results

The results presented are computed using the preceding even (1) and odd (2) Galerkin systems; the half space systems (3) and (4) are used to implement the solution using the approximate radiation for an exterior half space. Figure 2 shows an example of the center slot voltage at normal incidence for a cylindrical cavity near the first slot resonance using both a cylindrical exterior geometry (red curve) and an exterior half space approximation (blue curve). Figure 3 shows an example of the center slot voltage at normal incidence for a cylindrical cavity covering the first three slot resonances (even case at normal incidence) using both a cylindrical exterior geometry (red curve) and an exterior half space approximation (blue curve). The half space approximation is reasonably accurate here even though the ratio of slot length to half circumference is not extremely small $\ell/\left(\pi b\right)\approx 0.412$.

Figure 4 shows an example of the slot voltage one quarter of the distance from the end at oblique incidence for a cylindrical cavity near the first slot resonance using both a cylindrical exterior geometry (red curve)

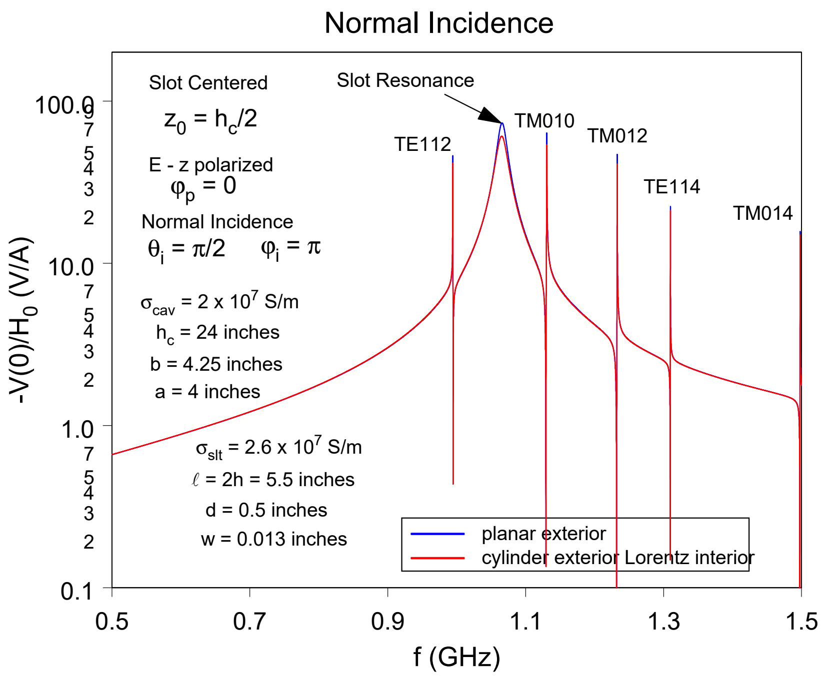

Figure 2: Example of center slot voltage (normalized by the incident magnetic field) at normal incidence, for a cylindrical cavity near the first resonance, using both the exterior cylindrical geometry (red curve) and the exterior half space approximation (blue curve). Cavity mode identies are denoted.

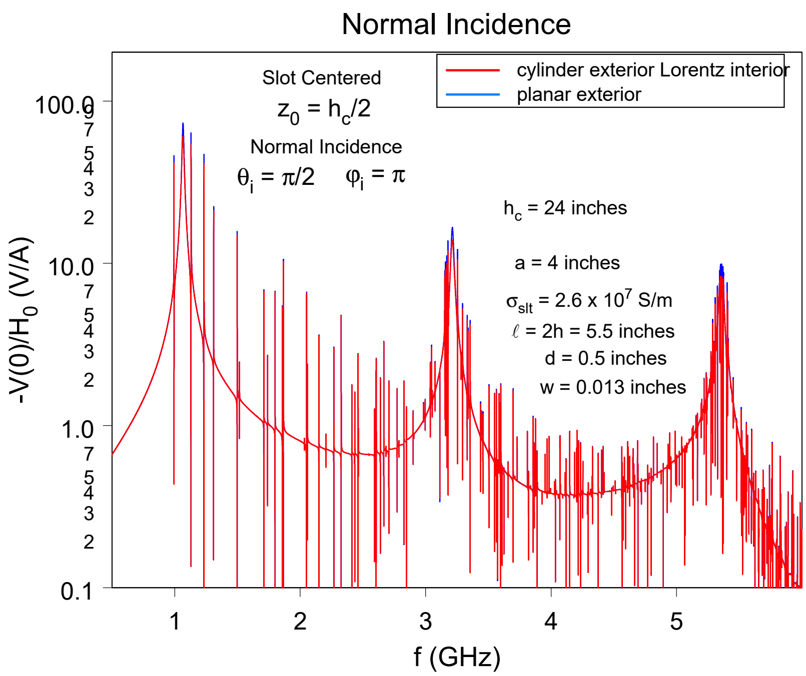

Figure 3: Example of center slot voltage normalized by incident magnetic field at normal incidence, for a cylindrical cavity covering the first three slot resonances (for the even case at normal incidence), using both the exterior cylindrical geometry (red curve) and the exterior half space approximation (blue curve).

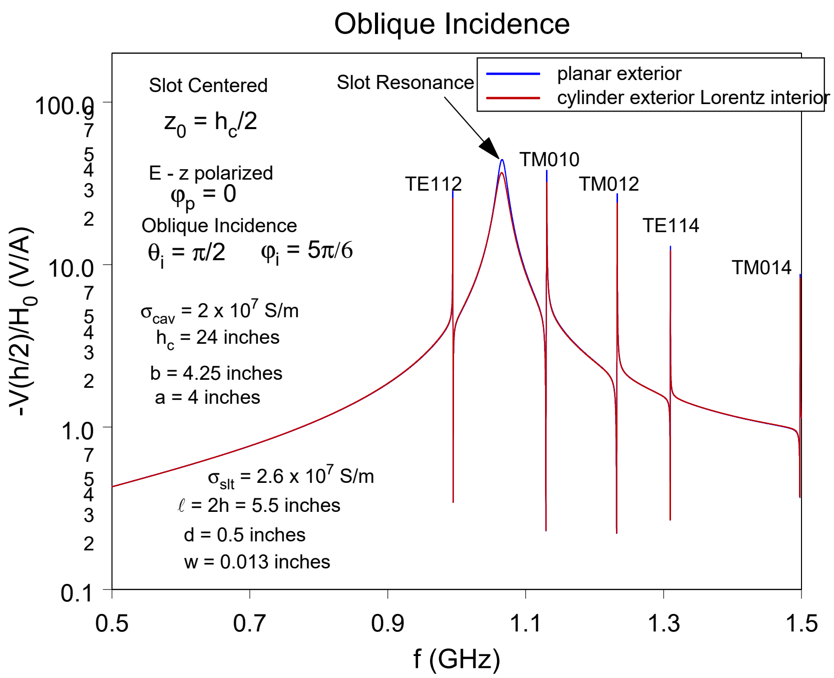

Figure 4: Example of slot voltage one quarter of the length from the end normalized by the incident magnetic field at oblique incidence, for a cylindrical cavity near the first resonance, using both the exterior cylindrical geometry (red curve) and the exterior half space approximation (blue curve). Cavity mode identies are denoted.

and an exterior half space approximation (blue curve). Figure 5 shows an example of the slot voltage one quarter of the distance from the end at oblique incidence covering the first five slot resonances using both a cylindrical exterior geometry (red curve) and an exterior half space approximation (blue curve). The half space approximation is reasonably accurate here as well. (The interior/exterior Lorentz gauge forms were used in these calculations.)

7.8 Electric Field

The TM potentials are

$$\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n} - k^2_{p,m,n}\right\} A^{TM}_{p,m,n} = \frac{\varepsilon_m \varepsilon_n}{k^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{j^3_{m,p} \cos\left(n\pi z_0/h_c\right)}{\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} \int_{-h}^{h} \cos\left(ms'/a\right) V_+\left(s'\right) ds'$$

$$= \frac{-i\omega\mu_0 \varepsilon_m \varepsilon_n}{k^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{j^3_{m,p} \cos\left(n\pi z_0/h_c\right)}{2\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} m^+_{es}$$

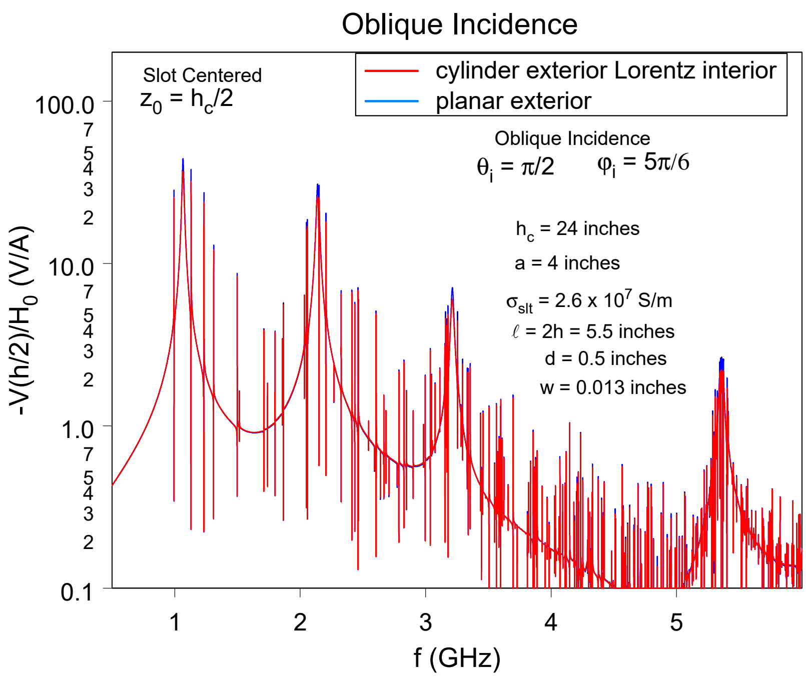

Figure 5: Example of slot voltage one quarter of the length from the end at normal incidence, for a cylindrical cavity covering the first five slot resonances, using both the exterior cylindrical geometry (red curve) and the exterior half space approximation (blue curve).

$$\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n} - k^2_{p,m,n}\right\} B^{TM}_{p,m,n} = \frac{\varepsilon_m \varepsilon_n}{k^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{j^3_{m,p}\cos\left(n\pi z_0/h_c\right)}{\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} \int_{-h}^{h} \sin\left(ms'/a\right) V_+\left(s'\right) ds'$$

$$= \frac{-i\omega\mu_0 \varepsilon_m \varepsilon_n}{k^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{j^3_{m,p}\cos\left(n\pi z_0/h_c\right)}{2\pi h_c a^5 J_{m-1}\left(j_{m,p}\right)} m^+_{os}$$

$$m^+_{es} = \frac{i2}{\omega\mu_0} \int_{-h}^{h} \cos\left(ms'/a\right) V_+\left(s'\right) ds'$$

$$m^+_{os} = \frac{i2}{\omega\mu_0} \int_{-h}^{h} \sin\left(ms'/a\right) V_+\left(s'\right) ds'$$

$$A^{(p,m,n)}_{e\rho} = \frac{\varepsilon_0}{k^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{1}{\rho} J_m\left(j_{m,p}\rho/a\right) m \left[A^{TM}_{p,m,n}\sin\left(m\varphi\right) - B^{TM}_{p,m,n}\cos\left(m\varphi\right)\right] \cos\left(n\pi z/h_c\right)$$

$$A^{(p,m,n)}_{e\varphi} = \frac{\varepsilon_0}{k^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{j_{m,p}}{a} J'_m\left(j_{m,p}\rho/a\right) \left[A^{TM}_{p,m,n}\cos\left(m\varphi\right) + B^{TM}_{p,m,n}\sin\left(m\varphi\right)\right] \cos\left(n\pi z/h_c\right)$$

The TE potentials are

$$\left\{k^2 + kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\} A^{TE}_{p,m,n}$$

$$= \frac{\varepsilon_m/\left(-i\omega\mu_0\right)}{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{2m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)} \int_{-h}^{h} \sin\left(ms'/a\right) V_+\left(s'\right) ds'$$

$$= \frac{\varepsilon_m \varepsilon_n}{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{2\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)} m^+_{os}$$

$$\left\{k^2 + kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n} - k'^2_{p,m,n}\right\} B^{TE}_{p,m,n}$$

$$= \frac{\varepsilon_m/\left(-i\omega\mu_0\right)}{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{2m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)} \int_{-h}^{h} \cos\left(ms'/a\right) V_+\left(s'\right) ds'$$

$$= \frac{\varepsilon_m \varepsilon_n}{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{m\pi n j'^2_{m,p}\cos\left(n\pi z_0/h_c\right)}{2\pi h_c^2 a^5 \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)} m^+_{es}$$

$$A^{(p,m,n)}_{e\rho} = \frac{1/\left(i\omega\right)}{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right) \left[A^{TE}_{p,m,n}\cos\left(m\varphi\right) - B^{TE}_{p,m,n}\sin\left(m\varphi\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$A^{(p,m,n)}_{e\varphi} = \frac{-1/\left(i\omega\right)}{k'^2_{p,m,n} - \left(n\pi/h_c\right)^2} \frac{m}{\rho} J_m\left(j'_{m,p}\rho/a\right) \left[A^{TE}_{p,m,n}\sin\left(m\varphi\right) + B^{TE}_{p,m,n}\cos\left(m\varphi\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$A_{ez}^{(p,m,n)} = \frac{1}{i\omega} J_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}^{TE}\cos(m\varphi) + B_{p,m,n}^{TE}\sin(m\varphi)\right]\sin(n\pi z/h_c)$$

The electric field is found as

$$\underline{D} = \varepsilon_0 \underline{E} = -\nabla \times \underline{A}_e$$

$$\varepsilon_0 E_\rho = \frac{\partial}{\partial z} A_{e\varphi} - \frac{1}{\rho}\frac{\partial}{\partial \varphi} A_{ez}$$

$$= \frac{-\varepsilon_0 (n\pi/h_c)}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{j_{m,p}}{a} J'_m(j_{m,p}\rho/a)\left[A_{p,m,n}^{TM}\cos(m\varphi) + B_{p,m,n}^{TM}\sin(m\varphi)\right]\sin(n\pi z/h_c)$$

$$-\frac{-1/(i\omega)}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{m}{\rho} J_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}^{TE}\sin(m\varphi) + B_{p,m,n}^{TE}\cos(m\varphi)\right]\left(\frac{n\pi}{h_c}\right)^2 \sin(n\pi z/h_c)$$

$$+\frac{m}{\rho}\frac{1}{i\omega} J_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}^{TE}\sin(m\varphi) - B_{p,m,n}^{TE}\cos(m\varphi)\right]\sin(n\pi z/h_c)$$

$$= \frac{-\varepsilon_0 (n\pi/h_c)}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{j_{m,p}}{a} J'_m(j_{m,p}\rho/a)\left[A_{p,m,n}^{TM}\cos(m\varphi) + B_{p,m,n}^{TM}\sin(m\varphi)\right]\sin(n\pi z/h_c)$$

$$+\frac{1/(i\omega)}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{m}{\rho} J_m\left(j'_{m,p}\rho/a\right)\left[k'^2_{p,m,n} A_{p,m,n}^{TE}\sin(m\varphi) - \left\{k'^2_{p,m,n} - 2(n\pi/h_c)^2\right\} B_{p,m,n}^{TE}\cos(m\varphi)\right]\sin(n\pi z/h_c)$$

$$\varepsilon_0 E_\varphi = \frac{\partial}{\partial \rho} A_{ez} - \frac{\partial}{\partial z} A_{e\rho}$$

$$= \frac{\varepsilon_0 (n\pi/h_c)}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{1}{\rho} J_m(j_{m,p}\rho/a)\, m\left[A_{p,m,n}^{TM}\sin(m\varphi) - B_{p,m,n}^{TM}\cos(m\varphi)\right]\sin(n\pi z/h_c)$$

$$+\frac{1}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{1}{i\omega} \frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}^{TE}\cos(m\varphi) - B_{p,m,n}^{TE}\sin(m\varphi)\right]\left(\frac{n\pi}{h_c}\right)^2 \sin(n\pi z/h_c)$$

$$+\frac{1}{i\omega}\frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right)\left[A_{p,m,n}^{TE}\cos(m\varphi) + B_{p,m,n}^{TE}\sin(m\varphi)\right]\sin(n\pi z/h_c)$$

$$= \frac{\varepsilon_0 (n\pi/h_c)}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{m}{\rho} J_m(j_{m,p}\rho/a)\left[A_{p,m,n}^{TM}\sin(m\varphi) - B_{p,m,n}^{TM}\cos(m\varphi)\right]\sin(n\pi z/h_c)$$

$$+\frac{1}{k'^2_{p,m,n} - (n\pi/h_c)^2} \frac{1}{i\omega} \frac{j'_{m,p}}{a} J'_m\left(j'_{m,p}\rho/a\right)\left[k'^2_{p,m,n} A_{p,m,n}^{TE}\cos(m\varphi) + \left\{k'^2_{p,m,n} - 2(n\pi/h_c)^2\right\} B_{p,m,n}^{TE}\sin(m\varphi)\right]\sin(n\pi z/h_c)$$

$$\varepsilon_0 E_z = \frac{1}{\rho}\frac{\partial}{\partial \varphi} A_\rho - \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\varphi)$$

$$= -\frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}\left(\frac{j_{m,p}}{a}\right)^2 \left[J_m''(j_{m,p}\rho/a) + \frac{1}{j_{m,p}\rho/a}J_m'(j_{m,p}\rho/a) - \frac{m^2}{(j_{m,p}\rho/a)^2}J_m(j_{m,p}\rho/a)\right]$$

$$\left[A_{p,m,n}^{TM}\cos(m\varphi) + B_{p,m,n}^{TM}\sin(m\varphi)\right]\cos(n\pi z/h_c)$$

$$= \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}\left(\frac{j_{m,p}}{a}\right)^2 J_m(j_{m,p}\rho/a)\left[A_{p,m,n}^{TM}\cos(m\varphi) + B_{p,m,n}^{TM}\sin(m\varphi)\right]\cos(n\pi z/h_c)$$

Then

$$E_\rho = \frac{-i\omega\mu_0}{2V_{cav}}\sum_{m=0}^{\infty}\varepsilon_m\left[m_{es}^+\cos(m\varphi) + m_{os}^+\sin(m\varphi)\right]\sum_{n=0}^{\infty}\varepsilon_n(n\pi/h_c)\cos(n\pi z_0/h_c)\sin(n\pi z/h_c)$$

$$\sum_{p=1}^{\infty}\left[\frac{-1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}}\frac{J_m'(j_{m,p}\rho/a)}{J_m'(j_{m,p})}\right.$$

$$\left.+\frac{1}{\left\{k^2 + kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right\}}\frac{a}{\rho}\frac{(n\pi/h_c)^2 m^2/j_{m,p}'^2}{k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\right]$$

$$-\frac{i\omega\mu_0}{2V_{cav}}\frac{1}{a\rho}\frac{\partial^2}{\partial\varphi^2}\sum_{m=0}^{\infty}\varepsilon_m\left[m_{es}^+\cos(m\varphi) - m_{os}^+\sin(m\varphi)\right]\sum_{n=0}^{\infty}\varepsilon_n(n\pi/h_c)\cos(n\pi z_0/h_c)\sin(n\pi z/h_c)$$

$$\sum_{p=1}^{\infty}\frac{1}{\left\{k^2 + kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right\}}\frac{1}{k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}$$

or

$$E_\rho = -\frac{i\omega\mu_0}{2V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\varepsilon_m\varepsilon_n\left(\frac{n\pi}{h_c}\right)\cos(n\pi z_0/h_c)\sin(n\pi z/h_c)$$

$$\left[\frac{m^2/\left(k_{p,m,n}'^2 a\rho\right)}{\left\{k^2 + kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right\}\left(1 - m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\right.$$

$$\left\{\left(\frac{n^2\pi^2/h_c^2}{j_{m,p}'^2/a^2} + 1\right)m_{os}^+\sin(m\varphi) + \left(\frac{n^2\pi^2/h_c^2}{j_{m,p}'^2/a^2} - 1\right)m_{es}^+\cos(m\varphi)\right\}$$

$$\left.+\frac{1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}}\frac{J_m'(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\left\{m_{es}^+\cos(m\varphi) + m_{os}^+\sin(m\varphi)\right\}\right]$$

Noting that

$$\frac{m}{a}m_{es}^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\frac{m}{a}\cos(ms'/a)V_+(s')\,ds' = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\frac{\partial}{\partial s'}\sin(ms'/a)V_+(s')\,ds'$$

$$= -\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)\frac{\partial}{\partial s'}V_+\left(s'\right)ds' = \frac{1}{\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)q_m^+\left(s'\right)ds'$$

$$\frac{m}{a}m_{os}^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\frac{m}{a}\sin\left(ms'/a\right)V_+\left(s'\right)ds' = -\frac{i2}{\omega\mu_0}\int_{-h}^{h}\frac{\partial}{\partial s'}\cos\left(ms'/a\right)V_+\left(s'\right)ds'$$

$$= \frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)\frac{\partial}{\partial s'}V_+\left(s'\right)ds' = -\frac{1}{\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)q_m^+\left(s'\right)ds'$$

$$\frac{1}{a\rho}\frac{\partial^2}{\partial\varphi^2}\sum_{m=0}^{\infty}\varepsilon_m\left[m_{es}^+\cos\left(m\varphi\right) - m_{os}^+\sin\left(m\varphi\right)\right] = -\frac{1}{\rho}\frac{\partial}{\partial\varphi}\sum_{m=0}^{\infty}\varepsilon_m\left[\frac{m}{a}m_{es}^+\sin\left(m\varphi\right) + \frac{m}{a}m_{os}^+\cos\left(m\varphi\right)\right]$$

$$= -\sum_{m=0}^{\infty}\frac{m}{\rho}\varepsilon_m\left[\frac{m}{a}m_{es}^+\cos\left(m\varphi\right) - \frac{m}{a}m_{os}^+\sin\left(m\varphi\right)\right]$$

$$\frac{\partial}{\partial s}I_m^+ = 2\frac{\partial}{\partial s}V_+ = i\omega q_m^+$$

Taking the even case as

$$V_e\left(s\right) = \sum_{m',odd}V_{em'}\cos\left(m'\pi s/\ell\right)$$

$$m_{es}^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos\left(ms'/a\right)V_+\left(s'\right)ds' = \frac{i2}{\omega\mu_0}\sum_{m',odd}V_{em'}\int_{-h}^{h}\cos\left(ms'/a\right)\cos\left(m'\pi s'/\ell\right)ds'$$

$$= \frac{i}{\omega\mu_0}\sum_{m',odd}V_{em'}\int_{-h}^{h}\left[\cos\left(\left(m/a - m'\pi/\ell\right)s'\right) + \cos\left(\left(m/a + m'\pi/\ell\right)s'\right)\right]ds'$$

$$= \frac{i2}{\omega\mu_0}\sum_{m',odd}V_{em'}\left[\frac{\sin\left(\left(m/a - m'\pi/\ell\right)h\right)}{\left(m/a - m'\pi/\ell\right)} + \frac{\sin\left(\left(m/a + m'\pi/\ell\right)h\right)}{\left(m/a + m'\pi/\ell\right)}\right]$$

$$\frac{\partial}{\partial s'}V_e\left(s'\right) = -\sum_{m',odd}\frac{m'\pi}{\ell}V_{em'}\sin\left(m'\pi s'/\ell\right)$$

In the odd case we take

$$V_o\left(s\right) = \sum_{m',even}V_{om'}\sin\left(m'\pi s/\ell\right)$$

$$m_{os}^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin\left(ms'/a\right)V_+\left(s'\right)ds' = \frac{i2}{\omega\mu_0}\sum_{m',even}V_{om'}\int_{-h}^{h}\sin\left(ms'/a\right)\sin\left(m'\pi s'/\ell\right)ds'$$

$$= \frac{i}{\omega\mu_0}\sum_{m',even}V_{om'}\int_{-h}^{h}\left[\cos\left(\left(m/a - m'\pi/\ell\right)s'\right) + \cos\left(\left(m/a + m'\pi/\ell\right)s'\right)\right]ds'$$

$$= \frac{i2}{\omega\mu_0}\sum_{m',even}V_{om'}\left[\frac{\sin\left(\left(m/a - m'\pi/\ell\right)h\right)}{\left(m/a - m'\pi/\ell\right)} - \frac{\sin\left(\left(m/a + m'\pi/\ell\right)h\right)}{\left(m/a + m'\pi/\ell\right)}\right]$$

$$\frac{\partial}{\partial s'} V_o(s') = \sum_{m',even} \frac{m'\pi}{\ell} V_{om'} \cos(m'\pi s'/\ell)$$

Then the axial field is

$$E_z = \frac{-i\omega\mu_0}{2V_{cav}} \sum_{m=0}^{\infty} \varepsilon_m \left[m_{es}^{+} \cos(m\varphi) + m_{os}^{+} \sin(m\varphi)\right] \sum_{n=0}^{\infty} \varepsilon_n \cos(n\pi z_0/h_c) \cos(n\pi z/h_c)$$

$$\sum_{p=1}^{\infty} \frac{1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}} \left(\frac{j_{m,p}}{a}\right) \frac{J_m(j_{m,p}\rho/a)}{J_m'(j_{m,p})}$$

and the radial field is

$$E_\rho(\rho = a) = \frac{-i\omega\mu_0}{2V_{cav}} \sum_{m=0}^{\infty} \varepsilon_m \left[m_{es}^{+} \cos(m\varphi) + m_{os}^{+} \sin(m\varphi)\right] \sum_{n=0}^{\infty} \varepsilon_n (n\pi/h_c) \cos(n\pi z_0/h_c) \sin(n\pi z/h_c)$$

$$\sum_{p=1}^{\infty} \left[\frac{-1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}} + \frac{1}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q_{p,m,n}^{TE} - k'^2_{p,m,n}\right\}} \frac{(n\pi/h_c)^2 m^2/j'^2_{m,p}}{k'^2_{p,m,n}\left(1 - m^2/j'^2_{m,p}\right)}\right]$$

$$-\frac{i\omega\mu_0}{2V_{cav}} \frac{1}{a^2} \frac{\partial^2}{\partial\varphi^2} \sum_{m=0}^{\infty} \varepsilon_m \left[m_{es}^{+} \cos(m\varphi) - m_{os}^{+} \sin(m\varphi)\right] \sum_{n=0}^{\infty} \varepsilon_n (n\pi/h_c) \cos(n\pi z_0/h_c) \sin(n\pi z/h_c)$$

$$\sum_{p=1}^{\infty} \frac{1}{\left\{k^2 + kk'_{p,m,n}(1+i)/Q_{p,m,n}^{TE} - k'^2_{p,m,n}\right\}} \frac{1}{k'^2_{p,m,n}\left(1 - m^2/j'^2_{m,p}\right)}$$

The results presented are again computed using the preceding even (1) and odd (2) Galerkin systems; the half space systems (3) and (4) are again used to implement the solution using the approximate radiation for an exterior half space. Figure 6 shows an example of the interior radial field at the cylindrical wall between the bottom end cap and the slot at the center for oblique incidence. The slot geometry and material properties are the same as in the preceding voltage example. The results with a cylindrical exterior are shown in red and the results using an exterior half space approximation are shown in blue. Figure 7 shows the corresponding axial field at the lower end of the cavity halfway between the center and the cylindrical wall. We see from these that the exterior half-space approximation is reasonable even for $\ell/(\pi b) \approx 0.412$.

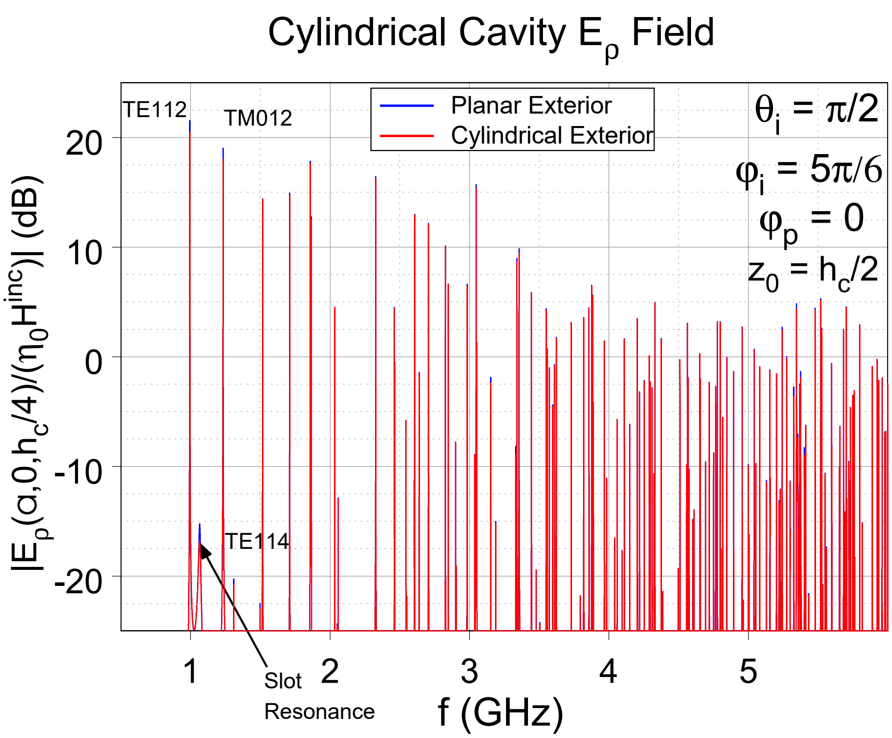

Figure 6: Radial electric field scaled by incident electric field (free space impedance times incident magnetic field) at cylindrical wall in a cylindrical cavity with an azimuthal slot at its center. A plane wave drives the slot impinging from an oblique angle. The red curve uses a cylindrical exterior and the blue curve uses an exterior half-space approximation.

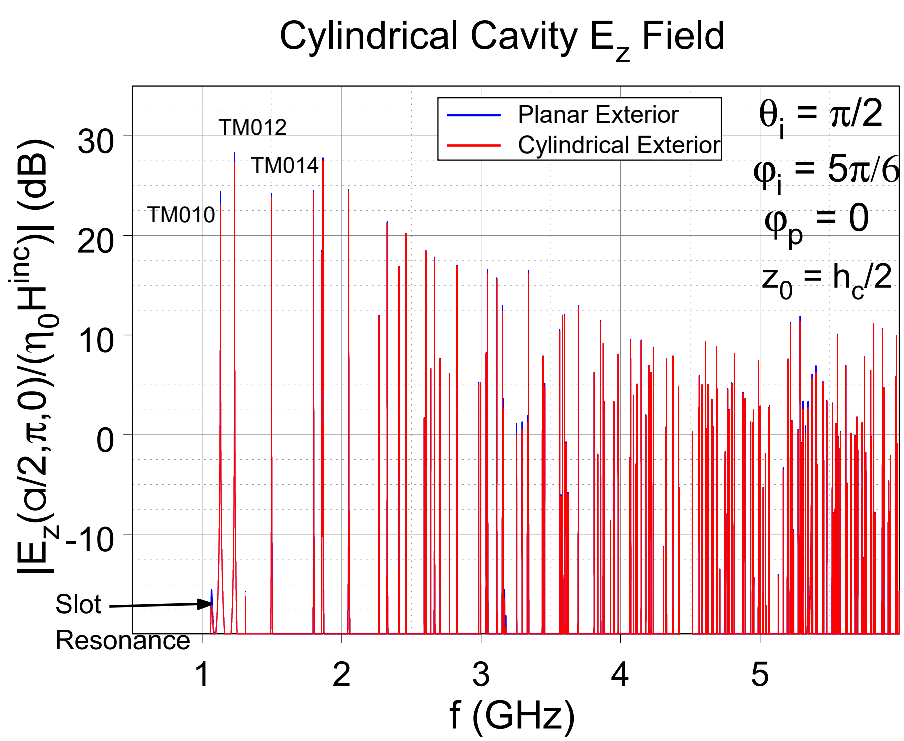

Figure 7: Axial electric field scaled by incident electric field (free space impedance times incident magnetic field) at bottom wall halfway between the center and cylindrical wall in a cylindrical cavity with an azimuthal slot at its center. A plane wave drives the slot impinging from an oblique angle. The red curve uses a cylindrical exterior and the blue curve uses an exterior half-space approximation.

8 POWER BALANCE APPROACH

This section discusses the conservation of net power flow in the steady state and how we can estimate shielding effectiveness of a cavity with apertures by using this conservation law. We initially discuss the high frequency region with overlapping cavity modes, where the slot aperture radiation properties to the interior are nearly the same as in free space. The input power to the cavity interior, or slot transmitted power, is set equal to (balanced between) cavity wall losses and slot aperture losses, including absorption and radiation (this approach assumes an incoherent power summation when the same slot is responsible for the drive and losses). We also discuss slot aperture received power cross sections with uniformly distributed matched loads, providing bounding net power cross sections (net power delivered into cavity), which can be applied over the entire frequency band when set equal to interior cavity wall losses. The uniformly distributed matched load is taken to have arbitrary values to maintain slot operation with a half sine voltage distribution for the maximum power received. The case is also examined when the uniformly distributed load properties are limited so the slot operates with the voltage distribution at the nearest resonance, where the oblique incidence angle is used to maximize power reception. We limit analysis to the uniformly distributed load not only for simplicity, but also because the slot walls are lossy and we expect any increases associated with nonuniform distributions to be very limited (this is similar to the problems which occur with supergain antenna current distributions).

We again focus on a cylindrical cavity having the absolute coordinate system with z axis along the cylinder axis having an azimuthal narrow slot, and the incident wave is in general taken to have an oblique angle with respect to the slot length for comparisons (the angles of incidence in the global coordinate system are the same as described in the initial sections on the incident wave). However, in this section the slot power reception in the power balance is set up with the simpler problem of a slot on a half space and uses a local slot coordinate system with z along the slot, with the voltage measured across the slot, with x being the direction of short circuit current from the positive voltage side to the reference side of the slot width. Some equations are numbered here to summarize what is illustrated in the figures.

8.1 Oblique Incidence Slot Cross Sections For Half Space

When the wave impinges on the slot at oblique incidence

$$K_x^{sc} = -H_z^{sc} = -2H_z^{inc} = -2H_0 e^{iq_0 z} \sin\theta_0$$

where θ_0 is the angle between the slot length (z direction) and the incident wavevector $\underline{k}$ with

$$q_0 = k\cos\theta_0$$

and here we are taking the incident magnetic field in the plane of incidence (plane containing $\underline{k}$ and the normal $\underline{n}$), where the $\underline{k}$ vector is also taken to be in the plane containing $\underline{e}_z$ and $\underline{n}$. The transmission line solution for the voltage, satisfying $V(\pm h) = 0$, is then [1]

$$V(z) = -\frac{Z2H_0\sin\theta_0}{q_0^2 - \Gamma^2}\left[\cos(q_0 z) - \cos(q_0 h)\frac{\cos(\Gamma z)}{\cos(\Gamma h)} + i\sin(q_0 z) - i\sin(q_0 h)\frac{\sin(\Gamma z)}{\sin(\Gamma h)}\right]$$

Note that the radiated power into both half spaces at the resonances $\Gamma h \approx \Gamma' h = n\pi/2$, with $\Gamma'' h << 1$, where $\Gamma = \Gamma' + i\Gamma''$ is the propagation constant along the slot transmission line,

$$P_{rad} = G_{rad}\int_{-h}^{h} |V(z)|^2 dz = hG_{rad}\left\{\begin{array}{l} |V_{0e}|^2 \ , \ n \text{ odd} \\ |V_{0o}|^2 \ , \ n \text{ even}\end{array}\right\}$$

where V_{0e} is the coefficient of the even term $\cos(\Gamma z) \approx \cos(\Gamma' z)$ in the voltage and V_{0o} is the coefficient of the odd term $\sin(\Gamma z) \approx \sin(\Gamma' z)$ in the voltage with radiation conductance [1] (which was derived integrating the scalar product of magnetic current and radiated magnetic field [1])

$$\pi\eta_0 h G_{rad} = \left(\frac{kh}{n\pi} + \frac{n\pi/4}{kh}\right)\{\mathrm{Cin}(n\pi + 2kh) - \mathrm{Cin}(n\pi - 2kh)\}$$

$$+\left(\frac{kh}{n\pi} - \frac{n\pi/4}{kh}\right) n\pi \{\mathrm{Si}(n\pi + 2kh) - \mathrm{Si}(n\pi - 2kh)\} - 2\sin^2(n\pi/2 + kh)$$

with asymptotic forms

$$\pi\eta_0 h G_{rad} \sim \frac{2}{3}\left(\frac{4kh}{n\pi}\right)^2 \ , \ kh << 1 \ , \ n \text{ odd}$$

$$\pi\eta_0 h G_{rad} \sim \frac{2}{15}\frac{(2kh)^4}{(n\pi)^2} \ , \ kh << 1 \ , \ n \text{ even}$$

$$\pi\eta_0 h G_{rad} \sim \pi kh \left[1 - \left(\frac{n\pi}{2kh}\right)^2\right] \ , \ kh >> 1$$

The reactive part of the radiation admittance $Y_{rad} = G_{rad} - iB_{rad}$ is [10]

$$\pi\eta_0 h B_{rad} = -\cos(n\pi)\sin(2kh) + \frac{kh}{n\pi}\left[1 + \left(\frac{n\pi}{2kh}\right)^2\right][\mathrm{Si}(n\pi + 2kh) + \mathrm{Si}(n\pi - 2kh)]$$

$$+kh\left[1 - \left(\frac{n\pi}{2kh}\right)^2\right][-C_e + 2\ln 2 - \mathrm{Cin}(n\pi + 2kh) - \mathrm{Cin}(n\pi - 2kh)]$$

with asymptotic forms

$$\pi\eta_0 h B_{rad} \sim -kh + \left(\frac{2kh}{n\pi} + \frac{n\pi}{2kh}\right)\mathrm{Si}(n\pi) + n\pi\left(\frac{2kh}{n\pi} - \frac{n\pi}{2kh}\right)[-C_e/2 + \ln 2 - \mathrm{Cin}(n\pi)] \ , \ kh << 1$$

$$\pi\eta_0 h B_{rad} \sim 2kh\left[1 - \left(\frac{n\pi}{2kh}\right)^2\right][-C_e/2 + \ln 2 - \gamma' - \ln(2kh)] + (n\pi)^2/(4kh) \ , \ kh >> 1$$

At the resonances, the transmitted power can be written as [1]

$$P_{trans} \approx \frac{8hG_{rad}\left|H^{inc}\right|^2 \sin^2\theta_0/(n\pi/2)^2}{\left\{1 - (k/\Gamma')^2\cos^2\theta_0\right\}^2 [G_{rad} + R_{int}\{\omega C + B_{rad}\}/(\omega L + X_{int})]^2}$$

$$\left\{\begin{array}{l} \cos^2(kh\cos\theta_0) \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ , \ n \text{ even} \end{array}\right\}$$

If a gasket were present with $k < \Gamma'$ the denominator factor in braces would not vanish for any θ_0; in such a case (particularly for $k/\Gamma' << 1$) the maximum occurs in the n odd case at normal incidence $\theta_0 = \pi/2$ [1], which decreases rapidly for high frequency $\Gamma' h = n\pi/2 >> 1$. In the case being studied here where no gasket is present $\Gamma' \to k$, and we have

$$P_{trans} = P_{rad}/2 \approx \frac{8hG_{rad}\left|H^{inc}\right|^2/(n\pi/2)^2}{\sin^2\theta_0\left[G_{rad}+R_{int}\left\{\omega C+B_{rad}\right\}/(\omega L+X_{int})\right]^2}\left\{\begin{array}{c}\cos^2(kh\cos\theta_0)\ ,\ n\text{ odd}\\ \sin^2(kh\cos\theta_0)\ ,\ n\text{ even}\end{array}\right\}$$

where now

$$\pi\eta_0 hG_{rad} = \mathrm{Cin}(4kh) = \mathrm{Cin}(2n\pi)\ ,\ kh = n\pi/2$$

$$\pi\eta_0 hG_{rad} = \mathrm{Cin}(4kh) = \mathrm{Cin}(2n\pi) \sim \gamma' + \ln(4kh)\ ,\ kh >> 1\ ,\ kh = n\pi/2$$

Note that extending this form to low frequencies

$$\pi\eta_0 hG_{rad} = \mathrm{Cin}(4kh) \sim 4(kh)^2\ ,\ kh << 1$$

results in nearly a factor of four overestimate. Also

$$\pi\eta_0 hB_{rad} = \mathrm{Si}(4kh) = \mathrm{Si}(2n\pi)\ ,\ kh = n\pi/2$$

$$\pi\eta_0 hB_{rad} = \mathrm{Si}(4kh) \sim \pi/2\ ,\ kh >> 1\ ,\ kh = n\pi/2$$

We can write this as [1]

$$P_{trans} = \sigma_{trans}\eta_0\left|\underline{H}^{inc}\right|^2 = \sigma_{trans}\eta_0\left|H_z^{inc}\right|^2/\sin^2\theta_0$$

where the transmission cross section is

$$\sigma_{trans} \approx q_{trans}^{tl}\frac{2\pi\ell^2}{(n\pi/2)^2\,\mathrm{Cin}(2n\pi)}\left\{\begin{array}{c}\cos^2(kh\cos\theta_0)\ ,\ n\text{ odd}\\ \sin^2(kh\cos\theta_0)\ ,\ n\text{ even}\end{array}\right\}/\sin^2\theta_0$$

but with mismatch factor

$$q_{trans}^{tl} = \frac{G_{rad}^2}{\left[G_{rad}+R_{int}\left\{\omega C+B_{rad}\right\}/(\omega L+X_{int})\right]^2} \approx \frac{G_{rad}^2}{\left[G_{rad}+R_{int}/(\eta_0\pi/\Omega_e)^2\right]^2}$$

where the final approximation ignores the corrections to the leading terms ωC and ωL. If the wall loss term becomes small as for PEC walls $R_{int} \to 0$ and $q_{trans}^{tl} \to 1$. We can maximize the cross section as a function of θ_0 resulting in the equation

$$\left\{\begin{array}{c}\cot(kh\cos\theta_0)\ ,\ n\text{ odd}\\ -\tan(kh\cos\theta_0)\ ,\ n\text{ even}\end{array}\right\}/(kh) = \frac{\sin^2\theta_0}{\cos\theta_0}$$

or

$$\frac{\tan(khu)}{khu} = \frac{2-u}{1-u}\ ,\ \cos\theta_0 = 1-u\ ,\ kh = n\pi/2$$

or with $x_r = khu \sim kh\theta_0^2/2$

$$\tan(x_r)/x_r = \frac{2-x_r/(kh)}{1-x_r/(kh)} \sim 2$$

For large kh (or n) we can take the right hand side to be 2 and we find the first root

$$x_r \approx 1.1656$$

and

$$\left\{ \begin{array}{c} \cos^2(kh\cos\theta_0) \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ , \ n \text{ even} \end{array} \right\} / \sin^2\theta_0 = \left\{ \begin{array}{c} \sin^2(khu) \ , \ n \text{ odd} \\ \sin^2(kh - khu) \ , \ n \text{ even} \end{array} \right\} \frac{1/(2u)}{1-u/2} = \sin^2(x_r)\frac{kh/(2x_r)}{1-x_r/(2kh)}$$

$$\sim \frac{1}{2}kh\frac{\sin^2 x_r}{x_r}$$

Then we obtain the maximum transmission cross section

$$\sigma_{trans} \sim q_{trans}^{tl}\frac{\pi\ell^2}{(kh)\,\mathrm{Cin}\,(2k\ell)}\frac{\sin^2 x_r}{x_r} = q_{trans}^{tl}\frac{\lambda\ell}{\mathrm{Cin}\,(4\pi\ell/\lambda)}\frac{4x_r}{4x_r^2+1}$$

$$\sim \frac{\lambda\ell}{1.38\,[\gamma' + \ln(4\pi\ell/\lambda)]}q_{trans}^{tl}$$

where $\gamma' \approx 0.5772$ is Euler's constant and the angle of the maximum is

$$\sqrt{\frac{2\lambda x_r}{\pi\ell}} \sim \theta_0$$

Recalling that the cross section of the linear array (given in the next subsection) is $\lambda\ell/\pi$ we see that the matched case (small wall loss) $q_{trans}^{tl} \to 1$ of this oblique result is less than the linear array beyond the first resonance $\ell/\lambda > 1/2$ (actually in the formula for $\ell/\lambda > 3/8$, although the asymptotic form requires $\ell/\lambda > 0.435$), because the denominator of the oblique cross section is slightly greater than π due to the slow logarithmic growth; nevertheless, the linear array is a reasonably tight bound on this oblique cross section. Figure 8 shows the normalized power transmission through the PEC slot between half spaces at oblique incidence angles computed by use of a Galerkin solution [9]. The dashed black curve represents the oblique transmission bound with the logarithmic denominator, and the dark gray curve is the linear array result. The light gray curve shows the λ^2 behavior (adjusted to match the first transmission resonance peak). Also shown are curves for a lossy gasket filling the slot (the gasket is lossy enough to eliminate the length resonances and thus has a maximum penetration at normal incidence) [1].

From the preceding radiated power we can write the absorbed power in the slot walls as

$$P_{abs} = \{R_{int}\{\omega C + B_{rad}\}/(\omega L + X_{int})\}\int_{-h}^{h}|V(z)|^2\,dz$$

$$\approx \frac{16h\{R_{int}\{\omega C + B_{rad}\}/(\omega L + X_{int})\}\left|H^{inc}\right|^2/(n\pi/2)^2}{\sin^2\theta_0\,[G_{rad} + R_{int}\{\omega C + B_{rad}\}/(\omega L + X_{int})]^2}\left\{ \begin{array}{c} \cos^2(kh\cos\theta_0) \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ , \ n \text{ even} \end{array} \right\}$$

We can write this as

$$P_{abs} = \sigma_{abs}\eta_0\left|\underline{H}^{inc}\right|^2 = \sigma_{abs}\eta_0\left|H_z^{inc}\right|^2/\sin^2\theta_0$$

where the absorption cross section is

$$\sigma_{abs} \approx q_{abs}^{tl}\frac{2\pi\ell^2}{(kh)^2\,\mathrm{Cin}\,(4kh)}\left\{ \begin{array}{c} \cos^2(kh\cos\theta_0) \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ , \ n \text{ even} \end{array} \right\} / \sin^2\theta_0$$

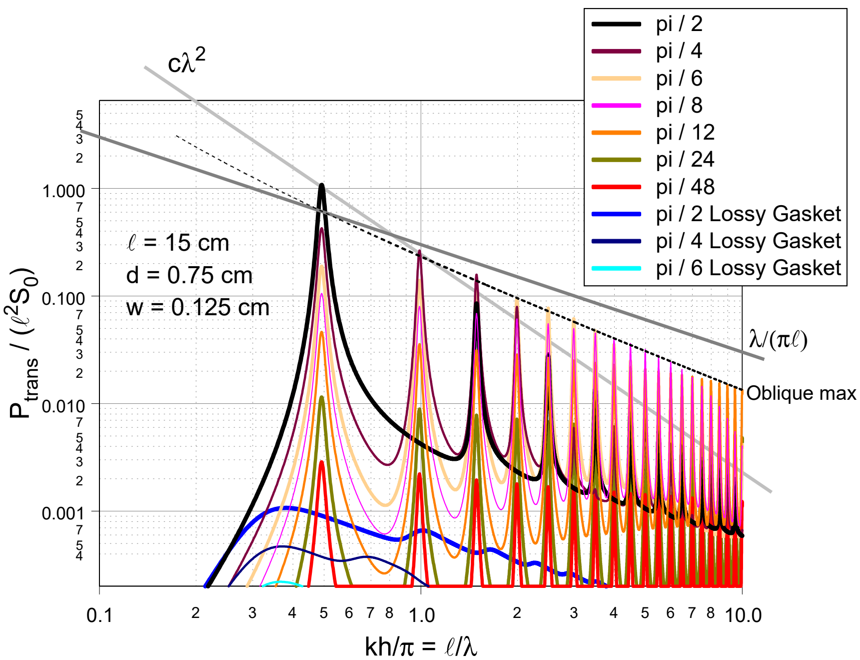

Figure 8: Power transmission through the PEC slot at oblique incidence angles from the Galerkin solution. The dashed black curve represents the oblique bound. The dark gray line is the linear array cross section. The light gray line shows λ^2 behavior with constant multiplier adjusted to match the first transmission peak. The case where a lossy gasket fills the slot is also shown illustrating the maximum penetration moving to normal incidence.

and

$$q_{abs}^{tl} = \frac{2G_{rad}\left\{R_{int}\left\{\omega C + B_{rad}\right\}/\left(\omega L + X_{int}\right)\right\}}{\left[G_{rad} + R_{int}\left\{\omega C + B_{rad}\right\}/\left(\omega L + X_{int}\right)\right]^2} \approx \frac{2G_{rad}R_{int}/Z_0^2}{\left[G_{rad} + R_{int}/Z_0^2\right]^2}$$

$$Z_0 = \sqrt{L/C} \sim \eta_0\pi/\Omega_e$$

$$Z_{int} = R_{int} - iX_{int} \approx \left(\frac{L}{L^{intr}}\right)^2 Z_{int}^{intr}$$

$$Z_{int}^{intr} = R_{int}^{intr} - iX_{int}^{intr} = 2Z_s/d$$

$$L = \mu_0\pi/\Omega_e$$

$$L^{intr} = \mu_0 w/d$$

$$\Omega_e = 2\ln\left(2h/a_e\right) + 2\left(\ln 2 - 7/3\right)$$

$$a_e \sim \frac{2w}{\pi e}e^{-\pi d/(2w)}\ ,\ d > 0.3w$$

$$Z_s = \left(1 - i\right)R_s$$

$$R_s = 1/\left(\sigma\delta\right)$$

$$\delta = \sqrt{2/\left(\omega\mu\sigma\right)}$$

8.1.1 Linear Array

The electrically longer slot becomes similar in behavior to a linear array of antenna elements (the array length is aligned with the slot length). Note that the broadside linear array cross section behavior is [1]

$$\sigma_{trans} = q_{trans}^{tl}\frac{\lambda^2}{4\pi}G_d^{LA} = q_{trans}^{tl}\lambda\ell/\pi = q_{trans}^{tl}2\ell/k$$

$$G_d^{LA} = 4\ell/\lambda = 2k\ell/\pi$$

8.1.2 Low Frequency Slot

We expect at low frequencies this gain to transition to the gain of a low frequency slot which is twice the low frequency dipole antenna gain of 3/2

$$G_d^0 \sim 2\left(3/2\right) = 3$$

$$\sigma_{trans} = q_{trans}^{tl}\frac{\lambda^2}{4\pi}G_d^0 = q_{trans}^{tl}\frac{\lambda^2}{4\pi}3 = q_{trans}^{tl}\frac{3\pi}{k^2}$$

8.1.3 Large Losses

For large wall or gasket losses the voltage tracks the short circuit current [1]

$$V(z) \sim V_0 e^{iq_0 z} \sim K_x^{sc}/(Y + Y_{rad})$$

$$Y = G - i\omega C$$

$$Y_{rad} = G_{rad} - iB_{rad}$$

Radiated power is then

$$P_{rad} = 2hG_{rad}|V_0|^2 = 2P_{trans}$$

Because in this lossy case the maximum transmission occurs for normal incidence $q_0 = k\cos\theta_0 = 0$, we restrict attention to this situation where the radiation conductance is [1]

$$\pi\eta_0 hG_{rad} = 2kh\,\mathrm{Si}(2kh) + \frac{\sin(2kh)}{2kh} + \cos(2kh) - 2$$

At higher frequencies

$$h\eta_0 G_{rad} \sim kh \ , \ kh >> 1$$

and

$$P_{trans} \sim \frac{kh}{\eta_0}|K_x^{sc}/(Y + Y_{rad})|^2 = \frac{4kh/\eta_0^2}{|Y + Y_{rad}|^2}\eta_0 \left|H^{inc}\right|^2 = \sigma_{trans}\eta_0 \left|H^{inc}\right|$$

If $\omega C >> G, Y_{rad}$ this can be written as

$$\sigma_{trans} \sim \frac{2\ell/k}{(C/\varepsilon_0)^2} \sim \frac{\lambda\ell/\pi}{(C/\varepsilon_0)^2}$$

This result has the frequency behavior of the broadside linear array, but for narrow slots, and particularly when the depth is large compared to the slot width $d >> w$, it has a much smaller level. Alternatively, if $G >> \omega C, Y_{rad}$ this result becomes

$$\sigma_{trans} \sim \frac{4kh/\eta_0^2}{G^2} = \frac{\lambda\ell}{\pi}\frac{(k/\eta_0)^2}{G^2} \sim \frac{\lambda\ell}{\pi}\frac{G_{rad}^2}{G^2}$$

which is again small for a conductive gasket. It is interesting that if G_{rad} is dominant we obtain the linear array cross section $\sigma_{trans} = 4kh/(\eta_0 G_{rad})^2 \approx \lambda\ell/\pi$ and hence this case can be included as

$$\sigma_{trans} \sim \frac{\lambda\ell}{\pi}\frac{G_{rad}^2}{(G + G_{rad})^2}$$

8.1.4 Fit For First Resonances & Low Frequency

The preceding asymptotic form of the transmission cross section underestimates the initial resonances. Using the solutions of

$$\tan\left(x_r\right)/x_r = \frac{2 - x_r/\left(kh\right)}{1 - x_r/\left(kh\right)}$$

at the first resonances

$$x_r = \pi/2\ ,\ kh = \pi/2$$

$$x_r = 1.291254\ ,\ kh = \pi$$

$$x_r = 1.24163\ ,\ kh = 3\pi/2$$

$$x_r = 1.220204\ ,\ kh = 2\pi$$

$$x_r \sim 1.165561\ ,\ kh >> 1$$

gives

$$\begin{aligned}
\sigma_{trans} &\approx q_{trans}^{tl}\frac{2\pi\ell^2}{\left(kh\right)^2 \mathrm{Cin}\left(4kh\right)}\left\{\begin{array}{l}\cos^2\left(kh\cos\theta_0\right)\ ,\ n \text{ odd}\\ \sin^2\left(kh\cos\theta_0\right)\ ,\ n \text{ even}\end{array}\right\}/\sin^2\theta_0\ ,\ kh = n\pi/2\\
&\approx q_{trans}^{tl}\frac{2\pi\ell^2}{\left(kh\right)^2 \mathrm{Cin}\left(4kh\right)}\sin^2\left(x_r\right)\frac{kh/\left(2x_r\right)}{1 - x_r/\left(2kh\right)}\\
&\approx q_{trans}^{tl}\frac{\ell\lambda}{\left(\pi/4\right)\mathrm{Cin}\left(4kh\right)}\ ,\ kh = \pi/2\\
&\approx q_{trans}^{tl}\frac{\ell\lambda}{1.110425\mathrm{Cin}\left(4kh\right)}\ ,\ kh = \pi\\
&\approx q_{trans}^{tl}\frac{\ell\lambda}{1.20385\mathrm{Cin}\left(4kh\right)}\ ,\ kh = 3\pi/2\\
&\approx q_{trans}^{tl}\frac{\ell\lambda}{1.2490615\mathrm{Cin}\left(4kh\right)}\ ,\ kh = 2\pi\\
&\sim q_{trans}^{tl}\frac{\lambda\ell}{1.38005014\mathrm{Cin}\left(4kh\right)}\ ,\ kh >> 1
\end{aligned}$$

A fit can thus be taken as

$$\sigma_{trans} \approx q_{trans}^{tl}\frac{\lambda\ell}{f_\sigma\left(kh\right)\mathrm{Cin}\left(4kh\right)} \approx q_{trans}^{tl}\frac{\lambda\ell}{f_\sigma\left(kh\right)\left[\gamma' + \ln\left(4\pi\ell/\lambda\right)\right]}\ ,\ kh \geq \pi/2\ ,\ \lambda \leq \ell/2$$

$$f_\sigma\left(kh\right) \approx 1.38 + \left(\pi/4 - 1.38\right)\frac{\pi/2}{kh}\ ,\ kh \geq \pi/2\ ,\ \lambda \leq \ell/2$$

The error in this fit is $f_\sigma(\pi) = 1.110425 \approx 1.0827\,(2.5\%)$, $f_\sigma(3\pi/2) = 1.20385 \approx 1.1818\,(1.8\%)$, $f_\sigma(2\pi) = 1.2490615 \approx 1.23135\,(1.4\%)$.

Noting that at the first resonance $kh = \pi/2$ we can write the cross section as

$$\sigma_{trans} \approx q_{trans}^{tl} \frac{\lambda\pi/k}{f_\sigma(kh)\,\mathrm{Cin}(4kh)} = q_{trans}^{tl} \frac{\lambda^2}{2 f_\sigma(kh)\,\mathrm{Cin}(4kh)} = q_{trans}^{tl} \frac{\lambda^2}{4\pi} \frac{2\pi}{(\pi/4)\,\mathrm{Cin}(2\pi)}$$

$$= q_{trans}^{tl} \frac{\lambda^2}{4\pi} \frac{8}{\mathrm{Cin}(2\pi)} = q_{trans}^{tl} \frac{\lambda^2}{4\pi} G_d$$

where

$$G_d = 8/\mathrm{Cin}(2\pi) \approx 2\,(1.64092237)$$

is the directivity gain of the slot at the first resonance (twice the gain of a dipole antenna). We expect at low frequencies this gain to transition to the gain of a low frequency slot

$$G_d^0 \sim 2\,(3/2) = 3$$

Thus we can take

$$\sigma_{trans} \approx q_{trans}^{tl} \frac{\lambda \ell}{(\pi/4)\,\mathrm{Cin}(2\pi)}\ ,\ k_0 h < kh < \pi/2$$

where

$$q_{trans}^{tl} \frac{4\ell^2}{k_0 h \mathrm{Cin}(2\pi)} = q_{trans}^{tl} \frac{\lambda_0 \ell}{(\pi/4)\,\mathrm{Cin}(2\pi)} = q_{trans}^{tl} \frac{\lambda_0^2}{4\pi} 3 = q_{trans}^{tl} \frac{\pi}{k_0^2} 3$$

or

$$k_0 h = \frac{\pi}{2} \frac{3}{8} \mathrm{Cin}(2\pi) \approx 0.914120026 \frac{\pi}{2} \approx 1.43589638$$

and take

$$\sigma_{trans} = q_{trans}^{tl} \frac{\lambda^2}{4\pi} G_d^0\ ,\ kh < k_0 h$$

8.2 Average Loss & Absorption Cross Sections

We sometimes run into the situation where a random field excites the slot, for example from within an overmoded cavity, where we have an interest in the cross section of a slot averaged over all incident and polarization angles [11], [1]. The preceding cross section for $kh = n\pi/2$ when averaged over all incident angles (2π solid angle for a half space and at high frequencies for a cavity) [1]

$$\frac{1}{2\pi} \int_0^\pi d\varphi \int_0^\pi \sigma_{trans} \sin\theta_0 d\theta_0 =$$

$$\frac{1}{2} \int_0^\pi \sigma_{trans} \sin\theta_0 d\theta_0 = q_{trans}^{tl} \frac{2\pi\ell^2}{(n\pi/2)^2\,\mathrm{Cin}(2n\pi)} \int_0^1 \left\{ \begin{array}{l} \cos^2(n\pi u/2)\ ,\ n \text{ odd} \\ \sin^2(n\pi u/2)\ ,\ n \text{ even} \end{array} \right\} \frac{du}{1-u^2}$$

$$= q_{trans}^{tl} \frac{\pi\ell^2}{(n\pi/2)^2\,\mathrm{Cin}(2n\pi)} \int_0^1 [1 \pm \cos(n\pi u)] \frac{du}{1-u^2}$$

$$= q_{trans}^{tl} \frac{\pi \ell^2/2}{(n\pi/2)^2 \operatorname{Cin}(2n\pi)} \int_0^1 [1 \pm \cos(n\pi u)] \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$$

$$= q_{trans}^{tl} \frac{\pi \ell^2/2}{(n\pi/2)^2 \operatorname{Cin}(2n\pi)} \int_0^2 \{1 - \cos(n\pi x)\} \frac{dx}{x} = q_{trans}^{tl} \frac{\pi \ell^2/2}{(n\pi/2)^2} = q_{trans}^{tl} \frac{\pi \ell^2/2}{(kh)^2} = q_{trans}^{tl} \frac{2\pi}{k^2} = q_{trans}^{tl} 2\frac{\lambda^2}{4\pi}$$

If we also average over the polarization angles we also introduce another factor of one half

$$\langle \sigma_{trans} \rangle = q_{trans}^{tl} 2 \frac{\lambda^2}{4\pi} \frac{1}{2}$$

and at the resonances where the reactive parts cancel, the mismatch ratio for transmission is [1] (this is the mismatch between radiation into the exterior half space $G_{rad}/2$ and the combination of radiation into the incident half space $G_{rad}/2$ in combination with slot wall losses R_{int}/Z_0^2)

$$q_{trans}^{tl} \approx \frac{G_{rad}^2}{(G_{rad} + R_{int}/Z_0^2)^2}$$

Similarly the average absorption cross section for losses in the slot walls

$$\langle \sigma_{abs} \rangle = q_{abs}^{tl} 2 \frac{\lambda^2}{4\pi} \frac{1}{2}$$

and at the resonances where the reactive parts cancel, the mismatch ratio for absorption is [1]

$$q_{abs}^{tl} \approx \frac{2 G_{rad} R_{int}/Z_0^2}{(G_{rad} + R_{int}/Z_0^2)^2}$$

The total loss cross section (losses for the interior cavity modes) for the slot is then the sum of these

$$\langle \sigma_{loss} \rangle = \langle \sigma_{trans} \rangle + \langle \sigma_{abs} \rangle$$

and $P_{loss} = \langle \sigma_{loss} \rangle S$, where S is given, for example, by

$$P_{in} = \sigma_{trans} S_0 = (\langle \sigma_{trans} \rangle + \langle \sigma_{abs} \rangle + \sigma_{wall}) S = P_{out} \tag{5}$$

where [1]

$$\sigma_{wall} = \frac{4}{3} S_{cav} R_s / \eta_0$$

$$S = \eta_0 \left\langle |\underline{H}|^2 \right\rangle_V = 3\eta_0 \left\langle |H_i|^2 \right\rangle_V \tag{6}$$

$$S_0 = \eta_0 \left| H^{inc} \right|^2$$

The quality factor is

$$Q = \frac{\omega W}{P_{in}} = \frac{\omega V_{cav} \left\langle \frac{1}{2}\varepsilon_0 |\underline{E}|^2 + \frac{1}{2}\mu_0 |\underline{H}|^2 \right\rangle_V}{(\langle \sigma_{trans} \rangle + \langle \sigma_{abs} \rangle + \sigma_{wall}) S} = \frac{\omega V_{cav} \left\langle \mu_0 |\underline{H}|^2 \right\rangle_V}{(\langle \sigma_{trans} \rangle + \langle \sigma_{abs} \rangle + \sigma_{wall}) S} = \frac{k^3 V_{cav}/\pi}{q_{trans}^{tl} + q_{abs}^{tl} + \frac{4}{3\pi} k^2 S_{cav} R_s/\eta_0}$$

8.3 Slot Cavity Received Power

A uniformly distributed matched load G_{ld} along the slot is used to mimic a worst case backing cavity [1] and the net input power P_{rec}. We first set up the case where the cavity load may or may not match the reactive slot terms. We start with the oblique incidence slot voltage distribution [1]

$$V(z) = -\frac{Z2H^{inc}\sin\theta_0}{q_0^2-\beta^2}\left[\cos(q_0 z) - \cos(q_0 h)\frac{\cos(\beta z)}{\cos(\beta h)} + i\sin(q_0 z) - i\sin(q_0 h)\frac{\sin(\beta z)}{\sin(\beta h)}\right]$$

$$= -\frac{Z2H^{inc}\sin\theta_0}{q_0^2-\beta^2}\left[\cos(q_0 z) + i\sin(q_0 z)\right] + V_{0e}\cos(\beta z) + V_{0o}\sin(\beta z)$$

$$\sim \left\{\begin{array}{c} V_{0e}\cos(\beta z)\ ,\ n \text{ even} \\ V_{0o}\sin(\beta z)\ ,\ n \text{ odd}\end{array}\right\} \approx \left\{\begin{array}{c} V_{0e}\cos(n\pi z/\ell)\ ,\ n \text{ even} \\ V_{0o}\sin(n\pi z/\ell)\ ,\ n \text{ odd}\end{array}\right\}$$

where

$$V_{0e} \approx \frac{Z2H^{inc}\sin\theta_0\cos(q_0 h)}{\left(q_0^2-\beta^2\right)\cos(\beta h)} = \frac{Z2H^{inc}\sin\theta_0\cos(kh\cos\theta_0)}{\left(k^2\cos^2\theta_0-\beta^2\right)\cos(\beta h)}$$

$$V_{0o} \approx i\frac{Z2H^{inc}\sin\theta_0\sin(q_0 h)}{\left(q_0^2-\beta^2\right)\sin(\beta h)} = i\frac{Z2H^{inc}\sin\theta_0\sin(kh\cos\theta_0)}{\left(k^2\cos^2\theta_0-\beta^2\right)\sin(\beta h)}$$

and [1]

$$P_{rec} \approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2\sin^2\theta_0}{\left|\beta^2-k^2\cos^2\theta_0\right|^2}\left\{\begin{array}{c} \cos^2(kh\cos\theta_0)/\left|\cos(\beta h)\right|^2\ ,\ n \text{ odd} \\ \sin^2(kh\cos\theta_0)/\left|\sin(\beta h)\right|^2\ ,\ n \text{ even}\end{array}\right\}$$

$$\approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2\sin^2\theta_0}{\left(\beta'^2-\beta''^2-k^2\cos^2\theta_0\right)^2+\left(2\beta'\beta''\right)^2}\left\{\begin{array}{c} \cos^2(kh\cos\theta_0)/\left|\cos(\beta h)\right|^2\ ,\ n \text{ odd} \\ \sin^2(kh\cos\theta_0)/\left|\sin(\beta h)\right|^2\ ,\ n \text{ even}\end{array}\right\}$$

$$\approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2\sin^2\theta_0}{\left(\beta'^2-k^2\cos^2\theta_0\right)^2+\beta''^4+2\beta'^2\beta''^2+2\beta''^2k^2\cos^2\theta_0}\left\{\begin{array}{c} \cos^2(kh\cos\theta_0)/\left|\cos(\beta h)\right|^2\ ,\ n \text{ odd} \\ \sin^2(kh\cos\theta_0)/\left|\sin(\beta h)\right|^2\ ,\ n \text{ even}\end{array}\right\}$$

$$\beta = \beta' + i\beta''$$

Then assuming $\beta'' << \beta'$ and $\beta'' h << 1$

$$\cos(\beta h) = \cos\left(\beta' h\right)\cosh\left(\beta'' h\right) - i\sin\left(\beta' h\right)\sinh\left(\beta'' h\right)$$

$$\approx \cos\left(\beta' h\right) - i\sin\left(\beta' h\right)\left(\beta'' h\right)$$

$$\left|\cos(\beta h)\right|^2 \approx \cos^2\left(\beta' h\right) + \sin^2\left(\beta' h\right)\left(\beta'' h\right)^2 \approx \cos^2\left(\beta' h\right) + \left\{1-\cos^2\left(\beta' h\right)\right\}\left(\beta'' h\right)^2$$

$$\approx \cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2$$

$$\sin(\beta h) = \sin\left(\beta' h\right)\cosh\left(\beta'' h\right) + i\cos\left(\beta' h\right)\sinh\left(\beta'' h\right)$$

$$\approx \sin\left(\beta' h\right) + i\cos\left(\beta' h\right)\left(\beta'' h\right)$$

$$\left|\sin\left(\beta h\right)\right|^2 \approx \sin^2\left(\beta' h\right) + \cos^2\left(\beta' h\right)\left(\beta'' h\right)^2 \approx \sin^2\left(\beta' h\right) + \left\{1 - \sin^2\left(\beta' h\right)\right\}\left(\beta'' h\right)^2$$

$$\approx \sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2$$

$$P_{rec} \approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2 \sin^2\theta_0}{\left(\beta'^2 - k^2\cos^2\theta_0\right)^2 + 2\beta''^2\left(\beta'^2 + k^2\cos^2\theta_0\right)}\left\{\begin{array}{l} \cos^2\left(kh\cos\theta_0\right) / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) / \left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ even} \end{array}\right\}$$

where

$$\beta'^2 \approx \left(\omega L + X_{int}\right)\left[\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right]$$

$$2\beta'\beta'' \approx R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\} + \left(\omega L + X_{int}\right)\left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)$$

Now for resonance we need $\beta' h \to n\pi/2$, but for gain approaching the linear array level we also need $\beta' h \to kh\cos\theta_0$, which would require that $kh\cos\theta_0 \to n\pi/2$; this combination pushes us toward grazing incidence $\theta_0 \to 0$ when $\beta' \approx k$ (although $\sin\theta_0 \to 0$). If β' is not near k, because we were forced to choose B_{ld} to be substantial in order to make $\beta' h \to n\pi/2$, then we will not achieve the linear array gain near, but below the normal slot resonance, when $B_{ld} > 0$ and $\beta' > k$

$$P_{rec} \approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2 \sin^2\theta_0}{\left(\beta'^2 - k^2\cos^2\theta_0\right)^2}\left\{\begin{array}{l} \cos^2\left(kh\cos\theta_0\right) / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) / \left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ even} \end{array}\right\}$$

and we might expect a maximum near normal $\theta_0 \to \pi/2$ incidence.

Alternatively, suppose $B_{ld} < 0$ because we are above the normal slot resonant frequency; in this case we can adjust the angle of incidence to make $\beta'^2/k^2 = \cos^2\theta_0$

$$P_{rec} \approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2\left(1 - \beta'^2/k^2\right)}{4\beta''^2\beta'^2}\left\{\begin{array}{l} \cos^2\left(\beta' h\right) / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ odd} \\ \sin^2\left(\beta' h\right) / \left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ even} \end{array}\right\}$$

or

$$P_{rec} \approx \frac{4hG_{ld}\left|H^{inc}\right|^2\left(1 - \beta'^2/k^2\right)}{\left[R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\} / \left(\omega L + X_{int}\right) + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2}$$
$$\left\{\begin{array}{l} \cos^2\left(\beta' h\right) / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ odd} \\ \sin^2\left(\beta' h\right) / \left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ even} \end{array}\right\}$$

$$\approx \frac{4hG_{ld}\left|H^{inc}\right|^2\left(1 - \beta'^2/k^2\right)}{\left[R_{int}\left(n\pi/2\right)^2 / \left\{\left(\omega L + X_{int}\right)h\right\}^2 + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2}\left\{\begin{array}{l} \cos^2\left(\beta' h\right) / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ odd} \\ \sin^2\left(\beta' h\right) / \left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ ,\ n \text{ even} \end{array}\right\}$$

where the factor in braces is less than unity. The presence of a gasket with $\Delta C_{gas}^{intr} > 0$ ordinarily increases β' relative to k, but in this case with the requirement that we choose $B_{ld} < 0$ to make $\beta' h = n\pi/2$, we need to make B_{ld} more negative to simultaneously achieve $kh\cos\theta_0 \to \beta' h = n\pi/2$; in a sense our choice of B_{ld} compensates for the presence of ΔC_{gas}^{intr}. If there is no gasket $\Delta C_{gas}^{intr} = 0 = G_{gas}^{intr}$ we can write

$$\beta'^2 \approx (\omega L + X_{int})(\omega C + B_{rad} + B_{ld})$$

$$2\beta'\beta'' \approx R_{int}(\omega C + B_{rad} + B_{ld}) + (\omega L + X_{int})(G_{rad}/2 + G_{ld})$$

and

$$P_{rec} \approx \frac{4hG_{ld}\left|H^{inc}\right|^2\left(1 - \beta'^2/k^2\right)}{\left[R_{int}(\omega C + B_{rad} + B_{ld})/(\omega L + X_{int}) + (G_{rad}/2 + G_{ld})\right]^2}\left\{\begin{array}{ll} \cos^2(\beta' h)/\left[\cos^2(\beta' h) + (\beta'' h)^2\right] & , n \text{ odd} \\ \sin^2(\beta' h)/\left[\sin^2(\beta' h) + (\beta'' h)^2\right] & , n \text{ even} \end{array}\right\}$$

The admittance mismatch exhibited in these equations for received power can be exploited to examine situations where the cavity does not match the slot [1]. The dual of antenna impedance is slot admittance. The statistical fluctuations of antenna impedance at high frequencies when attached to a cavity [14], can be interpreted through duality as the slot admittance fluctuations from the backing cavity. Limits on these fluctuations might be used in these power mismatch factors to limit the power reception in certain frequency bands similar to the antenna case [15].

8.3.1 Normal Incidence

If we look at the normal incidence result

$$P_{rec} \approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2}{|\beta|^4\left|\cos(\beta h)\right|^2} \approx hG_{ld}\frac{4\left|ZH^{inc}\right|^2}{\beta'^4\left[\cos^2(\beta' h) + (\beta'' h)^2\right]}\ ,\ n \text{ odd}$$

and at resonance

$$P_{rec} \approx hG_{ld}\frac{16\left|ZH^{inc}\right|^2/(\beta' h)^2}{(2\beta'\beta'')^2} \approx \frac{16hG_{ld}\left|H^{inc}\right|^2/(\beta' h)^2}{\left[R_{int}(\omega C + B_{rad} + B_{ld})/(\omega L + X_{int}) + (G_{rad}/2 + G_{ld})\right]^2}\ ,\ n \text{ odd}$$

$$\pi\eta_0 hG_{rad} = \left(\frac{kh}{n\pi} + \frac{n\pi/4}{kh}\right)\left\{\mathrm{Cin}(n\pi + 2kh) - \mathrm{Cin}(n\pi - 2kh)\right\}$$

$$+\left(\frac{kh}{n\pi} - \frac{n\pi/4}{kh}\right)n\pi\left\{\mathrm{Si}(n\pi + 2kh) - \mathrm{Si}(n\pi - 2kh)\right\} - 2\sin^2(n\pi/2 + kh)$$

Now maximizing with respect to G_{ld}

$$G_{ld} = R_{int}(\omega C + B_{rad} + B_{ld})/(\omega L + X_{int}) + G_{rad}/2$$

gives

$$P_{rec} \approx \frac{4h\left|H^{inc}\right|^2/(\beta' h)^2}{R_{int}(\omega C + B_{rad} + B_{ld})/(\omega L + X_{int}) + G_{rad}/2}\ ,\ n \text{ odd}$$

Now we can follow two paths. First we can use the approximation

$$\left(\omega C+B_{rad}+B_{ld}\right)/\left(\omega L+X_{int}\right)\approx 1/Z_0^2$$

to find

$$P_{rec}\approx\frac{4h\left|H^{inc}\right|^2/\left(\beta' h\right)^2}{R_{int}/Z_0^2+G_{rad}/2}\ ,\ n\text{ odd}$$

If we replace $\beta' h\rightarrow kh$

$$P_{rec}=\sigma_{rec}\eta_0\left|H^{inc}\right|^2\approx\frac{4h\left|H^{inc}\right|^2/\left(kh\right)^2}{R_{int}/Z_0^2+G_{rad}/2}=\frac{4\left|H^{inc}\right|^2/\left(k^2h\right)}{R_{int}/Z_0^2+G_{rad}/2}=\eta_0\left|H^{inc}\right|^2\frac{4/\left(k^2h\right)}{\eta_0R_{int}/Z_0^2+\eta_0G_{rad}/2}\ ,\ n\text{ odd}$$

and

$$\sigma_{rec}=\frac{4/k^2}{h\eta_0R_{int}/Z_0^2+h\eta_0G_{rad}/2}=\frac{\lambda^2}{\pi^2h\eta_0G_{rad}/2}\frac{h\eta_0G_{rad}/2}{h\eta_0R_{int}/Z_0^2+h\eta_0G_{rad}/2}\ ,\ n\text{ odd}$$

where we would typically use the $n=1$ form (in this case, when $k>>\beta'$, the replacement $\beta' h\rightarrow kh$ as well as the use of $\left(\omega C+B_{rad}+B_{ld}\right)/\left(\omega L+X_{int}\right)\approx 1/Z_0^2$ are questionable because B_{ld} must be taken as increasingly large and negative)

$$\pi\eta_0hG_{rad}=-2\sin^2\left(\pi/2+kh\right)+\left(\frac{kh}{\pi}+\frac{\pi}{4kh}\right)\left[\mathrm{Cin}\left(2kh+\pi\right)-\mathrm{Cin}\left(\pi-2kh\right)\right]$$
$$+kh\left[1-\left(\frac{\pi}{2kh}\right)^2\right]\left[\mathrm{Si}\left(2kh+\pi\right)-\mathrm{Si}\left(\pi-2kh\right)\right]$$

Alternatively, we can take

$$\left(\omega C+B_{rad}+B_{ld}\right)\left(\omega L+X_{int}\right)h^2\approx\left(n\pi/2\right)^2\approx\left(\beta' h\right)^2$$

which gives

$$P_{rec}\approx\frac{4h\left|H^{inc}\right|^2/\left(n\pi/2\right)^2}{R_{int}\left(n\pi/2\right)^2/\left\{\left(\omega L+X_{int}\right)^2h^2\right\}+G_{rad}/2}=\frac{2h\left|H^{inc}\right|^2 8/\left(n\pi\right)^2}{R_{int}\left(n\pi\right)^2/\left\{\left(\omega L+X_{int}\right)2h\right\}^2+G_{rad}/2}\ ,\ n\text{ odd}$$

If we take $n=1$ here (and we can also ignore X_{int}) we arrive at our prior matched result [1]

$$P_{rec}\approx\frac{2h\left|H^{inc}\right|^2 8/\pi^2}{R_{int}\pi^2/\left\{\left(\omega L+X_{int}\right)2h\right\}^2+G_{rad}/2}=\sigma_{rec}\eta_0\left|H^{inc}\right|^2$$

$$\sigma_{rec}=\frac{2h\left(8/\pi^2\right)/\eta_0}{R_{int}\pi^2/\left(\omega L2h\right)^2+G_{rad}/2}=\frac{G_{rad}/2}{R_{int}\pi^2/\left(\omega L2h\right)^2+G_{rad}/2}\frac{\ell^2\left(8/\pi^2\right)}{\eta_0hG_{rad}}=q_{rec}^{tl}\frac{\ell^2\left(8/\pi\right)}{\pi\eta_0hG_{rad}}$$

and we can define the mismatch due to the wall losses as

$$q_{rec}^{tl}=\frac{G_{rad}/2}{R_{int}\pi^2/\left(\omega L2h\right)^2+G_{rad}/2}$$

where we use the $n = 1$ form

$$\pi\eta_0 h G_{rad} = -2\sin^2(\pi/2 + kh) + \left(\frac{kh}{\pi} + \frac{\pi}{4kh}\right)[\mathrm{Cin}(2kh+\pi) - \mathrm{Cin}(\pi - 2kh)]$$
$$+kh\left[1 - \left(\frac{\pi}{2kh}\right)^2\right][\mathrm{Si}(2kh+\pi) - \mathrm{Si}(\pi - 2kh)]$$

and the asymptotic limits are

$$\pi\eta_0 h G_{rad} \sim \frac{2}{3}\left(\frac{4}{\pi}kh\right)^2 \ ,\ kh << 1$$
$$\pi\eta_0 h G_{rad} \sim \pi kh\left[1 - \left(\frac{\pi}{2kh}\right)^2\right] \ ,\ kh >> 1$$

In the limit $kh << 1$

$$\sigma_{rec} \sim \frac{(\omega L 2h/\pi)^2}{\eta_0 R_{int}}\ell\left(8/\pi^2\right) = 2\left(\eta_0/R_s\right)\left(\ell w^2/d\right)(kh/\pi)^2\left(8/\pi^2\right)$$

or

$$\sigma_{rec} \sim q_{rec}^{tl}\frac{3\pi}{k^2} = q_{rec}^{tl}\frac{\lambda^2}{4\pi}3$$

where because of the slot wall losses in the slot, the cross section behaves as $O\left(\omega^{3/2}\right)$, but in the case with perfectly conducting walls $q_{rec}^{tl} = 1$ and the behavior is then $O\left(\omega^{-2}\right)$. In the limit $kh >> 1$

$$\sigma_{rec} \sim q_{rec}^{tl}\frac{\ell^2(8/\pi)}{\pi\eta_0 h G_{rad}} \sim q_{rec}^{tl}\frac{\ell\lambda}{\pi}\left(8/\pi^2\right) \to \frac{\ell\lambda}{\pi}\left(8/\pi^2\right)$$

which is near the linear array level $\ell\lambda/\pi$ with $8/\pi^2 \approx 0.81$.

8.3.2 Linear Array

Note that the linear array cross section is useful as a comparison

$$\sigma_{trans} = q_{trans}^{tl}\frac{\lambda^2}{4\pi}G_d^{LA} = q_{trans}^{tl}\lambda\ell/\pi = q_{trans}^{tl}2\ell/k$$

$$G_d^{LA} = 4\ell/\lambda = 2k\ell/\pi$$

where we can take q_{rec}^{tl} to be given by the same result we are comparing this result with.

8.3.3 Low Frequency Slot

We expect at low frequencies this gain to transition to the gain of a low frequency slot which is twice the low frequency dipole antenna gain of 3/2, which again we can compare with another model

$$\sigma_{rec} = q_{rec}^{tl}\frac{\lambda^2}{4\pi}G_d^0 = q_{rec}^{tl}\frac{\lambda^2}{4\pi}3 = q_{rec}^{tl}\frac{3\pi}{k^2}$$

$$G_d^0 \sim 2(3/2) = 3$$

where we can take q_{rec}^{tl} to be given by the same result we are comparing this result with.

8.3.4 Large Losses

In the case of large losses normal incidence gives the maximum because it maximizes the projection of the incident magnetic field on the slot length (there is no cancellation in the denominator possible for these large losses). The slot voltage distribution in this case becomes

$$V(z) \sim V_0 \sim K_x^{sc} / (Y + Y_{rad}/2 + Y_{ld})$$

where

$$Y = G - i\omega C$$

$$Y_{rad} = G_{rad} - iB_{rad}$$

$$Y_{ld} = G_{ld} - iB_{ld}$$

$$\pi\eta_0 h G_{rad} = 2kh\,\mathrm{Si}(2kh) + \frac{\sin(2kh)}{2kh} + \cos(2kh) - 2$$

$$\pi\eta_0 h G_{rad} \sim \frac{1}{3}(2kh)^2 \left[1 - (2kh)^2/60\right] \ , \ kh << 1$$

$$\pi\eta_0 h G_{rad} \sim \pi kh - 2 \ , \ kh >> 1$$

(and if we include the reactive field in the transmitted half space we would replace $B_{rad}/2 \to B_{rad}$). Received power is then

$$P_{rec} = 2hG_{ld}|V_0|^2 = \frac{2hG_{ld}|K_x^{sc}|^2}{|Y + Y_{rad}/2 + Y_{ld}|^2} = \frac{8hG_{ld}\left|H^{inc}\right|^2}{(G + G_{rad}/2 + G_{ld})^2 + (\omega C + B_{rad}/2 + B_{ld})^2} = \sigma_{rec}\eta_0\left|H^{inc}\right|^2$$

$$\sigma_{rec} = q_{rec}^{tl}\frac{2\ell}{\eta_0 G_{rad}}$$

$$q_{rec}^{tl} = \frac{4G_{ld}(G_{rad}/2)}{(G + G_{rad}/2 + G_{ld})^2 + (\omega C + B_{rad}/2 + B_{ld})^2}$$

In the high frequency limit this becomes the linear array result with mismatch

$$\sigma_{rec} \sim q_{rec}^{tl}\frac{\lambda\ell}{\pi} \ , \ kh >> 1$$

and in the low frequency limit this becomes the low frequency slot limit with a gain of $G_0 = 3$ with mismatch

$$\sigma_{rec} = q_{rec}^{tl}\frac{3\pi}{k^2} = q_{rec}^{tl}\frac{\lambda^2}{4\pi}3 \ , \ kh << 1$$

In the matched case where the reactive terms cancel $\omega C + B_{rad}/2 + B_{ld} = 0$ and we take $G + G_{rad}/2 = G_{ld}$, we find

$$q_{rec}^{tl} = \frac{G_{rad}/2}{G + G_{rad}/2}$$

8.3.5 Oblique Incidence

Next we look further at the case of a fixed oblique incidence near the slot resonances $\beta' h = n\pi/2$

$$P_{rec} \approx hG_{ld} \frac{4\left|ZH^{inc}\right|^2 \sin^2\theta_0}{\left(\beta'^2 - k^2\cos^2\theta_0\right)^2 + 2\beta''^2\left(\beta'^2 + k^2\cos^2\theta_0\right)} \left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) / \left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] & , \ n \text{ even} \end{array} \right\}$$

and at resonance ($\cos^2\left(\beta' h\right) \to 0$ or $\sin^2\left(\beta' h\right) \to 0$, dropping the β''^2 term in the first denominator factor outside the braces)

$$P_{rec} \approx hG_{ld} \frac{16\left|ZH^{inc}\right|^2 \sin^2\theta_0}{\left(\beta' h\right)^2\left[1 - \left(k^2/\beta'^2\right)\cos^2\theta_0\right]^2\left(2\beta'\beta''\right)^2} \left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) & , \ n \text{ even} \end{array} \right\}$$

or

$$P_{rec} \approx \frac{16hG_{ld}\left|H^{inc}\right|^2 \sin^2\theta_0}{\left(\beta' h\right)^2\left[1 - \left(k^2/\beta'^2\right)\cos^2\theta_0\right]^2\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + \left(G_{rad}/2 + G_{ld}\right)\right]^2}$$
$$\left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) & , \ n \text{ even} \end{array} \right\}$$

Then choosing

$$G_{ld} = R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + G_{rad}/2$$

to maximize this factor

$$P_{rec} \approx \frac{4h\left|H^{inc}\right|^2 \sin^2\theta_0}{\left(\beta' h\right)^2\left[1 - \left(k^2/\beta'^2\right)\cos^2\theta_0\right]^2\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + G_{rad}/2\right]} \left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) & , \ n \text{ even} \end{array} \right\}$$

Then the first of our two approaches with

$$\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) \approx 1/Z_0^2$$

gives

$$P_{rec} \approx \frac{4h\left|H^{inc}\right|^2 \sin^2\theta_0}{\left(\beta' h\right)^2\left[1 - \left(k^2/\beta'^2\right)\cos^2\theta_0\right]^2\left(R_{int}/Z_0^2 + G_{rad}/2\right)} \left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) & , \ n \text{ even} \end{array} \right\}$$

If we replace $\beta' h \to kh$ (assuming we are near each of the slot resonances)

$$\begin{aligned} P_{rec} &\approx \frac{4h\left|H^{inc}\right|^2}{(kh)^2\left(R_{int}/Z_0^2 + G_{rad}/2\right)} \left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) & , \ n \text{ even} \end{array} \right\} / \sin^2\theta_0 \\ &\approx \frac{G_{rad}/2}{R_{int}/Z_0^2 + G_{rad}/2}\eta_0\left|H^{inc}\right|^2 \frac{4\pi/k^2}{\pi h\eta_0 G_{rad}/2} \left\{ \begin{array}{ll} \cos^2\left(kh\cos\theta_0\right) & , \ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right) & , \ n \text{ even} \end{array} \right\} / \sin^2\theta_0 \end{aligned}$$

or

$$\sigma_{rec} \approx q_{rec}^{tl} \frac{2\pi\ell^2}{(kh)^2 \operatorname{Cin}(2n\pi)} \left\{ \begin{array}{c} \cos^2(kh\cos\theta_0) \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ , \ n \text{ even} \end{array} \right\} / \sin^2\theta_0$$

$$\approx q_{rec}^{tl} \frac{2\pi\ell^2}{(kh)^2 \operatorname{Cin}(2n\pi)} \sin^2(x_r) \frac{kh/(2x_r)}{1 - x_r/(2kh)}$$

$$\approx q_{rec}^{tl} \frac{2\ell\lambda}{(kh)\operatorname{Cin}(2n\pi)} \sin^2(x_r) \frac{kh/(2x_r)}{1 - x_r/(2kh)}$$

or the asymptotic result

$$\sigma_{rec} \approx q_{rec}^{tl} \frac{\ell\lambda}{\operatorname{Cin}(2n\pi)} \frac{\sin^2(x_r)}{x_r} \approx q_{rec}^{tl} \frac{\ell\lambda}{1.38\left[\gamma' + \ln(4kh)\right]} \ , \ x_r \approx 1.1656 \ , \ kh >> 1$$

with

$$q_{rec}^{tl} = \frac{G_{rad}/2}{R_{int}/Z_0^2 + G_{rad}/2}$$

The preceding transmission fit used to accommodate the tilt of the maximum to normal incidence at the first resonance (which also occurs here), can also be used for this receiving case

$$\sigma_{rec} \approx \frac{\lambda\ell}{f_\sigma(kh)\operatorname{Cin}(4kh)} q_{rec}^{tl} \approx \frac{\lambda\ell}{f_\sigma(kh)\left[\gamma' + \ln(4\pi\ell/\lambda)\right]} q_{rec}^{tl} \ , \ kh \geq \pi/2 \ , \ \lambda \leq \ell/2$$

Even though this result was derived near the resonances $kh = n\pi/2$ we plan to use it continuously as a function of kh to interpolate between resonances.

8.3.6 Investigation Of Oblique Incidence

The alternative approach (used in the preceding normal case) gives

$$(\omega C + B_{rad} + B_{ld})(\omega L + X_{int}) h^2 \approx (n\pi/2)^2 \approx \left(\beta' h\right)^2$$

and

$$P_{rec} \approx \frac{4h\left|H^{inc}\right|^2 \sin^2\theta_0/(n\pi/2)^2}{\left[1 - \left\{(kh)/(n\pi/2)\right\}^2\cos^2\theta_0\right]^2 \left[R_{int}(n\pi/2)^2/\left\{(\omega L + X_{int})h\right\}^2 + G_{rad}/2\right]} \left\{ \begin{array}{c} \cos^2(kh\cos\theta_0) \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ , \ n \text{ even} \end{array} \right\}$$

Returning to the expression

$$P_{rec} \approx hG_{ld} \frac{4h^4\left|ZH^{inc}\right|^2 \sin^2\theta_0}{\left\{\left(\beta' h\right)^2 - (kh)^2\cos^2\theta_0\right\}^2 + 2\left(\beta'' h\right)^2 \left\{\left(\beta' h\right)^2 + (kh)^2\cos^2\theta_0\right\}}$$
$$\left\{ \begin{array}{l} \cos^2(kh\cos\theta_0)/\left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ , \ n \text{ odd} \\ \sin^2(kh\cos\theta_0)/\left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] \ , \ n \text{ even} \end{array} \right\}$$

we originally have variables B_{ld}, G_{ld}, kh, and θ_0, or in this expression $\beta' h$, $\beta'' h$, $kh\cos\theta_0$, $\sin^2\theta_0$. Assuming that $\beta'' h << 1$ let us take $\beta' h = n\pi/2$

$$P_{rec} \approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2 \sin^2\theta_0}{\left\{(n\pi/2)^2 - (kh)^2\cos^2\theta_0\right\}^2 + 2\left(\beta''h\right)^2 \left\{(n\pi/2)^2 + (kh)^2\cos^2\theta_0\right\}} \left\{ \begin{array}{l} \cos^2(kh\cos\theta_0)/\left(\beta''h\right)^2 \ \ ,\ n \text{ odd} \\ \sin^2(kh\cos\theta_0)/\left(\beta''h\right)^2 \ \ ,\ n \text{ even} \end{array} \right\}$$

we still have variables $\beta''h$, $kh\cos\theta_0$, $\sin^2\theta_0$. If we drop the $\left(\beta''h\right)^2$ term in the denominator (note that this term eventually plays a role for large n)

$$P_{rec} \approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2}{\left(\beta''h\right)^2} \frac{\sin^2\theta_0}{\left\{(n\pi/2)^2 - (kh)^2\cos^2\theta_0\right\}^2} \left\{ \begin{array}{l} \cos^2(kh\cos\theta_0) \ \ ,\ n \text{ odd} \\ \sin^2(kh\cos\theta_0) \ \ ,\ n \text{ even} \end{array} \right\}$$

Let us take

$$kh = n\pi/2 + \Delta_1$$

$$kh\cos\theta_0 = (n\pi/2 + \Delta_1)\cos\theta_0$$

$$\sin^2\theta_0 = 1 - \cos^2\theta_0$$

$$\cos\theta_0 = 1 - u$$

where $u = O(1)$

$$P_{rec} \approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2}{\left(\beta''h\right)^2} \frac{1 - \cos^2\theta_0}{\left\{(n\pi/2)^2 - (n\pi/2 + \Delta_1)^2\cos^2\theta_0\right\}^2} \left\{ \begin{array}{l} \cos^2((n\pi/2 + \Delta_1)\cos\theta_0) \ \ ,\ n \text{ odd} \\ \sin^2((n\pi/2 + \Delta_1)\cos\theta_0) \ \ ,\ n \text{ even} \end{array} \right\}$$

$$\approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2}{\left(\beta''h\right)^2} \frac{(1 - \cos\theta_0)(1 + \cos\theta_0)}{\{(n\pi/2) - (n\pi/2 + \Delta_1)\cos\theta_0\}^2 \{(n\pi/2) + (n\pi/2 + \Delta_1)\cos\theta_0\}^2}$$
$$\left\{ \begin{array}{l} \cos^2((n\pi/2 + \Delta_1)\cos\theta_0) \ \ ,\ n \text{ odd} \\ \sin^2((n\pi/2 + \Delta_1)\cos\theta_0) \ \ ,\ n \text{ even} \end{array} \right\}$$

$$\approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2}{\left(\beta''h\right)^2} \frac{u(2 - u)}{\{-\Delta_1 + (n\pi/2 + \Delta_1)u\}^2 \{2(n\pi/2) + \Delta_1 - (n\pi/2 + \Delta_1)u\}^2}$$
$$\left\{ \begin{array}{l} \cos^2((n\pi/2 + \Delta_1)(1 - u)) \ \ ,\ n \text{ odd} \\ \sin^2((n\pi/2 + \Delta_1)(1 - u)) \ \ ,\ n \text{ even} \end{array} \right\}$$

Now if u does not go to zero we are left with $O\left(|Z|^2/(n\pi/2)^4\right) = O\left(1/(n\pi/2)^2\right)$ behavior, noting that $|Z|^2 = O\left(\omega^2\right)$ (and we are investigating the case here where $\Delta_1 = O(1)$, in other words shifts in resonance in an interval between slot resonances). However, if $u \to 0$ this can be increased.

$$P_{rec} \approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2}{\left(2\beta'h\right)^2 \left(\beta''h\right)^2} \frac{(n\pi^2)\,u(2 - u)}{\{-\Delta_1 + (n\pi/2 + \Delta_1)u\}^2 \{2(n\pi/2) + \Delta_1 - (n\pi/2 + \Delta_1)u\}^2}$$

$$\left\{ \begin{array}{l} \cos^2\left(\left(n\pi/2+\Delta_1\right)\left(1-u\right)\right)\ ,\ n \text{ odd} \\ \sin^2\left(\left(n\pi/2+\Delta_1\right)\left(1-u\right)\right)\ ,\ n \text{ even} \end{array} \right\}$$

$$\approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2u}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}^2}\left\{ \begin{array}{l} \cos^2\left(\left(n\pi/2+\Delta_1\right)\left(1-u\right)\right)\ ,\ n \text{ odd} \\ \sin^2\left(\left(n\pi/2+\Delta_1\right)\left(1-u\right)\right)\ ,\ n \text{ even} \end{array} \right\}$$

Then

$$\left\{ \begin{array}{l} \cos^2\left(\left(n\pi/2+\Delta_1\right)\left(1-u\right)\right)\ ,\ n \text{ odd} \\ \sin^2\left(\left(n\pi/2+\Delta_1\right)\left(1-u\right)\right)\ ,\ n \text{ even} \end{array} \right\} \approx \left\{ \begin{array}{l} \cos^2\left(\left(n\pi/2\right)\left(1-u\right)+\Delta_1\right)\ ,\ n \text{ odd} \\ \sin^2\left(\left(n\pi/2\right)\left(1-u\right)+\Delta_1\right)\ ,\ n \text{ even} \end{array} \right\}$$

$$\approx \left\{ \begin{array}{l} \sin^2\left(-\left(n\pi/2\right)u+\Delta_1\right)\ ,\ n \text{ odd} \\ \sin^2\left(-\left(n\pi/2\right)u+\Delta_1\right)\ ,\ n \text{ even} \end{array} \right\}$$

and

$$P_{rec} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2u\sin^2\left(-\left(n\pi/2\right)u+\Delta_1\right)}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}^2}$$

Finding the maximum

$$0=\frac{d}{du}\left[\frac{2u\sin^2\left(-\left(n\pi/2\right)u+\Delta_1\right)}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}^2}\right]$$

$$0=\frac{2\left\{\left(n\pi/2\right)u-\Delta_1\right\}-2\left(n\pi\right)u}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}^3}\sin^2\left(-\left(n\pi/2\right)u+\Delta_1\right)-\frac{2u\left(n\pi\right)\sin\left(-\left(n\pi/2\right)u+\Delta_1\right)\cos\left(-\left(n\pi/2\right)u+\Delta_1\right)}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}^2}$$

or

$$\frac{\tan\left(-\left(n\pi/2\right)u+\Delta_1\right)}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}}=\frac{2u\left(n\pi\right)}{2\left\{-\left(n\pi/2\right)u-\Delta_1\right\}}$$

or

$$\frac{\tan\left(\left(n\pi/2\right)u-\Delta_1\right)}{\left\{\left(n\pi/2\right)u-\Delta_1\right\}}=\frac{2\left(n\pi/2\right)u}{\left(n\pi/2\right)u+\Delta_1}=\frac{2\left(n\pi/2\right)u-2\Delta_1+2\Delta_1}{\left(n\pi/2\right)u-\Delta_1+2\Delta_1}$$

or

$$\frac{\tan\left(x\right)}{x}=\frac{2x+2\Delta_1}{x+2\Delta_1}\ ,\ x=\left(n\pi/2\right)u-\Delta_1$$

$$\frac{\tan\left(u\pi/2-\Delta_1\right)}{u\pi/2-\Delta_1}=\frac{u\pi}{u\pi/2+\Delta_1}\ ,\ x=u\pi/2-\Delta_1$$

$$\frac{\tan\left(u\pi/2-\pi/2\right)}{u\pi/2-\pi/2}=\frac{u\pi}{u\pi/2+\pi/2}$$

$$\lim_{\zeta\to 0}\left(\frac{\tan\zeta}{\zeta}\right)=1\ ,\ \lim_{u\to 1}\left(\frac{u\pi}{u\pi/2+\pi/2}\right)=1$$

$$\frac{\tan\left(u\pi/2-\pi/4\right)}{u\pi/2-\pi/4}=\frac{u\pi}{u\pi/2+\pi/4}$$

$$\frac{\tan\left(\pi/4\right)}{\pi/4}=\frac{4}{\pi}\ ,\ \frac{\pi}{3\pi/4}=\frac{4}{3}$$

$$u=1/2\rightarrow\theta_0=\pi/3$$

$$P_{rec}\approx\frac{4hG_{ld}h^4\left|ZH^{inc}\right|^2u\left(2-u\right)}{\left[u\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2\left[\left(2-u\right)\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2+2\left(\beta''h\right)^2\left[\left(n\pi/2\right)^2+\left\{n\pi/2+\Delta_1-u\left(n\pi/2+\Delta_1\right)\right\}^2\right]}$$

$$\frac{\sin^2\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left(\beta''h\right)^2}$$

$$\approx\frac{4hG_{ld}h^4\left|ZH^{inc}\right|^2\left(2-1/2\right)/2}{\left[\left(\pi/2+\pi/4\right)/2-\pi/4\right]^2\left[\left(2-1/2\right)\left(\pi/2+\pi/4\right)-\pi/4\right]^2+2\left(\beta''h\right)^2\left[\left(\pi/2\right)^2+\left\{\pi/2+\pi/4-\left(\pi/2+\pi/4\right)/2\right\}^2\right]}$$

$$\frac{\sin^2\left\{\left(\pi/2+\pi/4\right)/2-\pi/4\right\}}{\left(\beta''h\right)^2}$$

$$\approx\frac{4hG_{ld}h^4\left|ZH^{inc}\right|^2 3/\pi^2}{\left(\pi/4\right)^2\left[7/8\right]^2+2\left(\beta''h\right)^2\left[1+\left(3/4\right)^2\right]}\frac{\sin^2\left(\pi/8\right)}{\left(\beta''h\right)^2}$$

$$\approx 4G_{ld}h^3\left|ZH^{inc}\right|^2/\left(\beta''\left(\pi/2\right)^2\right)^2\frac{3\sin^2\left(\pi/8\right)}{\left(7/8\right)^2}$$

$$\approx 4G_{ld}h^3\left|ZH^{inc}\right|^2/\left(\beta''\left(\pi/2\right)^2\right)^2\frac{3\left[1-\cos\left(\pi/4\right)\right]}{2\left(7/8\right)^2}$$

$$\approx 4G_{ld}h^3\left|ZH^{inc}\right|^2/\left(\beta''\left(\pi/2\right)^2\right)^2\frac{3\left(1-1/\sqrt{2}\right)}{2\left(7/8\right)^2}$$

$$\approx 4G_{ld}h^3\left|ZH^{inc}\right|^2/\left(\beta''\left(\pi/2\right)^2\right)^2 0.5738$$

If we use normal incidence $\theta_0=\pi/2$ for $n=1$

$$P_{rec}\approx\frac{4hG_{ld}\left|ZH^{inc}\right|^2/\left(\beta'\beta''\right)^2}{\left(\beta'h\right)^2+2\left(\beta''h\right)^2}\approx 4h^3G_{ld}\left|ZH^{inc}\right|^2/\left(\beta''\left(\pi/2\right)^2\right)^2$$

which is a larger result.

We see that for $\Delta_1=0$ we return to our prior solution for $x=x_r$

$$x_r=1.165561=\left(n\pi/2\right)u$$

$$\cos\theta_0 = 1 - u = 1 - \frac{1.165561}{n\pi/2}$$

$$P_{rec} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2u\sin^2\left(-\left(n\pi/2\right)u + \Delta_1\right)}{\left\{\left(n\pi/2\right)u - \Delta_1\right\}^2} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2\left(1.165561\right)\sin^2\left(-1.165561\right)}{\left(n\pi/2\right)\left\{1.165561\right\}^2}$$

$$\approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{1.4492227}{\left(n\pi/2\right)} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{1.4492227}{kh} \approx hG_{ld}\frac{h^2\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{1.4492227\lambda\ell}{\pi}$$

$$\approx hG_{ld}\frac{2h^2\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{\lambda\ell}{1.38005\pi}$$

We can write this as

$$P_{rec} \approx \frac{4hG_{ld}\left|ZH^{inc}\right|^2}{\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right) + \left(\omega L + X_{int}\right)\left(G_{rad}/2 + G_{ld}\right)\right]^2}\frac{1.4492227}{\left(n\pi/2\right)}$$

$$\approx \frac{4hG_{ld}\left|H^{inc}\right|^2}{\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + \left(G_{rad}/2 + G_{ld}\right)\right]^2}\frac{1.4492227}{\left(n\pi/2\right)}$$

For the matched case

$$P_{rec} \approx \frac{h\left|H^{inc}\right|^2}{\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + G_{rad}/2\right]}\frac{1.4492227}{\left(n\pi/2\right)}$$

Then either

$$P_{rec} \approx \frac{h/\eta_0}{\left[R_{int}\left(n\pi/2\right)^2/\left\{\left(\omega L + X_{int}\right)h\right\}^2 + G_{rad}/2\right]}\frac{1.4492227}{\left(n\pi/2\right)}\eta_0\left|H^{inc}\right|^2$$

or

$$P_{rec} \approx \frac{h/\eta_0}{\left(R_{int}/Z_0^2 + G_{rad}/2\right)}\frac{1.4492227}{\left(n\pi/2\right)}\eta_0\left|H^{inc}\right|^2$$

$$\pi h\eta_0 G_{rad} = \mathrm{Cin}\left(2n\pi\right)$$

or

$$P_{rec} \approx \frac{h/\eta_0}{\left(R_{int}/Z_0^2 + G_{rad}/2\right)}\frac{1.4492227}{kh}\eta_0\left|H^{inc}\right|^2$$

$$\pi h\eta_0 G_{rad} = \mathrm{Cin}\left(4kh\right)$$

Now if $\Delta_1 = \pi/4$

$$\frac{\tan\left(x\right)}{x} = \frac{2x + \pi/2}{x + \pi/2} = \frac{1 + x/\left(\pi/4\right)}{1 + x/\left(\pi/2\right)}$$

$$1.350646 = 1.35379\ ,\ x = 0.86$$

$$1.354146 = 1.35459\ ,\ x = 0.863^{*}$$

$$1.36244 = 1.35644\ ,\ x = 0.87$$

$$1.354146 = 1.35459\ ,\ x = 0.863^{*} = (n\pi/2)\,u - \pi/4$$

$$u = \frac{\pi/4 + 0.863}{n\pi/2}$$

$$\cos\theta_0 = 1 - u = 1 - \frac{\pi/4 + 0.863}{n\pi/2} = 1 - \frac{1.6484}{n\pi/2}$$

$$\sin^2\theta_0 = 1 - (1-u)^2$$

$$P_{rec} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2u\sin^2\left(-\left(n\pi/2\right)u + \Delta_1\right)}{\left\{\left(n\pi/2\right)u - \Delta_1\right\}^2} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2\left(\pi/4 + 0.863\right)\sin^2\left(0.863\right)}{\left\{0.863\right\}^2 n\pi/2}$$

$$\approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2.5554}{n\pi/2} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2.5554}{kh - \pi/4} \approx hG_{ld}\frac{h^2\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2.5554\lambda\ell}{\pi}$$

$$\approx hG_{ld}\frac{2h^2\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{\lambda\ell}{0.782656\pi}$$

Note that the solution

$$\frac{\tan\left(x\right)}{x} = \frac{2x + 2\Delta_1}{x + 2\Delta_1}\ ,\ x = \left(n\pi/2\right)u - \Delta_1$$

must have $x \geq \Delta_1$. If $\Delta_1 = 0$ we have $x = 1.165561$. If $\Delta_1 \to -\Delta_1$

$$\frac{\tan\left(x\right)}{x} = \frac{2x - 2\Delta_1}{x - 2\Delta_1}\ ,\ x = \left(n\pi/2\right)u + \Delta_1$$

we can change the sign of x

$$\frac{\tan\left(x\right)}{x} = \frac{2x + 2\Delta_1}{x + 2\Delta_1}\ ,\ \ -x = \left(n\pi/2\right)u + \Delta_1$$

which gives the same equation we had previously; however, this now yields $u < 0$ which implies $\cos\theta_0 > 1$.

Suppose we take the range

$$0 \leq \Delta_1 < \pi/2$$

At the top of the range

$$\frac{\tan(x)}{x} = \frac{2x+\pi}{x+\pi}\ ,\ x = (n\pi/2)\,u - \pi/2$$

$$1.169545 = 1.1714319\ ,\ x = 0.65$$

$$1.17334658 = 1.172741\ ,\ x = 0.656^{*}$$

$$1.17398637 = 1.17295879\ ,\ x = 0.657$$

$$1.1759165 = 1.1736114\ ,\ x = 0.66$$

$$x = 0.656^{*} = (n\pi/2)\,u - \pi/2$$

$$u = \frac{0.656 + \pi/2}{n\pi/2}$$

$$\cos\theta_0 = 1 - u = 1 - \frac{0.656 + \pi/2}{n\pi/2}$$

$$P_{rec} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2u\sin^2\left(-(n\pi/2)\,u + \Delta_1\right)}{\left\{(n\pi/2)\,u - \Delta_1\right\}^2} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{2\,(0.656+\pi/2)\sin^2(0.656)}{\{0.656\}^2\,n\pi/2}$$

$$\approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{3.8503}{n\pi/2} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{3.8503}{kh-\pi/2} \approx hG_{ld}\frac{h^2\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{3.8503\lambda\ell}{\pi}$$

$$\approx hG_{ld}\frac{2h^2\left|ZH^{inc}\right|^2}{\left(2\beta' h\right)^2\left(\beta'' h\right)^2}\frac{\lambda\ell}{0.519440\pi}$$

$$P_{rec} \approx \frac{4hG_{ld}\left|ZH^{inc}\right|^2}{\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right) + \left(\omega L + X_{int}\right)\left(G_{rad}/2 + G_{ld}\right)\right]^2}\frac{3.8503}{n\pi/2}$$

$$\approx \frac{4hG_{ld}\left|H^{inc}\right|^2}{\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + \left(G_{rad}/2 + G_{ld}\right)\right]^2}\frac{3.8503}{n\pi/2}$$

For the matched load

$$P_{rec} \approx \frac{h\left|H^{inc}\right|^2}{\left[R_{int}\left(\omega C + B_{rad} + B_{ld}\right)/\left(\omega L + X_{int}\right) + G_{rad}/2\right]}\frac{3.8503}{n\pi/2}$$

Then either

$$P_{rec} \approx \frac{h/\eta_0}{\left[R_{int}\left(n\pi/2\right)^2/\left\{\left(\omega L + X_{int}\right)h\right\}^2 + G_{rad}/2\right]}\frac{3.8503}{n\pi/2}\eta_0\left|H^{inc}\right|^2$$

$$P_{rec} \approx \frac{h/\eta_0}{\left(R_{int}/Z_0^2 + G_{rad}/2\right)} \frac{3.8503}{n\pi/2} \eta_0 \left|H^{inc}\right|^2$$

$$\pi\eta_0 h G_{rad} = \mathrm{Cin}\left(2n\pi + 2\pi\right) = \mathrm{Cin}\left(2\left(n+1\right)\pi\right)$$

Note that $3.8503/1.4492227 \approx 2.6568$.

We might fit the three values here as

$$f_{\sigma_m}\left(\Delta_1\right) \approx 1.4492227 + \left(3.8503 - 1.4492227\right) \frac{\Delta_1}{\pi/2}$$

$$f_{\sigma_m}\left(\pi/4\right) \approx 2.649761\left(3.7\%\right) = 2.5554$$

Then

$$P_{rec} \approx \frac{h/\eta_0}{\left[R_{int}\left(n\pi/2\right)^2 / \left\{\left(\omega L + X_{int}\right)h\right\}^2 + G_{rad}/2\right]} \frac{f_{\sigma_m}\left(kh - n\pi/2\right)}{n\pi/2} \eta_0 \left|H^{inc}\right|^2$$

$$P_{rec} \approx \frac{h/\eta_0}{\left(R_{int}/Z_0^2 + G_{rad}/2\right)} \frac{f_{\sigma_m}\left(kh - n\pi/2\right)}{n\pi/2} \eta_0 \left|H^{inc}\right|^2$$

where

$$\pi\eta_0 h G_{rad} = \left(\frac{kh}{n\pi} + \frac{n\pi/4}{kh}\right) \left\{\mathrm{Cin}\left(n\pi + 2kh\right) - \mathrm{Cin}\left(n\pi - 2kh\right)\right\}$$

$$+\left(\frac{kh}{n\pi} - \frac{n\pi/4}{kh}\right) n\pi \left\{\mathrm{Si}\left(n\pi + 2kh\right) - \mathrm{Si}\left(n\pi - 2kh\right)\right\} - 2\sin^2\left(n\pi/2 + kh\right)$$

Note that if we increment by kh by $\pi/2$ and n by 1

$$\pi\eta_0 h G_{rad} = \frac{1}{2}\left(\frac{2kh + \pi}{n\pi + \pi} + \frac{n\pi/2 + \pi/2}{kh + \pi/2}\right) \left\{\mathrm{Cin}\left(n\pi + 2kh + 2\pi\right) - \mathrm{Cin}\left(n\pi - 2kh\right)\right\}$$

$$+\frac{1}{2}\left(\frac{2kh + \pi}{n\pi + \pi} - \frac{n\pi/2 + \pi/2}{kh + \pi/2}\right) n\pi \left\{\mathrm{Si}\left(n\pi + 2kh + 2\pi\right) - \mathrm{Si}\left(n\pi - 2kh\right)\right\} - 2\sin^2\left(n\pi/2 + kh + \pi\right)$$

Then if we set $kh = n\pi/2$ we obtain

$$\pi\eta_0 h G_{rad} = \mathrm{Cin}\left(2n\pi + 2\pi\right) = \mathrm{Cin}\left(2\left(n+1\right)\pi\right)$$

which is the same as the smoothed value $\mathrm{Cin}(4kh)$.

Summary Of Approximations Because there are several approximations involved in the preceding, it is useful to briefly review them and check their validity. More generally taking

$$\beta' h = n\pi/2 + \delta_1$$

$$\cos^2\left(\beta' h\right) = \sin^2 \delta_1 = \sin^2\left(\beta' h\right)$$

$$kh = n\pi/2 + \Delta_1$$

$$kh\cos\theta_0 = \left(n\pi/2 + \Delta_1\right)\cos\theta_0 = \left(n\pi/2 + \Delta_1\right)(1-u) = n\pi/2 + \Delta_1 - u\left(n\pi/2 + \Delta_1\right)$$

$$\cos^2\left(kh\cos\theta_0\right) = \sin^2\left\{\Delta_1 - u\left(n\pi/2 + \Delta_1\right)\right\} = \sin^2\left(kh\cos\theta_0\right)$$

$$\left(\beta' h\right)^2 \pm (kh)^2\cos^2\theta_0 = \left(n\pi/2 + \delta_1\right)^2 \pm \left\{n\pi/2 + \Delta_1 - u\left(n\pi/2 + \Delta_1\right)\right\}^2$$

$$\left(\beta' h\right)^2 - (kh)^2\cos^2\theta_0 = \left[u\left(n\pi/2 + \Delta_1\right) - \Delta_1 + \delta_1\right]\left[n\pi - u\left(n\pi/2 + \Delta_1\right) + \Delta_1 + \delta_1\right]$$

$$\sin^2\theta_0 = 1 - \cos^2\theta_0 = 1 - (1-u)^2 = u(2-u)$$

$$\cos\theta_0 = 1 - u$$

gives

$$P_{rec} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2\sin^2\theta_0}{\left\{\left(\beta' h\right)^2 - (kh)^2\cos^2\theta_0\right\}^2 + 2\left(\beta'' h\right)^2\left\{\left(\beta' h\right)^2 + (kh)^2\cos^2\theta_0\right\}}$$

$$\left\{\begin{array}{ll} \cos^2\left(kh\cos\theta_0\right)/\left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] & ,\ n \text{ odd} \\ \sin^2\left(kh\cos\theta_0\right)/\left[\sin^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right] & ,\ n \text{ even} \end{array}\right\}$$

$$\approx \frac{4hG_{ld}h^4\left|ZH^{inc}\right|^2 u(2-u)}{\left[u\left(n\pi/2+\Delta_1\right) - \Delta_1 + \delta_1\right]^2\left[(2-u)\left(n\pi/2+\Delta_1\right) - \Delta_1 + \delta_1\right]^2 + 2\left(\beta'' h\right)^2\left[\left(n\pi/2+\delta_1\right)^2 + \left\{n\pi/2 + \Delta_1 - u\left(n\pi/2+\Delta_1\right)\right\}^2\right]}$$

$$\frac{\sin^2\left\{u\left(n\pi/2+\Delta_1\right) - \Delta_1\right\}}{\sin^2\delta_1 + \left(\beta'' h\right)^2}$$

If $\delta_1 = 0$ so that $\beta' h = n\pi/2$

$$P_{rec} \approx \frac{4hG_{ld}h^4\left|ZH^{inc}\right|^2 u(2-u)}{\left[u\left(n\pi/2+\Delta_1\right) - \Delta_1\right]^2\left[(2-u)\left(n\pi/2+\Delta_1\right) - \Delta_1\right]^2 + 2\left(\beta'' h\right)^2\left[\left(n\pi/2\right)^2 + \left\{n\pi/2 + \Delta_1 - u\left(n\pi/2+\Delta_1\right)\right\}^2\right]}$$

$$\frac{\sin^2\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left(\beta'' h\right)^2}$$

If we drop the $\left(\beta'' h\right)^2$ term in the denominator

$$P_{rec} \approx hG_{ld}\frac{4h^4\left|ZH^{inc}\right|^2}{\left(\beta'' h\right)^2}\frac{u\left(2-u\right)\sin^2\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left[u\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2\left[\left(2-u\right)\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2}$$

If we approximate the final factor (and the second factor in the numerator) of the denominator for $u \to 0$

$$P_{rec} \approx hG_{ld}\frac{8h^4\left|ZH^{inc}\right|^2}{\left(\beta'' h\right)^2\left[2\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2}\frac{u\sin^2\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left[u\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2}$$

Now if we maximize the final factor with respect to u

$$\frac{d}{du}\frac{u\sin^2\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left[u\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2} = \frac{\sin\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}\cos\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left[u\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2}\left\{u\left(n\pi/2+\Delta_1\right)+\Delta_1\right\}$$

$$\left[-\frac{\tan\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}{\left\{u\left(n\pi/2+\Delta_1\right)-\Delta_1\right\}}+\frac{2u\left(n\pi/2+\Delta_1\right)}{\left\{u\left(n\pi/2+\Delta_1\right)+\Delta_1\right\}}\right]=0$$

Taking

$$x = u\left(n\pi/2+\Delta_1\right)-\Delta_1$$

this becomes

$$\frac{\tan\left(x\right)}{x} = \frac{2\left(x+\Delta_1\right)}{x+2\Delta_1}$$

$$\cos\theta_0 = 1-u$$

For $\Delta_1 = 0$

$$x = 1.165561 = \left(n\pi/2\right)u$$

For $\Delta_1 = \pi/4$

$$x = 0.863 = \left(n\pi/2\right)u - \pi/4$$

For $\Delta_1 = \pi/2$

$$x = 0.656 = \left(n\pi/2\right)u - \pi/2$$

Now checking the dropping of the $\left(\beta'' h\right)^2$ term in the denominator

$$\left[u\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2\left[\left(2-u\right)\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2 >> 2\left(\beta'' h\right)^2\left[\left(n\pi/2\right)^2+\left\{n\pi/2+\Delta_1-u\left(n\pi/2+\Delta_1\right)\right\}^2\right]$$

or in terms of x

$$x^2\left[2\left(n\pi/2\right)-x\right]^2 >> 2\left(\beta''h\right)^2\left[\left(n\pi/2\right)^2+\left\{n\pi/2-x\right\}^2\right]$$

or

$$x^2\left(n\pi-x\right)^2 >> \left(\beta''h\right)^2\left[x^2\left(n\pi-x\right)^2+x^2\right]$$

or

$$1 >> \left(\beta''h\right)^2\left[1+1/\left(n\pi-x\right)^2\right]$$

This is clearly true for $\beta''h << 1$ even for $n=1$.

The other approximation which was to replace

$$x = u\left(n\pi/2+\Delta_1\right)-\Delta_1$$

$$\frac{(2-u)}{\left[(2-u)\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2} = \frac{2-\left(x+\Delta_1\right)/\left(n\pi/2+\Delta_1\right)}{\left(n\pi-x\right)^2} = \frac{2}{\left(n\pi+\Delta_1\right)^2}\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2}$$

$$\sim \frac{2}{\left(n\pi+\Delta_1\right)^2}\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{1-2\frac{x+\Delta_1}{n\pi+\Delta_1}}$$

by

$$\frac{2}{\left[2\left(n\pi/2+\Delta_1\right)-\Delta_1\right]^2} = \frac{2}{\left(n\pi+\Delta_1\right)^2}$$

$$\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2} \approx \frac{1-\frac{1.165561}{3\pi/2}}{\left(1-\frac{1.165561}{5\pi/4}\right)^2} \approx \frac{0.75266}{0.4945} \approx 1.522126$$

$$\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2} \approx \frac{1-\frac{1.648}{3\pi/2}}{\left(1-\frac{1.648}{5\pi/4}\right)^2} \approx \frac{0.6502}{0.336677} \approx 1.93123$$

$$\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2} \approx \frac{1-\frac{2.2268}{3\pi/2}}{\left(1-\frac{2.2268}{5\pi/4}\right)^2} \approx \frac{0.52746}{0.18745} \approx 2.8139$$

$$\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2} \approx \frac{1-\frac{1.165561}{5\pi/2}}{\left(1-\frac{1.165561}{9\pi/4}\right)^2} \approx \frac{0.8516}{0.6974} \approx 1.2211$$

$$\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2} \approx \frac{1-\frac{1.648}{5\pi/2}}{\left(1-\frac{1.648}{9\pi/4}\right)^2} \approx \frac{0.79012}{0.58798} \approx 1.34378$$

$$\frac{1-\frac{x+\Delta_1}{n\pi+2\Delta_1}}{\left(1-\frac{x+\Delta_1}{n\pi+\Delta_1}\right)^2} \approx \frac{1-\frac{2.2268}{5\pi/2}}{\left(1-\frac{2.2268}{9\pi/4}\right)^2} \approx \frac{0.716475}{0.46919} \approx 1.52705$$

If we use normal incidence $\theta_0 = \pi/2$ for $n = 1$

$$P_{rec} \approx hG_{ld} \frac{4h^4 \left|ZH^{inc}\right|^2 / \left(\beta' h\right)^2}{\left(\beta' h\right)^2 + 2\left(\beta'' h\right)^2} / \left[\cos^2\left(\beta' h\right) + \left(\beta'' h\right)^2\right]$$

Taking $\beta' h = \pi/2$

$$P_{rec} \approx \frac{4hG_{ld} \left|ZH^{inc}\right|^2 / \left(\beta'\beta''\right)^2}{\left(\beta' h\right)^2 + 2\left(\beta'' h\right)^2} = \frac{4h^3 G_{ld} \left|ZH^{inc}\right|^2 / \left(\beta''\left(\pi/2\right)^2\right)^2}{1 + 2\left(\beta'' h\right)^2 / \left(\pi/2\right)^2}$$

$$\approx \frac{4h^5 G_{ld} \left|ZH^{inc}\right|^2}{\left(\beta'' h\right)^2 \left(\beta' h\right)^2 \left(\pi/2\right)^2} = \frac{16hG_{ld} \left|ZH^{inc}\right|^2}{\left(2\beta'\beta''\right)^2 \left(\pi/2\right)^2}$$

where

$$2\beta'\beta'' \approx R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\} + \left(\omega L + X_{int}\right)\left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)$$

Thus

$$P_{rec} \approx \frac{16hG_{ld} \left|H^{inc}\right|^2 / \left(\pi/2\right)^2}{\left[R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\} / \left(\omega L + X_{int}\right) + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2}$$

$$\approx \frac{16hG_{ld} \left|H^{inc}\right|^2 / \left(\pi/2\right)^2}{\left[R_{int}\left(\pi/2\right)^2 / \left(\omega L + X_{int}\right)^2 + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2}$$

Above

$$P_{rec} \approx \frac{4hG_{ld} \left|H^{inc}\right|^2}{\left[R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\} / \left(\omega L + X_{int}\right) + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2} \frac{f_{\sigma_m}\left(\Delta_1\right)}{\left(\pi/2\right)}$$

$$\approx \frac{16hG_{ld} \left|H^{inc}\right|^2 / \left(\pi/2\right)^2}{\left[R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\} / \left(\omega L + X_{int}\right) + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2} \left(\pi/8\right) f_{\sigma_m}\left(\Delta_1\right)$$

$$\left(\pi/8\right) f_{\sigma_m}\left(\Delta_1\right) \approx 0.5691084 + \left(1.51201 - 0.5691084\right) \frac{\Delta_1}{\pi/2}$$

Note that the actual value at $\Delta_1 = \pi/4$ is $2.5554\pi/8 \approx 1.00350$. The value at $\Delta_1 = 0$ is 0.5691084 and at $\Delta_1 = \pi/2$ is 1.51201.

8.4 Summary Of Cross Sections & Cavity Electric Field Estimates

We now summarize the preceding results, explicitly listing the radiation conductance, and discuss the estimates of the interior electric fields. Fixed quantities needed below are

$$R_{int} \approx \left(\frac{L}{L^{intr}}\right)^2 2R_s/d$$

$$L = \mu_0 \pi / \Omega_e$$

$$L^{intr} = \mu_0 w / d$$

$$\Omega_e = 2 \ln (2h/a_e) + 2 (\ln 2 - 7/3)$$

$$a_e \sim \frac{2w}{\pi e} e^{-\pi d/(2w)} \ , \ d > 0.3w$$

$$R_s = 1/ (\sigma \delta)$$

$$\delta = \sqrt{2/ (\omega \mu \sigma)}$$

$$Z_0 = \eta_0 \pi / \Omega_e$$

$$S^{inc} = \eta_0 \left| H^{inc} \right|^2$$

The matched oblique slot cross section (with perfectly conducting walls) is fit by the formulas

$$\sigma^M_{obq} \approx \frac{\lambda \ell}{f_\sigma (kh) \operatorname{Cin} (4kh)} \approx \frac{\lambda \ell}{f_\sigma (kh) [\gamma' + \ln (4\pi \ell/\lambda)]} \ , \ kh \geq \pi/2 \ , \ \lambda \leq \ell/2$$

$$f_\sigma (kh) \approx 1.38 + \frac{\pi/4 - 1.38}{(kh)} \pi/2 \ , \ kh \geq \pi/2 \ , \ \lambda \leq \ell/2$$

$$\sigma^M_{obq} \approx \frac{\lambda \ell}{(\pi/4) \operatorname{Cin} (2\pi)} \ , \ k_0 h < kh < \pi/2$$

$$\sigma^M_{obq} = \frac{\lambda^2}{4\pi} 3 \ , \ kh < k_0 h = \frac{\pi}{2} \frac{3}{8} \operatorname{Cin} (2\pi) \approx 0.914120026 \frac{\pi}{2} \approx 1.43589638$$

$$\operatorname{Cin} (2\pi) \approx 2.4376534$$

$$\gamma' \approx 0.5772156649$$

There is negligible error if we let $k_0 h \to \pi/2$.

Using cavity dimensions $a = 4$ in, $h_c = 24$ in, the surface area and volume are

$$S_{cav} = 2\pi a^2 + 2\pi a h_c \approx 0.45401 \text{ m}^2$$

$$V_{cav} = \pi a^2 h_c \approx 0.01976888 \text{ m}^3$$

8.4.1 Matched Received Power At Normal Incidence

The matched power is taken as the uniformly loaded slot with maximum power transfer [1]

$$P_{rec} = \sigma_{rec} S^{inc}$$

$$\sigma_{rec} = \frac{2h\left(8/\pi^2\right)/\eta_0}{R_{int}\pi^2/\left(\omega L 2h\right)^2 + G_{rad}/2} = q^{tl}_{rec} \frac{\ell^2\left(8/\pi^2\right)}{\eta_0 h G_{rad}} \tag{7}$$

$$q^{tl}_{rec} = \frac{G_{rad}/2}{R_{int}\pi^2/\left(\omega L 2h\right)^2 + G_{rad}/2}$$

$$\pi\eta_0 h G_{rad} = -2\sin^2\left(\pi/2 + kh\right) + \left(\frac{kh}{\pi} + \frac{\pi}{4kh}\right)\left[\mathrm{Cin}\left(2kh+\pi\right) - \mathrm{Cin}\left(\pi - 2kh\right)\right]$$

$$+kh\left[1 - \left(\frac{\pi}{2kh}\right)^2\right]\left[\mathrm{Si}\left(2kh+\pi\right) - \mathrm{Si}\left(\pi - 2kh\right)\right]$$

Normal Alternative Form The alternative form of the normal matched cross section using the slot characteristic impedance is (this approximation is of questionable validity as noted in the preceding subsection where this is first discussed)

$$P_{rec} = \sigma_{rec} S^{inc}$$

$$\sigma_{rec} = \frac{2h\left(8/\pi^2\right)/\eta_0}{R_{int}/Z_0^2 + G_{rad}/2} = q^{tl}_{rec} \frac{\ell^2\left(8/\pi^2\right)}{\eta_0 h G_{rad}} \tag{8}$$

$$q^{tl}_{rec} = \frac{G_{rad}/2}{R_{int}/Z_0^2 + G_{rad}/2}$$

Matched Linear Array Received Power The linear array cross section is

$$P_{rec} = \sigma_{rec} S^{inc}$$

$$\sigma_{rec} = q^{tl}_{rec}\lambda\ell/\pi = q^{tl}_{rec} 2\ell/k$$

$$q^{tl}_{rec} = \frac{G_{rad}/2}{G_{rad}/2 + R_{int}/Z_0^2}$$

For comparison to the preceding results we can take the radiation conductance the same as in the preceding.

Large Slot Loss Received Power The large loss received power is (and if we include the reactive field in the transmitted half space we would replace $B_{rad}/2 \to B_{rad}$)

$$P_{rec} = \sigma_{rec} S^{inc}$$

$$\sigma_{rec} = \frac{8hG_{ld}/\eta_0}{\left(G + G_{rad}/2 + G_{ld}\right)^2 + \left(\omega C + B_{rad}/2 + B_{ld}\right)^2} = q_{rec}^{tl} \frac{\ell}{\eta_0 G_{rad}/2}$$

$$\pi\eta_0 h G_{rad} = 2kh\,\mathrm{Si}\left(2kh\right) + \frac{\sin\left(2kh\right)}{2kh} + \cos\left(2kh\right) - 2$$

$$q_{rec}^{tl} = \frac{4G_{ld}\left(G_{rad}/2\right)}{\left(G + G_{rad}/2 + G_{ld}\right)^2 + \left(\omega C + B_{rad}/2 + B_{ld}\right)^2}$$

For the matched case $\omega C + B_{rad}/2 + B_{ld} = 0$

$$G_{ld} = G + G_{rad}/2$$

$$q_{rec}^{tl} = \frac{G_{rad}/2}{G + G_{rad}/2}$$

8.4.2 Matched Oblique Received Power

The oblique angle matched received power is

$$P_{rec} = \sigma_{rec} S^{inc}$$

Then

$$\sigma_{rec} \approx \frac{h/\eta_0}{\left[R_{int}\left(n\pi/2\right)^2 / \left\{\left(\omega L + X_{int}\right)h\right\}^2 + G_{rad}/2\right]} \frac{f_{\sigma_m}\left(kh - n\pi/2\right)}{n\pi/2} \tag{9a}$$

$$f_{\sigma_m}\left(\Delta_1\right) \approx 1.4492227 + \left(3.8503 - 1.4492227\right)\frac{\Delta_1}{\pi/2}$$

$$\pi\eta_0 h G_{rad} = \left(\frac{kh}{n\pi} + \frac{n\pi/4}{kh}\right)\left\{\mathrm{Cin}\left(n\pi + 2kh\right) - \mathrm{Cin}\left(n\pi - 2kh\right)\right\}$$

$$+ \left(\frac{kh}{n\pi} - \frac{n\pi/4}{kh}\right) n\pi\left\{\mathrm{Si}\left(n\pi + 2kh\right) - \mathrm{Si}\left(n\pi - 2kh\right)\right\} - 2\sin^2\left(n\pi/2 + kh\right)$$

The first resonance uses normal incidence

$$\sigma_{rec} \approx \frac{4h/\eta_0}{\left[R_{int}\left(\pi/2\right)^2 / \left\{\left(\omega L + X_{int}\right)h\right\}^2 + G_{rad}/2\right]\left(\pi/2\right)^2}$$

$$n\pi/2 \le kh < \left(n+1\right)\pi/2$$

$$P_{rec} \approx \frac{16h\left(G_{ld}/\eta_0\right)/\left(\pi/2\right)^2}{\left[R_{int}\left\{\omega\left(C + \Delta C_{gas}^{intr}\right) + B_{rad} + B_{ld}\right\}/\left(\omega L + X_{int}\right) + \left(G_{gas}^{intr} + G_{rad}/2 + G_{ld}\right)\right]^2}\eta_0\left|H^{inc}\right|^2$$

$$\approx \frac{16h\left(G_{ld}/\eta_0\right)/\left(\pi/2\right)^2}{\left[R_{int}\left(\pi/2\right)^2/\left(\omega L+X_{int}\right)^2+\left(G_{gas}^{intr}+G_{rad}/2+G_{ld}\right)\right]^2}\eta_0\left|H^{inc}\right|^2$$

$$G_{ld}=R_{int}\left\{\omega\left(C+\Delta C_{gas}^{intr}\right)+B_{rad}+B_{ld}\right\}/\left(\omega L+X_{int}\right)+\left(G_{gas}^{intr}+G_{rad}/2\right)$$

$$\sigma_{rec}\approx\frac{4\left(h/\eta_0\right)/\left(\pi/2\right)^2}{\left[R_{int}\left\{\omega\left(C+\Delta C_{gas}^{intr}\right)+B_{rad}+B_{ld}\right\}/\left(\omega L+X_{int}\right)+\left(G_{gas}^{intr}+G_{rad}/2\right)\right]}$$

$$\approx\frac{4\left(h/\eta_0\right)/\left(\pi/2\right)^2}{\left[R_{int}\left(\pi/2\right)^2/\left(\omega L+X_{int}\right)^2+\left(G_{gas}^{intr}+G_{rad}/2\right)\right]}$$

Oblique Alternate Smooth Form The maximum oblique angle received power is

$$P_{rec}=\sigma_{rec}S^{inc}$$

$$\sigma_{rec}\approx\sigma_{obq}^{M}q_{rec}^{tl} \tag{10}$$

$$q_{rec}^{tl}=\frac{4G_{ld}G_{rad}/2}{\left[G_{ld}+G_{rad}/2+R_{int}\left(\omega C+B_{rad}+B_{ld}\right)/\left(\omega L+X_{int}\right)\right]^2}$$

$$G_{ld}=G_{rad}/2+R_{int}\left(\omega C+B_{rad}+B_{ld}\right)/\left(\omega L+X_{int}\right)\approx G_{rad}/2+R_{int}/Z_0^2$$

$$q_{rec}^{tl}=\frac{G_{rad}/2}{G_{rad}/2+R_{int}/Z_0^2}$$

$$\pi\eta_0 hG_{rad}=\mathrm{Cin}\left(4kh\right)\ ,\ kh=n\pi/2$$

where we intend to use this for general kh, interpreting the result as an interpolation between resonances.

8.4.3 Electric Field Estimates

Balancing the bound on the net power received through the slot aperture into the cavity with the cavity wall loss can now be used to estimate the cavity field [1]. The net power received can be written as

$$P_{rec}=\sigma_{rec}S^{inc}=P_{wall}$$

In a general cavity the power absorbed in the cavity walls can be written as [11], [1]

$$P_{wall}=\sigma_{wall}S^{cavfld}=R_s\left\langle\left|\underline{H}\right|^2\right\rangle_S=R_sS_{cav}2\left\langle\left|H_j\right|^2\right\rangle_S$$

where the subscript S of the angular brackets indicates an average over the cavity metallic surface, and each of the two components of the magnetic field on the wall are denoted by H_j. The cavity field power density S^{cavfld} is [11]

$$S^{cavfld}=\left\langle\left|\underline{E}\right|^2\right\rangle_V/\eta_0=\eta_0\left\langle\left|\underline{H}\right|^2\right\rangle_V=3\eta_0\left\langle\left|H_j\right|^2\right\rangle_V$$

where the subscript V indicates average over the cavity volume and in the general cavity each of the three volume components is denoted by H_j. In the general cavity the component averages at the cavity wall versus in the cavity volume are related by [1]

$$\left\langle |H_j|^2 \right\rangle_S = 2 \left\langle |H_j|^2 \right\rangle_V$$

The wall cross section is [1]

$$\sigma_{wall} = \frac{4}{3} S_{cav} R_s / \eta_0$$

From these we can write the mean shielding effectiveness

$$\langle SE \rangle = S^{cavfld}/S^{inc} = \left\langle \left|\underline{E}/E^{inc}\right|^2 \right\rangle_V = 3 \left\langle \left|E_j/E^{inc}\right|^2 \right\rangle_V = \left\langle \left|\underline{H}/H^{inc}\right|^2 \right\rangle_V = 3 \left\langle \left|H_j/H^{inc}\right|^2 \right\rangle_V = \sigma_{rec}/\sigma_{wall} \tag{11}$$

Extreme values for a component will be estimated as a three-dimensional standing wave (with a factor of two for the peak to average in each of the three directions)

$$\left|E_j/E^{inc}\right|^2_{\max} = 8 \left\langle \left|E_j/E^{inc}\right|^2 \right\rangle_V = \left|H_j/H^{inc}\right|^2_{\max} = 8 \left\langle \left|H_j/H^{inc}\right|^2 \right\rangle_V = \frac{8}{3} \sigma_{rec}/\sigma_{wall} \tag{12}$$

We can then write the mean wall value of a component as

$$\left\langle \left|E_n/E^{inc}\right|^2 \right\rangle_S = \left\langle \left|H_j/H^{inc}\right|^2 \right\rangle_S = \frac{2}{3} \sigma_{rec}/\sigma_{wall} \tag{13}$$

The plots below show electric fields on the wall and include a 3 dB increase from the volume field. There is a question whether we should also include the 3 dB wall enhancement on top of this extreme value in the volume, even though the wall value is being averaged in two-dimensions. However, there are additional factor of two increases in expected extreme levels in special cases, as discussed in the next subsection, and therefore we will include this factor of two in the comparisons below. In addition, we also note that in the region (generally higher frequencies with more complex cavities in the undermoded region as well as the overmoded region) where cavity statistics apply, we expect the three standard deviation level to be a factor of nine above the mean squared component level; furthermore, on the wall we expect another factor of 2 [19].

8.4.4 Cavity Modes

Here we review special choices of indices. If we choose indices such that there is only a single component of the electric field, then

$$S^{cavfld} = \eta_0 \left\langle |\underline{H}|^2 \right\rangle_V = \left\langle |\underline{E}|^2 \right\rangle_V / \eta_0 = \left\langle |E_j|^2 \right\rangle_V / \eta_0$$

$$\left|E_j/E^{inc}\right|^2_{\max} = 8 \left\langle \left|E_j/E^{inc}\right|^2 \right\rangle_V = 8 \sigma_{rec}/\sigma_{wall}$$

Rectangular Cavity TE Modes The eigenmode electric field is

$$\varepsilon_0 E_x^{(n,j,m)} = (j\pi/b)\, A_{njm}^{TE} \cos(n\pi x/a) \sin(j\pi y/b) \sin(m\pi z/c)$$

$$\varepsilon_0 E_y^{(n,j,m)} = -\,(n\pi/a)\, A_{njm}^{TE} \sin(n\pi x/a) \cos(j\pi y/b) \sin(m\pi z/c)$$

$$\varepsilon_0 E_z^{(njm)} = 0$$

and the eigenmode magnetic field is

$$H_x^{(n,j,m)} = \frac{-i}{\omega_{njm}\mu_0\varepsilon_0} A_{njm}^{TE} (n\pi/a)(m\pi/c)\sin(n\pi x/a)\cos(j\pi y/b)\cos(m\pi z/c)$$

$$H_y^{(n,j,m)} = \frac{-i}{\omega_{njm}\mu_0\varepsilon_0} A_{njm}^{TE} (j\pi/b)(m\pi/c)\cos(n\pi x/a)\sin(j\pi y/b)\cos(m\pi z/c)$$

$$H_z^{(n,j,m)} = \frac{i}{\omega_{njm}\mu_0\varepsilon_0} A_{njm}^{TE} \left(k_{njm}^2 - m^2\pi^2/c^2\right)\cos(n\pi x/a)\cos(j\pi y/b)\sin(m\pi z/c)$$

If we set $n = 0$ we only have E_x, H_y and H_z. If we set $j = 0$ we only have E_y, H_x, H_z. Note that each component distribution can be written as

$$\frac{|E_x|^2}{\left\langle |E_x|^2 \right\rangle_V} = \varepsilon_n \cos^2(n\pi x/a)\,\varepsilon_j \sin^2(j\pi y/b)\,\varepsilon_m \sin^2(m\pi z/c)$$

Cylindrical Cavity TM Modes The TM modes in the cylindrical cavity are

$$E_z^{(p,m,n)} = A_{p,m,n} J_m(j_{m,p}\rho/a)\cos[m(\varphi-\varphi_0)]\cos(n\pi z/h_c)$$

where the roots of the Bessel function $j_{m,p}$ satisfy

$$J_m(j_{m,p}) = 0$$

$$E_\rho^{(p,m,n)} = \frac{-1}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n} \frac{j_{m,p}}{a} J_m'(j_{m,p}\rho/a)\cos[m(\varphi-\varphi_0)]\frac{n\pi}{h_c}\sin(n\pi z/h_c)$$

$$E_\varphi^{(p,m,n)} = \frac{1}{k_{p,m,n}^2 - (n\pi/h_c)^2}\frac{1}{\rho} A_{p,m,n} J_m(j_{m,p}\rho/a)\, m\sin[m(\varphi-\varphi_0)]\frac{n\pi}{h_c}\sin(n\pi z/h_c)$$

$$H_\rho^{(p,m,n)} = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2}\frac{1}{\rho} A_{p,m,n} J_m(j_{m,p}\rho/a)\, m\sin[m(\varphi-\varphi_0)]\cos(n\pi z/h_c)$$

$$H_\varphi^{(p,m,n)} = \frac{i\omega\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n}\frac{j_{m,p}}{a} J_m'(j_{m,p}\rho/a)\cos[m(\varphi-\varphi_0)]\cos(n\pi z/h_c)$$

$$H_z^{(p,m,n)} = 0$$

Note that for $n = 0$ there is only an E_z electric field component with H_ρ, H_φ components. If $m = 0$ there is only the H_φ component, along with E_ρ, E_z components. If both $n = 0 = m$ we only have E_z and H_φ components. Note that we can write the axial field distribution as

$$\frac{|E_z|^2}{\left\langle |E_z|^2 \right\rangle_V} = \frac{J_m^2(j_{m,p}\rho/a)}{J_m^2(j_{m,p})}\varepsilon_m \cos^2(m\varphi)\,\varepsilon_n \cos^2(n\pi z/h_c)$$

where the extreme values of the leading radial factor are discussed later in this report.

The preceding wall cross section for these TM modes with only two magnetic field components is

$$\sigma_{wall} = \frac{R_s}{\eta_0} \left\langle |\underline{H}|^2 \right\rangle_S / \left\langle |\underline{H}|^2 \right\rangle_V = \frac{R_s}{\eta_0} \left(\varepsilon_n 2\pi a^2 + 2\pi a h_c \right) = \frac{R_s}{\eta_0} \left(\varepsilon_n a/h_c + 1 \right) 2\pi a h_c$$

The quality factor is

$$Q^{TM}_{p,m,n} = \frac{\omega\mu_0 V_{cav} \left\langle |\underline{H}|^2 \right\rangle_V}{R_s S_{cav} \left\langle |\underline{H}|^2 \right\rangle_S} = \frac{k V_{cav}}{\sigma_{wall}} = \frac{k_{p,m,n} a}{\varepsilon_n a/h_c + 1} \frac{\eta_0}{2R_s}$$

The special case $n = 0$ with E_z, H_ρ, and H_φ

$$\sigma_{wall} = \frac{R_s}{\eta_0} S_{cav}$$

and

$$S^{cavfld} = \left\langle |E_z|^2 \right\rangle_V / \eta_0$$

$$\langle SE \rangle = S^{cavfld}/S^{inc} = \left\langle \left| \underline{E}/E^{inc} \right|^2 \right\rangle_V = \left\langle \left| E_z/E^{inc} \right|^2 \right\rangle_V = \sigma_{rec}/\sigma_{wall} = \sigma_{rec}/ \left(\frac{R_s}{\eta_0} S_{cav} \right) = \frac{4}{3} \sigma_{rec}/ \left(\frac{4}{3} \frac{R_s}{\eta_0} S_{cav} \right)$$

In the special case where $n = 0$ we loose one factor of two since the variation in the z direction is then constant, so the extreme level would be expected to be

$$\left| E_z/E^{inc} \right|^2_{\max} = 4 \left\langle \left| E_z/E^{inc} \right|^2 \right\rangle_V = 4\sigma_{rec}/\sigma_{wall} = 4\sigma_{rec}/ \left(\frac{R_s}{\eta_0} S_{cav} \right) = 2 \left[\frac{8}{3} \sigma_{rec}/ \left(\frac{4}{3} \frac{R_s}{\eta_0} S_{cav} \right) \right]$$

Writing it as the final expression, we see that overall there is a factor of $2 = 3$ dB increase over the general case. Hence we see that these special cases lead to further enhancement of the field extreme.

Cylindrical Cavity TE Modes The TE modes of interest (driven by a slot at the center of the cylindrical cavity) are odd in z with axial field

$$H_z^{(p,m,n)} = A_{p,m,n} J_m \left(j'_{m,p} \rho/a \right) \cos \left[m \left(\varphi - \varphi_0 \right) \right] \sin \left(n\pi z/h_c \right)$$

where the roots of the derivative of the Bessel function $j'_{m,p}$ satisfy

$$J'_m \left(j'_{m,p} \right) = 0$$

$$H_\rho^{(p,m,n)} = \frac{1}{k'^2_{p,m,n} - \left(n\pi/h_c \right)^2} A_{p,m,n} \frac{j'_{m,p}}{a} J'_m \left(j'_{m,p} \rho/a \right) \cos \left[m \left(\varphi - \varphi_0 \right) \right] \frac{n\pi}{h_c} \cos \left(n\pi z/h_c \right)$$

$$H_\varphi^{(p,m,n)} = \frac{-1}{k'^2_{p,m,n} - \left(n\pi/h_c \right)^2} \frac{m}{\rho} A_{p,m,n} J_m \left(j'_{m,p} \rho/a \right) \sin \left[m \left(\varphi - \varphi_0 \right) \right] \frac{n\pi}{h_c} \cos \left(n\pi z/h_c \right)$$

$$E_\rho^{(p,m,n)} = \frac{-i\omega\mu_0}{k'^2_{p,m,n} - \left(n\pi/h_c \right)^2} \frac{m}{\rho} A_{p,m,n} J_m \left(j'_{m,p} \rho/a \right) \sin \left[m \left(\varphi - \varphi_0 \right) \right] \sin \left(n\pi z/h_c \right)$$

$$E_\varphi^{(p,m,n)} = \frac{-i\omega\mu_0}{k'^2_{p,m,n} - \left(n\pi/h_c \right)^2} A_{p,m,n} \frac{j'_{m,p}}{a} J'_m \left(j'_{m,p} \rho/a \right) \cos \left[m \left(\varphi - \varphi_0 \right) \right] \sin \left(n\pi z/h_c \right)$$

We cannot have $n = 0$ because the field vanishes. If $m = 0$ we have a symmetric field with H_z, H_ρ, and E_φ. If $m = 0$ and $p = 1$, where $j'_{0,1} = 0$ we have a symmetric field with H_z only.

8.4.5 Oblique Power Sum

Now we summarize results for the case where we estimate the net power into the cavity by using the combination of the transmitted power through the slot, in addition to the power radiated out of the cavity and absorbed in the slot walls (the same slot) when driven by an average of plane waves at all angles. This approach, which is an incoherent power summation and has been used in the past [11], [1], underestimates the interior field at the lower frequencies. We write

$$P_{trans} = P_{loss} + P_{wall}$$

where the power transmitted through the slot from the exterior is

$$P_{trans} = \sigma_{trans} S^{inc}$$

with

$$\sigma_{trans} \approx \sigma_{obq}^{M} q_{trans}^{tl} \tag{14}$$

where the transmission mismatch factor is taken as

$$q_{trans}^{tl} = \frac{4\left(G_{rad}/2\right)^2}{\left[G_{rad}/2 + G_{rad}/2 + R_{int}\left(\omega C + B_{rad}\right)/\left(\omega L + X_{int}\right)\right]^2} \approx \frac{G_{rad}^2}{\left(G_{rad} + R_{int}/Z_0^2\right)^2}$$

The power lost to the slot is

$$P_{loss} = \left\langle \sigma_{loss} \right\rangle S^{cavfld}$$

where the cavity power density is

$$S^{cavfld} = \eta_0 \left\langle \left|\underline{H}\right|^2 \right\rangle_V = 3\eta_0 \left\langle \left|H_j\right|^2 \right\rangle_V$$

The slot loss cross section (slot driven by the interior cavity field, approximated by cross section for average of plane waves) is taken to be averaged over all angles and polarization

$$\left\langle \sigma_{loss} \right\rangle = \left\langle \sigma_{trans} \right\rangle + \left\langle \sigma_{abs} \right\rangle \tag{15}$$

where the average slot transmission cross section is

$$\left\langle \sigma_{trans} \right\rangle = q_{trans}^{tl} 2 \frac{\lambda^2}{4\pi} \frac{1}{2}$$

and the average slot absorption cross section is

$$\left\langle \sigma_{abs} \right\rangle = q_{abs}^{tl} 2 \frac{\lambda^2}{4\pi} \frac{1}{2}$$

with mismatch factor taken as

$$q_{abs}^{tl} = \frac{4\left(G_{rad}/2\right)\left\{R_{int}\left(\omega C + B_{rad}\right)/\left(\omega L + X_{int}\right)\right\}}{\left[G_{rad} + R_{int}\left(\omega C + B_{rad}\right)/\left(\omega L + X_{int}\right)\right]^2} \approx \frac{2G_{rad}R_{int}/Z_0^2}{\left(G_{rad} + R_{int}/Z_0^2\right)^2}$$

Power balance is then used as

$$P_{in} = \sigma_{trans} S^{inc} = \left(\left\langle \sigma_{loss} \right\rangle + \sigma_{wall}\right) S^{cavfld} = P_{loss} + P_{wall}$$

again with cavity wall absorption cross section

$$\sigma_{wall} = \frac{4}{3} S_{cav} R_s / \eta_0$$

Then we find the ratio of the interior average field to incident field

$$\langle SE \rangle = S^{cavfld} / S^{inc} = \left\langle \left| \underline{H} / H^{inc} \right|^2 \right\rangle_V = 3 \left\langle \left| H_j / H^{inc} \right|^2 \right\rangle_V = \sigma_{trans} / \left(\langle \sigma_{trans} \rangle + \langle \sigma_{abs} \rangle + \sigma_{wall} \right) \qquad (16)$$

The extreme level is found in the same manner as in the preceding subsection.

8.5 Comparisons Of Power Balance Results And Rigorous Cylindrical Cavity Solution

Here we compare the various bounding curves with the cylindrical cavity electric field calculations; we note again that Figure 9 shows five curves illustrating the cylindrical cavity (with cylindrical exterior) versus normal and oblique bounding curves for the interior wall electric field using (11), (12), and (13). The solid black curve shows the field when using the normal matched resonant received cross section (7). The dashed black curve shows the field when using the normal matched received cross section (8). These two results are similar to one another and appear to bound the cavity field results except for a resonance just above 1 GHz, which is a couple of dB higher (this could be a result of a higher projections on the slot of the short circuit current drive of the cylinder exterior as mentioned at the end of this subsection). The light blue curve uses the received oblique cross section (9a); note that approximations used in the derivation of the light blue curve may not be accurate at the lower frequencies (smaller values of $n = 1, 2$) This is in a sense the generalization of the normal black curve, but the mode number $n \le kh$ changes value in the blue curve with frequency, whereas it remains fixed at $n = 1$ in the black curve. The dark blue curve uses received cross section (10) and is a smoother version of the oblique result.

Figure 10 shows the preceding bounding curves using the normal matched resonant received cross section (7) (black curve), using the alternate normal matched received cross section (8) (dashed black curve), using the received oblique cross section (10) (blue curve); the blue curve is near but somewhat below the E_z field peaks but seems to bound the E_ρ peaks (this may have to do with the TM nature of the cavity modes for E_z versus the general cavity assumption in the power balance). The bounding green curve uses the transmission cross section (14) along with the average loss cross section (15) in the balance (16) (green curve); notice that this result fails to bound the cavity results, particularly at the lower frequencies near the first slot resonances. This is not too surprising since the incoherent sum is not rigorously justified. Figure 11 shows the same comparison for the cylindrical cavity E_ρ field.

It is important to note that the Thick slot model was used here to generate the cylindrical cavity results (purple curves) which assumes the slot depth is less than one half wavelength even though the frequency is taken higher than is valid for the depth dimension; hence it must be assumed that the results at frequencies above about 10 GHz actually describe a case where depth and width (and slot wall conductivity) are decreased, maintaining similar reactive (and slot wall loss) terms. If the more general Deep slot model were used in the setup we would expect more slot resonances at the higher frequencies due to the depth modes.

The cylindrical cavity model has been setup to handle arbitrary angles of incidence. We should examine other azimuth angles φ_i of oblique incidence, including normal incidence at the higher frequencies (because we are on a cylinder there are phase variations of the drive current along the slot even at normal incidence, which might modify the cross section). We should also examine other co-latitude angles of incidence θ_i, including near grazing along the cylinder, which should lead to larger current drives of the slot even at

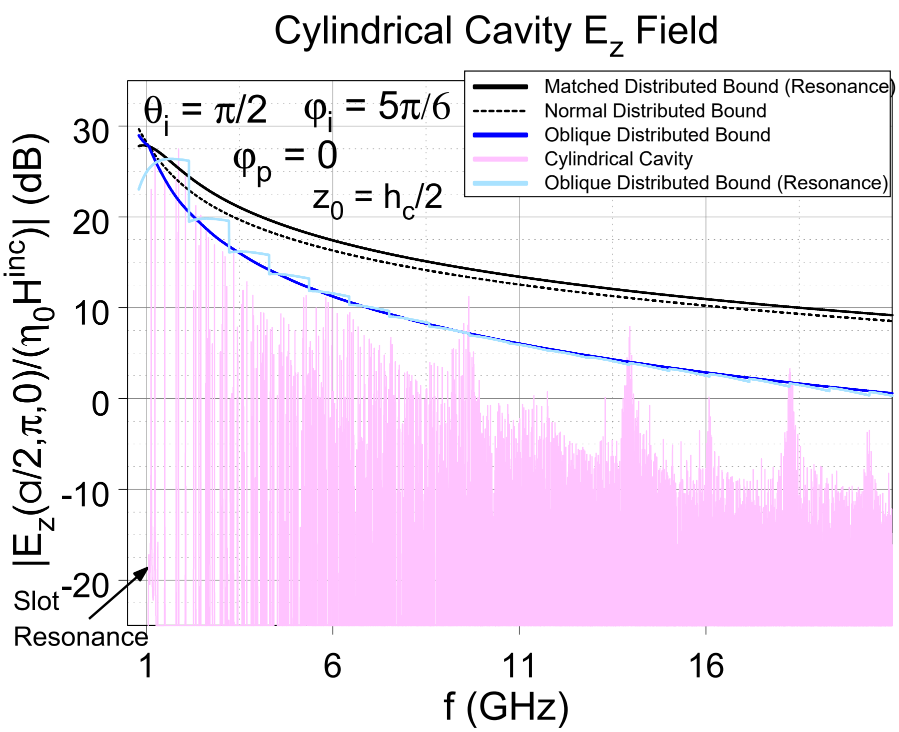

Figure 9: Comparison of interior cylindrical cavity and infinite cylinder exterior to various bounding results. The curves are the result of matching the slot with a uniformly distributed load for power reception. The black curve uses the normally incident plane wave matched to maintain a half wave voltage distribution; the dashed black curve is similar but approximates the ratio of distributed slot impedance to slot plus load distrbuted admittances by the slot characteristic impedance. The light blue jagged curve uses the preceding discontinuous form of the oblique incidence bounding result (is in a sense the oblique form of the black curve normal result); the dark blue curve approximates the ratio of distributed slot impedance to slot plus load distrbuted admittances by the slot characteristic impedance. The purple curve is the cylindrical cavity response for E_z.

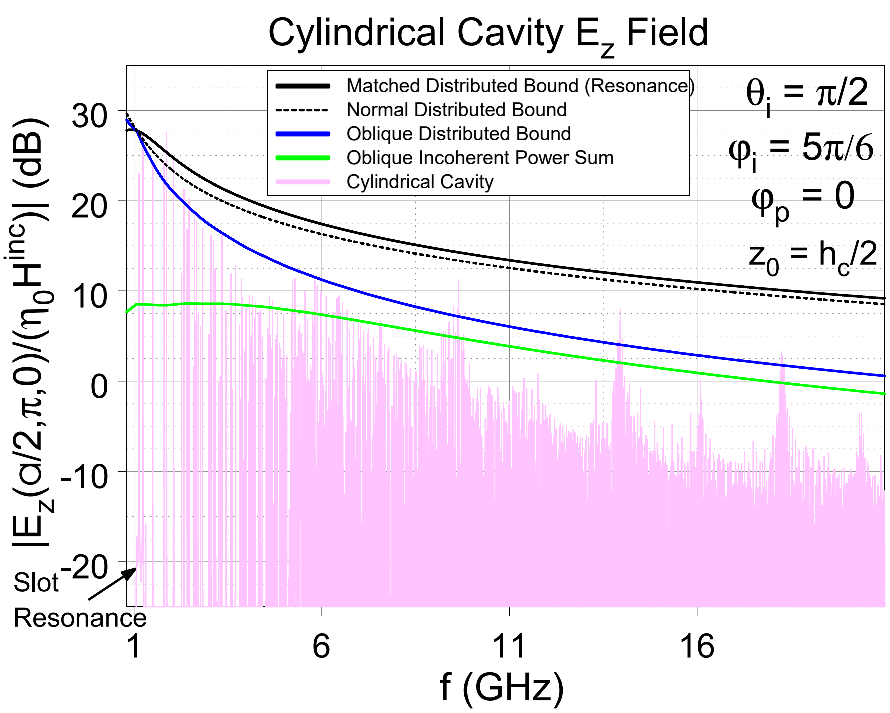

Figure 10: Comparison of interior cylindrical cavity and infinite cylinder exterior to various bounding results. The top three curves are the result of matching the slot with a uniformly distributed load for power reception. The black curve uses the normally incident plane wave matched to maintain a half wave voltage distribution; the dashed black curve is similar but approximates the ratio of distributed slot impedance to slot plus load distrbuted admittances by the slot characteristic impedance. The green curve uses an incoherent power sum of half space cross sections. The purple curve is the cylindrical cavity response for E_z.

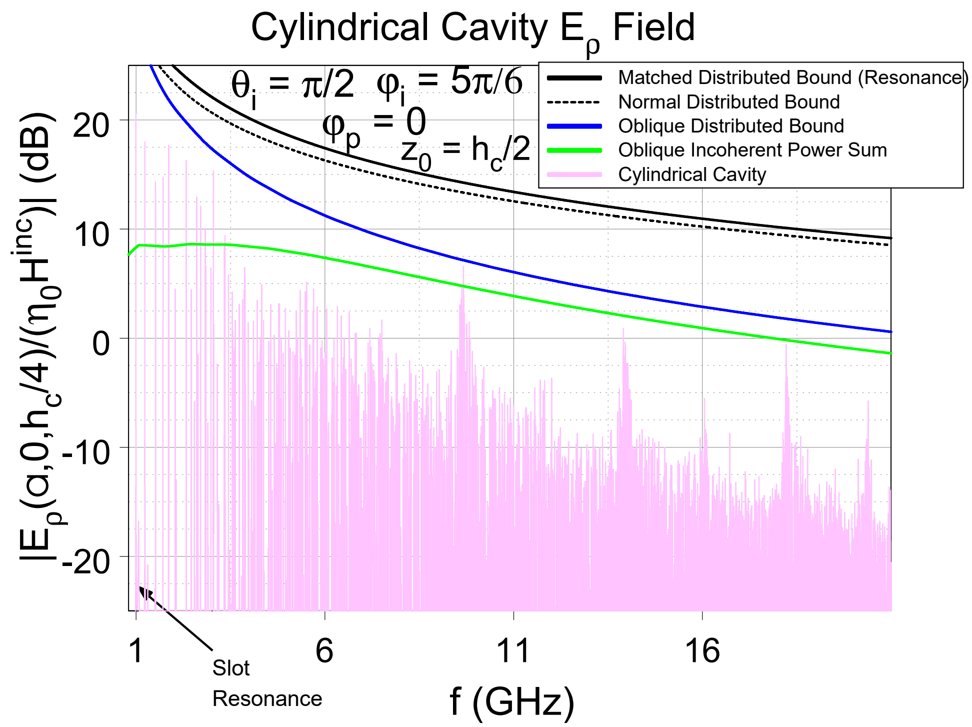

Figure 11: Comparison of interior cylindrical cavity and infinite cylinder exterior to various bounding results. The top three curves are the result of matching the slot with a uniformly distributed load for power reception. The black curve uses the normally incident plane wave matched to maintain a half wave voltage distribution; the dashed black curve is similar but approximates the ratio of distributed slot impedance to slot plus load distrbuted admittances by the slot characteristic impedance. The green curve uses an incoherent power sum of half space cross sections. The purple curve is the cylindrical cavity response for E_ρ.

the slot resonance frequencies (which we have referred to as connected to exterior object gain). We can also include exterior short circuit current drives from the cylinder in the power balance estimates (in the preceding analysis we used the planar short circuit current drive as an approximation).

9 SLOT RADIATION ON CYLINDER

Here we examine the slot radiation on the cylinder exterior and compare to the half space assumption. Our goal here is to use the simple transmission line voltage solution for constant current drive and determine the radiated power on the cylinder and for the half space. The actual voltage beyond the first resonance will deviate somewhat from this value due to the variation of the drive current along the slot length. The idea in this section is to develop corroborating insight into the previous cylindrical versus half space exterior results.

9.1 Exterior Coupling Formulation

Based on the interior TM potentials

$$A_{e\rho}^{(p,m,n)} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} \frac{1}{\rho} A_{p,m,n} J_m\left(j_{m,p}\rho/a\right) m \sin\left[m\left(\varphi-\varphi_0\right)\right] \cos\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{\varepsilon_0}{k_{p,m,n}^2 - (n\pi/h_c)^2} A_{p,m,n} \frac{j_{m,p}}{a} J_m'\left(j_{m,p}\rho/a\right) \cos\left[m\left(\varphi-\varphi_0\right)\right] \cos\left(n\pi z/h_c\right)$$

the exterior TM representation is taken to be

$$A_{e\rho} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} \frac{m}{\rho} A_m^{TM}(\alpha) H_m^{(1)}(k_t\rho) \sin\left[m\left(\varphi-\varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t} A_m^{TM}(\alpha) H_m^{(1)\prime}(k_t\rho) \cos\left[m\left(\varphi-\varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

where

$$k_t^2 = k^2 - \alpha^2$$

Similarly, based on the interior TE potentials

$$A_{ez}^{(p,m,n)} = \frac{A_{p,m,n}}{i\omega} J_m\left(j_{m,p}'\rho/a\right) \cos\left[m\left(\varphi-\varphi_0\right)\right] \sin\left(n\pi z/h_c\right)$$

$$A_{e\varphi}^{(p,m,n)} = \frac{-1/(i\omega)}{k_{p,m,n}'^2 - (n\pi/h_c)^2} \frac{m}{\rho} A_{p,m,n} J_m\left(j_{m,p}'\rho/a\right) \sin\left[m\left(\varphi-\varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

$$A_{e\rho}^{(p,m,n)} = \frac{1/(i\omega)}{k_{p,m,n}'^2 - (n\pi/h_c)^2} A_{p,m,n} \frac{j_{m,p}'}{a} J_m'\left(j_{m,p}'\rho/a\right) \cos\left[m\left(\varphi-\varphi_0\right)\right] \frac{n\pi}{h_c} \cos\left(n\pi z/h_c\right)$$

the exterior TE representation is taken to be

$$A_{ez} = \sum_m \int_0^\infty A_m^{TE}(\alpha) H_m^{(1)}(k_t\rho) \cos\left[m\left(\varphi-\varphi_m^{TE}\right)\right] \left\{ \begin{array}{c} \sin(\alpha z) \\ -\cos(\alpha z) \end{array} \right\} d\alpha$$

$$A_{e\varphi} = \sum_m \int_0^\infty \frac{-1}{k_t^2} \frac{m}{\rho} A_m^{TE}(\alpha) H_m^{(1)}(k_t\rho) \sin\left[m\left(\varphi-\varphi_m^{TE}\right)\right] \alpha \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$A_{e\rho} = \sum_m \int_0^\infty \frac{1}{k_t} A_m^{TE}(\alpha) H_m^{(1)\prime}(k_t\rho) \cos\left[m\left(\varphi-\varphi_m^{TE}\right)\right] \alpha \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

These representations along with orthogonality on the $\rho = b$ exterior cylinder surface in z and φ can be used to expand the field. Matching the tangential electric field on the surface (along with the normal magnetic field) can be used to find the coefficients $A_m^{TM}(\alpha)$ and $A_m^{TE}(\alpha)$. The azimuthal field on the surface for a narrow slot is taken to be near zero

$$\varepsilon_0 E_\varphi = -\frac{\partial}{\partial z} A_{e\rho} + \frac{\partial}{\partial \rho} A_{ez} \approx 0$$

$$\to \sum_m \int_0^\infty \frac{k^2}{k_t} A_m^{TE}(\alpha) H_m^{(1)\prime}(k_t b) \cos\left[m\left(\varphi - \varphi_m^{TE}\right)\right] \left\{ \begin{array}{c} \sin(\alpha z) \\ -\cos(\alpha z) \end{array} \right\} d\alpha$$

$$= -\sum_m \int_0^\infty \frac{\alpha}{k_t^2} \frac{m}{b} \varepsilon_0 A_m^{TM}(\alpha) H_m^{(1)}(k_t b) \sin\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \sin(\alpha z) \\ -\cos(\alpha z) \end{array} \right\} d\alpha$$

or setting $m\varphi_m^{TE} = m\varphi_m^{TM} + \pi/2$

$$\frac{k^2}{k_t} A_m^{TE}(\alpha) H_m^{(1)\prime}(k_t b) = -\frac{\alpha}{k_t^2} \frac{m}{b} \varepsilon_0 A_m^{TM}(\alpha) H_m^{(1)}(k_t b)$$

The axial electric field is

$$\varepsilon_0 E_z = -\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_{e\varphi}) + \frac{1}{\rho} \frac{\partial}{\partial \varphi} A_{e\rho}$$

$$= \sum_m \int_0^\infty \frac{1}{k_t} \frac{m}{\rho} A_m^{TE}(\alpha) H_m^{(1)\prime}(k_t \rho) \sin\left[m\left(\varphi - \varphi_m^{TE}\right)\right] \alpha \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$+ \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} A_m^{TM}(\alpha) \left(k_t^2 - \frac{m^2}{\rho^2}\right) H_m^{(1)}(k_t \rho) \cos\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$- \sum_m \int_0^\infty \frac{1}{k_t} A_m^{TE}(\alpha) \frac{m}{\rho} H_m^{(1)\prime}(k_t \rho) \sin\left[m\left(\varphi - \varphi_m^{TE}\right)\right] \alpha \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$+ \sum_m \int_0^\infty \frac{\varepsilon_0}{k_t^2} \frac{m^2}{\rho^2} A_m^{TM}(\alpha) H_m^{(1)}(k_t \rho) \cos\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

$$= \sum_m \int_0^\infty \varepsilon_0 A_m^{TM}(\alpha) H_m^{(1)}(k_t \rho) \cos\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \left\{ \begin{array}{c} \cos(\alpha z) \\ \sin(\alpha z) \end{array} \right\} d\alpha$$

Now for a specified slot voltage distribution we take the axial field to be (the slot here is taken to be at $z = 0$ since for the infinite length exterior there is nothing to be gained by shifting its position)

$$E_z(b, \varphi, z) = -\delta(z) V(s) = \sum_m \int_0^\infty A_m^{TM}(\alpha) H_m^{(1)}(k_t b) \cos\left[m\left(\varphi - \varphi_m^{TM}\right)\right] \cos(\alpha z) \, d\alpha$$

Noting that the unknown function $A_m^{TM}(\alpha)$ is not really function of α because of the result

$$\delta(z) = \frac{1}{\pi} \int_0^\infty \cos(\alpha z) \, d\alpha = \lim_{R\to\infty} \frac{1}{\pi} \int_0^R \cos(\alpha z) \, d\alpha = \lim_{R\to\infty} \left\{ \frac{\sin(zR)}{\pi z} \right\} = \lim_{R\to\infty} \left\{ \frac{\sin(\pi z R/\pi)}{\pi z} \right\} = \lim_{R\to\infty} \{R \mathrm{sinc}(Rz)\}$$

where

$$\mathrm{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\delta(t) = \lim_{X \to \infty} \left[X \mathrm{sinc}\,(Xt)\right]$$

we can then reduce this electric field condition to

$$-\frac{1}{\pi} V(s) = \sum_m A_m^{TM} H_m^{(1)}(k_t b) \cos\left[m\left(\varphi - \varphi_m^{TM}\right)\right]$$

Now if we take the trigonometric slot distribution (for constant drive current density)

$$V(s) \approx V_0 \left\{\cos(ks) - \cos(kh)\right\}$$

then with $\varphi_m^{TM} = 0$

$$-\frac{1}{\pi} V_0 \left\{\cos(ks) - \cos(kh)\right\} = \sum_{m=0}^{\infty} A_m^{TM} H_m^{(1)}(k_t b) \cos(m\varphi)\ \ ,\ \ 0 < \varphi < h/b$$

$$0 = \sum_{m=0}^{\infty} A_m^{TM} H_m^{(1)}(k_t b) \cos(m\varphi)\ \ ,\ \ h/b < \varphi < \pi$$

Then using the integral

$$\int_0^{\pi} \cos(m\varphi) \cos(m'\varphi)\, d\varphi = \frac{1}{2} \int_0^{\pi} \left[\cos(m - m')\varphi + \cos(m + m')\varphi\right] d\varphi = \frac{1}{2} \left[\frac{\sin(m - m')\varphi}{(m - m')} + \frac{\sin(m + m')\varphi}{(m + m')}\right]_0^{\pi}$$

$$= \frac{\pi}{\varepsilon_m} \delta_{mm'}$$

to invert the Fourier series

$$-\frac{1}{\pi} V_0 \int_0^{h/b} \cos(m\varphi) \left\{\cos(ks) - \cos(kh)\right\} d\varphi = -\frac{V_0}{\pi b} \int_0^{h} \cos(m\varphi) \left\{\cos(ks) - \cos(kh)\right\} ds = \frac{\pi}{\varepsilon_m} A_m^{TM} H_m^{(1)}(k_t b)$$

or

$$\frac{\pi}{\varepsilon_m} A_m^{TM} H_m^{(1)}(k_t b) = -\frac{1}{\pi} V_0 \int_0^{h/b} \cos(m\varphi) \left\{\cos(kb\varphi) - \cos(kh)\right\} d\varphi$$

$$= -\frac{1}{2}\frac{1}{\pi} V_0 \left\{\frac{\sin((kb - m)\, h/b)}{(kb - m)} + \frac{\sin((kb + m)\, h/b)}{(kb + m)} - \frac{2}{m} \sin(mh/b) \cos(kh)\right\}$$

$$= -\frac{1}{2}\frac{1}{\pi} V_0 \left\{\frac{\sin(kh - mh/b)}{(kb - m)} + \frac{\sin(kh + mh/b)}{(kb + m)} - \frac{2}{m} \sin(mh/b) \cos(kh)\right\}$$

$$= -\frac{1}{2}\frac{1}{\pi} V_0 \left\{\frac{\sin(kh)\cos(mh/b) - \cos(kh)\sin(mh/b)}{(kb - m)} + \frac{\sin(kh)\cos(mh/b) + \cos(kh)\sin(mh/b)}{(kb + m)} - \frac{2}{m} \sin(mh/b) \cos(kh)\right\}$$

$$= -\frac{1}{\pi} V_0 \left\{ \sin(kh) \cos(mh/b) \left(\frac{kb}{k^2 b^2 - m^2} \right) - \cos(kh) \sin(mh/b) \left(\frac{m}{k^2 b^2 - m^2} \right) - \frac{1}{m} \sin(mh/b) \cos(kh) \right\}$$

$$= -\frac{1}{\pi} V_0 \left\{ \sin(kh) \cos(mh/b) - (kb/m) \sin(mh/b) \cos(kh) \right\} \left(\frac{kb}{k^2 b^2 - m^2} \right)$$

or

$$A_m^{TM}(\alpha) H_m^{(1)}(k_t b) = -\frac{\varepsilon_m}{\pi^2} V_0 \left\{ \sin(kh) \cos(mh/b) - (kb/m) \sin(mh/b) \cos(kh) \right\} \left(\frac{kb}{k^2 b^2 - m^2} \right)$$

and

$$A_m^{TE}(\alpha) H_m^{(1)\prime}(k_t b) = -\frac{\alpha}{k^2 k_t} \frac{m}{b} \frac{\varepsilon_m}{\pi^2} \varepsilon_0 V_0 \left\{ \sin(kh) \cos(mh/b) - (kb/m) \sin(mh/b) \cos(kh) \right\} \left(\frac{kb}{k^2 b^2 - m^2} \right)$$

The azimuthal magnetic field is then

$$H_\varphi = i\omega A_{e\varphi} - \frac{1}{\rho} \frac{\partial}{\partial \varphi} \phi_m$$

where the total potential is

$$A_{e\varphi} = A_{e\varphi}^{TM} + A_{e\varphi}^{TE}$$

with

$$A_{e\varphi}^{TM} = \sum_{m=0}^{\infty} \int_0^\infty \frac{\varepsilon_0}{k_t} A_m^{TM}(\alpha) H_m^{(1)\prime}(k_t \rho) \cos(m\varphi) \cos(\alpha z)\, d\alpha$$

$$= -\frac{\varepsilon_0 V_0}{\pi^2} \sum_{m=0}^{\infty} \varepsilon_m \left\{ \sin(kh) \cos(mh/b) - (kb/m) \sin(mh/b) \cos(kh) \right\} \left(\frac{kb}{k^2 b^2 - m^2} \right) \cos(m\varphi)$$

$$\int_0^\infty \frac{H_m^{(1)\prime}(k_t \rho)}{H_m^{(1)}(k_t b)} \cos(\alpha z) \frac{d\alpha}{k_t}$$

and

$$A_{e\varphi}^{TE} = \sum_{m=0}^{\infty} \int_0^\infty \frac{\alpha}{k_t^2} \frac{m}{\rho} A_m^{TE}(\alpha) H_m^{(1)}(k_t \rho) \cos(m\varphi) \cos(\alpha z)\, d\alpha$$

$$= \frac{\varepsilon_0 V_0}{\pi^2 (k^2 b \rho)} \sum_{m=0}^{\infty} \varepsilon_m \left\{ \sin(kh) \cos(mh/b) - (kb/m) \sin(mh/b) \cos(kh) \right\} \left(\frac{kb}{k^2 b^2 - m^2} \right) m^2 \cos(m\varphi)$$

$$\int_0^\infty \frac{H_m^{(1)}(k_t \rho)}{H_m^{(1)\prime}(k_t b)} \cos(\alpha z) \frac{\alpha^2}{k_t^2} \frac{d\alpha}{k_t}$$

We note that for $\alpha > k$

$$k_t = i\left(-ik_t\right)$$

$$\frac{H_m^{(1)\prime}\left(k_t\rho\right)}{H_m^{(1)}\left(k_t b\right)} = \frac{-iK_m'\left(-ik_t\rho\right)}{K_m\left(-ik_t b\right)}$$

$$\frac{H\left(k_t\rho\right)}{H_m^{(1)\prime}\left(k_t b\right)} = \frac{K_m\left(-ik_t\rho\right)}{-iK_m'\left(-ik_t b\right)}$$

and therefore the integrand becomes real and the contribution to the azimuthal magnetic field becomes pure imaginary (taking V_0 to be real). Thus the real contribution to the azimuthal magnetic field, which is the part required to evaluate the average radiated power, is from the integrand region $\alpha < k$. Then

$$\operatorname{Im}\left[A_{e\varphi}^{TM}\left(\rho=b\right)\right] = -\frac{\varepsilon_0 V_0}{\pi^2}\sum_{m=0}^{\infty}\varepsilon_m\left\{\sin\left(kh\right)\cos\left(mh/b\right) - \left(kb/m\right)\sin\left(mh/b\right)\cos\left(kh\right)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)\cos\left(m\varphi\right)$$

$$\int_0^k \operatorname{Im}\left\{\frac{H_m^{(1)\prime}\left(k_t b\right)}{H_m^{(1)}\left(k_t b\right)}\right\}\cos\left(\alpha z\right)\frac{d\alpha}{k_t}$$

$$\operatorname{Im}\left[A_{e\varphi}^{TE}\left(\rho=b\right)\right] = \frac{\varepsilon_0 V_0}{\pi^2 k^2 b^2}\sum_{m=0}^{\infty}\varepsilon_m\left\{\sin\left(kh\right)\cos\left(mh/b\right) - \left(kb/m\right)\sin\left(mh/b\right)\cos\left(kh\right)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)m^2\cos\left(m\varphi\right)$$

$$\int_0^k \operatorname{Im}\left\{\frac{H_m^{(1)}\left(k_t b\right)}{H_m^{(1)\prime}\left(k_t b\right)}\right\}\cos\left(\alpha z\right)\frac{\alpha^2}{k_t^2}\frac{d\alpha}{k_t}$$

or

$$\operatorname{Im}\left[A_{e\varphi}^{TM}\left(\rho=b\right)\right] = -\frac{\varepsilon_0 V_0}{\pi^2}\sum_{m=0}^{\infty}\varepsilon_m\left\{\sin\left(kh\right)\cos\left(mh/b\right) - \left(kb/m\right)\sin\left(mh/b\right)\cos\left(kh\right)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)\cos\left(m\varphi\right)$$

$$\int_0^k \operatorname{Im}\left[\frac{\left\{J_m'\left(k_t b\right)+iY_m'\left(k_t b\right)\right\}\left\{J_m\left(k_t b\right)-iY_m\left(k_t b\right)\right\}}{J_m^2\left(k_t b\right)+Y_m^2\left(k_t b\right)}\right]\cos\left(\alpha z\right)\frac{d\alpha}{k_t}$$

$$\operatorname{Im}\left[A_{e\varphi}^{TE}\left(\rho=b\right)\right] = \frac{\varepsilon_0 V_0}{\pi^2 k^2 b^2}\sum_{m=0}^{\infty}\varepsilon_m\left\{\sin\left(kh\right)\cos\left(mh/b\right) - \left(kb/m\right)\sin\left(mh/b\right)\cos\left(kh\right)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)m^2\cos\left(m\varphi\right)$$

$$\int_0^k \operatorname{Im}\left[\frac{\left\{J_m\left(k_t b\right)+iY_m\left(k_t b\right)\right\}\left\{J_m'\left(k_t b\right)-iY_m'\left(k_t b\right)\right\}}{J_m'^2\left(k_t b\right)+Y_m'^2\left(k_t b\right)}\right]\cos\left(\alpha z\right)\frac{\alpha^2}{k_t^2}\frac{d\alpha}{k_t}$$

or

$$\operatorname{Im}\left[A_{e\varphi}^{TM}\left(\rho=b\right)\right] = -\frac{\varepsilon_0 V_0}{\pi^2}\sum_{m=0}^{\infty}\varepsilon_m\left\{\sin\left(kh\right)\cos\left(mh/b\right) - \left(kb/m\right)\sin\left(mh/b\right)\cos\left(kh\right)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)\cos\left(m\varphi\right)$$

$$\int_0^k \frac{J_m(k_t b)\, Y_m'(k_t b) - J_m'(k_t b)\, Y_m(k_t b)}{J_m^2(k_t b) + Y_m^2(k_t b)} \cos(\alpha z)\, \frac{d\alpha}{k_t}$$

$$\operatorname{Im}\left[A_{e\varphi}^{TE}(\rho = b)\right] = -\frac{\varepsilon_0 V_0}{\pi^2 k^2 b^2} \sum_{m=0}^{\infty} \varepsilon_m \left\{\sin(kh)\cos(mh/b) - (kb/m)\sin(mh/b)\cos(kh)\right\} \left(\frac{kb}{k^2 b^2 - m^2}\right) m^2 \cos(m\varphi)$$

$$\int_0^k \frac{J_m'(k_t b)\, Y_m(k_t b) - J_m(k_t b)\, Y_m'(k_t b)}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \cos(\alpha z)\, \frac{\alpha^2}{k_t^2} \frac{d\alpha}{k_t}$$

or

$$\operatorname{Im}\left[A_{e\varphi}^{TM}(\rho = b, \varphi, z = 0)\right] = -\frac{2\varepsilon_0 V_0}{\pi^3 b} \sum_{m=0}^{\infty} \varepsilon_m \left\{\sin(kh)\cos(mh/b) - (kb/m)\sin(mh/b)\cos(kh)\right\} \left(\frac{kb}{k^2 b^2 - m^2}\right) \cos(m\varphi)$$

$$\int_0^k \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} \frac{d\alpha}{k_t^2}$$

$$\operatorname{Im}\left[A_{e\varphi}^{TE}(\rho = b, \varphi, z = 0)\right] = -\frac{2\varepsilon_0 V_0}{\pi^3 k^2 b^3} \sum_{m=0}^{\infty} \varepsilon_m$$

$$\left\{\sin(kh)\cos(mh/b)\left(\frac{kb}{k^2 b^2 - m^2}\right) - \cos(kh)\sin(mh/b)\left(\frac{m}{k^2 b^2 - m^2}\right) - \frac{1}{m}\sin(mh/b)\cos(kh)\right\} m^2 \cos(m\varphi)$$

$$\int_0^k \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \frac{\alpha^2}{k_t^2} \frac{d\alpha}{k_t^2}$$

where we used the Wronskian

$$J_m(k_t b)\, Y_m'(k_t b) - J_m'(k_t b)\, Y_m(k_t b) = \frac{2}{\pi k_t b}$$

The transverse wavenumber

$$k_t^2 = k^2 - \alpha^2$$

has derivative

$$\frac{dk_t}{d\alpha} = -\frac{\alpha}{k_t} = -\frac{\sqrt{k^2 - k_t^2}}{k_t}$$

and therefore

$$\int_0^k \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} \frac{d\alpha}{k_t^2} = \int_0^k \frac{1}{J_m^2(k_t b) + Y_m^2(k_t b)} \left(-\frac{d\alpha}{dk_t}\right) \frac{dk_t}{k_t^2} = \int_0^k \frac{1/\sqrt{k^2 - k_t^2}}{J_m^2(k_t b) + Y_m^2(k_t b)} \frac{dk_t}{k_t}$$

$$= b \int_0^k \frac{1/\sqrt{k^2 b^2 - k_t^2 b^2}}{J_m^2(k_t b) + Y_m^2(k_t b)} \frac{dk_t b}{k_t b} = \frac{1}{k} \int_0^{kb} \frac{1/\sqrt{1 - u^2/(kb)^2}}{J_m^2(u) + Y_m^2(u)} \frac{du}{u}$$

$$\int_0^k \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \frac{\alpha^2}{k_t^2} \frac{d\alpha}{k_t^2} = \int_0^k \frac{1}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \frac{\alpha^2}{k_t^2} \left(-\frac{d\alpha}{dk_t}\right) \frac{dk_t}{k_t^2} = \int_0^k \frac{\sqrt{k^2 - k_t^2}}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \frac{dk_t}{k_t^3}$$

$$= b \int_0^k \frac{\sqrt{k^2 b^2 - k_t^2 b^2}}{J_m'^2(k_t b) + Y_m'^2(k_t b)} \frac{dk_t b}{k_t^3 b^3} = kb^2 \int_0^{kb} \frac{\sqrt{1 - u^2/(kb)^2}}{J_m'^2(u) + Y_m'^2(u)} \frac{du}{u^3}$$

and therefore

$$\omega \operatorname{Im}\left[A_{e\varphi}^{TM}(\rho = b, \varphi, z = 0)\right]$$

$$= -\frac{2V_0/(\pi\eta_0)}{\pi^2 b} \sum_{m=0}^{\infty} \varepsilon_m \left\{\sin(kh)\cos(mh/b) - (kb/m)\sin(mh/b)\cos(kh)\right\} \left(\frac{kb}{k^2b^2 - m^2}\right) \cos(m\varphi)$$

$$\int_0^{kb} \frac{1/\sqrt{1 - u^2/(kb)^2}}{J_m^2(u) + Y_m^2(u)} \frac{du}{u}$$

$$\omega \operatorname{Im}\left[A_{e\varphi}^{TE}(\rho = b, \varphi, z = 0)\right]$$

$$= -\frac{2V_0/(\pi\eta_0)}{\pi^2 b} \sum_{m=0}^{\infty} \varepsilon_m \left\{\sin(kh)\cos(mh/b) - (kb/m)\sin(mh/b)\cos(kh)\right\} \left(\frac{kb}{k^2b^2 - m^2}\right) m^2 \cos(m\varphi)$$

$$\int_0^{kb} \frac{\sqrt{1 - u^2/(kb)^2}}{J_m'^2(u) + Y_m'^2(u)} \frac{du}{u^3}$$

9.2 EMF Method For Radiated Power

Using the Poynting theorem we can write

$$P_{rad} = \operatorname{Re}\left[-\oint_S E_z H_\varphi^* dS\right] = \operatorname{Re}\left[-\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} E_z(b, \varphi, z) H_\varphi^*(b, \varphi, z)\, b d\varphi dz\right]$$

$$= 2bV_0 \operatorname{Re}\left[\int_0^{h/b} \left\{\cos(ks) - \cos(kh)\right\} H_\varphi^*(b, \varphi, 0)\, d\varphi\right]$$

$$= 2bV_0 \operatorname{Re}\left[\int_0^{h/b} \left\{\cos(ks) - \cos(kh)\right\} (i\omega)^* i^* \operatorname{Im}\left\{A_{e\varphi}(\rho = b, \varphi, z = 0)\right\} d\varphi\right]$$

$$P_{rad} = -2\omega b V_0 \left[\int_0^{h/b} \left\{\cos(ks) - \cos(kh)\right\} \operatorname{Im}\left\{A_{e\varphi}(\rho = b, \varphi, z = 0)\right\} d\varphi\right]$$

Because this vector potential was set up in the Coulomb gauge, do we need to consider the scalar potential contribution? Writing the scalar potential contribution as

$$\mathrm{Re}\left[\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} V_0\left\{\cos(ks)-\cos(kh)\right\}\frac{\partial\phi_m^*}{\partial\varphi}d\varphi dz\right]$$

where from the continuity equation

$$\frac{\partial}{\partial s}I_m = 2\frac{\partial}{\partial s}V(s) = i\omega q_m(s)$$

The magnetic charge is thus in phase quadrature with the voltage. But the magnetic scalar potential is in phase with the magnetic charge. Hence the scalar potential part has no net radiated power. The scalar potential may play a role in the induction zone reactive power.

Then continuing with the calculation we need

$$V_0\int_0^{h/b}\left\{\cos(ks)-\cos(kh)\right\}\cos(m\varphi)\,d\varphi =$$

$$= V_0\left\{\sin(kh)\cos(mh/b)-(kb/m)\sin(mh/b)\cos(kh)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)$$

where

$$P_{rad} = \left\{|V_0|^2/(2\pi\eta_0)\right\}\frac{8}{\pi^2}\sum_{m=0}^{\infty}\varepsilon_m^2\left\{\sin(kh)\cos(mh/b)-(kb/m)\sin(mh/b)\cos(kh)\right\}^2\left(\frac{kb}{k^2b^2-m^2}\right)^2$$

$$\int_0^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)}\frac{du}{u}$$

$$+\left\{|V_0|^2/(2\pi\eta_0)\right\}\frac{8}{\pi^2}\sum_{m=1}^{\infty}\varepsilon_m m^2\left\{\sin(kh)\cos(mh/b)-(kb/m)\sin(mh/b)\cos(kh)\right\}^2\left(\frac{kb}{k^2b^2-m^2}\right)^2$$

$$\int_0^{kb}\frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)}\frac{du}{u^3}$$

Now for small argument

$$J_0^2(u)+Y_0^2(u)\sim 1+\frac{4}{\pi^2}\ln^2(ue^{\gamma}/2)$$

where $\gamma \approx 0.5772$ is Euler's constant. Then taking $\Delta << 1$ and $\Delta << kb$

$$\int_0^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}\approx\int_0^{\Delta}\frac{1}{1+\frac{4}{\pi^2}\ln^2(ue^{\gamma}/2)}\frac{du}{u}+\int_{\Delta}^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}$$

$$\approx\int_0^{\Delta e^{\gamma}/2}\frac{1}{1+\frac{4}{\pi^2}\ln^2(u)}\frac{du}{u}+\int_{\Delta}^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}$$

$$\approx\frac{\pi}{2}\int_{-(2/\pi)\ln(\Delta e^{\gamma}/2)}^{\infty}\frac{dv}{1+v^2}+\int_{\Delta}^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}$$

$$\approx \frac{\pi}{2}\int_{\arctan\{-(2/\pi)\ln(\Delta e^{\gamma}/2)\}}^{\pi/2} d\xi + \int_{\Delta}^{kb} \frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}$$

$$\approx \frac{\pi^2}{4} + \frac{\pi}{2}\arctan\{(2/\pi)\ln(\Delta e^{\gamma}/2)\} + \int_{\Delta}^{kb} \frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}$$

where $\tan(\xi) = v = -\frac{2}{\pi}\ln(u)$ and $dv = -\frac{2}{\pi}du/u$. Finally we can allow $\Delta \to kb$

$$\int_0^{kb} \frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u} = \frac{\pi^2}{4} + \frac{\pi}{2}\arctan\{(2/\pi)\ln(\Delta e^{\gamma}/2)\}$$

$$+\int_0^{\Delta}\left[\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)} - \frac{1}{1+\frac{4}{\pi^2}\ln^2(ue^{\gamma}/2)}\right]\frac{du}{u} + \int_{\Delta}^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u}$$

In addition

$$J_m^2(u) \sim \{(u/2)^m/m!\}^2 \ , \ m \geq 1 \ , \ u << m$$

$$Y_m^2(u) \sim \left\{-\frac{1}{\pi}(m-1)!\,(2/u)^m\right\}^2 \ , \ m \geq 1 \ , \ u << m$$

$$J_m'^2(u) \sim \left\{\frac{1}{2}(u/2)^{m-1}/(m-1)!\right\} \ , \ m \geq 1 \ , \ u << m$$

$$Y_m'^2(u) \sim \left\{\frac{1}{2\pi}m!\,(2/u)^{m+1}\right\}^2 \ , \ m \geq 1 \ , \ u << m$$

and for $\Delta << m \geq 1$ and assuming $\Delta << kb$

$$\int_0^{kb} \frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)}\frac{du}{u} \approx \left\{\frac{\pi}{2^m(m-1)!}\right\}^2 \int_0^{\Delta} u^{2m-1}du + \int_{\Delta}^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)}\frac{du}{u}$$

$$\approx \frac{m}{2}\left(\frac{\pi\Delta^m}{2^m m!}\right)^2 + \int_{\Delta}^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)}\frac{du}{u}$$

$$\int_0^{kb} \frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)}\frac{du}{u^3} \approx \left(\frac{\pi}{2^m m!}\right)^2 \int_0^{\Delta} u^{2m-1}du + \int_{\Delta}^{kb}\frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)}\frac{du}{u^3}$$

$$\approx \frac{1}{2m}\left(\frac{\pi\Delta^m}{2^m m!}\right)^2 + \int_{\Delta}^{kb}\frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)}\frac{du}{u^3}$$

Finally we can allow $\Delta \to kb$

$$\int_0^{kb} \frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)}\frac{du}{u} = \frac{\pi^2}{4}+\frac{\pi}{2}\arctan\{(2/\pi)\ln(kbe^{\gamma}/2)\}+\int_0^{kb}\left\{\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)} - \frac{1}{1+\frac{4}{\pi^2}\ln^2(ue^{\gamma}/2)}\right\}\frac{du}{u}$$

$$\int_0^{kb} \frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)}\frac{du}{u} \approx \frac{m}{2}\left\{\frac{\pi (kb)^m}{2^m m!}\right\}^2 + \int_0^{kb}\left\{\frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)} - \left(\frac{\pi}{2^m (m-1)!}\right)^2 u^{2m}\right\}\frac{du}{u}\ ,\ m \geq 1$$

$$m^2 \int_0^{kb} \frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)}\frac{du}{u^3} = \frac{m}{2}\left\{\frac{\pi (kb)^m}{2^m m!}\right\}^2 + \int_0^{kb}\left\{\frac{m^2}{u^2}\frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)} - \left(\frac{\pi}{2^m (m-1)!}\right)^2 u^{2m}\right\}\frac{du}{u}\ ,\ m \geq 1$$

We not only improve convergence with these subtractions, but we also see for large values of m that the convergence of the Fourier series are powers of kb times inverse m factorials squared. Putting this together gives

$$P_{rad} = \left\{|V_0|^2/(2\pi\eta_0)\right\}\frac{8}{\pi^2}\sum_{m=0}^{\infty}\varepsilon_m\left\{\sin(kh)\cos(mh/b) - (kb/m)\sin(mh/b)\cos(kh)\right\}^2\left(\frac{kb}{k^2b^2-m^2}\right)^2$$

$$\left[\int_0^{kb}\frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)}\frac{du}{u} + m^2\int_0^{kb}\frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)}\frac{du}{u^3}\right]$$

Separating out the $m=0$ term

$$P_{rad} = \left\{|V_0|^2/(2\pi\eta_0)\right\}\frac{8}{\pi^2}\left\{\sin(kh)\left(\frac{1}{kb}\right) - \cos(kh)\,\frac{h}{b}\right\}^2$$

$$\left[\frac{\pi^2}{4} + \frac{\pi}{2}\arctan\left\{(2/\pi)\ln\left(kbe^{\gamma}/2\right)\right\} + \int_0^{kb}\left\{\frac{1/\sqrt{1-u^2/(kb)^2}}{J_0^2(u)+Y_0^2(u)} - \frac{1}{1+\frac{4}{\pi^2}\ln^2\left(ue^{\gamma}/2\right)}\right\}\frac{du}{u}\right]$$

$$+\left\{|V_0|^2/(2\pi\eta_0)\right\}\frac{8}{\pi^2}\sum_{m=1}^{\infty}2\left\{\sin(kh)\cos(mh/b) - (kb/m)\sin(mh/b)\cos(kh)\right\}^2\left(\frac{kb}{k^2b^2-m^2}\right)^2$$

$$\left[m\left\{\frac{\pi (kb)^m}{2^m m!}\right\}^2 + \int_0^{kb}\left\{\frac{1/\sqrt{1-u^2/(kb)^2}}{J_m^2(u)+Y_m^2(u)} + \frac{m^2}{u^2}\frac{\sqrt{1-u^2/(kb)^2}}{J_m'^2(u)+Y_m'^2(u)} - 2\left(\frac{\pi u^m}{2^m (m-1)!}\right)^2\right\}\frac{du}{u}\right]$$

Note that

$$\ln\left(\frac{\pi u^m}{2^m (m-1)!}\right)^2 = \ln\left(\frac{\pi u^m}{2^m \Gamma(m)}\right)^2 \sim 2m\ln\left(\frac{u}{2m}\right) + 2m - \ln(2\pi) + \ln(m) + 2\ln(\pi)$$

$$\sim (2m-1)\ln\left(\frac{u}{2m}\right) + \ln(\pi u/4) + 2m$$

Note that

$$\lim_{kb\to m}\left[\left\{\sin(kh)\cos(mh/b)-(kb/m)\sin(mh/b)\cos(kh)\right\}\left(\frac{kb}{k^2b^2-m^2}\right)\right]$$

$$=\lim_{kb\to m}\left\{\frac{\sin(kh)\cos\left(\frac{m}{kb}kh\right)-(kb/m)\sin\left(\frac{m}{kb}kh\right)\cos(kh)}{k^2b^2-m^2}\right\}(kb)$$

$$=\frac{1}{2}\lim_{kb\to m}\left\{-\sin(kh)\sin\left(\frac{m}{kb}kh\right)\frac{m}{k^2b^2}kh+(1/m)\cos\left(\frac{m}{kb}kh\right)\frac{m}{kb}kh\cos(kh)-(1/m)\sin\left(\frac{m}{kb}kh\right)\cos(kh)\right\}$$

$$=\frac{1}{2m}\left\{kh\cos^2(kh)-kh\sin^2(kh)-\sin(kh)\cos(kh)\right\}=\frac{1}{4m}\left\{2kh\cos(2kh)-\sin(2kh)\right\}$$

We wish to compare these results with (the one half is introduced to obtain the half space value) [9], [9]

$$P_{rad}=\frac{1}{2}hG_{rad}D_0\left|V_0\right|^2$$

$$D_0=1+\cos(kh)\left\{2\cos(kh)-3\sin(kh)/(kh)\right\}$$

$$\pi h\eta_0G_{rad}D_0=\mathrm{Cin}(4kh)+4\,\mathrm{Si}(2kh)\cos(kh)\left\{kh\cos(kh)-\sin(kh)\right\}-\sin^2(2kh)$$

or

$$P_{rad}=\left[\mathrm{Cin}(4kh)+4\,\mathrm{Si}(2kh)\cos(kh)\left\{kh\cos(kh)-\sin(kh)\right\}-\sin^2(2kh)\right]\left|V_0\right|^2/(2\pi\eta_0)$$

Figures 12, 13, 14 show a comparison of the power radiated on a cylinder versus the half space radiation for a collection of cylinder radii. We see that in this frequency range the cylinder radiates somewhat more for a fixed voltage amplitude. As the slot length to cylinder radius shrinks, the result approaches the half space result (black curve). The discrepancy between the half space (black curve) and the $\ell = 5.5$ inch slot (green curve) near the first slot resonance $kh \approx \pi/2$ appears to be larger here than the discrepancies in the previous section slot voltage and field comparisons. However, slot losses in the previous section, which are the same in both the exterior cylinder and exterior half space approximation, tend to reduce the differences in the peak heights.

The following sections present approximate results for the slot penetration at low frequencies and then near the first slot resonance using the exterior half space approximation, in addition to simplifying the interior by including single resonant modes.

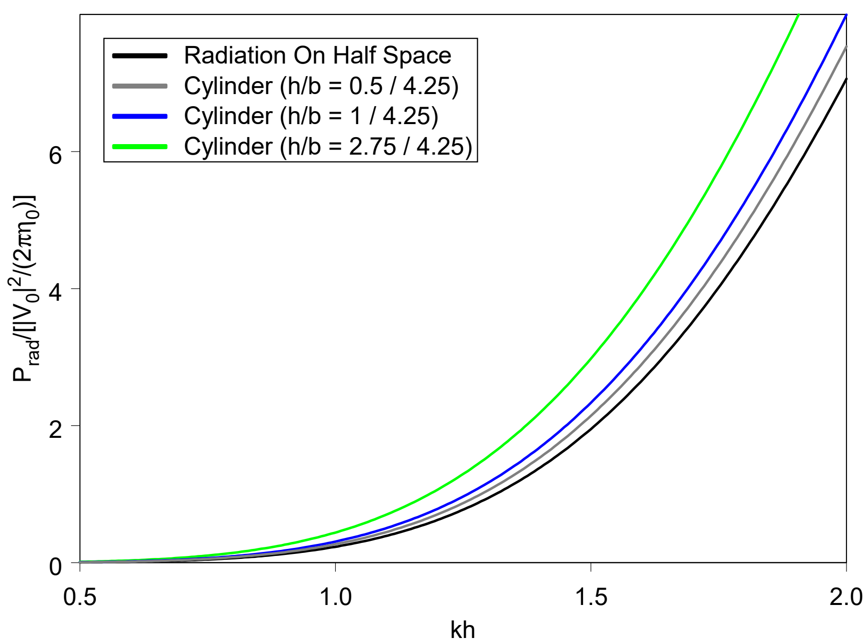

Figure 12: Comparison of radiated power on a cylinder versus the half space radiation for the smaller range of frequencies.

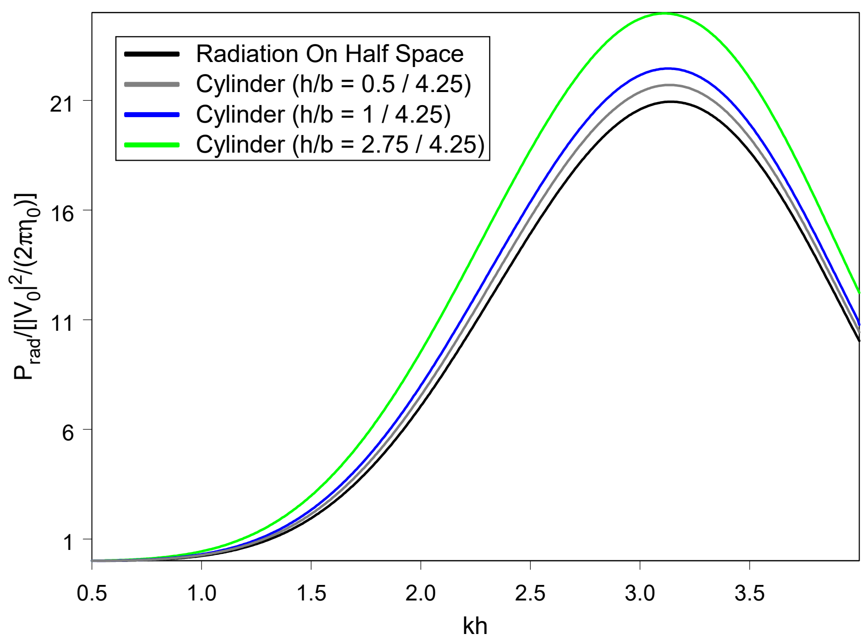

Figure 13: Comparison of radiated power on a cylinder versus the half space radiation.

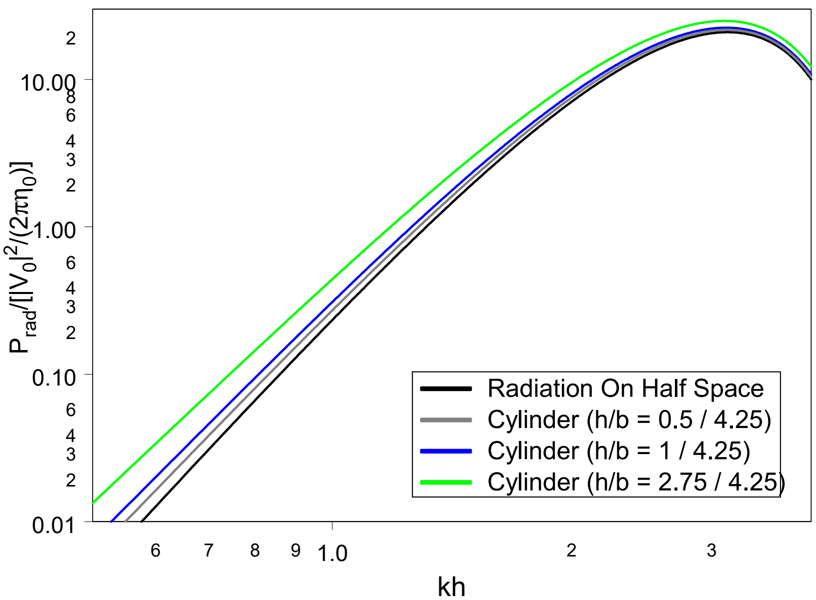

Figure 14: Comparison of radiated power on a cylinder versus the half space radiation on a logarithmic scale.

10 ELECTRICALLY SMALL SLOT LENGTHS

We examine a finite length cylindrical cavity driven by an electrically short narrow azimuthal slot. Because this slot aperture is operating below its resonant frequency we can approximate the slot voltage solution by its low frequency inductive form. We take the exterior to be an infinite cylinder in the z direction, although if desired the exterior drive current density can be replaced by the current on a finite length cylinder. The approach taken here is similar to that found in [1] for a rectangular cavity.

10.1 Single Slot Source

Consider the modes of a cylindrical cavity driven by a single circumferential slot. The slot is assumed to operate at low frequency with magnetic currents

$$I_m^{\pm}(s) = I_m^{\pm}(0)\left(1 - s^2/h^2\right) \ , \ \ -h < s < h$$

where the arc length is

$$s = a\varphi$$

and center magnetic current

$$I_m^{\pm}(0) = \pm 2V_{\pm}(0)$$

and distribution

$$I_m^{\pm}(s) = I_m^{\pm}(0)\left(1 - s^2/h^2\right) = \pm 2V_{\pm}(s) = \pm 2V_{\pm}(0)\left(1 - s^2/h^2\right) \ , \ \ -h < s < h$$

where the plus side is the interior cavity side and the minus side is the exterior side

$$I_m(0) = I_m^{-}(0) = -2V_{-}(0) = -2V(0)$$

where for $w, d << \ell = 2h$ the two sides become approximately equal $V_{+}(s) \approx V_{-}(s) = V(s)$. First we ignore interior loading of the slots and also ignore the wall losses and write [9]

$$V_{+}(0) \approx V_{-}(0) = V(0) = -i\omega L_{slt} I_N(0)$$

where the positive reference for the voltage will be taken on the negative z side of the slot and the slot inductance in this case is

$$\mu_0 (h/2) / L_{slt} \approx d/w + \frac{2}{\pi}\left\{\ln(16h/w) - 7/3\right\}$$

The short circuit current at the slot center is (where $\rho = b$ is the exterior radius)

$$I_N(0) = -hK_z^{sc}(0, z_0) = -hH_{\varphi}^{sc}(b, 0, z_0)$$

where K_z^{sc} is the exterior short circuit current density with short circuit field H_{φ}^{sc}, and the center magnetic current is

$$I_m(0) = -2V(0) \approx -2V_{+}(0) = i2\omega L_{slt} I_N(0) = -i2\omega L_{slt} h K_z^{sc}(0, z_0)$$

where the short circuit current on the infinite cylinder is

$$K_z^{sc}(0, z_0) = -2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[\frac{\cos(n\varphi_i)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m \sin(m\varphi_i)\cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right]$$

$$k_{\rho_i} = k \sin\theta_i$$

The potential is then

$$A_{e\varphi} = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty} \left[A_{e\varphi}^{TM(p,m,n)} + A_{e\varphi}^{TE(p,m,n)} \right]$$

The projection on the TM modes is then

$$A_{e\varphi}^{TM(p,m,n)} = \frac{\varepsilon_0}{(j_{m,p}/a)} J_m'(j_{m,p}\rho/a) \left[A_{p,m,n}^{TM} \cos(m\varphi) + B_{p,m,n}^{TM} \sin(m\varphi) \right] \cos(n\pi z/h_c)$$

$$\approx \frac{\varepsilon_0}{(j_{m,p}/a)} J_m'(j_{m,p}\rho/a) A_{p,m,n}^{TM} \cos(m\varphi)\cos(n\pi z/h_c)$$

$$\left[k^2 + k k_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right] A_{p,m,n}^{TM} = \varepsilon_m \varepsilon_n \frac{(j_{m,p}/a)\cos(n\pi z_0/h_c)}{V_{cav} J_{m-1}(j_{m,p})} \int_{-h}^{h} \cos(ms'/a) V_+(s')\, ds'$$

$$\approx \frac{\varepsilon_m \varepsilon_n (j_{m,p}/a)}{V_{cav} J_{m-1}(j_{m,p})} \cos(n\pi z_0/h_c) V_+(0) \frac{4}{3} h$$

$$\left[k^2 + k k_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right] B_{p,m,n}^{TM} = \varepsilon_m \varepsilon_n \frac{(j_{m,p}/a)\cos(n\pi z_0/h_c)}{V_{cav} J_{m-1}(j_{m,p})} \int_{-h}^{h} \sin(ms'/a) V_+(s')\, ds'$$

$$k_{pmn}^2 - (n\pi/h_c)^2 = j_{m,p}^2/a^2$$

$$Q_{p,m,n}^{TM} = \frac{k_{p,m,n} a}{\varepsilon_n a/h_c + 1} \frac{\eta_0}{2R_s^{(p,m,n)}}$$

The projection on the TE modes is then

$$A_{e\varphi}^{TE(p,m,n)} = \frac{-1/(i\omega)}{k_{p,m,n}'^2 - (n\pi/h_c)^2} \frac{m}{\rho} J_m\left(j_{m,p}'\rho/a\right) \left[A_{p,m,n}^{TE} \sin(m\varphi) + B_{p,m,n}^{TE} \cos(m\varphi) \right] \frac{n\pi}{h_c} \cos(n\pi z/h_c)$$

$$\approx \frac{-1/(i\omega)}{k_{p,m,n}'^2 - (n\pi/h_c)^2} \frac{m}{\rho} J_m\left(j_{m,p}'\rho/a\right) B_{p,m,n}^{TE} \cos(m\varphi) \frac{n\pi}{h_c} \cos(n\pi z/h_c)$$

$$\left[k^2 + k k_{p,m,n}'(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right] A_{j,m,n}^{TE}$$

$$= \frac{2m\pi n \cos(n\pi z_0/h_c)\, \varepsilon_m/(-i\omega\mu_0)}{V_{cav} h_c a \left(k_{j,m,n}'/k\right)^2 \left(1 - m^2/j_{m,j}'^2\right) J_m\left(j_{m,j}'\right)} \int_{-h}^{h} \sin(ms'/a) V_+(s')\, ds'$$

$$\left[k^2 + k k_{p,m,n}'(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right] B_{p,m,n}^{TE}$$

$$= \frac{2m\pi n \cos(n\pi z_0/h_c)\,\varepsilon_m/(-i\omega\mu_0)}{V_{cav}h_c a \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)} \int_{-h}^{h} \cos(ms'/a)\, V_+(s')\, ds'$$

$$\approx \frac{2m\pi n \cos(n\pi z_0/h_c)\,\varepsilon_m/(-i\omega\mu_0)}{V_{cav}h_c a \left(k'_{p,m,n}/k\right)^2 \left(1 - m^2/j'^2_{m,p}\right) J_m\left(j'_{m,p}\right)} V_+(0)\,\frac{4}{3}h$$

$$k'^2_{pmn} - (n\pi/h_c)^2 = j'^2_{m,p}/a^2$$

$$Q^{TE}_{p,m,n} = \frac{\left(1 - m^2/j'^2_{m,p}\right)}{j'^2_{m,p}/a^2 + 2\,(a/h_c)\,(n\pi/h_c)^2 \left(1 - m^2/j'^2_{m,p}\right) + (n\pi/h_c)^2\, m^2/j'^2_{m,p}} k'^3_{p,m,n}\, a \frac{\eta_0}{2R_s^{(p,m,n)}}$$

where we can define

$$m^+_{es} = \frac{i2}{\omega\mu_0}\int_{-h}^{h} \cos(ms'/a)\, V_+(s')\, ds' \approx \frac{i2}{\omega\mu_0}\int_{-h}^{h} V_+(s')\, ds' \approx \frac{i2}{\omega\mu_0}\frac{4}{3}hV_+(0)$$

$$m^+_{os} = \frac{i2}{\omega\mu_0}\int_{-h}^{h} \sin(ms'/a)\, V_+(s')\, ds'$$

The axial component of the electric field is

$$E_z = -\frac{1}{\varepsilon_0}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_{e\varphi}) - \frac{1}{\rho}\frac{\partial}{\partial\varphi}A_{e\rho}\right] = \sum_{p,m,n}\left[A^{TM}_{p,m,n}\cos(m\varphi) + B^{TM}_{p,m,n}\sin(m\varphi)\right] J_m(j_{m,p}\rho/a)\cos(n\pi z/h_c)$$

$$\approx \sum_{p,m,n} A^{TM}_{p,m,n}\cos(m\varphi)\, J_m(j_{m,p}\rho/a)\cos(n\pi z/h_c)$$

but near a resonance we can take a single term

$$E_z \approx A^{TM}_{p,m,n} J_m(j_{m,p}\rho/a)\cos(m\varphi)\cos(n\pi z/h_c)$$

with

$$A^{TM}_{p,m,n} \approx \frac{\varepsilon_m\varepsilon_n\,(j_{m,p}/a)}{V_{cav}J_{m-1}(j_{m,p})\left[k^2 + kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n} - k^2_{p,m,n}\right]}\cos(n\pi z_0/h_c)\, V_+(0)\,\frac{4}{3}h$$

$$V_{cav} = \pi a^2 h_c$$

gives

$$E_z \approx \frac{\varepsilon_m\varepsilon_n/V_{cav}}{k^2 + kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n} - k^2_{p,m,n}} V_+(0)\,\frac{4}{3}h\frac{(j_{m,p}/a)\, J_m(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\cos(m\varphi)\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

The slot dipole moment is

$$m^+_s = -2\alpha^+_{m,ss}H^{sc}_\varphi = \frac{1}{\mu_0}\int_{-h}^{h} q^+_m(s)\, s ds = \frac{1}{i\omega\mu_0}\int_{-h}^{h}\left(\frac{\partial}{\partial s}I^+_m\right) s ds = \frac{h}{i\omega\mu_0}\left[I^+_m(h) + I^+_m(-h)\right] - \frac{1}{i\omega\mu_0}\int_{-h}^{h} I^+_m(s)\, ds$$

$$= \frac{i}{\omega\mu_0}\int_{-h}^{h} I_m^+(s)\,ds = \frac{i2}{\omega\mu_0}\int_{-h}^{h} V_+(s)\,ds$$

where we used the continuity equation

$$\frac{\partial}{\partial s} I_m^+ = i\omega q_m^+$$

Therefore at low frequencies

$$m_s^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h} V_+(s)\,ds \approx \frac{i2}{\omega\mu_0} V_+(0)\int_{-h}^{h}\left(1 - s^2/h^2\right) ds = \frac{i2}{\omega\mu_0} V_+(0)\,\frac{4}{3}h$$

$$= \frac{i2}{\omega\mu_0} i\omega L_{slt} h K_z^{sc}(0, z_0)\,\frac{4}{3}h$$

$$\alpha_{m,ss}^+ = \frac{1}{\mu_0} L_{slt} h\frac{4}{3}h = \frac{1}{\mu_0} L\frac{h}{2}h\frac{4}{3}h \sim \frac{\pi\ell^3}{12\Omega_e} \sim \alpha_{m,ss}^0$$

which is the polarizability of the slot in a half space [17], [9]; in other words at low frequencies the slot dipole moment and voltage becomes the same as that of a slot in a half space (the cavity loading for typical values of cavity quality factor is insufficient to change the slot response). The axial electric field in this low frequency case is thus

$$E_z \approx \frac{i\omega\mu_0 K_z^{sc}(0, z_0)\,\varepsilon_m\varepsilon_n\left(\alpha_{m,ss}^0/V_{cav}\right)}{k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2}\,\frac{(j_{m,p}/a)\,J_m(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\cos(m\varphi)\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)$$

10.2 Electrically Small Slot In Rectangular Cavity

It is interesting to compare the axial electric field in the cylindrical cavity with the field in the rectangular cavity. In the rectangular cavity the interior TE potential, in the Lorentz gauge, with respect to the slot length direction z is [1]

$$A_{ez} \approx \frac{\varepsilon_0}{V_{cav}}\frac{(\varepsilon_n\varepsilon_j\varepsilon_m/2)\,i\omega\mu_0 m_z^+}{k^2 + kk_{n,j,m}(1+i)/Q_{n,j,m}^{TE} - k_{n,j,m}^2}\cos(n\pi/2)\cos(n\pi x/a)\cos(j\pi y/b)\sin(m\pi/2)\sin(m\pi z/c)$$

$$\approx \frac{\varepsilon_0}{V_{cav}}\frac{\varepsilon_n\varepsilon_j\varepsilon_m i\omega\mu_0\alpha_{m,zz}^+ K_x^{sc}(0, z_0)}{k^2 + kk_{n,j,m}(1+i)/Q_{n,j,m}^{TE} - k_{n,j,m}^2}\cos(n\pi/2)\cos(n\pi x/a)\cos(j\pi y/b)\sin(m\pi/2)\sin(m\pi z/c)$$

$$\approx \frac{\varepsilon_0}{V_{cav}}\frac{\varepsilon_n\varepsilon_j\varepsilon_m i\omega\mu_0\alpha_{m,zz}^0 K_x^{sc}(0, z_0)}{k^2 + kk_{n,j,m}(1+i)/Q_{n,j,m}^{TE} - k_{n,j,m}^2}\cos(n\pi/2)\cos(n\pi x/a)\cos(j\pi y/b)\sin(m\pi/2)\sin(m\pi z/c)$$

$$V_{cav} = abc$$

field is TE with respect to z with, for example the field in the direction of the slot width, is

$$E_x(x, y, z) = -\frac{1}{\varepsilon_0}\frac{\partial}{\partial y} A_{ez}$$

$$\approx \frac{-i\omega\mu_0 K_x^{sc}(0,z_0)\,\varepsilon_n\varepsilon_j\varepsilon_m\left(\alpha^0_{m,zz}/V_{cav}\right)}{k^2+kk_{n,j,m}(1+i)/Q^{TE}_{n,j,m}-k^2_{n,j,m}}(j\pi/b)\cos(n\pi/2)\cos(n\pi x/a)\sin(j\pi y/b)\sin(m\pi/2)\sin(m\pi z/c)$$

$$k_{njm}=\sqrt{(n\pi/a)^2+(j\pi/b)^2+(m\pi/c)^2}$$

$$Q^{TE}_{n,j,m}=\frac{1}{2}\left(\frac{\eta_0}{R_s}\right)k_{njm}\frac{(m/c)^2(n/a)^2+(m/c)^2(j/b)^2+\left\{(n/a)^2+(j/b)^2\right\}^2}{(\varepsilon_j/b+2/c)(m/c)^2(n/a)^2+(\varepsilon_n/a+2/c)(m/c)^2(j/b)^2+(\varepsilon_n/a+\varepsilon_j/b)\left\{(n/a)^2+(j/b)^2\right\}^2}$$

$$\sim\left(1+\frac{2}{\varepsilon_n\varepsilon_j}\right)\frac{V_{cav}k\eta_0}{2R_sS_{cav}}$$

Noting that the wavenumbers normal to the slot are $k_\rho = j_{m,p}/a$ in the cylindrical case and $-\varepsilon_j k_y = -j\pi/(b/\varepsilon_j)$ in the rectangular case (where $b/2$ is the half y dimension of the rectangular cavity similar to radius a of the cylindrical cavity), we see that these fields are essentially the same with $\sin(j\pi y/b)$ replaced by $J_m(j_{m,p}\rho/a)/J'_m(j_{m,p})$.

This page left blank

11 HALF SPACE APPROXIMATION & RESONANT SLOTS

In this section we take the slots to be operating in the range of, or below, their first resonant frequencies but we take into account exterior radiation. We assume in this section that we can approximate the radiation characteristics of the slot, as well as other local characteristics, by those of a half space; since this section is approximate we usually only include the nearest resonant interior cavity mode in the formulation.

11.1 Continuity And Approximate Integro-differential Equation

If the slot length is similar to the depth and/or the depth is similar to the wavelength (the "Deep" case), we must use the transfer relations of the slot fields [10] in order to set up the integro-differential equations for the slot-cavity problem. Instead, here we will restrict attention to the "Thick" case where the slot length is much larger than the slot depth and width $\ell >> d, w$, where we can directly enforce continuity of the azimuthal magnetic field from exterior to interior to set up the integro-differential equation [1] for the magnetic current or voltage

$$I_m^{\pm}(s) = \pm 2V_{\pm}(s) \approx \pm 2V(s)$$

where we often use the incident side notation $I_m^-(s) = I_m(s) = -2V(s) = -2V_-(s)$. We are not only assuming the slot length is much larger than the slot depth and width, but also that the slot length is small compared to the cylinder half perimeter $\ell << \pi a$. To enforce approximate continuity of the magnetic field across the slot we write [1]

$$H_s^>\left(a_{eq}^-, s\right) + \frac{1}{2}\left(\Delta Y_L \frac{\partial^2}{\partial s^2} I_m^- - \Delta Y_C I_m^-\right) + H_\varphi^{sc}\left(b, \varphi = s/b, z_0\right) \approx H_s^>\left(a_{eq}^+, s\right) + H_\varphi^{(p,m,n)}\left(a, \varphi = s/a, z_0\right)$$

where again a is the interior cavity radius and b is the exterior cylinder radius, $H_s^>\left(a_{eq}^-, s\right)$ is the exterior magnetic field generated by the slot magnetic current evaluated near the slot, $H_\varphi^{sc}\left(b, \varphi = s/b, z_0\right)$ is the exterior short circuit magnetic field evaluated at the slot, $H_s^>\left(a_{eq}^+, s\right)$ is the local interior magnetic field near the slot, and $H_\varphi^{(p,m,n)}\left(a, \varphi = s/a, z_0\right)$ is the magnetic field of the nearest interior resonant mode at the slot. The approximate total exterior magnetic azimuthal field is on the left side (plus the terms ΔY_L and ΔY_C accounting for additional slot properties of interior wall losses and gaskets, if present) and the approximate total interior azimuthal magnetic field is on the right side.

On the slot exterior we take the magnetic field to be approximately the filament contribution on a half space (the minus sign superscript on the equivalent radius indicates that it is the exterior field half space contribution) [9]

$$H_s^>\left(a_{eq}^-, s\right) = \frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2} + k^2\right)\int_{-h}^{h} I_m^-(s') \frac{e^{ik\sqrt{a_{eq}^2 + (s-s')^2}}}{4\pi\sqrt{a_{eq}^2 + (s-s')^2}} ds'$$

with equivalent slot radius [9]

$$a_{eq} \sim \frac{2w}{\pi e} e^{-\pi d/(2w)}\ ,\ d > w/3$$

plus the short circuit azimuthal magnetic field of the cylinder, and short circuit axial current density are

$$K_z^{sc}\left(\varphi = s/b, z_0\right) =$$

$$H_{\varphi}^{sc}\left(b,\varphi=s/b,z_0\right)=2H_0e^{ik_{z_i}z_0}\sum_{m''=0}^{\infty}\frac{\varepsilon_{m''}i^{m''}}{\pi k_{\rho_i}b}\left[-\frac{\cos\left(m''\left(s/b-\varphi_i\right)\right)\cos\varphi_p}{H_{m''}^{(1)}\left(k_{\rho_i}b\right)}+\frac{m''\sin\left(m''\left(s/b-\varphi_i\right)\right)\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_{m''}^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

On the interior cavity in this section we use the Coulomb gauge form of the field along with the local approximation of the half space. From the continuity equation

$$\nabla\cdot\underline{J}_m=i\omega\rho_m$$

applied to the slot

$$\frac{\partial}{\partial s}I_m^{\pm}=i\omega q_m^{\pm}$$

where $q_m^{\pm}(s)$ is the magnetic charge per unit length and $I_m^{\pm}(s)$ is the slot magnetic current. Then from

$$\underline{H}=-\nabla\phi_m+i\omega\underline{A}_e$$

we have the azimuthal component

$$H_s=-\frac{\partial}{\partial s}\phi_m+i\omega A_{es}$$

From the free space (or half space) solution of the Poisson equation

$$\nabla^2\phi_m=-\rho_m/\mu_0$$

or

$$\phi_m\left(\underline{r}\right)=\frac{1}{\mu_0}\int_V\frac{\rho_m\left(\underline{r}'\right)}{4\pi\left|\underline{r}-\underline{r}'\right|}dV'$$

and we can write

$$\phi_m^h\left(\rho_0,s\right)=\frac{1}{\mu_0}\int_{-h}^{h}\frac{q_m^+\left(s'\right)}{4\pi\sqrt{\rho_0^2+\left(s-s'\right)^2}}ds'=\frac{1}{i\omega\mu_0}\int_{-h}^{h}\frac{1}{4\pi\sqrt{\rho_0^2+\left(s-s'\right)^2}}\frac{\partial}{\partial s'}I_m^+\left(s'\right)ds'$$

$$=\frac{1}{i\omega\mu_0}\left[\frac{I_m^+\left(s'\right)}{4\pi\sqrt{\rho_0^2+\left(s-s'\right)^2}}\right]_{-h}^{h}-\frac{1}{i\omega\mu_0}\int_{-h}^{h}I_m^+\left(s'\right)\frac{\partial}{\partial s'}\frac{1}{4\pi\sqrt{\rho_0^2+\left(s-s'\right)^2}}ds'=\frac{1}{i\omega\mu_0}\frac{\partial}{\partial s}\int_{-h}^{h}\frac{I_m^+\left(s'\right)}{4\pi\sqrt{\rho_0^2+\left(s-s'\right)^2}}ds'$$

where we have integrated by parts and used $I_m^{\pm}(\pm h)=0$. The distance from the slot line source, we evaluate at the slot equivalent radius $\rho_0\to a_{eq}$. In principle, the magnetic scalar potential expansion in the cavity

$$\phi_m\left(a,\varphi,z\right)=\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\phi_m^{(p,m,n)}=\frac{1}{i\omega\mu_0}\frac{\partial}{\partial s}\frac{1}{V_{cav}}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\frac{\varepsilon_n\varepsilon_m\cos\left(n\pi z_0/h_c\right)}{k_{pmn}'^2\left(1-m^2/j_{m,p}'^2\right)}\cos\left(n\pi z/h_c\right)$$

$$\left[\cos\left(ms/a\right)\int_{-h}^{h}V_+\left(s'\right)\cos\left(ms'/a\right)ds'+\sin\left(ms/a\right)\int_{-h}^{h}V_+\left(s'\right)\sin\left(ms'/a\right)ds'\right]$$

should be used to evaluate the scalar potential. However, we can accelerate the convergence of this cavity modal form ϕ_m by subtracting the modal form of the preceding half space representation $\phi_m^{h(p,m,n)}$ term-by-term, and then add back the integral form of the half space representation ϕ_m^h (the integral form ϕ_m^h and the modal expansion of the integral form ϕ_m^h cancel, leaving the cavity potential expansion ϕ_m). In this section, we can obtain the approximate Coulomb gauge scalar potential near the slot ϕ_m by simply using the preceding half space approximation for the magnetic scalar potential ϕ_m^h, and neglecting the series difference terms

$$\phi_m = \phi_m^h + \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\left[\phi_m^{(p,m,n)} - \phi_m^{h(p,m,n)}\right] \approx \phi_m^h$$

In the region of separated modes, the modal solution for the electric vector potential usually has a term near resonance plus the remainder of the modal series. We intend to approximate this remainder as a local contribution, which we take to be the half space form [1]

$$A_{es}^h(\rho_0, s) \approx \varepsilon_0 \int_{-h}^{h} \frac{I_m^+(s')}{4\pi\sqrt{\rho_0^2 + (s-s')^2}} ds'$$

Thus similar to the preceding scalar potential approach we expand the half space form in modes $A_{es}^{h(p,m,n)}$ and write the total electric vector potential near the slot as

$$A_{e\varphi} = A_{es}^h + \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\left[A_{e\varphi}^{TM(p,m,n)} + A_{e\varphi}^{TE(p,m,n)} - A_{es}^{h(p,m,n)}\right] \approx A_{es}^h + \left[A_{e\varphi}^{TM(p,m,n)} + A_{e\varphi}^{TE(p,m,n)} - A_{es}^{h(p,m,n)}\right]$$

where the final approximation retains the nearest resonant mode term. Adding the half-space scalar potential contribution ϕ_m^h for the interior to this local quasi-static electric vector potential contribution A_{es}^h, gives the local field

$$H_s^{>}\left(a_{eq}^+, s\right) \approx i\omega A_{es}^h - \frac{\partial}{\partial s}\phi_m^h = -\frac{1}{i\omega\mu_0}\left(\frac{\partial^2}{\partial s^2} + k^2\right)\int_{-h}^{h} \frac{I_m^+(s')}{4\pi\sqrt{a_{eq}^2 + (s-s')^2}} ds'$$

The total interior azimuthal magnetic field at the wall near the slot can then be written as

$$H_\varphi^{<} = i\omega A_{e\varphi} - \frac{\partial}{\partial\varphi}\phi_m \approx H_s^{>}\left(a_{eq}^+, s\right) + i\omega\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\left[A_{e\varphi}^{TM(p,m,n)} + A_{e\varphi}^{TE(p,m,n)} - A_{e\varphi}^{h(p,m,n)}\right]$$

$$\approx \frac{i\omega\varepsilon_0}{2\pi k^2}\left(\frac{\partial^2}{\partial s^2} + k^2\right)\int_{-h}^{h} \frac{V_+(s')}{\sqrt{\rho_0^2 + (s-s')^2}} ds' + i\omega\left[A_{e\varphi}^{TM(p,m,n)} + A_{e\varphi}^{TE(p,m,n)} - A_{e\varphi}^{h(p,m,n)}\right]$$

with the nearest resonant term

$$A_{e\varphi}^{TM(p,m,n)} + A_{e\varphi}^{TE(p,m,n)} - A_{e\varphi}^{h(p,m,n)} = -\left(\varepsilon_m\varepsilon_n\varepsilon_0/V_{cav}\right)$$

$$\left[\frac{1}{\left\{k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}}\right.$$

$$+\frac{(n\pi/h_c)^2 m^2/j'^2_{m,p}}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}\left(1-m^2/j'^2_{m,p}\right)}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\Bigg]$$

$$\cos(n\pi z_0/h_c)\cos(n\pi z/h_c)\left[\cos(ms/a)\int_{-h}^{h}V_+(s')\cos(ms'/a)\,ds'+\sin(ms/a)\int_{-h}^{h}V_+(s')\sin(ms'/a)\,ds'\right]$$

where the TM contributions have

$$k^2_{p,m,n}=\omega^2_{p,m,n}\mu_0\varepsilon_0=\left(\frac{j_{m,p}}{a}\right)^2+(n\pi/h_c)^2$$

$$Q^{TM}_{p,m,n}=\frac{k_{p,m,n}a}{\varepsilon_n a/h_c+1}\frac{\eta_0}{2R_s^{(p,m,n)}}$$

$$V_{cav}=h_c\pi a^2$$

and the TE contributions have

$$k'^2_{p,m,n}=\omega'^2_{p,m,n}\mu_0\varepsilon_0=\left(\frac{j'_{m,p}}{a}\right)^2+(n\pi/h_c)^2$$

$$Q^{TE}_{p,m,n}=\frac{\left(1-m^2/j'^2_{m,p}\right)}{j'^2_{m,p}/a^2+2(a/h_c)(n\pi/h_c)^2\left(1-m^2/j'^2_{m,p}\right)+(n\pi/h_c)^2 m^2/j'^2_{m,p}}k'^3_{p,m,n}a\frac{\eta_0}{2R_s^{(p,m,n)}}$$

The first two terms in brackets of the electric vector potential are the TM and TE modal contributions. The third term is a local approximation of minus the quasistatic form of the electric vector potential (using the preceding cavity expansion of the magnetic scalar potential as a guide for $A^{h(p,m,n)}_{e\varphi}$), included to approximately remove the local contribution already included as part of $A^h_{es}\left(a^+_{eq},s\right)$. Note that the bracketed form in the preceding expression can be written for $k_{p,m,n}$ dominant over k as

$$\left[\frac{1}{-k^2_{p,m,n}}+\frac{1}{k'^2_{p,m,n}}+\frac{\left(-j'^2_{m,p}/a^2\right)}{k'^2_{p,m,n}k'^2_{p,m,n}}\left(1-\frac{1}{1-m^2/j'^2_{m,p}}\right)-\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\right],\ k_{p,m,n}>>k$$

from which we see that the convergence is improved by including the final quasistatic term (note that the first two terms nearly cancel for large mode numbers). In terms of the interior field this can be written

$$H^<_\varphi(a,\varphi,z_0)\approx H^>_s\left(a^+_{eq},s\right)+H^{(p,m,n)}_\varphi(a,\varphi=s/a,z_0)$$

with the field taken as the sum of TM and TE contributions

$$H^{(p,m,n)}_\varphi=H^{TM(p,m,n)}_\varphi+H^{TE(p,m,n)}_\varphi=i\omega\left[A^{TM(p,m,n)}_{e\varphi}+A^{TE(p,m,n)}_{e\varphi}-A^{h(p,m,n)}_{e\varphi}\right]=-\left(\varepsilon_m\varepsilon_n/V_{cav}\right)k^2$$

$$\Bigg[\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}}$$

$$+\frac{(n\pi/h_c)^2 m^2/j'^2_{m,p}}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}\left(1-m^2/j'^2_{m,p}\right)}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\Bigg]$$

$$\cos^2(n\pi z_0/h_c)\frac{1}{2}\left[m^+_{es}\cos(ms/a)+m^+_{os}\sin(ms/a)\right]$$

and

$$m^+_{es}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\cos(ms'/a)\,V_+(s')\,ds'$$

$$m^+_{os}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}\sin(ms'/a)\,V_+(s')\,ds'$$

11.2 Transmission Line Approximation

Using a leading order Hallen-type approximate form for the local integrals (half space approximations near the slot)

$$\int_{-h}^{h}I^\pm_m(s')\frac{ds'}{\sqrt{a^2_{eq}+(s-s')^2}}\approx\Omega_e I^\pm_m(s)\approx\int_{-h}^{h}I^\pm_m(s')\frac{e^{ik\sqrt{a^2_{eq}+(s-s')^2}}}{\sqrt{a^2_{eq}+(s-s')^2}}ds'$$

$$\approx\int_{-h}^{h}I^\pm_m(s')\frac{\cos\left[k\sqrt{a^2_{eq}+(s-s')^2}\right]}{\sqrt{a^2_{eq}+(s-s')^2}}ds'$$

where

$$\Omega_e=2\ln(2h/a_{eq})+C_e$$

and if we choose to preserve the low frequency dipole moment of the slot we take

$$C_e=2(\ln 2-7/3)$$

Making use of the leading Hallen approximations we can write the matching equation as

$$\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\frac{\Omega_e}{4\pi}I^-_m(s)+\frac{1}{2}\left(\Delta Y_L\frac{\partial^2}{\partial s^2}I^-_m-\Delta Y_C I^-_m\right)+K^{sc}_z(\varphi=s/b,z_0)$$

$$=\frac{i}{\omega\mu_0}\left(\frac{\partial^2}{\partial s^2}+k^2\right)\frac{\Omega_e}{4\pi}I^+_m(s)+H^{(p,m,n)}_\varphi(a,\varphi=s/a,z_0)$$

or with $I^-_m=I_m$

$$\frac{1}{2}\left[\left(\frac{1}{-i\omega\mu_0\pi/\Omega_e}+\Delta Y_L\right)\frac{\partial^2}{\partial s^2}-\left(-i\omega\varepsilon_0\Omega_e/\pi+\Delta Y_C\right)\right]I_m(s)=-K^{sc}_z(\varphi=s/b,z_0)+H^{(p,m,n)}_\varphi(a,\varphi=s/a,z_0)$$

Then we define the transmission line impedance per unit length Z by means of

$$\frac{1}{Z} = \Delta Y_L + \frac{1}{-i\omega L}$$

where the inductance per unit length is

$$L = \mu_0 \pi / \Omega_e$$

and if wall losses are present we include

$$\Delta Y_L = Y^{int} = G^{int} - iB^{int} \approx \frac{2Z_s/d}{i\omega\left(\mu_0 w/d\right)\left(2Z_s/d - i\omega\mu_0 w/d\right)} \ , \ d >> w$$

where we can write

$$Z = -i\omega L + \frac{\left(\omega L\right)^2 Y^{int}}{1 - i\omega L Y^{int}} \sim -i\omega L + \left(\omega L\right)^2 Y^{int}$$

and

$$G^{int} - iB^{int} \approx \frac{2Z_s/d}{\left(\omega\mu_0 w/d\right)^2} \ , \ 2R_s << \omega\mu_0 w \text{ or } \frac{\mu}{\mu_0}\delta << w$$

where

$$Z_s = \left(1 - i\right) R_s$$

$$R_s = 1/\left(\sigma\delta\right)$$

$$\delta = \sqrt{2/\left(\omega\mu\sigma\right)}$$

and the admittance per unit length Y by

$$Y = \Delta Y_C - i\omega C$$

where the capacitance per unit length is

$$C = \varepsilon_0 \Omega_e / \pi$$

and if no gasket is present the extra term vanishes

$$\Delta Y_C = 0 \text{ , no gasket}$$

To include radiation we add the perturbation of the radiation admittance per unit length Y_{rad} to the admittance per unit length

$$Y \rightarrow Y + Y_{rad}$$

where

$$Y_{rad} = G_{rad} - iB_{rad}$$

This can be evaluated for the full space approximation by means of

$$Y_{rad}\int_{-h}^{h}\left|V\left(s\right)\right|^{2}ds=-\frac{1}{\pi}\frac{i}{\omega\mu_{0}}\int_{-h}^{h}V^{*}\left(s\right)\left(\frac{\partial^{2}}{\partial s^{2}}+k^{2}\right)$$

$$\left[V\left(s\right)\left\{-C_{e}+\ln\left(1-s^{2}/h^{2}\right)\right\}+\int_{-h}^{h}\frac{V\left(s'\right)e^{ik\left|s-s'\right|}-V\left(s\right)}{\left|s-s'\right|}ds'\right]ds$$

For a single half space (the slot exterior) we set

$$Y\rightarrow Y+Y_{rad}/2$$

and the one half is inserted because Y_{rad} is defined here to include two half spaces; we can also use

$$Y\rightarrow Y+G_{rad}/2-iB_{rad}$$

to include reactive power on both sides of the slot but radiation damping only on the exterior (this is a good choice here since we are not including terms of the interior representation other than the resonant term). The transmission line equation with $I_m=-2V$ then becomes

$$\frac{1}{2}\left[\frac{1}{Z}\frac{\partial^{2}}{\partial s^{2}}-\left(Y+G_{rad}/2-iB_{rad}\right)\right]I_{m}\left(s\right)=-K_{z}^{sc}\left(\varphi=s/b,z_{0}\right)+H_{\varphi}^{(p,m,n)}\left(a,\varphi=s/a,z_{0}\right)$$

or

$$\left[\frac{1}{Z}\frac{\partial^{2}}{\partial s^{2}}-\left(Y+G_{rad}/2-iB_{rad}\right)\right]V\left(s\right)=K_{z}^{sc}\left(\varphi=s/b,z_{0}\right)-H_{\varphi}^{(p,m,n)}\left(a,\varphi=s/a,z_{0}\right)$$

or

$$\frac{\partial^{2}}{\partial s^{2}}V\left(s\right)-Z\left(Y+G_{rad}/2-iB_{rad}\right)V\left(s\right)=\left(\frac{\partial^{2}}{\partial s^{2}}+\Gamma^{2}\right)V\left(s\right)=ZK_{z}^{sc}\left(\varphi=s/b,z_{0}\right)-ZH_{\varphi}^{(p,m,n)}\left(a,\varphi=s/a,z_{0}\right)$$

where

$$\Gamma=\Gamma'+i\Gamma''=\sqrt{-Z\left(Y+G_{rad}/2-iB_{rad}\right)}\text{ or }\sqrt{-Z\left(Y+Y_{rad}/2\right)}\text{ or }\sqrt{-Z\left(Y+Y_{rad}\right)}\text{ or }\sqrt{-ZY}$$

The homogeneous solutions are

$$V_{H}=a_{0}\cos\left(\Gamma s\right)+a_{1}\sin\left(\Gamma s\right)$$

where the resonant modes on the right hand side involve the sum of

$$H_{\varphi}^{(p,m,n)}=H_{\varphi}^{TM(p,m,n)}+H_{\varphi}^{TE(p,m,n)}=i\omega\left[A_{e\varphi}^{TM(p,m,n)}+A_{e\varphi}^{TE(p,m,n)}-A_{e\varphi}^{h(p,m,n)}\right]$$

$$=-i\omega\varepsilon_{0}\left(\varepsilon_{m}\varepsilon_{n}/V_{cav}\right)\left[\frac{1}{\left\{k^{2}+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^{2}\right\}}\right.$$

$$\left.+\frac{\left(n\pi/h_{c}\right)^{2}m^{2}/j_{m,p}'^{2}}{k_{p,m,n}'^{2}\left\{k^{2}+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^{2}\right\}\left(1-m^{2}/j_{m,p}'^{2}\right)}+\frac{1}{k_{p,m,n}'^{2}\left(1-m^{2}/j_{m,p}'^{2}\right)}\right]$$

$$\cos^2\left(n\pi z_0/h_c\right)\left[\cos\left(ms/a\right)\int_{-h}^{h}V_+\left(s'\right)\cos\left(ms'/a\right)ds'+\sin\left(ms/a\right)\int_{-h}^{h}V_+\left(s'\right)\sin\left(ms'/a\right)ds'\right]$$

and the short circuit current density is

$$K_z^{sc}\left(\varphi=s/b,z_0\right)=2H_0e^{ik_{z_i}z_0}\sum_{m''=0}^{\infty}\frac{\varepsilon_{m''}i^{m''}}{\pi k_{\rho_i}b}\left[-\frac{\cos\left\{m''\left(s/b-\varphi_i\right)\right\}\cos\varphi_p}{H_{m''}^{(1)}\left(k_{\rho_i}b\right)}+\frac{m''\sin\left\{m''\left(s/b-\varphi_i\right)\right\}\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_{m''}^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

The solution will have the form of the homogeneous solutions plus the particular solutions driven by the right hand side

$$V\left(s\right)=V_H\left(s\right)+V_P\left(s\right)=a_0\cos\left(\Gamma s\right)+a_1\sin\left(\Gamma s\right)$$

$$+\frac{b_m}{\Gamma^2-m^2/a^2}\begin{matrix}\cos\\ \sin\end{matrix}\left(ms/a\right)+\sum_{m''}\frac{c_{m''}}{\Gamma^2-m''^2/b^2}\begin{matrix}\cos\\ \sin\end{matrix}\left\{m''\left(s/b-\varphi_i\right)\right\}$$

$$=a_0\cos\left(\Gamma s\right)+a_1\sin\left(\Gamma s\right)$$

$$+\frac{i\omega\varepsilon_0Z\left(\varepsilon_m\varepsilon_n/V_{cav}\right)}{\Gamma^2-m^2/a^2}\left[\frac{1}{\left\{k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}\right.$$

$$\left.+\frac{\left(n\pi/h_c\right)^2m^2/j_{m,p}'^2}{k_{p,m,n}'^2\left\{k^2+kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}\left(1-m^2/j_{m,p}'^2\right)}+\frac{1}{k_{p,m,n}'^2\left(1-m^2/j_{m,p}'^2\right)}\right]$$

$$\cos^2\left(n\pi z_0/h_c\right)\left[\cos\left(ms/a\right)\int_{-h}^{h}V\left(s'\right)\cos\left(ms'/a\right)ds'+\sin\left(ms/a\right)\int_{-h}^{h}V\left(s'\right)\sin\left(ms'/a\right)ds'\right]$$

$$+2ZH_0e^{ik_{z_i}z_0}\sum_{m''=0}^{\infty}\frac{\varepsilon_{m''}i^{m''}}{\pi k_{\rho_i}b\left(\Gamma^2-m''^2/b^2\right)}\left[-\frac{\cos\left\{m''\left(s/b-\varphi_i\right)\right\}\cos\varphi_p}{H_{m''}^{(1)}\left(k_{\rho_i}b\right)}+\frac{m''\sin\left\{m''\left(s/b-\varphi_i\right)\right\}\cos\theta_i\sin\varphi_p}{k_{\rho_i}bH_{m''}^{(1)\prime}\left(k_{\rho_i}b\right)}\right]$$

The homogeneous coefficients are determined to make $V\left(\pm h\right)=0$. However this voltage must be substituted into the integrals to determine the normalization of these terms from $H_\varphi^{(p,m,n)}\left(a,\varphi=s/a,z_0\right)$. This result is fairly complicated and we are also required to evaluate the radiation admittance Y_{rad}. In the subsections below we simplify this in several ways near the first slot resonance.

If we wanted to go to higher frequencies we could in principle substitute the form

$$V\left(s\right)=V_H\left(s\right)+V_P\left(s\right)=a_0\cos\left(\Gamma s\right)+a_1\sin\left(\Gamma s\right)$$

$$+\sum_m\frac{b_m}{\Gamma^2-m^2/a^2}\begin{matrix}\cos\\ \sin\end{matrix}\left(ms/a\right)+\sum_{m''}\frac{c_{m''}}{\Gamma^2-m''^2/b^2}\begin{matrix}\cos\\ \sin\end{matrix}\left\{m''\left(s/b-\varphi_i\right)\right\}$$

to account for the effect of the multiple resonant modes on the distributions of voltage. However, because of the series form of the short circuit current drive in addition now to the series contributions of the response to the resonant modes, this is not really any simpler than the previous Fourier basis. In addition at higher frequencies there may develop some near dependency between terms over the slot interval, possibly resulting in some conditioning issues. If this approach is pursued the same Galerkin setup as in the previous Fourier basis would be used for the radiation term.

11.3 Example Below Or Near First Slot Resonance

The preceding form of the solution involves an infinite series resulting from the short circuit current drive. To simplify the solution in this section we use our shorter slot length $h/a << \pi/2$, and consider expanding the drives in their first two terms

$$K_z^{sc}\left(\varphi = s/b << 1, z_0\right)$$

$$\approx 2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[-\frac{\left\{\cos\left(m\varphi_i\right) + \left(ms/b\right)\sin\left(m\varphi_i\right)\right\}\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m\left\{\left(ms/b\right)\cos\left(m\varphi_i\right) - \sin\left(m\varphi_i\right)\right\}\cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right]$$

$$\begin{aligned}
&\approx -2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[\frac{\cos\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m \sin\left(m\varphi_i\right)\cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right] \\
&-2sH_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{n\varepsilon_m i^m}{\pi k_{\rho_i} b^2} \left[\frac{\sin\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} - \frac{m \cos\left(m\varphi_i\right)\cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right] \\
&= K_0^{sc} + sK_1^{sc}
\end{aligned}$$

and

$$\begin{aligned}
H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right) = -i\omega\varepsilon_0\left(\varepsilon_m\varepsilon_n/V_{cav}\right) &\left[\frac{1}{\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}} \right. \\
&+ \left. \frac{\left(n\pi/h_c\right)^2 m^2/j_{m,p}^{\prime 2}}{k_{p,m,n}^{\prime 2}\left\{k^2 + kk_{p,m,n}^{\prime}\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}^{\prime 2}\right\}\left(1 - m^2/j_{m,p}^{\prime 2}\right)} + \frac{1}{k_{p,m,n}^{\prime 2}\left(1 - m^2/j_{m,p}^{\prime 2}\right)} \right] \\
&\cos^2\left(n\pi z_0/h_c\right)\left[\int_{-h}^{h} V_+\left(s'\right)ds' + s\left(m/a\right)^2 \int_{-h}^{h} V_+\left(s'\right)s'ds' \right] \\
&= H_0^{(p,m,n)} + sH_1^{(p,m,n)}
\end{aligned}$$

We can write these as

$$\begin{aligned}
H_0^{(p,m,n)} = -\frac{k^2}{2}\left(\varepsilon_m\varepsilon_n/V_{cav}\right)\cos^2\left(n\pi z_0/h_c\right)m_s^+ &\left[\frac{1}{\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}} \right. \\
&+ \left. \frac{\left(n\pi/h_c\right)^2 m^2/j_{m,p}^{\prime 2}}{k_{p,m,n}^{\prime 2}\left\{k^2 + kk_{p,m,n}^{\prime}\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}^{\prime 2}\right\}\left(1 - m^2/j_{m,p}^{\prime 2}\right)} + \frac{1}{k_{p,m,n}^{\prime 2}\left(1 - m^2/j_{m,p}^{\prime 2}\right)} \right]
\end{aligned}$$

where the slot magnetic dipole moment is

$$m_s^+ = \frac{i2}{\omega\mu_0}\int_{-h}^{h} V_+ (s')\, ds'$$

and as

$$H_1^{(p,m,n)} = -\frac{k^2}{4}\left(\varepsilon_m \varepsilon_n / V_{cav}\right)\cos^2\left(n\pi z_0/h_c\right) Q_{mss}^+ \left[\frac{1}{\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}}\right.$$
$$\left. + \frac{\left(n\pi/h_c\right)^2 m^2/j_{m,p}'^2}{k_{p,m,n}'^2\left\{k^2 + kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right\}\left(1 - m^2/j_{m,p}'^2\right)} + \frac{1}{k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}\right]$$

where the slot magnetic quadrupole moment is

$$Q_{mss}^+ = \frac{1}{\mu_0}\int_{-h}^{h} q_m^+ (s)\, s ds = \frac{1}{i\omega\mu_0}\int_{-h}^{h}\left(\frac{\partial}{\partial s} I_m^+\right) s^2 ds = \frac{h^2}{i\omega\mu_0}\left[I_m^+ (h) - I_m^+ (-h)\right] - \frac{2}{i\omega\mu_0}\int_{-h}^{h} I_m^+ (s)\, s ds$$
$$= \frac{i2}{\omega\mu_0}\int_{-h}^{h} I_m^+ (s)\, s ds = \frac{i4}{\omega\mu_0}\int_{-h}^{h} V_+ (s)\, s ds$$

and we used the continuity equation $\partial I_m^+/\partial s = i\omega q_m^+$ with boundary conditions $I_m(\pm h) = 0$.

In this case the solution of the transmission line equation

$$\left(\frac{\partial^2}{\partial s^2} + \Gamma^2\right) V(s) \approx Z\left(K_0^{sc} + sK_1^{sc}\right) - Z\left(H_0^{(p,m,n)} + sH_1^{(p,m,n)}\right)$$

with a voltage vanishing at the end points $s = \pm h$, is

$$V(s) \approx V_0\left[\cos(\Gamma s) - \cos(\Gamma h)\right] + V_1\left[h\sin(\Gamma s) - s\sin(\Gamma h)\right]$$

where

$$V_0 = Z\left(-K_0^{sc} + H_0^{(p,m,n)}\right)/\left[\Gamma^2\cos(\Gamma h)\right]$$

$$V_1 = Z\left(-K_1^{sc} + H_1^{(p,m,n)}\right)/\left[\Gamma^2\sin(\Gamma h)\right]$$

The dipole moment is

$$m_s^+ = \frac{i4V_0 h}{\omega\mu_0}\left[\frac{1}{\Gamma h}\sin(\Gamma h) - \cos(\Gamma h)\right]$$
$$= \frac{i4hZ}{\omega\mu_0\Gamma^2\cos(\Gamma h)}\left[\frac{1}{\Gamma h}\sin(\Gamma h) - \cos(\Gamma h)\right]\left(-K_0^{sc} + H_0^{(p,m,n)}\right)$$

and the quadrupole moment is

$$Q_{mss}^+ = \frac{i4V_1}{\omega\mu_0}\int_{-h}^{h}\left[hs\sin(\Gamma s) - s^2\sin(\Gamma h)\right] ds = \frac{i8V_1 h^3}{\omega\mu_0}\left[\frac{1}{\Gamma h}\left\{\frac{\sin(\Gamma h)}{\Gamma h} - \cos(\Gamma h)\right\} - \frac{1}{3}\sin(\Gamma h)\right]$$

$$= \frac{i8h^3 Z}{\omega\mu_0 \Gamma^2 \sin(\Gamma h)} \left[\frac{1}{\Gamma h} \left\{ \frac{\sin(\Gamma h)}{\Gamma h} - \cos(\Gamma h) \right\} - \frac{1}{3}\sin(\Gamma h) \right] \left(-K_1^{sc} + H_1^{(p,m,n)} \right)$$

If we specialize to the vicinity of a TM mode resonance

$$m_s^+ = \frac{i4hZ}{\omega\mu_0 \Gamma^2 \cos(\Gamma h)} \left[\frac{1}{\Gamma h}\sin(\Gamma h) - \cos(\Gamma h) \right]$$

$$\left[-K_0^{sc} - \frac{\cos(m\varphi_0)\cos(n\pi z_0/h_c)\,\varepsilon_m\varepsilon_n k_{p,m,n}^2/2}{V_{cav}\left[k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right]} \left\{ \cos(m\varphi_0)\, m_s^+ + \frac{m}{2a}\sin(m\varphi_0)\, Q_{mss}^+ \right\} \right]$$

$$Q_{mss}^+ = \frac{i8h^3 Z}{\omega\mu_0 \Gamma^2 \sin(\Gamma h)} \left[\frac{1}{\Gamma h} \left\{ \frac{\sin(\Gamma h)}{\Gamma h} - \cos(\Gamma h) \right\} - \frac{1}{3}\sin(\Gamma h) \right]$$

$$\left[-K_1^{sc} - \frac{\sin(m\varphi_0)(m/a)\cos(n\pi z_0/h_c)\,\varepsilon_m\varepsilon_n k_{p,m,n}^2/2}{V_{cav}\left[k^2 + kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right]} \left\{ \cos(m\varphi_0)\, m_s^+ + \frac{m}{2a}\sin(m\varphi_0)\, Q_{mss}^+ \right\} \right]$$

If we specialize to the vicinity of a TE mode resonance

$$m_s^+ = \frac{i4hZ}{\omega\mu_0 \Gamma^2 \cos(\Gamma h)} \left[\frac{1}{\Gamma h}\sin(\Gamma h) - \cos(\Gamma h) \right]$$

$$\left[-K_0^{sc} - \frac{m^2}{\left(m^2 - j_{m,p}'^2\right)} \frac{\cos(m\varphi_0)(m/a)\cos(n\pi z_0/h_c)\,\varepsilon_n\varepsilon_m (n\pi/h_c)^2/2}{V_{cav}\left[k^2 + kk'_{p,m,n}(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right]} \left\{ \frac{m}{2a}\cos(m\varphi_0)\, Q_{mss}^+ - \sin(m\varphi_0)\, m_s^+ \right\} \right]$$

$$Q_{mss}^+ = \frac{i8h^3 Z}{\omega\mu_0 \Gamma^2 \sin(\Gamma h)} \left[\frac{1}{\Gamma h} \left\{ \frac{\sin(\Gamma h)}{\Gamma h} - \cos(\Gamma h) \right\} - \frac{1}{3}\sin(\Gamma h) \right]$$

$$\left[-K_1^{sc} - \frac{m^2}{\left(m^2 - j_{m,p}'^2\right)} \frac{\cos(m\varphi_0)(m/a)\cos(n\pi z_0/h_c)\,\varepsilon_n\varepsilon_m (n\pi/h_c)^2/2}{V_{cav}\left[k^2 + kk'_{p,m,n}(1+i)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right]} \left\{ \frac{m}{2a}\cos(m\varphi_0)\, Q_{mss}^+ - \sin(m\varphi_0)\, m_s^+ \right\} \right]$$

Note that for small losses

$$\cos(\Gamma h) = \cos(\Gamma' h)\cosh(\Gamma'' h) - i\sin(\Gamma' h)\sinh(\Gamma'' h) \approx \cos(\Gamma' h) - i\Gamma'' h \sin(\Gamma' h)$$

$$\sin(\Gamma h) = \sin(\Gamma' h)\cosh(\Gamma'' h) + i\cos(\Gamma' h)\sinh(\Gamma'' h) \approx \sin(\Gamma' h) + i\Gamma'' h \cos(\Gamma' h)$$

$$\cos(\Gamma' h) = \cos(kh)\cos(\Gamma' h - kh) - \sin(kh)\sin(\Gamma' h - kh) \approx \cos(kh) - (\Gamma' h - kh)\sin(kh)$$

$$\sin(\Gamma' h) = \sin(kh)\cos(\Gamma' h - kh) + \cos(kh)\sin(\Gamma' h - kh) \approx \sin(kh) + (\Gamma' h - kh)\cos(kh)$$

and without a gasket

$$\Gamma = \Gamma' + i\Gamma'' = \sqrt{-Z\left(Y - iB_{rad} + G_{rad}/2\right)} \approx \sqrt{\omega^2 LC \left\{1 + \omega L\left(B^{int} + iG^{int}\right)\right\} \left\{1 + \left(B_{rad} + iG_{rad}/2\right)/(\omega C)\right\}}$$

$$\approx k + \sqrt{L/C}\left\{\left(k^2 B^{int} + B_{rad}\right) + i\left(k^2 G^{int} + G_{rad}/2\right)\right\}/2$$

$$k = \omega\sqrt{LC}$$

$$\Gamma' - k \approx \sqrt{L/C}\left(k^2 B^{int} + B_{rad}\right)/2$$

$$\Gamma'' \approx \sqrt{L/C}\left(k^2 G^{int} + G_{rad}/2\right)/2$$

$$\cos(\Gamma h) \approx \cos(kh) - (\Gamma' h - kh)\sin(kh) - i\Gamma'' h \sin(kh)$$

$$\sin(\Gamma h) \approx \sin(kh) + (\Gamma' h - kh)\cos(kh) + i\Gamma'' h \cos(kh)$$

Breaking the voltage into even and odd parts

$$V(s) = V_0(s) + V_1(s)$$

$$V_0(-s) = V_0(s) \text{ and } V_1(-s) = -V_1(s)$$

Using the leading approximate voltage without a gasket

$$V_0(s) \approx V_0\{\cos(ks) - \cos(kh)\}$$

$$V_1(s) \approx V_1\{h\sin(ks) - s\sin(kh)\}$$

gives the dipole moment

$$m_s^+ \approx \frac{i4V_0 h}{\omega\mu_0}\left\{\frac{\sin(kh)}{kh} - \cos(kh)\right\}$$

and the quadrupole moment is

$$Q_{mss}^+ \approx \frac{i8V_1 h^3}{\omega\mu_0}\left[\frac{1}{kh}\left\{\frac{\sin(kh)}{kh} - \cos(kh)\right\} - \frac{1}{3}\sin(kh)\right]$$

11.4 Radiation Admittance On Half Space

To evaluate the radiation admittance [10], [9]

$$Y_{rad}\int_{-h}^{h} V^*(s)\,V(s)\,ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0}\int_{-h}^{h} V^*(s)\left(\frac{\partial^2}{\partial s^2} + k^2\right)$$

$$\left[V(s)\left\{-C_e + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h}\frac{V(s')\,e^{ik|s-s'|} - V(s)}{|s-s'|}ds'\right] ds$$

Integration by parts gives

$$\int_{-h}^{h} V^*(s)\left(\frac{\partial^2}{\partial s^2}+k^2\right)F(s)\,ds = \left[V^*(s)\frac{\partial}{\partial s}F(s)\right]_{-h}^{h} + \int_{-h}^{h}\left[-\frac{\partial}{\partial s}V^*(s)\frac{\partial}{\partial s}+k^2V^*(s)\right]F(s)\,ds$$

$$= \left[V^*(s)\frac{\partial}{\partial s}F(s) - \frac{\partial}{\partial s}V^*(s)F(s)\right]_{-h}^{h} + \int_{-h}^{h}\left[\frac{\partial^2}{\partial s^2}V^*(s)+k^2V^*(s)\right]F(s)\,ds$$

where

$$F(s) = V(s)\left\{-C_e + \ln\left(1-s^2/h^2\right)\right\} + \int_{-h}^{h}\frac{V(s')\,e^{ik|s-s'|}-V(s)}{|s-s'|}ds'$$

Then noting that for the logarithmic (and constant) terms

$$\left[V^*(s)\frac{\partial}{\partial s}F(s) - \frac{\partial}{\partial s}V^*(s)F(s)\right]_{-h}^{h}$$

$$\to \left[V^*(s)\frac{\partial}{\partial s}\left\{V(s)\left(-C_e+\ln\left(1-s^2/h^2\right)\right)\right\} - \left\{V(s)\left(-C_e^o+\ln\left(1-s^2/h^2\right)\right)\right\}\frac{\partial}{\partial s}V^*(s)\right]_{-h}^{h}$$

$$= \left[V^*(s)V(s)\frac{\partial}{\partial s}\left\{\ln(1-s/h)+\ln(1+s/h)\right\}\right]_{-h}^{h} = \left[V^*(s)V(s)\left\{-\frac{1/h}{1-s/h}+\frac{1/h}{1+s/h}\right\}\right]_{-h}^{h} = 0$$

We expect the first term for the integral to vanish but not the second term. Hence we can write

$$\int_{-h}^{h} V^*(s)\left(\frac{\partial^2}{\partial s^2}+k^2\right)F(s)\,ds = \left[-\frac{\partial}{\partial s}V^*(s)F(s)\right]_{-h}^{h} + \int_{-h}^{h}\left[\frac{\partial^2}{\partial s^2}V^*(s)+k^2V^*(s)\right]F(s)\,ds$$

We then find

$$Y_{rad}\int_{-h}^{h}V^*(s)V(s)\,ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{V^*(s)\frac{\partial}{\partial s}F(s)-\frac{\partial}{\partial s}V^*(s)F(s)\right\}_{-h}^{h} + \int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V^*(s)+k^2V^*(s)\right\}F(s)\,ds\right]$$

$$= -\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{-\frac{\partial}{\partial s}V^*(s)F(s)\right\}_{-h}^{h} + \int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V^*(s)+k^2V^*(s)\right\}F(s)\,ds\right]$$

Now substituting the even term $V_0(s)$ plus the odd term $V_1(s)$

$$\int_{-h}^{h}V^*(s)V(s)\,ds = \int_{-h}^{h}\left[V_0^*(s)+V_1^*(s)\right]\left[V_0(s)+V_1(s)\right]ds = \int_{-h}^{h}V_0^*(s)V_0(s)\,ds + \int_{-h}^{h}V_1^*(s)V_1(s)\,ds$$

Evaluation using the leading terms

$$\int_{-h}^{h}V^*(s)V(s)\,ds \approx |V_0|^2\int_{-h}^{h}\left\{\cos(ks)-\cos(kh)\right\}^2 ds + |V_1|^2\int_{-h}^{h}\left\{h\sin(ks)-s\sin(kh)\right\}^2 ds$$

$$\approx |V_0|^2 \int_{-h}^{h} \left\{\cos^2(ks) - 2\cos(kh)\cos(ks) + \cos^2(kh)\right\} ds$$

$$+ |V_1|^2 \int_{-h}^{h} \left\{h^2 \sin^2(ks) - 2s\sin(ks)\, h\sin(kh) + s^2 \sin^2(kh)\right\} ds$$

$$\approx \frac{1}{2} |V_0|^2 \int_{-h}^{h} \left\{1 + \cos(2ks) - 4\cos(kh)\cos(ks) + 2\cos^2(kh)\right\} ds$$

$$+\frac{1}{2} |V_1|^2 \int_{-h}^{h} \left\{h^2 - h^2\cos(2ks) - 4s\sin(ks)\, h\sin(kh) + 2s^2 \sin^2(kh)\right\} ds$$

$$\approx \frac{1}{2} |V_0|^2 \left\{s + \frac{1}{2k}\sin(2ks) - \frac{4}{k}\cos(kh)\sin(ks) + 2s\cos^2(kh)\right\}_{-h}^{h}$$

$$+\frac{1}{2} |V_1|^2 \left\{h^2 s - \frac{h^2}{2k}\sin(2ks) + \frac{4}{k} s\cos(ks)\, h\sin(kh) - \frac{4}{k^2}\sin(ks)\, h\sin(kh) + \frac{2}{3} s^3 \sin^2(kh)\right\}_{-h}^{h}$$

$$\approx |V_0|^2 h \left\{1 + 2\cos^2(kh) - 3\frac{\sin(kh)}{kh}\cos(kh)\right\}$$

$$+ |V_1|^2 h^3 \left\{1 + 3\cos(kh)\frac{\sin(kh)}{kh} - 4\left(1 - \frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

Note that

$$\left(\frac{\partial^2}{\partial s^2} + k^2\right) V^*(s) \approx \left(\frac{\partial^2}{\partial s^2} + k^2\right) \left[V_0^*(s) + V_1^*(s)\right] \approx -k^2 V_0^* \cos(kh) - k^2 V_1^* s\sin(kh)$$

Inserting these voltages

$$Y_{rad}^{e} \int_{-h}^{h} |V_0(s)|^2 ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0} \int_{-h}^{h} V_0^*(s) \left(\frac{\partial^2}{\partial s^2} + k^2\right)$$

$$\left[V_0(s)\left\{-C_e^e + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h} \frac{V_0(s')\, e^{ik|s-s'|} - V_0(s)}{|s-s'|} ds'\right] ds$$

and

$$Y_{rad}^{o} \int_{-h}^{h} |V_1(s)|^2 ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0} \int_{-h}^{h} V_1^*(s) \left(\frac{\partial^2}{\partial s^2} + k^2\right)$$

$$\left[V_1(s)\left\{-C_e^o + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h} \frac{V_1(s')\, e^{ik|s-s'|} - V_1(s)}{|s-s'|} ds'\right] ds$$

or

$$Y_{rad}^{e} \int_{-h}^{h} |V_0(s)|^2 ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0} \left[\left\{V_0^*(s)\frac{\partial}{\partial s}F_0(s) - \frac{\partial}{\partial s}V_0^*(s)\, F_0(s)\right\}_{-h}^{h} + \int_{-h}^{h} \left\{\frac{\partial^2}{\partial s^2}V_0^*(s) + k^2 V_0^*(s)\right\} F_0(s)\, ds\right]$$

$$= -\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{-\frac{\partial}{\partial s}V_0^*\left(s\right)F_0\left(s\right)\right\}_{-h}^{h} + \int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V_0^*\left(s\right) + k^2 V_0^*\left(s\right)\right\}F_0\left(s\right)ds\right]$$

or

$$Y_{rad}^{e}\left|V_0\right|^2 h\left\{1 + 2\cos^2\left(kh\right) - 3\frac{\sin\left(kh\right)}{kh}\cos\left(kh\right)\right\}$$

$$= -\frac{1}{\pi}\frac{i}{\omega\mu_0}V_0^* k\sin\left(kh\right)\left\{F_0\left(h\right) + F_0\left(-h\right)\right\} + \frac{1}{\pi}i\omega\varepsilon_0 V_0^*\cos\left(kh\right)\int_{-h}^{h}F_0\left(s\right)ds$$

$$F_0\left(s\right) = V_0\left(s\right)\left\{-C_e^e + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h}\frac{V_0\left(s'\right)e^{ik\left|s-s'\right|} - V_0\left(s\right)}{\left|s-s'\right|}ds'$$

and

$$Y_{rad}^{o}\int_{-h}^{h}\left|V_1\left(s\right)\right|^2 ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{V_1^*\left(s\right)\frac{\partial}{\partial s}F_1\left(s\right) - \frac{\partial}{\partial s}V_1^*\left(s\right)F_1\left(s\right)\right\}_{-h}^{h} + \int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V_1^*\left(s\right) + k^2 V_1^*\left(s\right)\right\}F_1\left(s\right)ds\right]$$

$$= -\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{-\frac{\partial}{\partial s}V_1^*\left(s\right)F_1\left(s\right)\right\}_{-h}^{h} + \int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V_1^*\left(s\right) + k^2 V_1^*\left(s\right)\right\}F_1\left(s\right)ds\right]$$

or

$$Y_{rad}^{o}\left|V_1\right|^2 h^3\left\{1 + 3\cos\left(kh\right)\frac{\sin\left(kh\right)}{kh} - 4\left(1 - \frac{1}{6}k^2h^2\right)\frac{\sin^2\left(kh\right)}{k^2h^2}\right\}$$

$$= \frac{1}{\pi}\frac{i}{\omega\mu_0}V_1^*\left\{kh\cos\left(kh\right) - \sin\left(kh\right)\right\}\left\{F_1\left(h\right) - F_1\left(-h\right)\right\} + \frac{1}{\pi}i\omega\varepsilon_0 V_1^*\sin\left(kh\right)\int_{-h}^{h}sF_1\left(s\right)ds$$

$$F_1\left(s\right) = V_1\left(s\right)\left\{-C_e^o + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h}\frac{V_1\left(s'\right)e^{ik\left|s-s'\right|} - V_1\left(s\right)}{\left|s-s'\right|}ds'$$

These leading-order even and odd cases are carried out in the Appendix and summarized in the next two subsections. Following these two subsections we continue with some examples focusing on the dipole moment drive.

11.4.1 Constant Drive Radiation Admittance On Half Space

For a near constant drive (an even case) and voltage distribution

$$V\left(s\right) \approx V_0\left\{\cos\left(ks\right) - \cos\left(kh\right)\right\}$$

the full space radiation admittance parameters [10], [9] are

$$\pi h\eta_0 G_{rad}^{e} = \frac{\mathrm{Cin}\left(4kh\right) + 4\,\mathrm{Si}\left(2kh\right)\cos\left(kh\right)\left\{kh\cos\left(kh\right) - \sin\left(kh\right)\right\} - \sin^2\left(2kh\right)}{1 + \cos\left(kh\right)\left\{2\cos\left(kh\right) - 3\sin\left(kh\right)/kh\right\}}$$

$$\sim \frac{10}{9}\left(kh\right)^2\ ,\ kh << 1$$

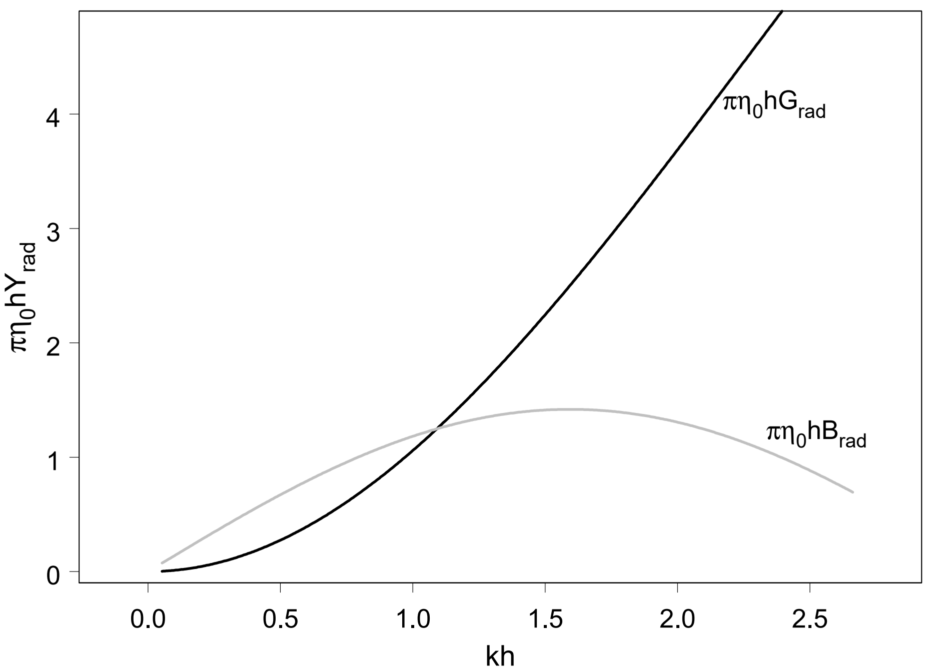

Figure 15: Behavior of radiation admittance resulting from an even slot voltage from a constant drive current as a function of kh.

$$\pi h\eta_0 B^e_{rad} = \frac{4\cos(kh)\{kh\cos(kh) - \sin(kh)\}\{\ln 2 - C^e_e/2 - \mathrm{Cin}(2kh)\} + \mathrm{Si}(4kh) - 2\cos^2(kh)\sin(2kh)}{1 + \cos(kh)\{2\cos(kh) - 3\sin(kh)/kh\}}$$

$$\sim \frac{7}{5}(kh)\ ,\ kh << 1$$

where

$$C^e_e = 2[\ln(2) - 7/3] \approx 2[\ln(2) - 2.33333] \approx -3.280$$

These quantities are shown in Figure 15. The first slot resonance $kh = \pi/2$ gives

$$\pi h\eta_0 G^e_{rad} = \mathrm{Cin}(2\pi) \approx 2.43765$$

$$\pi h\eta_0 B^e_{rad} = \mathrm{Si}(2\pi) \approx 1.41815$$

11.4.2 Linear Drive Radiation Admittance On Half Space

For a linear drive (an odd case) and voltage distribution

$$V_1(s) \approx V_1 \{h \sin(ks) - s \sin(kh)\}$$

the full space radiation admittance parameters (see Appendix) are

$$\pi\eta_0 h G^o_{rad} \left\{1 + 3\cos(kh)\frac{\sin(kh)}{kh} - 4\left(1 - \frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$= \mathrm{Cin}(4kh) + \sin(kh)\left\{\cos(kh) - \left(\frac{1}{kh} - \frac{1}{3}kh\right)\sin(kh)\right\} 4\,\mathrm{Si}(2kh)$$

$$-\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh) - \frac{1}{kh}\sin(kh) + \frac{1}{2}kh\sin(kh)\right\}\sin^3(kh) - \frac{4}{3}\sin^2(kh)$$

$$\pi\eta_0 h B^o_{rad} \left\{1 + 3\cos(kh)\frac{\sin(kh)}{kh} - 4\left(1 - \frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$= -2\sin(kh)\left\{\cos(kh) - \left(\frac{1}{kh} - \frac{1}{3}kh\right)\sin(kh)\right\}\{C^o_e - 2\ln(2)\}$$

$$-\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh) - \frac{1}{kh}\sin(kh) + \frac{1}{2}kh\sin(kh)\right\}\sin^2(kh)\cos(kh) - \frac{4}{3}\frac{1}{kh}\sin^2(kh)$$

$$+\,\mathrm{Si}(4kh) - \sin(kh)\left\{\cos(kh) - \left(\frac{1}{kh} - \frac{1}{3}kh\right)\sin(kh)\right\} 4\mathrm{Cin}(2kh)$$

and we take

$$C^o_e = 2[\ln(2) - 43/15] \approx 2[\ln(2) - 2.86666] \approx -4.347$$

This radiation admittance is shown in Figure 16. We see that this contribution is relatively small for small kh.

11.5 Parabolic Approximation For Voltage Distribution

Note that the low frequency formula of the preceding subsection yields $\pi h\eta_0 G_{rad} \sim 10(kh)^2/9 \to 2.7416$ at $kh = \pi/2$, which is not too far above the preceding resonant result 2.43765; the corresponding low frequency parabolic form of the voltage (expanding the difference of cosines)

$$V(s) \sim V_0 k^2 \left(h^2 - s^2\right)/2\ ,\ kh << \pi/2$$

is also a good approximation to the voltage distribution up to near the first slot resonance. The corresponding radiation terms are

$$\pi h\eta_0 G^e_{rad} \sim \frac{10}{9}(kh)^2\ \ ,\ kh << 1$$

$$\pi h\eta_0 B^e_{rad} \sim \frac{7}{5}(kh)\ \ ,\ kh << 1$$

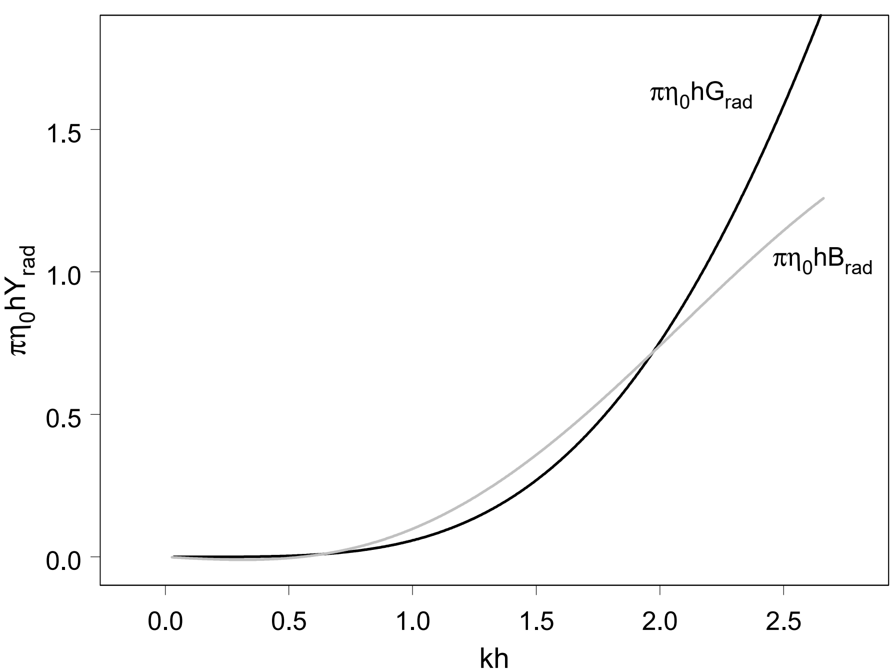

Figure 16: Behavior of radiation admittance from odd voltage distribution resulting from linear current drive.

For frequencies below the first slot resonance we can approximate the drive short circuit current density and interior cavity field at the slot by the values at the slot center assuming the slot half length is small compared to half the cylinder circumference $h << \pi a < \pi b$. The short circuit current density at the slot center is

$$K_z^{sc}\left(\varphi = s/b << 1, z_0\right) \approx -2H_0 e^{ik_{z_i} z_0} \sum_{m=0}^{\infty} \frac{\varepsilon_m i^m}{\pi k_{\rho_i} b} \left[\frac{\cos\left(m\varphi_i\right)\cos\varphi_p}{H_m^{(1)}\left(k_{\rho_i} b\right)} + \frac{m \sin\left(m\varphi_i\right)\cos\theta_i \sin\varphi_p}{k_{\rho_i} b H_m^{(1)\prime}\left(k_{\rho_i} b\right)} \right] = K_0^{sc}$$

and the nearest resonant mode contribution is

$$H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right) = H_0^{(p,m,n)}$$

$$= -k^2 \frac{1}{2}\left(\varepsilon_m \varepsilon_n / V_{cav}\right)\cos^2\left(n\pi z_0/h_c\right) m_s^+ \left[\frac{1}{\left\{k^2 + k k_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}} \right.$$
$$\left. + \frac{\left(n\pi/h_c\right)^2 m^2 / j_{m,p}^{\prime 2}}{k_{p,m,n}^{\prime 2}\left\{k^2 + k k_{p,m,n}^{\prime}\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}^{\prime 2}\right\}\left(1 - m^2/j_{m,p}^{\prime 2}\right)} + \frac{1}{k_{p,m,n}^{\prime 2}\left(1 - m^2/j_{m,p}^{\prime 2}\right)} \right]$$

where the slot magnetic dipole moment is

$$m_s^+ = -2\alpha_{m,ss}^+ H_s^{sc} = \frac{i2}{\omega\mu_0}\int_{-h}^{h} V_+\left(s'\right) ds' \sim i\omega\varepsilon_0 V_0 \frac{4}{3} h^3 = i\left(V_0/\eta_0\right)\frac{4}{3} k h^3$$

Note that the ratio of this parabolic voltage $V\left(s\right) \sim V_0 k^2\left(h^2 - s^2\right)/2$ result for the dipole moment m_s^+, to the approximate result using $V\left(s\right) \approx V_0\left\{\cos\left(ks\right) - \cos\left(kh\right)\right\}$ is $\left(k^2h^2/3\right)/\left\{\sin\left(kh\right)/\left(kh\right) - \cos\left(kh\right)\right\} \to 1$ at low frequencies; even at $kh = \pi/2$ this ratio is not too much greater than unity $\left(\pi^3/24\right) \approx 31/24 \approx 1.29$.

Applying the operator

$$\int_{-h}^{h} V^*\left(s\right) ds$$

to the differential equation

$$\frac{\partial^2}{\partial s^2}V\left(s\right) - Z\left(Y + G_{rad}/2 - iB_{rad}\right)V\left(s\right) \approx \left(\frac{\partial^2}{\partial s^2} + \Gamma^2\right)V\left(s\right)$$

$$= Z K_z^{sc}\left(\varphi = s/b << 1, z_0\right) - Z H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right)$$

gives

$$\int_{-h}^{h} V^*\left(s\right)\frac{\partial^2}{\partial s^2}V\left(s\right) ds + \Gamma^2 \int_{-h}^{h} V^*\left(s\right) V\left(s\right) ds = -\int_{-h}^{h}\frac{\partial}{\partial s}V^*\left(s\right)\frac{\partial}{\partial s}V\left(s\right) ds + \Gamma^2 \int_{-h}^{h} V^*\left(s\right) V\left(s\right) ds$$

$$= Z K_z^{sc}\left(\varphi = s/b << 1, z_0\right)\int_{-h}^{h} V^*\left(s\right) ds - Z H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right)\int_{-h}^{h} V^*\left(s\right) ds$$

where we integrated by parts using $V^*\left(\pm h\right) = 0$

$$-\int_{-h}^{h}\left|\frac{\partial}{\partial s}V\left(s\right)\right|^2 ds + \Gamma^2\int_{-h}^{h}\left|V\left(s\right)\right|^2 ds + Z H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right)\int_{-h}^{h} V^*\left(s\right) ds$$

$$\approx ZK_z^{sc}\left(\varphi = s/b << 1, z_0\right) \int_{-h}^{h} V^*\left(s\right) ds$$

Inserting the parabolic approximation

$$V_0 k^2 \left(-1 + \frac{2}{5}\Gamma^2 h^2\right) + ZH_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right) \approx ZK_z^{sc}\left(\varphi = s/b << 1, z_0\right)$$

or

$$V_0 k^2 \left\{1/Z + \left(Y + G_{rad}/2 - iB_{rad}\right)\frac{2}{5}h^2\right\} - H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right) \approx -K_z^{sc}\left(\varphi = s/b << 1, z_0\right)$$

or

$$V_0 k^2 \left\{\frac{1}{-i\omega L} + \Delta Y_L + \left(-i\omega C + \Delta Y_C + G_{rad}/2 - iB_{rad}\right)\frac{2}{5}h^2\right\} - H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right)$$

$$\approx -K_z^{sc}\left(\varphi = s/b << 1, z_0\right)$$

where

$$Y = -i\omega C + \Delta Y_C$$

$$1/Z = \frac{1}{-i\omega L} + \Delta Y_L$$

Now without a gasket $\Delta Y_C = 0$ and

$$V_0 k^2 \left\{\frac{1}{-i\omega L} + G^{int} - iB^{int} + \left(-i\omega C + G_{rad}/2 - iB_{rad}\right)\frac{2}{5}h^2\right\} - H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right)$$

$$\approx -K_z^{sc}\left(\varphi = s/b << 1, z_0\right)$$

where

$$\Delta Y_L = G^{int} - iB^{int}$$

Using the form

$$-H_\varphi^{(p,m,n)}\left(a, \varphi = s/a << 1, z_0\right)$$

$$= \varepsilon_m \varepsilon_n \cos^2\left(n\pi z_0/h_c\right)\left(\alpha_{m,ss}^0/V_{cav}\right) k^2 \left[\frac{1}{\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}}\right.$$

$$\left. + \frac{\left(n\pi/h_c\right)^2 m^2/j_{m,p}'^2}{k_{p,m,n}'^2\left\{k^2 + kk_{p,m,n}'\left(1+i\right)/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right\}\left(1 - m^2/j_{m,p}'^2\right)} + \frac{1}{k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}\right] \frac{V_0 k^2}{-i\omega L}$$

where

$$C \sim \varepsilon_0 \Omega_e / \pi$$

$$L \sim \mu_0 \pi / \Omega_e$$

$$\sqrt{L/C} \sim \eta_0 \pi / \Omega_e$$

$$\omega \sqrt{LC} = k$$

and the half space dipole moment is

$$m_s^0 = -2\alpha_{m,ss}^0 H_s^{sc}$$

with low frequency polarizability

$$\alpha_{m,ss}^0 \sim \frac{\pi \ell^3}{12\Omega_e}$$

gives

$$\begin{aligned}
&\left\{ \left(1 - \frac{2}{5} k^2 h^2 \right) - i\omega L \left(G^{int} - iB^{int} \right) - i\omega L \left(G_{rad}/2 - iB_{rad} \right) \frac{2}{5} h^2 \right\} \frac{V_0 k^2}{-i\omega L} \\
&+ \varepsilon_m \varepsilon_n \cos^2 \left(n\pi z_0 / h_c \right) \left(\alpha_{m,ss}^0 / V_{cav} \right) k^2 \left[\frac{1}{\left\{ k^2 + k k_{p,m,n} \left(1 + i \right) / Q_{p,m,n}^{TM} - k_{p,m,n}^2 \right\}} \right. \\
&\left. + \frac{\left(n\pi / h_c \right)^2 \left(m^2 / j_{m,p}'^2 \right) / \left(1 - m^2 / j_{m,p}'^2 \right)}{k_{p,m,n}'^2 \left\{ k^2 + k k_{p,m,n}' \left(1 + i \right) / Q_{p,m,n}^{TE} - k_{p,m,n}'^2 \right\}} + \frac{1}{k_{p,m,n}'^2 \left(1 - m^2 / j_{m,p}'^2 \right)} \right] \frac{V_0 k^2}{-i\omega L} \\
&\approx -K_z^{sc} \left(\varphi = s/b << 1, z_0 \right)
\end{aligned}$$

or replacing the voltage amplitude by the slot dipole moment

$$\begin{aligned}
&\left\{ \left(1 - \frac{2}{5} k^2 h^2 \right) - i\omega L \left(G^{int} - iB^{int} \right) - i\omega L \left(G_{rad}/2 - iB_{rad} \right) \frac{2}{5} h^2 \right\} \frac{m_s^+}{2\alpha_{m,ss}^0} \\
&+ \varepsilon_m \varepsilon_n \cos^2 \left(n\pi z_0 / h_c \right) \left(\alpha_{m,ss}^0 / V_{cav} \right) k^2 \left[\frac{1}{\left\{ k^2 + k k_{p,m,n} \left(1 + i \right) / Q_{p,m,n}^{TM} - k_{p,m,n}^2 \right\}} \right. \\
&\left. + \frac{\left(n\pi / h_c \right)^2 \left(m^2 / j_{m,p}'^2 \right) / \left(1 - m^2 / j_{m,p}'^2 \right)}{k_{p,m,n}'^2 \left\{ k^2 + k k_{p,m,n}' \left(1 + i \right) / Q_{p,m,n}^{TE} - k_{p,m,n}'^2 \right\}} + \frac{1}{k_{p,m,n}'^2 \left(1 - m^2 / j_{m,p}'^2 \right)} \right] \frac{m_s^+}{2\alpha_{m,ss}^0} \\
&\approx -K_z^{sc} \left(\varphi = s/b << 1, z_0 \right)
\end{aligned}$$

or replacing the slot dipole moment by the polarizability

$$\alpha_{m,ss}^0 / \alpha_{m,ss}^+ \approx \left(1 - \frac{2}{5} k^2 h^2 \right) - i\omega L \left(G^{int} - iB^{int} \right) - i\omega L \left(G_{rad}/2 - iB_{rad} \right) \frac{2}{5} h^2$$

$$+\varepsilon_m\varepsilon_n\cos^2(n\pi z_0/h_c)\left(\alpha^0_{m,ss}/V_{cav}\right)k^2\left[\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}}\right.$$
$$\left.+\frac{(n\pi/h_c)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\right]$$

We can identify the normally small parameter $\alpha^0_{m,ss}/V_{cav} << 1$ in the cavity resonance terms. Resonance occurs when the real part of the right hand side becomes small (or vanishes, when possible).

As the frequency is decreased we let

$$1/Q_{slt}=1/Q_{wall}+1/Q_{rad}\approx\omega Lh\left(G^{int}/h+G_{rad}h/5\right)$$

and noting that

$$k^2h\eta_0G^{int}\approx(R_s/\eta_0)\,\ell d/w^2\approx k^2h\eta_0B^{int}$$

taking the definition [1]

$$1/Q^{p,m,n}_0=\varepsilon_m\varepsilon_n\cos^2(n\pi z_0/h_c)\left(\alpha^0_{m,ss}/V_{cav}\right)$$

and

$$\Delta kh^2_{slt}=\omega L\left(B^{int}+\frac{2}{5}h^2B_{rad}\right)$$

we can write

$$\alpha^0_{m,ss}/\alpha^+_{m,ss}\approx\left(1-\frac{2}{5}k^2h^2-\Delta kh^2_{slt}\right)-i/Q_{slt}$$
$$+\left(k^2/Q^{p,m,n}_0\right)\left[\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}}\right.$$
$$\left.+\frac{(n\pi/h_c)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\right]$$

For the real part of the right side to vanish, we must have

$$0=\left(1-\frac{2}{5}k^2h^2-\Delta kh^2_{slt}\right)$$
$$+\left(k^2/Q^{p,m,n}_0\right)\left[\frac{k^2+kk_{p,m,n}/Q^{TM}_{p,m,n}-k^2_{p,m,n}}{\left|k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right|^2}\right.$$
$$\left.+\frac{\left\{k^2+kk'_{p,m,n}/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}(n\pi/h_c)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}\left|k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right|^2}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\right]$$

Near the slot resonance $kh\approx\pi/2$ and thus the first term on the right is $1-2k^2h^2/5\approx1-\pi^2/10$, which is approaching zero; hence, even if the other terms remain small this quantity can vanish leading to a resonant enhancement. However, as the frequency decreases we eventually require

$$1 \approx \left(k^2/Q_0^{p,m,n}\right) \frac{k^2 + kk_{p,m,n}/Q_{p,m,n}^{TM} - k_{p,m,n}^2}{\left(k^2 + kk_{p,m,n}/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right)^2 + \left(kk_{p,m,n}/Q_{p,m,n}^{TM}\right)^2} \text{ , TM modes}$$

$$1 \approx \left(k^2/Q_0^{p,m,n}\right) \frac{\left(n\pi/h_c\right)^2}{k_{p,m,n}'^2} \frac{k^2 + kk_{p,m,n}'/Q_{p,m,n}^{TE} - k_{p,m,n}'^2}{\left(k^2 + kk_{p,m,n}'/Q_{p,m,n}^{TE} - k_{p,m,n}'^2\right)^2 + \left(kk_{p,m,n}'/Q_{p,m,n}^{TE}\right)^2} \frac{\left(m^2/j_{m,p}'^2\right)}{\left(1 - m^2/j_{m,p}'^2\right)} \text{ , TE modes}$$

or because

$$\frac{x}{x^2 + x_0^2} \leq \frac{1}{2x_0}$$

this means for resonance

$$1 \approx \frac{\left(k/Q_0^{p,m,n}\right)}{2k_{p,m,n}/Q_{p,m,n}^{TM}} \approx \frac{1}{2Q_0^{p,m,n}/Q_{p,m,n}^{TM}} \text{ , TM modes}$$

$$1 \approx \frac{\left(n\pi/h_c\right)^2}{k_{p,m,n}'^2} \frac{\left(k/Q_0^{p,m,n}\right)}{2k_{p,m,n}'/Q_{p,m,n}^{TE}} \frac{\left(m^2/j_{m,p}'^2\right)}{\left(1 - m^2/j_{m,p}'^2\right)} \approx \frac{1}{2Q_0^{p,m,n}/Q_{p,m,n}^{TE}} \frac{\left(n\pi/h_c\right)^2 \left(m^2/j_{m,p}'^2\right)}{k_{p,m,n}'^2 \left(1 - m^2/j_{m,p}'^2\right)} \text{ , TE modes}$$

The axial electric field is then

$$E_z \approx \frac{\varepsilon_m \varepsilon_n / V_{cav}}{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2} V_+\left(0\right) \frac{4}{3} h \frac{\left(j_{m,p}/a\right) J_m\left(j_{m,p}\rho/a\right)}{J_{m-1}\left(j_{m,p}\right)} \cos\left(m\varphi\right) \cos\left(n\pi z_0/h_c\right) \cos\left(n\pi z/h_c\right)$$

$$\approx \frac{-i\omega\mu_0 m_s^+ \varepsilon_m \varepsilon_n / \left(2V_{cav}\right)}{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2} \frac{\left(j_{m,p}/a\right) J_m\left(j_{m,p}\rho/a\right)}{J_{m-1}\left(j_{m,p}\right)} \cos\left(m\varphi\right) \cos\left(n\pi z_0/h_c\right) \cos\left(n\pi z/h_c\right)$$

$$\approx \frac{i\omega\mu_0 \varepsilon_m \varepsilon_n H_s^{sc}\left(\alpha_{m,ss}^+/V_{cav}\right)}{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2} \frac{\left(j_{m,p}/a\right) J_m\left(j_{m,p}\rho/a\right)}{J_{m-1}\left(j_{m,p}\right)} \cos\left(m\varphi\right) \cos\left(n\pi z_0/h_c\right) \cos\left(n\pi z/h_c\right)$$

The magnitude squared is then

$$\left|E_z\right|^2 \approx \frac{\varepsilon_m^2 \varepsilon_n^2 \left|\alpha_{m,ss}^+/V_{cav}\right|^2 \omega^2 \mu_0^2 \left|H_\varphi^{sc}\left(b,0,z_0\right)\right|^2}{\left|k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right|^2}$$

$$\frac{\left(j_{m,p}/a\right)^2 J_m^2\left(j_{m,p}\rho/a\right)}{J_{m-1}^2\left(j_{m,p}\right)} \cos^2\left(m\varphi\right) \cos^2\left(n\pi z_0/h_c\right) \cos^2\left(n\pi z/h_c\right)$$

At the TM cavity resonances

$$\alpha_{m,ss}^0/\alpha_{m,ss}^+ \approx \left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right) - i/Q_{slt}$$

$$+ \left(k^2/Q_0^{p,m,n}\right)$$

$$\left[\frac{1}{k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)}{k^2_{p,m,n}-k'^2_{p,m,n}}+1\right\}\right]$$

or

$$\frac{\left|k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right|^2}{\left|\alpha^+_{m,ss}/\alpha^0_{m,ss}\right|^2}$$

$$\approx\left|\left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh^2_{slt}\right)-i/Q_{slt}\right\}\left\{k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}\right.$$

$$\left.+\left(k^2/Q^{p,m,n}_0\right)\left[1+\frac{k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)}{k^2_{p,m,n}-k'^2_{p,m,n}}+1\right\}\right]\right|^2$$

Setting the real part inside the absolute value to zero

$$\left(1-\frac{2}{5}k^2h^2-\Delta kh^2_{slt}\right)\left(k^2+kk_{p,m,n}/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right)+kk_{p,m,n}/\left(Q_{slt}Q^{TM}_{p,m,n}\right)$$

$$+\left(k^2/Q^{p,m,n}_0\right)\left[1+\frac{\left(k^2+kk_{p,m,n}/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right)}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)}{k^2_{p,m,n}-k'^2_{p,m,n}}+1\right\}\right]=0$$

or

$$-\left(k^2+kk_{p,m,n}/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right)$$

$$=\frac{kk_{p,m,n}/\left(Q_{slt}Q^{TM}_{p,m,n}\right)+\left(k^2/Q^{p,m,n}_0\right)}{\left(1-\frac{2}{5}k^2h^2-\Delta kh^2_{slt}\right)+\frac{\cos^2\left(n\pi z_0/h_c\right)\left(k^2/Q_0\right)}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)}{k^2_{p,m,n}-k'^2_{p,m,n}}+1\right\}}$$

Using this in the previous expression

$$\frac{\left|k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right|^2}{\left|\alpha^+_{m,ss}/\alpha^0_{m,ss}\right|^2}\approx$$

$$\left|\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh^2_{slt}\right)-i/Q_{slt}\right.\right.$$

$$\left.+\frac{\left(k^2/Q^{p,m,n}_0\right)}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)}{k^2_{p,m,n}-k'^2_{p,m,n}}+1\right\}\right]\left\{k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}$$

$$\left.+\left(k^2/Q^{p,m,n}_0\right)\right|^2$$

or

$$\frac{\left|k^2+kk_{p,m,n}\left(1+i\right)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right|^2}{\left|\alpha^+_{m,ss}/\alpha^0_{m,ss}\right|^2}\approx$$

$$\left|\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)\right.\right.$$

$$\left.+\frac{\left(k^2/Q_0^{p,m,n}\right)}{k_{p,m,n}'^2\left(1-m^2/j_{m,p}'^2\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j_{m,p}'^2\right)}{k_{p,m,n}^2-k_{p,m,n}'^2}+1\right\}\right]\left\{k^2+kk_{p,m,n}/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}$$

$$+\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)\right.$$

$$\left.+\frac{\left(k^2/Q_0^{p,m,n}\right)}{k_{p,m,n}'^2\left(1-m^2/j_{m,p}'^2\right)}\left\{\frac{\left(n\pi/h_c\right)^2\left(m^2/j_{m,p}'^2\right)}{k_{p,m,n}^2-k_{p,m,n}'^2}+1\right\}\right]ikk_{p,m,n}/Q_{p,m,n}^{TM}$$

$$\left.-\left(i/Q_{slt}\right)\left(k^2+kk_{p,m,n}/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right)+kk_{p,m,n}/\left(Q_{p,m,n}^{TM}Q_{slt}\right)+\left(k^2/Q_0^{p,m,n}\right)\right|^2$$

gives

$$\frac{\left|k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right|^2}{\left|\alpha_{m,ss}^{+}/\alpha_{m,ss}^{0}\right|^2}\approx$$

$$\left[\left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)+\frac{\left(k^2/Q_0^{p,m,n}\right)}{k_{p,m,n}'^2\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{\left(n\pi/h_c\right)^2\left(m^2/j_{m,p}'^2\right)}{k_{p,m,n}^2-k_{p,m,n}'^2}+1\right)\right\}\right.$$

$$\left.kk_{p,m,n}/Q_{p,m,n}^{TM}+\frac{kk_{p,m,n}/\left(Q_{slt}^2Q_{p,m,n}^{TM}\right)+k^2/\left(Q_0^{p,m,n}Q_{slt}\right)}{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)+\frac{\left(k^2/Q_0^{p,m,n}\right)}{k_{p,m,n}'^2\left(1-m^2/j_{m,p}'^2\right)}\left(\frac{\left(n\pi/h_c\right)^2\left(m^2/j_{m,p}'^2\right)}{k_{p,m,n}^2-k_{p,m,n}'^2}+1\right)}\right]^2$$

Dropping the $O\left(\frac{1}{Q_0^{p,m,n}Q_{p,m,n}^{TM}}\right),O\left[\frac{1}{\left(Q_0^{p,m,n}\right)^2Q_{slt}}\right]$ terms

$$\frac{\left|k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right|^2}{\left|\alpha_{m,ss}^{+}/\alpha_{m,ss}^{0}\right|^2}$$

$$\approx\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)kk_{p,m,n}/Q_{p,m,n}^{TM}+\frac{kk_{p,m,n}/\left(Q_{slt}^2Q_{p,m,n}^{TM}\right)+k^2/\left(Q_0^{p,m,n}Q_{slt}\right)}{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)}\right]^2$$

Then

$$\left|E_z\right|^2\approx\frac{\varepsilon_m^2\varepsilon_n^2\left(\alpha_{m,ss}^0/V_{cav}\right)^2\left|\alpha_{m,ss}^{+}/\alpha_{m,ss}^{0}\right|^2\omega^2\mu_0^2\left|H_\varphi^{sc}\left(b,0,z_0\right)\right|^2}{\left|k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right|^2}$$

$$\frac{\left(j_{m,p}/a\right)^2J_m^2\left(j_{m,p}\rho/a\right)}{J_{m-1}^2\left(j_{m,p}\right)}\cos^2\left(m\varphi\right)\cos^2\left(n\pi z_0/h_c\right)\cos^2\left(n\pi z/h_c\right)$$

Note that the low frequency result of the prior section on ELECTRICALLY SMALL SLOT LENGTHS (subsection Single Slot Source), near the cavity TM resonances was

$$|E_z|^2 \approx \frac{\omega^2\mu_0^2\left|H_\varphi^{sc}(b,0,z_0)\right|^2\varepsilon_m^2\varepsilon_n^2\left(\alpha_{m,ss}^0/V_{cav}\right)^2}{\left|k^2+kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right|^2}\frac{(j_{m,p}/a)^2 J_m^2(j_{m,p}\rho/a)}{J_{m-1}^2(j_{m,p})}\cos^2(m\varphi)\cos^2(n\pi z_0/h_c)\cos^2(n\pi z/h_c)$$

This is the same as the result in this subsection except here we have the additional factor

$$\left|\alpha_{m,ss}^+/\alpha_{m,ss}^0\right|^2$$

since the slot voltage amplitude and polarizability can be modified by the cavity mode. Using the preceding result we can write the result near resonance as

$$\left|\frac{E_z}{\eta_0 H_\varphi^{sc}(b,0,z_0)}\right|^2 \approx \frac{1}{\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)Q_0^{p,m,n}/Q_{p,m,n}^{TM}+\frac{1/Q_{slt}}{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)}\right]^2}$$

$$\frac{(j_{m,p}/a)^2 J_m^2(j_{m,p}\rho/a)}{k^2 J_{m-1}^2(j_{m,p})}\cos^2(m\varphi)\cos^2(n\pi z/h_c)/\cos^2(n\pi z_0/h_c)$$

We see from this that the field saturates as a function the cavity quality factor for

$$\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)Q_0^{p,m,n}/Q_{p,m,n}^{TM} << \frac{1/Q_{slt}}{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)}$$

or

$$Q_{p,m,n}^{TM} >> Q_0^{p,m,n}Q_{slt}^2\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2$$

If we define

$$1/Q_{eff}^{TM} = \left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)/Q_{p,m,n}^{TM}+\frac{1/\left(Q_0^{p,m,n}Q_{slt}\right)}{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)}$$

we can then write this as

$$\left|\frac{E_z}{\eta_0 H_\varphi^{sc}(b,0,z_0)}\right|^2 \approx \varepsilon_m^2\varepsilon_n^2\left(Q_{eff}^{TM}\alpha_{m,ss}^0/V_{cav}\right)^2$$

$$\frac{(j_{m,p}/a)^2 J_m^2(j_{m,p}\rho/a)}{k^2 J_{m-1}^2(j_{m,p})}\cos^2(m\varphi)\cos^2(n\pi z/h_c)/\cos^2(n\pi z_0/h_c)$$

The radial field is

$$E_\rho \approx -\frac{i\omega\mu_0}{2V_{cav}}\varepsilon_m\varepsilon_n\left(\frac{n\pi}{h_c}\right)\cos(n\pi z_0/h_c)\sin(n\pi z/h_c)\cos(m\varphi)\,m_s^+$$

$$\left[\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}\frac{J_m'(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\right.$$

$$\left.+\frac{m^2/\left(k_{p,m,n}'^2 a\rho\right)}{\left\{k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n^2\pi^2/h_c^2}{j_{m,p}'^2/a^2}-1\right)\right]$$

or

$$E_\rho \approx i\omega\mu_0\varepsilon_m\varepsilon_n H_s^{sc}\left(\alpha_{m,ss}^{+}/V_{cav}\right)(n\pi/h_c)\cos(m\varphi)\cos(n\pi z_0/h_c)\sin(n\pi z/h_c)$$

$$\left[\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}\frac{J_m'(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\right.$$

$$\left.+\frac{m^2/\left(k_{p,m,n}'^2 a\rho\right)}{\left\{k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n^2\pi^2/h_c^2}{j_{m,p}'^2/a^2}-1\right)\right]$$

The magnitude squared is then

$$|E_\rho|^2 \approx \omega^2\mu_0^2\varepsilon_m^2\varepsilon_n^2\left|H_s^{sc}(b,0,z_0)\right|^2\left|\alpha_{m,ss}^{+}/V_{cav}\right|^2(n\pi/h_c)^2\cos^2(m\varphi)\cos^2(n\pi z_0/h_c)\sin^2(n\pi z/h_c)$$

$$\left|\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}\frac{J_m'(j_{m,p}\rho/a)}{J_{m-1}(j_{m,p})}\right.$$

$$\left.+\frac{m^2/\left(k_{p,m,n}'^2 a\rho\right)}{\left\{k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}\left(1-m^2/j_{m,p}'^2\right)}\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\left(\frac{n^2\pi^2/h_c^2}{j_{m,p}'^2/a^2}-1\right)\right|^2$$

Near the TE resonance

$$\left|\frac{E_\rho}{\eta_0 H_s^{sc}(b,0,z_0)}\right|^2 \approx \varepsilon_m^2\varepsilon_n^2\cos^2(m\varphi)\cos^2(n\pi z_0/h_c)\sin^2(n\pi z/h_c)\left\{\frac{J_m\left(j_{m,p}'\rho/a\right)}{J_m\left(j_{m,p}'\right)}\right\}^2$$

$$\frac{\left|\alpha_{m,ss}^{+}/V_{cav}\right|^2(n\pi/h_c)^2 m^4k^2/\left\{\left(k_{p,m,n}'^2 a\rho\right)\left(1-m^2/j_{m,p}'^2\right)\right\}^2}{\left|k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right|^2}\left(\frac{n^2\pi^2/h_c^2}{j_{m,p}'^2/a^2}-1\right)^2$$

where

$$\left\{k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}\alpha_{m,ss}^{0}/\alpha_{m,ss}^{+} \approx \left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)-i/Q_{slt}\right\}$$

$$\left\{k^2+kk_{p,m,n}'(1+i)/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}+\left(k^2/Q_0^{p,m,n}\right)\frac{(n\pi/h_c)^2\left(m^2/j_{m,p}'^2\right)/\left(1-m^2/j_{m,p}'^2\right)}{k_{p,m,n}'^2}$$

Setting the real part to zero

$$\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)\left\{k^2+kk_{p,m,n}'/Q_{p,m,n}^{TE}-k_{p,m,n}'^2\right\}$$

$$+kk_{p,m,n}'/\left(Q_{p,m,n}^{TE}Q_{slt}\right)+\left(k^2/Q_0^{p,m,n}\right)\frac{(n\pi/h_c)^2\left(m^2/j_{m,p}'^2\right)/\left(1-m^2/j_{m,p}'^2\right)}{k_{p,m,n}'^2}=0$$

Then

$$\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TE}-k'^2_{p,m,n}\right\}\alpha_{m,ss}^0/\alpha_{m,ss}^+\approx i\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2kk'_{p,m,n}/Q_{p,m,n}^{TE}$$

$$-i\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)\left\{k^2+kk'_{p,m,n}/Q_{p,m,n}^{TE}-k'^2_{p,m,n}\right\}/Q_{slt}$$

$$\approx i\left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2+1/Q_{slt}^2\right\}kk'_{p,m,n}/Q_{p,m,n}^{TE}+i\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}}k^2/\left(Q_0^{p,m,n}Q_{slt}\right)$$

and

$$\left|\frac{E_\rho}{\eta_0H_s^{sc}\left(b,0,z_0\right)}\right|^2\approx\cos^2\left(m\varphi\right)\left\{\sin^2\left(n\pi z/h_c\right)/\cos^2\left(n\pi z_0/h_c\right)\right\}\left(j'_{m,p}/a\right)^2\left\{\frac{J_m\left(j'_{m,p}\rho/a\right)}{J_m\left(j'_{m,p}\right)}\right\}^2\left(\frac{n^2\pi^2/h_c^2}{j'^2_{m,p}/a^2}-1\right)^2\left(m/\rho\right)^2$$

$$\frac{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2\left(1/Q_0^{p,m,n}\right)^2\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)k^2/\left\{\left(k'^2_{p,m,n}\right)\left(1-m^2/j'^2_{m,p}\right)\right\}^2}{\left[\left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2+1/Q_{slt}^2\right\}kk'_{p,m,n}/Q_{p,m,n}^{TE}+\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}}k^2/\left(Q_0^{p,m,n}Q_{slt}\right)\right]^2}$$

Dropping $1/Q_{slt}^2$

$$\left|\frac{E_\rho}{\eta_0H_s^{sc}\left(b,0,z_0\right)}\right|^2\approx\cos^2\left(m\varphi\right)\left\{\sin^2\left(n\pi z/h_c\right)/\cos^2\left(n\pi z_0/h_c\right)\right\}\left(j'_{m,p}/a\right)^2\left\{\frac{J_m\left(j'_{m,p}\rho/a\right)}{J_m\left(j'_{m,p}\right)}\right\}^2\left(\frac{n^2\pi^2/h_c^2}{j'^2_{m,p}/a^2}-1\right)^2\left(m/\rho\right)^2$$

$$\frac{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2\left(1/Q_0^{p,m,n}\right)^2\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)k^2/\left\{\left(k'^2_{p,m,n}\right)\left(1-m^2/j'^2_{m,p}\right)\right\}^2}{\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2kk'_{p,m,n}/Q_{p,m,n}^{TE}+\frac{\left(n\pi/h_c\right)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}}k^2/\left(Q_0^{p,m,n}Q_{slt}\right)\right]^2}$$

11.6 Power Transmission

The power transmission through the slot is

$$P_{trans}=-\operatorname{Re}\int_{-h}^{h}V\left(s\right)H_\varphi^*\left(a,\varphi,z_0\right)ds=-\operatorname{Re}\int_{-h}^{h}V^*\left(s\right)H_\varphi\left(a,\varphi,z_0\right)ds$$

$$V\left(s\right)\sim V_0k^2\left(h^2-s^2\right)/2\ ,\ kh<<\pi/2$$

$$H_\varphi^{(p,m,n)}\left(a,\varphi=s/a<<1,z_0\right)=H_0^{(p,m,n)}$$

$$=-k^2\frac{1}{2}\left(\varepsilon_m\varepsilon_n/V_{cav}\right)\cos^2\left(n\pi z_0/h_c\right)m_s^+\left[\frac{1}{\left\{k^2+kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM}-k_{p,m,n}^2\right\}}\right.$$

$$+\frac{(n\pi/h_c)^2 m^2/j'^2_{m,p}}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}\left(1-m^2/j'^2_{m,p}\right)}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\Bigg]$$

$$P_{trans}\approx-\operatorname{Re}\int_{-h}^{h}V^*(s)\,H_\varphi^{(p,m,n)}(a,\varphi=s/a<<1,z_0)\,ds$$

$$\approx k^2\frac{1}{4}(\varepsilon_m\varepsilon_n/V_{cav})\cos^2(n\pi z_0/h_c)\,k^2\int_{-h}^{h}\left(h^2-s^2\right)ds\,\operatorname{Re}\left[\frac{V_0^*m_s^+}{\left\{k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}}\right.$$

$$\left.+\frac{V_0^*m_s^+(n\pi/h_c)^2 m^2/j'^2_{m,p}}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}\left(1-m^2/j'^2_{m,p}\right)}+\frac{V_0^*m_s^+}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\right]$$

$$m_s^+=-2\alpha^+_{m,ss}H_s^{sc}=\frac{i2}{\omega\mu_0}\int_{-h}^{h}V_+(s')\,ds'\sim i\omega\varepsilon_0V_0\frac{4}{3}h^3=i\,(V_0/\eta_0)\,\frac{4}{3}kh^3$$

$$i2\alpha^+_{m,ss}H_s^{sc}\approx(V_0/\eta_0)\,\frac{4}{3}kh^3$$

$$P_{trans}\approx\frac{1}{4}(\varepsilon_m\varepsilon_n/V_{cav})\cos^2(n\pi z_0/h_c)\,\frac{k}{\eta_0}\left(\frac{4}{3}\right)^2|V_0|^2k^4h^6\left[\frac{kk_{p,m,n}/Q^{TM}_{p,m,n}}{\left|k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right|^2}\right.$$

$$\left.+\frac{(n\pi/h_c)^2\left(m^2/j'^2_{m,p}\right)kk'_{p,m,n}/Q^{TE}_{p,m,n}}{k'^2_{p,m,n}\left|k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right|^2\left(1-m^2/j'^2_{m,p}\right)}\right]$$

$$\alpha^0_{m,ss}/\alpha^+_{m,ss}\approx\left(1-\frac{2}{5}k^2h^2\right)-i\omega L\left(G^{int}-iB^{int}\right)-i\omega L\left(G_{rad}/2-iB_{rad}\right)\frac{2}{5}h^2$$

$$+\varepsilon_m\varepsilon_n\cos^2(n\pi z_0/h_c)\left(\alpha^0_{m,ss}/V_{cav}\right)k^2\left[\frac{1}{\left\{k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right\}}\right.$$

$$\left.+\frac{(n\pi/h_c)^2\left(m^2/j'^2_{m,p}\right)/\left(1-m^2/j'^2_{m,p}\right)}{k'^2_{p,m,n}\left\{k^2+kk'_{p,m,n}(1+i)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}}+\frac{1}{k'^2_{p,m,n}\left(1-m^2/j'^2_{m,p}\right)}\right]$$

We may want to split up the results for near TM resonances and then near TE resonances.

11.6.1 TM Resonances

For the TM resonances

$$\frac{P_{trans}}{\eta_0|H_s^{sc}|^2}\approx\frac{1}{4}(\varepsilon_m\varepsilon_n/V_{cav})\cos^2(n\pi z_0/h_c)\,k\left(\frac{4}{3}\right)^2\frac{|V_0|^2}{\eta_0^2|H_s^{sc}|^2}k^4h^6\frac{kk_{p,m,n}/Q^{TM}_{p,m,n}}{\left|k^2+kk_{p,m,n}(1+i)/Q^{TM}_{p,m,n}-k^2_{p,m,n}\right|^2}$$

$$\left|\alpha^+_{m,ss}\right|^2\approx\frac{|V_0|^2}{\eta_0^2|H_s^{sc}|^2}\frac{4}{9}k^2h^6$$

$$\frac{P_{trans}}{\eta_0 \left|H_s^{sc}\right|^2} \approx \alpha_{m,ss}^0 \left(\varepsilon_m \varepsilon_n \alpha_{m,ss}^0 / V_{cav}\right) \cos^2\left(n\pi z_0/h_c\right) k^4 \frac{\left|\alpha_{m,ss}^+/\alpha_{m,ss}^0\right|^2}{\left|k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right|^2} k_{p,m,n}/Q_{p,m,n}^{TM}$$

$$\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\} \alpha_{m,ss}^0/\alpha_{m,ss}^+$$

$$\approx \left[\left(1 - \frac{2}{5}k^2h^2\right) - i\omega L\left(G^{int} - iB^{int}\right) - i\omega L\left(G_{rad}/2 - iB_{rad}\right)\frac{2}{5}h^2\right]\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}$$

$$+\varepsilon_m \varepsilon_n \cos^2\left(n\pi z_0/h_c\right)\left(\alpha_{m,ss}^0/V_{cav}\right) k^2$$

Letting

$$1/Q_0^{p,m,n} = \varepsilon_m \varepsilon_n \cos^2\left(n\pi z_0/h_c\right)\alpha_{m,ss}^0/V_{cav}$$

$$\Delta k h_{slt}^2 = \omega L\left(B^{int} + \frac{2}{5}h^2 B_{rad}\right)$$

$$1/Q_{slt} = \omega L\left(G^{int} + \frac{2}{5}h^2 G_{rad}/2\right)$$

$$\frac{P_{trans}}{\eta_0 \left|H_s^{sc}\right|^2} \approx \frac{\alpha_{m,ss}^0 k^4 k_{p,m,n}/\left(Q_0^{p,m,n} Q_{p,m,n}^{TM}\right)}{\left|k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right|^2 \left|\alpha_{m,ss}^0/\alpha_{m,ss}^+\right|^2}$$

$$\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\} \alpha_{m,ss}^0/\alpha_{m,ss}^+$$

$$\approx \left[\left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right) - i/Q_{slt}\right]\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\} + k^2/Q_0^{p,m,n}$$

Setting the real part of the right hand side to zero

$$\left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right)\left\{k^2 + kk_{p,m,n}/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\} + kk_{p,m,n}/\left(Q_{slt}Q_{p,m,n}^{TM}\right) + k^2/Q_0^{p,m,n} = 0$$

or

$$\left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right)\left\{k^2 + kk_{p,m,n}\left(1+i\right)/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\} \alpha_{m,ss}^0/\alpha_{m,ss}^+$$

$$\approx \left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right)^2 ikk_{p,m,n}/Q_{p,m,n}^{TM} - i\left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right)\left\{k^2 + kk_{p,m,n}/Q_{p,m,n}^{TM} - k_{p,m,n}^2\right\}/Q_{slt}$$

$$\approx \left(1 - \frac{2}{5}k^2h^2 - \Delta k h_{slt}^2\right)^2 ikk_{p,m,n}/Q_{p,m,n}^{TM} + i\left\{kk_{p,m,n}/\left(Q_{slt}Q_{p,m,n}^{TM}\right) + k^2/Q_0^{p,m,n}\right\}/Q_{slt}$$

$$\approx i\left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2+1/Q_{slt}^2\right\}kk_{p,m,n}/Q_{p,m,n}^{TM}+ik^2/\left(Q_0^{p,m,n}Q_{slt}\right)$$

Then

$$\frac{P_{trans}}{\eta_0\left|H_s^{sc}\right|^2}\approx\frac{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2\alpha_{m,ss}^0k_{p,m,n}/\left(Q_0^{p,m,n}Q_{p,m,n}^{TM}\right)}{\left[\left\{\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2+1/Q_{slt}^2\right\}\left(k_{p,m,n}/k\right)/Q_{p,m,n}^{TM}+1/\left(Q_0^{p,m,n}Q_{slt}\right)\right]^2}$$

If we drop the $1/Q_{slt}^2$ term in the denominator (assuming we are de-tuned off of the slot resonance) we can write this as

$$\frac{P_{trans}}{\eta_0\left|H_s^{sc}\right|^2/4}\approx q_{p,m,n}^{TM}\alpha_{m,ss}^0kQ_{slt}$$

where we denote the mismatch factor by

$$q_{p,m,n}^{TM}=\frac{4\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2\left(k_{p,m,n}/k\right)/\left(Q_{slt}Q_0^{p,m,n}Q_{p,m,n}^{TM}\right)}{\left[\left(1-\frac{2}{5}k^2h^2-\Delta kh_{slt}^2\right)^2\left(k_{p,m,n}/k\right)/Q_{p,m,n}^{TM}+1/\left(Q_0^{p,m,n}Q_{slt}\right)\right]^2}$$

$$Q_{p,m,n}^{TM}=\frac{k_{p,m,n}a}{\varepsilon_na/h_c+1}\frac{\eta_0}{2R_s^{(p,m,n)}}$$

$$\left|\frac{E_z}{\eta_0H_s^{sc}/2}\right|^2\approx\left(q_{p,m,n}^{TM}\alpha_{m,ss}^0kQ_{slt}\right)\frac{Q_{p,m,n}^{TM}}{k_{p,m,n}V_{cav}}\frac{\left(j_{m,p}/a\right)^2}{k^2}\frac{J_m^2\left(j_{m,p}\rho/a\right)}{J_{m-1}^2\left(j_{m,p}\right)}\varepsilon_m\varepsilon_n\cos^2\left(m\varphi\right)\cos^2\left(n\pi z/h_c\right)$$

Note that

$$\frac{1}{V_{cav}}\int_0^{h_c}\cos^2\left(n\pi z/h_c\right)dz\int_{-\pi}^{\pi}\cos^2\left(m\varphi\right)d\varphi\int_0^a\frac{J_m^2\left(j_{m,p}\rho/a\right)}{J_{m-1}^2\left(j_{m,p}\right)}\rho d\rho$$

$$=\frac{1}{V_{cav}}\frac{2\pi a^2h_c}{\varepsilon_m\varepsilon_n}\frac{1}{J_{m-1}^2\left(j_{m,p}\right)}\int_0^1J_m^2\left(j_{m,p}u\right)udu=\frac{1}{V_{cav}}\frac{2\pi a^2h_c}{\varepsilon_m\varepsilon_n}\frac{\frac{1}{2}J_m'^2\left(j_{m,p}\right)}{J_{m-1}^2\left(j_{m,p}\right)}=\frac{1}{V_{cav}}\frac{\pi a^2h_c}{\varepsilon_m\varepsilon_n}=\frac{1}{\varepsilon_m\varepsilon_n}$$

The enhancement above the mean square is thus

$$\frac{\left|E_z\right|^2}{\left\langle\left|E_z\right|^2\right\rangle_V}=\varepsilon_m\cos^2\left(m\varphi\right)\varepsilon_n\cos^2\left(n\pi z/h_c\right)\frac{J_m^2\left(j_{m,p}\rho/a\right)}{J_{m-1}^2\left(j_{m,p}\right)}$$

Noting for $m=0$ that the final ratio at $\rho=0$ is

$$\frac{J_0^2\left(j_{0,p}0\right)}{J_1^2\left(j_{0,p}\right)}=\frac{1}{J_1^2\left(j_{0,p}\right)}\approx\frac{1}{0.5191474973^2},\frac{1}{0.3402648065^2},\frac{1}{0.2714522999^2},\frac{1}{0.2324598314^2},...$$

$$\approx 3.7104,8.63706,13.71034,18.506,...$$

$$\frac{J_0^2\left(j_{0,2}'\right)}{J_1^2\left(j_{0,p}\right)}\approx\frac{0.40276^2}{0.51915^2},\frac{0.40276^2}{0.34026^2},\frac{0.40276^2}{0.27145^2},\frac{0.40276^2}{0.23246^2},...$$

$$\approx 0.60188, 1.4011, 2.2015, 3.0019, ...$$

we see that there are large enhancements at the center. Furthermore, for other values of m

$$\frac{d}{d\rho} J_m\left(j_{m,p}\rho/a\right) = \left(j_{m,p}/a\right) J_m'\left(j_{m,p}\rho/a\right) = 0 \to \rho/a = j_{m,p'}'/j_{m,p}$$

with maxima

$$\frac{J_m^2\left(j_{m,p'}'\right)}{J_{m-1}^2\left(j_{m,p}\right)} = \frac{J_m^2\left(j_{m,p'}'\right)}{J_m'^2\left(j_{m,p}\right)}$$

$$\frac{J_m^2\left(j_{m,p'}'\right)}{J_m'^2\left(j_{m,p}\right)} \sim \left[-\frac{m^{1/3}\mathrm{Ai}\left(a_p'\right) z\left(\zeta'\right) h\left(\zeta\right) h\left(\zeta'\right)}{2\mathrm{Ai}'\left(a_p\right)}\right]^2 = \sqrt{a_p a_p'}\left\{\frac{\mathrm{Ai}\left(a_p'\right)}{\mathrm{Ai}'\left(a_p\right)}\right\}^2 \frac{z^2\left(\zeta'\right)}{\sqrt{\left(1-z^2\left(\zeta\right)\right)\left(1-z^2\left(\zeta'\right)\right)}}$$

$$\frac{2}{3}\left(-\zeta\right)^{3/2} = \sqrt{z^2-1} - \arccos\left(1/z\right)$$

$$\mathrm{Ai}\left(a_p\right) = 0 = \mathrm{Ai}'\left(a_p'\right)$$

$$a_1 \approx -2.33810741$$

$$a_1' \approx -1.01879297$$

$$\mathrm{Ai}\left(a_1'\right) \approx 0.53565666$$

$$\mathrm{Ai}'\left(a_1\right) \approx 0.70121082$$

$$\zeta = a_p/m^{2/3}$$

$$\zeta' = a_p'/m^{2/3}$$

$$\frac{2}{3}\left(-a_1/m^{2/3}\right)^{3/2} = \sqrt{\left(z-1\right)\left(z-1+2\right)} - \arccos\left(\frac{1}{z-1+1}\right)$$

$$\frac{2}{3}\left(-a_1'/m^{2/3}\right)^{3/2} = \sqrt{\left(z-1\right)\left(z-1+2\right)} - \arccos\left(\frac{1}{z-1+1}\right)$$

$$\arccos\left(x\right) = \frac{\pi}{2} - \arcsin\left(x\right)$$

$$\arcsin\left(1-x\right) = \frac{\pi}{2} - \sqrt{2x}\left[1 + x/12 + ...\right]$$

$$\arccos\left(1-x\right) = \frac{\pi}{2} - \arcsin\left(1-x\right) = \sqrt{2\left(1-x\right)}\left[1 + \left(1-x\right)/12 + ...\right]$$

$$\arccos\left(\frac{1}{z-1+1}\right)\sim$$

For $m=1$ at $\rho = aj'_{1,1}/j_{1,p} \approx 0.4805a, 0.26244a, 0.18098a, 0.13819a, ...$ the final factor is

$$\frac{J_1^2\left(j'_{1,1}\right)}{J_1'^2\left(j_{1,p}\right)} \approx \frac{0.58187^2}{0.40276^2}, \frac{0.58187^2}{0.30012^2}, \frac{0.58187^2}{0.24970^2}, \frac{0.58187^2}{0.21836^2}, ...$$

$$\approx 2.08718, 3.75891, 5.4302, 7.100775, ...$$

$$\frac{J_1^2\left(j'_{1,2}\right)}{J_1'^2\left(j_{1,p}\right)} \approx \frac{0.34613^2}{0.40276^2}, \frac{0.34613^2}{0.30012^2}, \frac{0.34613^2}{0.24970^2}, \frac{0.34613^2}{0.21836^2}, ...$$

$$\approx 0.73856, 1.33011, 1.9215, 2.51265, ...$$

For $m=2$ at $\rho = aj'_{2,1}/j_{2,p} \approx 0.59472a, 0.362855a, 0.262847a, 0.206424a, ...$ the final factor is

$$\frac{J_2^2\left(j'_{2,1}\right)}{J_2'^2\left(j_{2,p}\right)} \approx \frac{0.48650^2}{0.33967^2}, \frac{0.48650^2}{0.27138^2}, \frac{0.48650^2}{0.23244^2}, \frac{0.48650^2}{0.20654^2}, ...$$

$$\approx 2.00514, 3.21373, 4.38070, 5.54827, ...$$

$$\frac{J_2^2\left(j'_{2,2}\right)}{J_2'^2\left(j_{2,p}\right)} \approx \frac{0.31353^2}{0.33967^2}, \frac{0.31353^2}{0.27138^2}, \frac{0.31353^2}{0.23244^2}, \frac{0.31353^2}{0.20654^2}, ...$$

$$\approx 0.85201, 1.33476, 1.81943, 2.3044, ...$$

again indicating large enhancements near the center for higher-order modes. However, cavity losses, inducing modal overlap at higher frequencies, tend to mitigate this enhancement at the higher orders because the power is distributed among many modes, most having null behaviors near the center. This type of increase near the cylinder center is consistent with general interior enhancements in axisymmetric geometry.

11.6.2 TE Resonances

For the TE resonances

$$\frac{P_{trans}}{\eta_0\left|H_s^{sc}\right|^2} \approx \left(\varepsilon_m\varepsilon_n\alpha^0_{m,ss}/V_{cav}\right)\cos^2\left(n\pi z_0/h_c\right)k^3\left(\frac{4}{9}\right)\frac{\left|V_0\right|^2}{\eta_0^2\left|H_s^{sc}\right|^2}k^2h^6$$
$$\frac{\left(n\pi/h_c\right)^2 kk'_{p,m,n}/Q^{TE}_{p,m,n}}{k'^2_{p,m,n}\left|k^2+kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right|^2\alpha^0_{m,ss}}\frac{\left(m^2/j'^2_{m,p}\right)}{\left(1-m^2/j'^2_{m,p}\right)}$$

where

$$\left|\alpha^+_{m,ss}\right|^2 \approx \frac{\left|V_0\right|^2}{\eta_0^2\left|H_s^{sc}\right|^2}\frac{4}{9}k^2h^6$$

and

$$\left\{k^2+kk'_{p,m,n}\left(1+i\right)/Q^{TE}_{p,m,n}-k'^2_{p,m,n}\right\}\alpha^0_{m,ss}/\alpha^+_{m,ss}$$

$$\approx \left\{ \left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right) - i/Q_{slt} \right\} \left\{ k^2 + k k'_{p,m,n} \left(1 + i \right) / Q_{p,m,n}^{TE} - k'^2_{p,m,n} \right\} + \left(k^2 / Q_0^{p,m,n} \right) \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)}$$

Setting the real part of the right hand side to zero

$$\left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right) \left\{ k^2 + k k'_{p,m,n} / Q_{p,m,n}^{TE} - k'^2_{p,m,n} \right\} + \left(k^2 / Q_0^{p,m,n} \right) \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)} + k k'_{p,m,n} / \left(Q_{p,m,n}^{TE} Q_{slt} \right) = 0$$

then

$$\left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right) \left\{ k^2 + k k'_{p,m,n} \left(1 + i \right) / Q_{p,m,n}^{TE} - k'^2_{p,m,n} \right\} \alpha_{m,ss}^0 / \alpha_{m,ss}^+$$

$$\approx i \left\{ \left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 + 1/Q_{slt}^2 \right\} k k'_{p,m,n} / Q_{p,m,n}^{TE} + i \left\{ \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)} \right\} k^2 / \left(Q_0^{p,m,n} Q_{slt} \right)$$

Then

$$\frac{P_{trans}}{\eta_0 \left| H_s^{sc} \right|^2}$$

$$\approx \frac{\alpha_{m,ss}^0 \left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 \left(n\pi/h_c \right)^2 \left(1/k'_{p,m,n} \right) / \left(Q_{p,m,n}^{TE} Q_0^{p,m,n} \right)}{\left[\left\{ \left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 + 1/Q_{slt}^2 \right\} \left(k'_{p,m,n}/k \right) / Q_{p,m,n}^{TE} + \left\{ \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)} \right\} / \left(Q_0^{p,m,n} Q_{slt} \right) \right]^2} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)}$$

If we drop the $1/Q_{slt}^2$ term in the denominator (assuming we are de-tuned off of the slot resonance) we can write this as

$$\frac{P_{trans}}{\eta_0 \left| H_s^{sc} \right|^2} \approx \frac{\alpha_{m,ss}^0 \left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 \left(n\pi/h_c \right)^2 \left(k/k'^2_{p,m,n} \right) / \left(Q_{p,m,n}^{TE} Q_0^{p,m,n} \right)}{\left[\left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 \left(k'_{p,m,n}/k \right) / Q_{p,m,n}^{TE} + \left\{ \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)} \right\} / \left(Q_0^{p,m,n} Q_{slt} \right) \right]^2} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)}$$

or

$$\frac{P_{trans}}{\eta_0 \left| H_s^{sc} \right|^2 / 4} \approx q_{p,m,n}^{TE} \alpha_{m,ss}^0 k Q_{slt}$$

where the TE mismatch factor is

$$q_{p,m,n}^{TE} = \frac{4 \left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 \left(k'_{p,m,n}/k \right) \left\{ \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)} \right\} / \left(Q_{p,m,n}^{TE} Q_0^{p,m,n} Q_{slt} \right)}{\left[\left(1 - \frac{2}{5} k^2 h^2 - \Delta k h_{slt}^2 \right)^2 \left(k'_{p,m,n}/k \right) / Q_{p,m,n}^{TE} + \left\{ \frac{\left(n\pi/h_c \right)^2}{k'^2_{p,m,n}} \frac{\left(m^2 / j'^2_{m,p} \right)}{\left(1 - m^2 / j'^2_{m,p} \right)} \right\} / \left(Q_0^{p,m,n} Q_{slt} \right) \right]^2}$$

$$Q_{p,m,n}^{TE} = \frac{\left(1 - m^2/j_{m,p}'^2\right)}{j_{m,p}'^2/a^2 + 2\left(a/h_c\right)\left(n\pi/h_c\right)^2\left(1 - m^2/j_{m,p}'^2\right) + \left(n\pi/h_c\right)^2 m^2/j_{m,p}'^2} k_{p,m,n}'^3 a \frac{\eta_0}{2R_s^{(p,m,n)}}$$

$$= \frac{k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}{k_{p,m,n}'^2 + \left(2a/h_c - 1\right)\left(n\pi/h_c\right)^2\left(1 - m^2/j_{m,p}'^2\right)} k_{p,m,n}' a \frac{\eta_0}{2R_s^{(p,m,n)}}$$

$$1/Q_{p,m,n}^{TE} = \left[\frac{1}{\left(1 - m^2/j_{m,p}'^2\right)} + \frac{\left(2a/h_c - 1\right)\left(n\pi/h_c\right)^2}{k_{p,m,n}'^2}\right]\left(k_{p,m,n}' a \frac{\eta_0}{2R_s^{(p,m,n)}}\right)^{-1}$$

The field is

$$\left|\frac{E_\rho}{\eta_0 H_s^{sc}/2}\right|^2 \approx \left(q_{p,m,n}^{TE}\alpha_{m,ss}^0 k Q_{slt}\right)\frac{Q_{p,m,n}^{TE}}{k_{p,m,n}' V_{cav}}\varepsilon_m\varepsilon_n\cos^2\left(m\varphi\right)\sin^2\left(n\pi z/h_c\right)$$

$$\left\{\frac{J_m\left(j_{m,p}'\rho/a\right)}{\left(j_{m,p}'\rho/a\right)J_m\left(j_{m,p}'\right)}\right\}^2\left(n^2\pi^2/h_c^2 - j_{m,p}'^2/a^2\right)^2\frac{m^2}{k^2 k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}$$

or

$$\left|\frac{E_\rho}{\eta_0 H_s^{sc}/2}\right|^2 \approx \left(q_{p,m,n}^{TE}\alpha_{m,ss}^0 k Q_{slt}\right)\frac{Q_{p,m,n}^{TE}}{k_{p,m,n}' V_{cav}}\varepsilon_m\varepsilon_n\cos^2\left(m\varphi\right)\sin^2\left(n\pi z/h_c\right)$$

$$\left\{\frac{J_m\left(j_{m,p}'\rho/a\right)}{\left(j_{m,p}'\rho/a\right)J_m\left(j_{m,p}'\right)}\right\}^2\left(k_{p,m,n}'^2 - 2j_{m,p}'^2/a^2\right)^2\frac{m^2}{k^2 k_{p,m,n}'^2\left(1 - m^2/j_{m,p}'^2\right)}$$

Note that

$$\frac{1}{V_{cav}}\int_0^{h_c}\sin^2\left(n\pi z/h_c\right)dz\int_{-\pi}^{\pi}\cos^2\left(m\varphi\right)d\varphi\int_0^a\left\{\frac{J_m\left(j_{m,p}'\rho/a\right)}{\left(j_{m,p}'\rho/a\right)J_m\left(j_{m,p}'\right)}\right\}^2\rho d\rho$$

$$= \frac{\pi a^2 h_c/2}{V_{cav}}\int_0^1\left\{\frac{J_m\left(j_{m,p}'u\right)}{j_{m,p}'J_m\left(j_{m,p}'\right)}\right\}^2\frac{du}{u} = \frac{\pi a^2 h_c/2}{V_{cav}}\frac{1}{2m j_{m,p}'^2 J_m^2\left(j_{m,p}'\right)}\left[1 - \sum_{m'=0}^{m-1}\varepsilon_{m'}J_{m'}^2\left(j_{m,p}'\right) - J_m^2\left(j_{m,p}'\right)\right]$$

where we used [16]

$$\int J_m^2\left(z\right)\frac{dz}{z} = -\frac{1}{2m}\left[J_0^2\left(z\right) + \sum_{m'=1}^{m-1}2J_{m'}^2\left(z\right) + J_m^2\left(z\right)\right]$$

or

$$\int_0^{j_{m,p}'}J_m^2\left(z\right)\frac{dz}{z} = \int_0^1 J_m^2\left(j_{m,p}'u\right)\frac{du}{u} = \frac{1}{2m}\left[1 - \sum_{m'=1}^{m-1}\varepsilon_{m'}J_{m'}^2\left(j_{m,p}'\right) - J_m^2\left(j_{m,p}'\right)\right]$$

This page left blank

12 CONCLUSIONS

The interior electric field is determined inside a resonant finite length cylindrical cavity with finitely conducting walls driven by a narrow azimuthal slot. The exterior is treated rigorously as an infinite cylinder or as a planar half space (the short circuit current density driving the slot is from the infinite cylinder in both cases). The model is set up rigorously, by extracting the slot transmission line operator from the integro-differential equation, allowing slot depth, metal losses, and gaskets to be included. The model includes modifications of the slot voltage distribution from the cavity mode resonances. A finite Fourier basis is used for the slot voltage distribution. The results with the cylinder exterior versus the planar exterior are not significantly different.

Simplified analytic models are examined for the electrically short azimuthal slot aperture, where the inductive slot voltage is not significantly modified . Coupling to both cylindrical modes, TM and TE with respect to the cylinder axis, are treated. The exterior drive is again taken as the current density on an infinite cylinder.

The case where the slot is near the first resonance is also considered. The exterior radiation is first approximated as that of a slot on a half space. Later, the half space radiation is compared to the radiation on an infinite cylinder and shown to somewhat underestimate the cylinder radiation, and this comparison is shown to be consistent with the previous field comparison.

Finally, a power balance bounding approach is considered. We set up bounding receiving cross sections for the slot aperture, based on uniformly distributed matched loads and balance the net power in with the cavity wall absorption to estimate the cavity fields. Both a normal incidence bound [1] and oblique incidence bound are constructed. We also consider an incoherent form of the power balance, where the power transmitted into the cavity is balanced with the power absorbed by the cavity walls plus the power lost due to the slot.

This page left blank

13 APPENDIX - HALF SPACE RADIATION

This appendix carries out integrals for the Y_{rad} estimates for the even voltage distribution resulting from a constant current drive and for the odd voltage distribution resulting from a linear current drive.

13.1 Even Voltage Distribution

For the even voltage distribution we need to evaluate

$$Y_{rad}\int_{-h}^{h}V_0^*\left(s\right)V_0\left(s\right)ds=-\frac{1}{\pi}\frac{i}{\omega\mu_0}\int_{-h}^{h}V_0^*\left(s\right)\left(\frac{\partial^2}{\partial s^2}+k^2\right)F_0\left(s\right)ds$$

$$F_0\left(s\right)=V_0\left(s\right)\left\{-C_e^e+\ln\left(1-s^2/h^2\right)\right\}+\int_{-h}^{h}\frac{V_0\left(s'\right)e^{ik\left|s-s'\right|}-V_0\left(s\right)}{\left|s-s'\right|}ds'$$

Integration by parts gives

$$\int_{-h}^{h}V_0^*\left(s\right)\left(\frac{\partial^2}{\partial s^2}+k^2\right)F_0\left(s\right)ds=\left[V_0^*\left(s\right)\frac{\partial}{\partial s}F_0\left(s\right)\right]_{-h}^{h}+\int_{-h}^{h}\left[-\frac{\partial}{\partial s}V_0^*\left(s\right)\frac{\partial}{\partial s}+k^2V_0^*\left(s\right)\right]F_0\left(s\right)ds$$

$$=\left[V_0^*\left(s\right)\frac{\partial}{\partial s}F_0\left(s\right)-\frac{\partial}{\partial s}V_0^*\left(s\right)F_0\left(s\right)\right]_{-h}^{h}+\int_{-h}^{h}\left[\frac{\partial^2}{\partial s^2}V_0^*\left(s\right)+k^2V_0^*\left(s\right)\right]F_0\left(s\right)ds$$

$$\left[V^*\left(s\right)\frac{\partial}{\partial s}F_0\left(s\right)-\frac{\partial}{\partial s}V^*\left(s\right)F_0\left(s\right)\right]_{-h}^{h}=\left[-\frac{\partial}{\partial s}V^*\left(s\right)F_0\left(s\right)\right]_{-h}^{h}=2V_0^*k\sin\left(kh\right)F_0\left(h\right)$$

so we find

$$Y_{rad}^e\left|V_0\right|^2h\left\{1+2\cos^2\left(kh\right)-3\frac{\sin\left(kh\right)}{kh}\cos\left(kh\right)\right\}=-\frac{1}{\pi}\frac{i}{\omega\mu_0}2V_0^*k\sin\left(kh\right)F_0\left(h\right)+\frac{1}{\pi}i\omega\varepsilon_0V_0^*\cos\left(kh\right)\int_{-h}^{h}F_0\left(s\right)ds$$

Carrying out the inner integral

$$\int_{-h}^{h}\frac{V_0\left(s'\right)e^{ik\left|s-s'\right|}-V_0\left(s\right)}{\left|s-s'\right|}ds'$$

$$=\int_{-h}^{s}\frac{V_0\left\{\cos\left(ks'\right)-\cos\left(kh\right)\right\}\left[\cos\left(k\left(s-s'\right)\right)+i\sin\left(k\left(s-s'\right)\right)\right]-V_0\left\{\cos\left(ks\right)-\cos\left(kh\right)\right\}}{s-s'}ds'$$

$$+\int_{s}^{h}\frac{V_0\left\{\cos\left(ks'\right)-\cos\left(kh\right)\right\}\left[\cos\left(k\left(s'-s\right)\right)+i\sin\left(k\left(s'-s\right)\right)\right]-V_0\left\{\cos\left(ks\right)-\cos\left(kh\right)\right\}}{s'-s}ds'$$

$$=V_0\int_{0}^{s+h}\frac{\left\{\cos\left(k\left(s-u\right)\right)-\cos\left(kh\right)\right\}\left[\cos\left(ku\right)+i\sin\left(ku\right)\right]-\left\{\cos\left(ks\right)-\cos\left(kh\right)\right\}}{u}du$$

$$+V_0\int_{0}^{h-s}\frac{\left\{\cos\left(k\left(s+u\right)\right)-\cos\left(kh\right)\right\}\left[\cos\left(ku\right)+i\sin\left(ku\right)\right]-\left\{\cos\left(ks\right)-\cos\left(kh\right)\right\}}{u}du$$

$$= V_0 \int_0^{s+h} \frac{\{\cos(ks)\cos(ku)+\sin(ks)\sin(ku)-\cos(kh)\}\cos(ku)-\cos(ks)+\cos(kh)}{u}du$$

$$+V_0 \int_0^{h-s} \frac{\{\cos(ks)\cos(ku)-\sin(ks)\sin(ku)-\cos(kh)\}\cos(ku)-\cos(ks)+\cos(kh)}{u}du$$

$$+iV_0 \int_0^{s+h} \frac{\{\cos(ks)\cos(ku)+\sin(ks)\sin(ku)-\cos(kh)\}\sin(ku)}{u}du$$

$$+iV_0 \int_0^{h-s} \frac{\{\cos(ks)\cos(ku)-\sin(ks)\sin(ku)-\cos(kh)\}\sin(ku)}{u}du$$

$$= V_0 \int_0^{s+h} \frac{-\cos(ks)\left\{1-\cos^2(ku)\right\}+\sin(ks)\sin(ku)\cos(ku)+\cos(kh)\{1-\cos(ku)\}}{u}du$$

$$+V_0 \int_0^{h-s} \frac{-\cos(ks)\left\{1-\cos^2(ku)\right\}-\sin(ks)\sin(ku)\cos(ku)+\cos(kh)\{1-\cos(ku)\}}{u}du$$

$$+iV_0 \int_0^{s+h} \frac{\cos(ks)\sin(ku)\cos(ku)+\sin(ks)\sin^2(ku)-\cos(kh)\sin(ku)}{u}du$$

$$+iV_0 \int_0^{h-s} \frac{\cos(ks)\sin(ku)\cos(ku)-\sin(ks)\sin^2(ku)-\cos(kh)\sin(ku)}{u}du$$

$$= \frac{1}{2}V_0 \int_0^{s+h} \frac{-\cos(ks)\{1-\cos(2ku)\}+\sin(ks)\sin(2ku)+2\cos(kh)\{1-\cos(ku)\}}{u}du$$

$$+\frac{1}{2}V_0 \int_0^{h-s} \frac{-\cos(ks)\{1-\cos(2ku)\}-\sin(ks)\sin(2ku)+2\cos(kh)\{1-\cos(ku)\}}{u}du$$

$$+\frac{i}{2}V_0 \int_0^{s+h} \frac{\cos(ks)\sin(2ku)+\sin(ks)\{1-\cos(2ku)\}-2\cos(kh)\sin(ku)}{u}du$$

$$+\frac{i}{2}V_0 \int_0^{h-s} \frac{\cos(ks)\sin(2ku)-\sin(ks)\{1-\cos(2ku)\}-2\cos(kh)\sin(ku)}{u}du$$

$$= \frac{1}{2}V_0 \{2\cos(kh)\,\mathrm{Cin}(k(h+s))-\cos(ks)\,\mathrm{Cin}(2k(h+s))+\sin(ks)\,\mathrm{Si}(2k(h+s))\}$$

$$+\frac{1}{2}V_0 \{2\cos(kh)\,\mathrm{Cin}(k(h-s))-\cos(ks)\,\mathrm{Cin}(2k(h-s))-\sin(ks)\,\mathrm{Si}(2k(h-s))\}$$

$$+\frac{i}{2}V_0 \{\cos(ks)\,\mathrm{Si}(2k(h+s))+\sin(ks)\,\mathrm{Cin}(2k(h+s))-2\cos(kh)\,\mathrm{Si}(k(h+s))\}$$

$$+\frac{i}{2}V_0 \{\cos(ks)\,\mathrm{Si}(2k(h-s))-\sin(ks)\,\mathrm{Cin}(2k(h-s))-2\cos(kh)\,\mathrm{Si}(k(h-s))\}$$

Then we have

$$F_0(s) = V_0(s)\left\{-C_e^e + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h} \frac{V_0(s')\, e^{ik|s-s'|} - V_0(s)}{|s-s'|} ds'$$

$$= V_0\{\cos(ks) - \cos(kh)\}\{-C_e^e - 2\ln(h) + \ln(h+s) + \ln(h-s)\}$$

$$+\frac{1}{2}V_0\{2\cos(kh)\operatorname{Cin}(k(h+s)) - \cos(ks)\operatorname{Cin}(2k(h+s)) + \sin(ks)\operatorname{Si}(2k(h+s))\}$$

$$+\frac{1}{2}V_0\{2\cos(kh)\operatorname{Cin}(k(h-s)) - \cos(ks)\operatorname{Cin}(2k(h-s)) - \sin(ks)\operatorname{Si}(2k(h-s))\}$$

$$+\frac{i}{2}V_0\{\cos(ks)\operatorname{Si}(2k(h+s)) + \sin(ks)\operatorname{Cin}(2k(h+s)) - 2\cos(kh)\operatorname{Si}(k(h+s))\}$$

$$+\frac{i}{2}V_0\{\cos(ks)\operatorname{Si}(2k(h-s)) - \sin(ks)\operatorname{Cin}(2k(h-s)) - 2\cos(kh)\operatorname{Si}(k(h-s))\}$$

with boundary values

$$F_0(\pm h) = \frac{1}{2}V_0\{2\cos(kh)\operatorname{Cin}(2kh) - \cos(kh)\operatorname{Cin}(4kh) + \sin(kh)\operatorname{Si}(4kh)\}$$

$$+\frac{i}{2}V_0\{\cos(kh)\operatorname{Si}(4kh) + \sin(kh)\operatorname{Cin}(4kh) - 2\cos(kh)\operatorname{Si}(2kh)\}$$

We also note that

$$\frac{\partial}{\partial s}F_0(s) = -V_0 k\sin(ks)\{-C_e^e - 2\ln(h) + \ln(h+s) + \ln(h-s)\}$$

$$+V_0\{\cos(ks) - \cos(kh)\}\left\{\frac{1}{h+s} - \frac{1}{h-s}\right\}$$

$$+\frac{1}{2}V_0\left\{2\cos(kh)\frac{1-\cos(k(h+s))}{(h+s)} - \cos(ks)\frac{1-\cos(2k(h+s))}{(h+s)} + \sin(ks)\frac{\sin(2k(h+s))}{(h+s)}\right\}$$

$$+\frac{1}{2}V_0\{k\sin(ks)\operatorname{Cin}(2k(h+s)) + k\cos(ks)\operatorname{Si}(2k(h+s))\}$$

$$-\frac{1}{2}V_0\left\{2\cos(kh)\frac{1-\cos(k(h-s))}{(h-s)} - \cos(ks)\frac{1-\cos(2k(h-s))}{h-s} - \sin(ks)\frac{\sin(2k(h-s))}{(h-s)}\right\}$$

$$+\frac{1}{2}V_0\{k\sin(ks)\operatorname{Cin}(2k(h-s)) - k\cos(ks)\operatorname{Si}(2k(h-s))\}$$

$$+\frac{i}{2}V_0\left\{\cos(ks)\frac{\sin(2k(h+s))}{(h+s)} + \sin(ks)\frac{1-\cos(2k(h+s))}{(h+s)} - 2\cos(kh)\frac{\sin(k(h+s))}{(h+s)}\right\}$$

$$+\frac{i}{2}V_0\{-k\sin(ks)\operatorname{Si}(2k(h+s)) + k\cos(ks)\operatorname{Cin}(2k(h+s))\}$$

$$-\frac{i}{2}V_0\left\{\cos(ks)\frac{\sin(2k(h-s))}{(h-s)}-\sin(ks)\frac{1-\cos(2k(h-s))}{(h-s)}-2\cos(kh)\frac{\sin(k(h-s))}{(h-s)}\right\}$$

$$+\frac{i}{2}V_0\left\{-k\sin(ks)\,\mathrm{Si}(2k(h-s))-k\cos(ks)\,\mathrm{Cin}(2k(h-s))\right\}$$

with boundary behavior

$$\left[\frac{\partial}{\partial s}F_0(s)\right]_{s\to\pm h}=\mp V_0k\sin(kh)\left\{1-C_e^e-2\ln(h)+\ln(h+s)+\ln(h-s)\right\}$$

$$\pm\frac{1}{2}V_0\left\{2\cos(kh)\frac{1-\cos(2kh)}{2h}-\cos(kh)\frac{1-\cos(4kh)}{2h}+\sin(kh)\frac{\sin(4kh)}{2h}\right\}$$

$$\mp V_0k\sin(kh)$$

$$\pm\frac{1}{2}V_0\left\{k\sin(kh)\,\mathrm{Cin}(4kh)+k\cos(kh)\,\mathrm{Si}(4kh)\right\}$$

$$\pm\frac{i}{2}V_0\left\{\cos(kh)\frac{\sin(4kh)}{2h}+\sin(kh)\frac{1-\cos(4kh)}{2h}-2\cos(kh)\frac{\sin(2kh)}{2h}\right\}$$

$$\pm\frac{i}{2}V_0\left\{-k\sin(kh)\,\mathrm{Si}(4kh)+k\cos(kh)\,\mathrm{Cin}(4kh)\right\}$$

Note then that the boundary terms in integration by parts are

$$\left[V^*(s)\frac{\partial}{\partial s}F_0(s)-\frac{\partial}{\partial s}V^*(s)F_0(s)\right]_{-h}^{h}=\left[-\frac{\partial}{\partial s}V^*(s)F_0(s)\right]_{-h}^{h}=2V_0^*k\sin(kh)F_0(h)$$

We now need to evaluate

$$Y_{rad}^e|V_0|^2h\left\{1+2\cos^2(kh)-3\frac{\sin(kh)}{kh}\cos(kh)\right\}=-\frac{1}{\pi}\frac{i}{\eta_0}2V_0^*\sin(kh)F_0(h)+\frac{1}{\pi}i\frac{k}{\eta_0}V_0^*\cos(kh)\int_{-h}^{h}F_0(s)\,ds$$

with

$$F_0(\pm h)=\frac{1}{2}V_0\left\{2\cos(kh)\,\mathrm{Cin}(2kh)-\cos(kh)\,\mathrm{Cin}(4kh)+\sin(kh)\,\mathrm{Si}(4kh)\right\}$$

$$+\frac{i}{2}V_0\left\{\cos(kh)\,\mathrm{Si}(4kh)+\sin(kh)\,\mathrm{Cin}(4kh)-2\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

Carrying out the second integration

$$\int_{-h}^{h}F_0(s)\,ds$$

$$=-2hV_0\left\{C_e^e+2\ln(h)\right\}\left\{\frac{1}{kh}\sin(kh)-\cos(kh)\right\}+V_0\int_{-h}^{h}\left\{\cos(ks)-\cos(kh)\right\}\left\{\ln(h+s)+\ln(h-s)\right\}ds$$

$$+\frac{1}{2}V_0\int_{-h}^{h}\left\{2\cos\left(kh\right)\mathrm{Cin}\left(k\left(h+s\right)\right)-\cos\left(ks\right)\mathrm{Cin}\left(2k\left(h+s\right)\right)+\sin\left(ks\right)\mathrm{Si}\left(2k\left(h+s\right)\right)\right\}ds$$

$$+\frac{1}{2}V_0\int_{-h}^{h}\left\{2\cos\left(kh\right)\mathrm{Cin}\left(k\left(h-s\right)\right)-\cos\left(ks\right)\mathrm{Cin}\left(2k\left(h-s\right)\right)-\sin\left(ks\right)\mathrm{Si}\left(2k\left(h-s\right)\right)\right\}ds$$

$$+\frac{i}{2}V_0\int_{-h}^{h}\left\{\cos\left(ks\right)\mathrm{Si}\left(2k\left(h+s\right)\right)+\sin\left(ks\right)\mathrm{Cin}\left(2k\left(h+s\right)\right)-2\cos\left(kh\right)\mathrm{Si}\left(k\left(h+s\right)\right)\right\}ds$$

$$+\frac{i}{2}V_0\int_{-h}^{h}\left\{\cos\left(ks\right)\mathrm{Si}\left(2k\left(h-s\right)\right)-\sin\left(ks\right)\mathrm{Cin}\left(2k\left(h-s\right)\right)-2\cos\left(kh\right)\mathrm{Si}\left(k\left(h-s\right)\right)\right\}ds$$

$$=-2hV_0\left\{C_e^e+2\ln\left(h\right)\right\}\left\{\frac{1}{kh}\sin\left(kh\right)-\cos\left(kh\right)\right\}$$

$$+V_0\int_{0}^{2h}\left\{\cos\left(k\left(u-h\right)\right)-\cos\left(kh\right)\right\}\ln\left(u\right)du+V_0\int_{0}^{2h}\left\{\cos\left(k\left(h-u\right)\right)-\cos\left(kh\right)\right\}\ln\left(u\right)du$$

$$+\frac{1}{2}V_0\int_{0}^{2h}\left\{2\cos\left(kh\right)\mathrm{Cin}\left(ku\right)-\cos\left(k\left(u-h\right)\right)\mathrm{Cin}\left(2ku\right)+\sin\left(k\left(u-h\right)\right)\mathrm{Si}\left(2ku\right)\right\}du$$

$$+\frac{1}{2}V_0\int_{0}^{2h}\left\{2\cos\left(kh\right)\mathrm{Cin}\left(ku\right)-\cos\left(k\left(h-u\right)\right)\mathrm{Cin}\left(2ku\right)-\sin\left(k\left(h-u\right)\right)\mathrm{Si}\left(2ku\right)\right\}du$$

$$+\frac{i}{2}V_0\int_{0}^{2h}\left\{\cos\left(k\left(u-h\right)\right)\mathrm{Si}\left(2ku\right)+\sin\left(k\left(u-h\right)\right)\mathrm{Cin}\left(2ku\right)-2\cos\left(kh\right)\mathrm{Si}\left(ku\right)\right\}du$$

$$+\frac{i}{2}V_0\int_{0}^{2h}\left\{\cos\left(k\left(h-u\right)\right)\mathrm{Si}\left(2ku\right)-\sin\left(k\left(h-u\right)\right)\mathrm{Cin}\left(2ku\right)-2\cos\left(kh\right)\mathrm{Si}\left(ku\right)\right\}du$$

$$=-2hV_0\left\{C_e^e+2\ln\left(h\right)\right\}\left\{\frac{1}{kh}\sin\left(kh\right)-\cos\left(kh\right)\right\}$$

$$+2V_0\int_{0}^{2h}\left\{\cos\left(k\left(u-h\right)\right)-\cos\left(kh\right)\right\}\ln\left(u\right)du$$

$$+V_0\int_{0}^{2h}\left\{2\cos\left(kh\right)\mathrm{Cin}\left(ku\right)-\cos\left(k\left(u-h\right)\right)\mathrm{Cin}\left(2ku\right)+\sin\left(k\left(u-h\right)\right)\mathrm{Si}\left(2ku\right)\right\}du$$

$$+iV_0\int_{0}^{2h}\left\{\cos\left(k\left(u-h\right)\right)\mathrm{Si}\left(2ku\right)+\sin\left(k\left(u-h\right)\right)\mathrm{Cin}\left(2ku\right)-2\cos\left(kh\right)\mathrm{Si}\left(ku\right)\right\}du$$

$$=-2hV_0\left\{C_e^e+2\ln\left(h\right)\right\}\left\{\frac{1}{kh}\sin\left(kh\right)-\cos\left(kh\right)\right\}$$

$$+2V_0\left[\left\{\frac{1}{k}\sin\left(k\left(u-h\right)\right)-u\cos\left(kh\right)+\frac{1}{k}\sin\left(kh\right)\right\}\ln\left(u\right)\right]_0^{2h}$$

$$-2V_0\int_{0}^{2h}\left\{\frac{1}{k}\sin\left(k\left(u-h\right)\right)-u\cos\left(kh\right)+\frac{1}{k}\sin\left(kh\right)\right\}\frac{du}{u}$$

$$+V_0\left\{2\cos(kh)\,u\mathrm{Cin}(ku)-\frac{1}{k}\sin(k(u-h))\,\mathrm{Cin}(2ku)-\frac{1}{k}\cos(k(u-h))\,\mathrm{Si}(2ku)\right\}_0^{2h}$$

$$-V_0\int_0^{2h}\left\{2\cos(kh)(1-\cos(ku))-\frac{1}{k}\sin(k(u-h))\frac{1-\cos(2ku)}{u}-\frac{1}{k}\cos(k(u-h))\frac{\sin(2ku)}{u}\right\}du$$

$$+iV_0\left\{\frac{1}{k}\sin(k(u-h))\,\mathrm{Si}(2ku)-\frac{1}{k}\cos(k(u-h))\,\mathrm{Cin}(2ku)-2\cos(kh)\,u\,\mathrm{Si}(ku)\right\}_0^{2h}$$

$$-iV_0\int_0^{2h}\left\{\frac{1}{k}\sin(k(u-h))\frac{\sin(2ku)}{u}-\frac{1}{k}\cos(k(u-h))\frac{1-\cos(2ku)}{u}-2\cos(kh)\sin(ku)\right\}du$$

$$=-2hV_0\left\{C_e^e-2\ln(2)\right\}\left\{\frac{1}{kh}\sin(kh)-\cos(kh)\right\}$$

$$-2V_0\int_0^{2h}\left\{\frac{\sin(ku)}{u}\frac{1}{k}\cos(kh)+\frac{1-\cos(ku)}{u}\frac{1}{k}\sin(kh)-\cos(kh)\right\}du$$

$$+V_0\left\{4h\cos(kh)\,\mathrm{Cin}(2kh)-\frac{1}{k}\sin(kh)\,\mathrm{Cin}(4kh)-\frac{1}{k}\cos(kh)\,\mathrm{Si}(4kh)\right\}$$

$$-V_0\int_0^{2h}\left\{2\cos(kh)(1-\cos(ku))-\frac{1}{ku}\sin(k(u-h))-\frac{1}{ku}\sin(k(u+h))\right\}du$$

$$+iV_0\left\{\frac{1}{k}\sin(kh)\,\mathrm{Si}(4kh)-\frac{1}{k}\cos(kh)\,\mathrm{Cin}(4kh)-4h\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

$$-iV_0\int_0^{2h}\left\{\frac{1}{ku}\cos(k(u+h))-\frac{1}{ku}\cos(k(u-h))-2\cos(kh)\sin(ku)\right\}du$$

$$=-2hV_0\left\{C_e^e-2\ln(2)\right\}\left\{\frac{1}{kh}\sin(kh)-\cos(kh)\right\}$$

$$-2V_0\left\{\frac{1}{k}\cos(kh)\,\mathrm{Si}(2kh)-2h\cos(kh)+\frac{1}{k}\sin(kh)\,\mathrm{Cin}(2kh)\right\}$$

$$+V_0\left\{4h\cos(kh)\,\mathrm{Cin}(2kh)-\frac{1}{k}\sin(kh)\,\mathrm{Cin}(4kh)-\frac{1}{k}\cos(kh)\,\mathrm{Si}(4kh)\right\}$$

$$-2V_0\cos(kh)\int_0^{2h}\left\{1-\cos(ku)-\frac{1}{k}\frac{\sin(ku)}{u}\right\}du$$

$$+iV_0\left\{\frac{1}{k}\sin(kh)\,\mathrm{Si}(4kh)-\frac{1}{k}\cos(kh)\,\mathrm{Cin}(4kh)-4h\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

$$+i2V_0\int_0^{2h}\left\{\frac{1}{k}\sin(kh)\frac{\sin(ku)}{u}+\cos(kh)\sin(ku)\right\}du$$

$$= -2hV_0 \left\{C_e^e - 2\ln(2)\right\} \left\{\frac{1}{kh}\sin(kh) - \cos(kh)\right\}$$

$$-2V_0 \left\{\frac{1}{k}\cos(kh)\,\mathrm{Si}(2kh) - 2h\cos(kh) + \frac{1}{k}\sin(kh)\,\mathrm{Cin}(2kh)\right\}$$

$$+V_0 \left\{4h\cos(kh)\,\mathrm{Cin}(2kh) - \frac{1}{k}\sin(kh)\,\mathrm{Cin}(4kh) - \frac{1}{k}\cos(kh)\,\mathrm{Si}(4kh)\right\}$$

$$-2V_0\cos(kh) \left\{2h - \frac{1}{k}\sin(2kh) - \frac{1}{k}\mathrm{Si}(2kh)\right\}$$

$$+iV_0 \left\{\frac{1}{k}\sin(kh)\,\mathrm{Si}(4kh) - \frac{1}{k}\cos(kh)\,\mathrm{Cin}(4kh) - 4h\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

$$+i2V_0 \left\{\frac{1}{k}\sin(kh)\,\mathrm{Si}(2kh) + \frac{1}{k}\cos(kh) - \frac{1}{k}\cos(kh)\cos(2kh)\right\}$$

$$= -2hV_0 \left\{C_e^e - 2\ln(2)\right\} \left\{\frac{1}{kh}\sin(kh) - \cos(kh)\right\} + V_0\frac{2}{k}\cos(kh)\sin(2kh)$$

$$-V_0\frac{1}{k}\sin(kh)\,\mathrm{Cin}(4kh) - V_0\frac{1}{k}\cos(kh)\,\mathrm{Si}(4kh) - V_0\left\{\frac{2}{k}\sin(kh) - 4h\cos(kh)\right\}\mathrm{Cin}(2kh)$$

$$+iV_0 \left\{\frac{1}{k}\sin(kh)\,\mathrm{Si}(4kh) - \frac{1}{k}\cos(kh)\,\mathrm{Cin}(4kh) - 4h\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

$$+i2V_0 \left\{\frac{1}{k}\sin(kh)\,\mathrm{Si}(2kh) + \frac{1}{k}\cos(kh) - \frac{1}{k}\cos(kh)\cos(2kh)\right\}$$

Then the integral contribution to the radiation conductance and radiation susceptance are

$$G_{rad}^e h \left\{1 + 2\cos^2(kh) - 3\frac{\sin(kh)}{kh}\cos(kh)\right\} \to \frac{k}{\pi\eta_0}\cos(kh)$$

$$\left[\frac{1}{k}\cos(kh)\,\mathrm{Cin}(4kh) - \frac{1}{k}\sin(kh)\,\mathrm{Si}(4kh) + \left\{2h\cos(kh) - \frac{1}{k}\sin(kh)\right\}2\,\mathrm{Si}(2kh) - \frac{4}{k}\cos(kh)\sin^2(kh)\right]$$

$$B_{rad}^e h \left\{1 + 2\cos^2(kh) - 3\frac{\sin(kh)}{kh}\cos(kh)\right\} \to \frac{k}{\pi\eta_0}\cos(kh)\left[2h\left\{C_e^e - 2\ln(2)\right\}\left\{\frac{1}{kh}\sin(kh) - \cos(kh)\right\}\right.$$

$$\left.-\frac{2}{k}\cos(kh)\sin(2kh) + \left\{\frac{2}{k}\sin(kh) - 4h\cos(kh)\right\}\mathrm{Cin}(2kh) + \frac{1}{k}\sin(kh)\,\mathrm{Cin}(4kh) + \frac{1}{k}\cos(kh)\,\mathrm{Si}(4kh)\right]$$

Using

$$-i2V_0^* F_0(\pm h) = -i\left|V_0\right|^2 \left\{2\cos(kh)\,\mathrm{Cin}(2kh) - \cos(kh)\,\mathrm{Cin}(4kh) + \sin(kh)\,\mathrm{Si}(4kh)\right\}$$

$$+\left|V_0\right|^2\left\{\cos(kh)\,\mathrm{Si}(4kh)+\sin(kh)\,\mathrm{Cin}(4kh)-2\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

the boundary contribution the radiation conductance and radiation susceptance are

$$G^e_{rad}h\left\{1+2\cos^2(kh)-3\frac{\sin(kh)}{kh}\cos(kh)\right\}\to\frac{1}{\pi\eta_0}\sin(kh)\left\{\cos(kh)\,\mathrm{Si}(4kh)+\sin(kh)\,\mathrm{Cin}(4kh)-2\cos(kh)\,\mathrm{Si}(2kh)\right\}$$

$$B^e_{rad}h\left\{1+2\cos^2(kh)-3\frac{\sin(kh)}{kh}\cos(kh)\right\}\to\frac{1}{\pi\eta_0}\sin(kh)\left\{2\cos(kh)\,\mathrm{Cin}(2kh)-\cos(kh)\,\mathrm{Cin}(4kh)+\sin(kh)\,\mathrm{Si}(4kh)\right\}$$

Combining these two the total radiation conductance is

$$\pi\eta_0 G^e_{rad}h\left\{1+2\cos^2(kh)-3\frac{\sin(kh)}{kh}\cos(kh)\right\}=$$

$$\cos^2(kh)\,\mathrm{Cin}(4kh)-\sin(kh)\cos(kh)\,\mathrm{Si}(4kh)+\left\{2kh\cos^2(kh)-\sin(kh)\cos(kh)\right\}2\,\mathrm{Si}(2kh)-4\cos^2(kh)\sin^2(kh)$$

$$+\sin(kh)\cos(kh)\,\mathrm{Si}(4kh)+\sin^2(kh)\,\mathrm{Cin}(4kh)-2\sin(kh)\cos(kh)\,\mathrm{Si}(2kh)$$

$$=\mathrm{Cin}(4kh)+\cos(kh)\left\{kh\cos(kh)-\sin(kh)\right\}4\,\mathrm{Si}(2kh)-4\cos^2(kh)\sin^2(kh)$$

which is our former answer [9]. The total radiation susceptance is

$$\pi\eta_0 B^e_{rad}h\left\{1+2\cos^2(kh)-3\frac{\sin(kh)}{kh}\cos(kh)\right\}=\cos(kh)\left[2\left\{C^e_e-2\ln(2)\right\}\left\{\sin(kh)-kh\cos(kh)\right\}\right.$$

$$\left.-2\cos(kh)\sin(2kh)+\left\{\sin(kh)-2kh\cos(kh)\right\}2\mathrm{Cin}(2kh)+\sin(kh)\,\mathrm{Cin}(4kh)+\cos(kh)\,\mathrm{Si}(4kh)\right]$$

$$+\sin(kh)\left\{2\cos(kh)\,\mathrm{Cin}(2kh)-\cos(kh)\,\mathrm{Cin}(4kh)+\sin(kh)\,\mathrm{Si}(4kh)\right\}$$

$$=2\left\{C^e_e-2\ln(2)+2\mathrm{Cin}(2kh)\right\}\left\{\sin(kh)-kh\cos(kh)\right\}\cos(kh)+\mathrm{Si}(4kh)-2\cos^2(kh)\sin(2kh)$$

$$=4\left\{\ln(2)-C^e_e/2-\mathrm{Cin}(2kh)\right\}\left\{kh\cos(kh)-\sin(kh)\right\}\cos(kh)+\mathrm{Si}(4kh)-2\cos^2(kh)\sin(2kh)$$

which is also our former answer [9].

13.2 Odd Voltage Distribution

$$Y_{rad}^{o}\int_{-h}^{h}\left|V_1\left(s\right)\right|^2 ds=-\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{V_1^*\left(s\right)\frac{\partial}{\partial s}F_1\left(s\right)-\frac{\partial}{\partial s}V_1^*\left(s\right)F_1\left(s\right)\right\}_{-h}^{h}+\int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V_1^*\left(s\right)+k^2V_1^*\left(s\right)\right\}F_1\left(s\right)ds\right]$$

$$=-\frac{1}{\pi}\frac{i}{\omega\mu_0}\left[\left\{-\frac{\partial}{\partial s}V_1^*\left(s\right)F_1\left(s\right)\right\}_{-h}^{h}+\int_{-h}^{h}\left\{\frac{\partial^2}{\partial s^2}V_1^*\left(s\right)+k^2V_1^*\left(s\right)\right\}F_1\left(s\right)ds\right]$$

or

$$Y_{rad}^{o}\left|V_1\right|^2 h^3\left\{1+3\cos\left(kh\right)\frac{\sin\left(kh\right)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2\left(kh\right)}{k^2h^2}\right\}$$

$$=\frac{1}{\pi}\frac{i}{\omega\mu_0}V_1^*\left\{kh\cos\left(kh\right)-\sin\left(kh\right)\right\}\left\{F_1\left(h\right)-F_1\left(-h\right)\right\}+\frac{1}{\pi}i\omega\varepsilon_0V_1^*\sin\left(kh\right)\int_{-h}^{h}sF_1\left(s\right)ds$$

$$F_1\left(s\right)=V_1\left(s\right)\left\{-C_e^o+\ln\left(1-s^2/h^2\right)\right\}+\int_{-h}^{h}\frac{V_1\left(s'\right)e^{ik\left|s-s'\right|}-V_1\left(s\right)}{\left|s-s'\right|}ds'$$

For the odd voltage distribution we need to evaluate

$$Y_{rad}^{o}\left|V_1\right|^2 h^3\left\{1+3\cos\left(kh\right)\frac{\sin\left(kh\right)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2\left(kh\right)}{k^2h^2}\right\}=\frac{1}{\pi}i\omega\varepsilon_0V_1^*\sin\left(kh\right)\int_{-h}^{h}sF_1\left(s\right)ds$$

where

$$F_1\left(s\right)=V_1\left(s\right)\left\{-C_e^o+\ln\left(1-s^2/h^2\right)\right\}+\int_{-h}^{h}\frac{V_1\left(s'\right)e^{ik\left|s-s'\right|}-V_1\left(s\right)}{\left|s-s'\right|}ds'$$

Carrying out the inner integral for the V_1 term

$$\int_{-h}^{h}\frac{V_1\left(s'\right)e^{ik\left|s-s'\right|}-V_1\left(s\right)}{\left|s-s'\right|}ds'$$

$$=\int_{-h}^{s}\frac{V_1\left\{h\sin\left(ks'\right)-s'\sin\left(kh\right)\right\}\left[\cos\left(k\left(s-s'\right)\right)+i\sin\left(k\left(s-s'\right)\right)\right]-V_1\left\{h\sin\left(ks\right)-s\sin\left(kh\right)\right\}}{s-s'}ds'$$

$$+\int_{s}^{h}\frac{V_1\left\{h\sin\left(ks'\right)-s'\sin\left(kh\right)\right\}\left[\cos\left(k\left(s'-s\right)\right)+i\sin\left(k\left(s'-s\right)\right)\right]-V_1\left\{h\sin\left(ks\right)-s\sin\left(kh\right)\right\}}{s'-s}ds'$$

$$=\int_{0}^{s+h}\frac{V_1\left\{h\sin\left(k\left(s-u\right)\right)-\left(s-u\right)\sin\left(kh\right)\right\}\left[\cos\left(ku\right)+i\sin\left(ku\right)\right]-V_1\left\{h\sin\left(ks\right)-s\sin\left(kh\right)\right\}}{u}du$$

$$+\int_{0}^{h-s}\frac{V_1\left\{h\sin\left(k\left(s+u\right)\right)-\left(s+u\right)\sin\left(kh\right)\right\}\left[\cos\left(ku\right)+i\sin\left(ku\right)\right]-V_1\left\{h\sin\left(ks\right)-s\sin\left(kh\right)\right\}}{u}du$$

$$= V_1 \int_0^{s+h} \frac{\{h\sin(ks)\cos(ku) - h\cos(ks)\sin(ku) - (s-u)\sin(kh)\}[\cos(ku) + i\sin(ku)] - \{h\sin(ks) - s\sin(kh)\}}{u} du$$

$$+V_1 \int_0^{h-s} \frac{\{h\sin(ks)\cos(ku) + h\cos(ks)\sin(ku) - (s+u)\sin(kh)\}[\cos(ku) + i\sin(ku)] - \{h\sin(ks) - s\sin(kh)\}}{u} du$$

$$= V_1 \int_0^{s+h} \frac{-(h/2)\cos(ks)\sin(2ku) + u\sin(kh)\cos(ku) - (h/2)\sin(ks)\{1-\cos(2ku)\} + s\sin(kh)\{1-\cos(ku)\}}{u} du$$

$$+V_1 \int_0^{h-s} \frac{(h/2)\cos(ks)\sin(2ku) - u\sin(kh)\cos(ku) - (h/2)\sin(ks)\{1-\cos(2ku)\} + s\sin(kh)\{1-\cos(ku)\}}{u} du$$

$$+iV_1 \int_0^{s+h} \frac{(h/2)\sin(ks)\sin(2ku) - (h/2)\cos(ks)\{1-\cos(2ku)\} - (s-u)\sin(kh)\sin(ku)}{u} du$$

$$+iV_1 \int_0^{h-s} \frac{(h/2)\sin(ks)\sin(2ku) + (h/2)\cos(ks)\{1-\cos(2ku)\} - (s+u)\sin(kh)\sin(ku)}{u} du$$

$$= V_1 \left[-(h/2)\cos(ks)\,\mathrm{Si}(2k(s+h)) + \sin(kh)\frac{1}{k}\sin(k(s+h)) \right.$$

$$\left. -(h/2)\sin(ks)\,\mathrm{Cin}(2k(s+h)) + s\sin(kh)\,\mathrm{Cin}(k(s+h)) \right]$$

$$+V_1 \left[(h/2)\cos(ks)\,\mathrm{Si}(2k(h-s)) - \sin(kh)\frac{1}{k}\sin(k(h-s)) \right.$$

$$\left. -(h/2)\sin(ks)\,\mathrm{Cin}(2k(h-s)) + s\sin(kh)\,\mathrm{Cin}(k(h-s)) \right]$$

$$+iV_1 \left[(h/2)\sin(ks)\,\mathrm{Si}(2k(s+h)) - (h/2)\cos(ks)\,\mathrm{Cin}(2k(s+h)) \right.$$

$$\left. -s\sin(kh)\,\mathrm{Si}(k(s+h)) + \sin(kh)\frac{1}{k}\{1-\cos(k(s+h))\} \right]$$

$$+iV_1 \left[(h/2)\sin(ks)\,\mathrm{Si}(2k(h-s)) + (h/2)\cos(ks)\,\mathrm{Cin}(2k(h-s)) \right.$$

$$\left. -s\sin(kh)\,\mathrm{Si}(k(h-s)) - \sin(kh)\frac{1}{k}\{1-\cos(k(h-s))\} \right]$$

Therefore

$$F_1(s) = V_1(s)\left\{-C_e^o + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h} \frac{V_1(s')\, e^{ik|s-s'|} - V_1(s)}{|s-s'|} ds'$$

$$= V_1\left\{h\sin(ks) - s\sin(kh)\right\}\left\{-C_e^o + \ln\left(1 - s^2/h^2\right)\right\}$$

$$+V_1\left[-(h/2)\cos(ks)\,\mathrm{Si}(2k(s+h)) + \sin(kh)\frac{1}{k}\sin(k(s+h)) - (h/2)\sin(ks)\,\mathrm{Cin}(2k(s+h)) + s\sin(kh)\,\mathrm{Cin}(k(s+h))\right]$$

$$+V_1\left[(h/2)\cos(ks)\,\mathrm{Si}(2k(h-s)) - \sin(kh)\frac{1}{k}\sin(k(h-s)) - (h/2)\sin(ks)\,\mathrm{Cin}(2k(h-s)) + s\sin(kh)\,\mathrm{Cin}(k(h-s))\right]$$

$$+iV_1\left[(h/2)\sin(ks)\,\mathrm{Si}(2k(s+h)) - (h/2)\cos(ks)\,\mathrm{Cin}(2k(s+h))\right.$$

$$\left.-s\sin(kh)\,\mathrm{Si}(k(s+h)) + \sin(kh)\frac{1}{k}\left\{1 - \cos(k(s+h))\right\}\right]$$

$$+iV_1\left[(h/2)\sin(ks)\,\mathrm{Si}(2k(h-s)) + (h/2)\cos(ks)\,\mathrm{Cin}(2k(h-s))\right.$$

$$\left.-s\sin(kh)\,\mathrm{Si}(k(h-s)) - \sin(kh)\frac{1}{k}\left\{1 - \cos(k(h-s))\right\}\right]$$

and

$$\pm F_1(\pm h) = V_1\left[-(h/2)\cos(kh)\,\mathrm{Si}(4kh) + \sin(kh)\frac{1}{k}\sin(2kh) - (h/2)\sin(kh)\,\mathrm{Cin}(4kh) + h\sin(kh)\,\mathrm{Cin}(2kh)\right]$$

$$+iV_1\left[(h/2)\sin(kh)\,\mathrm{Si}(4kh) - (h/2)\cos(kh)\,\mathrm{Cin}(4kh) - h\sin(kh)\,\mathrm{Si}(2kh) + \sin(kh)\frac{1}{k}\left\{1 - \cos(2kh)\right\}\right]$$

We now need to evaluate

$$Y_{rad}^o \int_{-h}^{h} |V_1(s)|^2 ds = -\frac{1}{\pi}\frac{i}{\omega\mu_0}\int_{-h}^{h} V_1^*(s)\left(\frac{\partial^2}{\partial s^2} + k^2\right)$$

$$\left[V_1(s)\left\{-C_e^o + \ln\left(1 - s^2/h^2\right)\right\} + \int_{-h}^{h} \frac{V_1(s')\, e^{ik|s-s'|} - V_1(s)}{|s-s'|} ds'\right] ds$$

$$Y_{rad}^o |V_1|^2 h^3 \left\{1 + 3\cos(kh)\frac{\sin(kh)}{kh} - 4\left(1 - \frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$= \frac{2}{\pi}\frac{i}{\omega\mu_0} V_1^*\left\{kh\cos(kh) - \sin(kh)\right\} F_1(h) + \frac{1}{\pi} i\omega\varepsilon_0 V_1^* \sin(kh)\int_{-h}^{h} sF_1(s)\, ds$$

or

$$\pi\eta_0 h Y^o_{rad}\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=2\frac{i}{kh}\left\{kh\cos(kh)-\sin(kh)\right\}\frac{1}{hV_1}F_1(h)+i\frac{k}{h^2V_1}\sin(kh)\int_{-h}^{h}sF_1(s)\,ds$$

For the moment ignoring the boundary term

$$Y^o_{rad}\left|V_1\right|^2h^3\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}=\frac{1}{\pi}i\omega\varepsilon_0V_1^*\sin(kh)\int_{-h}^{h}sF_1(s)\,ds$$

$$F_1(s)=V_1(s)\left\{-C^o_e+\ln\left(1-s^2/h^2\right)\right\}+\int_{-h}^{h}\frac{V_1(s')\,e^{ik|s-s'|}-V_1(s)}{|s-s'|}ds'$$

Carrying out the integration

$$\int_{-h}^{h}sF_1(s)\,ds=V_1\int_{-h}^{h}\left\{h\sin(ks)-s\sin(kh)\right\}\left\{-C^o_e-2\ln(h)+\ln(h+s)+\ln(h-s)\right\}sds$$

$$+V_1\int_{-h}^{h}\left[-(h/2)\cos(ks)\,\mathrm{Si}(2k(s+h))+\sin(kh)\frac{1}{k}\sin(k(s+h))\right.$$

$$\left.-(h/2)\sin(ks)\,\mathrm{Cin}(2k(s+h))+s\sin(kh)\,\mathrm{Cin}(k(s+h))\right]sds$$

$$+V_1\int_{-h}^{h}\left[(h/2)\cos(ks)\,\mathrm{Si}(2k(h-s))-\sin(kh)\frac{1}{k}\sin(k(h-s))\right.$$

$$\left.-(h/2)\sin(ks)\,\mathrm{Cin}(2k(h-s))+s\sin(kh)\,\mathrm{Cin}(k(h-s))\right]sds$$

$$+iV_1\int_{-h}^{h}\left[(h/2)\sin(ks)\,\mathrm{Si}(2k(s+h))-(h/2)\cos(ks)\,\mathrm{Cin}(2k(s+h))\right.$$

$$\left.-s\sin(kh)\,\mathrm{Si}(k(s+h))+\sin(kh)\frac{1}{k}\left\{1-\cos(k(s+h))\right\}\right]sds$$

$$+iV_1\int_{-h}^{h}\left[(h/2)\sin(ks)\,\mathrm{Si}(2k(h-s))+(h/2)\cos(ks)\,\mathrm{Cin}(2k(h-s))\right.$$

$$\left.-s\sin(kh)\,\mathrm{Si}(k(h-s))-\sin(kh)\frac{1}{k}\left\{1-\cos(k(h-s))\right\}\right]sds$$

$$=V_1\int_{-h}^{h}\left\{h\sin(ks)-s\sin(kh)\right\}\left\{-C^o_e-2\ln(h)\right\}sds+2V_1\int_{-h}^{h}\left\{h\sin(ks)-s\sin(kh)\right\}\ln(h+s)\,sds$$

$$+2V_1\int_{-h}^{h}\left[-(h/2)\cos(ks)\,\mathrm{Si}(2k(s+h))+\sin(kh)\frac{1}{k}\sin(k(s+h))\right.$$

$$-\left(h/2\right)\sin\left(ks\right)\mathrm{Cin}\left(2k\left(s+h\right)\right)+s\sin\left(kh\right)\mathrm{Cin}\left(k\left(s+h\right)\right)\right]sds$$

$$+i2V_1\int_{-h}^{h}\left[\left(h/2\right)\sin\left(ks\right)\mathrm{Si}\left(2k\left(s+h\right)\right)-\left(h/2\right)\cos\left(ks\right)\mathrm{Cin}\left(2k\left(s+h\right)\right)\right.$$

$$\left.-s\sin\left(kh\right)\mathrm{Si}\left(k\left(s+h\right)\right)+\sin\left(kh\right)\frac{1}{k}\left\{1-\cos\left(k\left(s+h\right)\right)\right\}\right]sds$$

$$=-V_1\left\{C_e^o+2\ln\left(h\right)\right\}\left\{-\frac{h}{k}s\cos\left(ks\right)+\frac{h}{k^2}\sin\left(ks\right)-\frac{1}{3}s^3\sin\left(kh\right)\right\}_{-h}^{h}$$

$$+2V_1\int_0^{2h}\left\{h\left(u-h\right)\sin\left(k\left(u-h\right)\right)-\left(u-h\right)^2\sin\left(kh\right)\right\}\ln\left(u\right)du$$

$$+2V_1\int_0^{2h}\left[-\left(h/2\right)\cos\left(k\left(u-h\right)\right)\mathrm{Si}\left(2ku\right)+\sin\left(kh\right)\frac{1}{k}\sin\left(ku\right)\right.$$

$$\left.-\left(h/2\right)\sin\left(k\left(u-h\right)\right)\mathrm{Cin}\left(2ku\right)+\left(u-h\right)\sin\left(kh\right)\mathrm{Cin}\left(ku\right)\right]\left(u-h\right)du$$

$$+i2V_1\int_0^{2h}\left[\left(h/2\right)\sin\left(k\left(u-h\right)\right)\mathrm{Si}\left(2ku\right)-\left(h/2\right)\cos\left(k\left(u-h\right)\right)\mathrm{Cin}\left(2ku\right)\right.$$

$$\left.-\left(u-h\right)\sin\left(kh\right)\mathrm{Si}\left(ku\right)+\sin\left(kh\right)\frac{1}{k}\left\{1-\cos\left(ku\right)\right\}\right]\left(u-h\right)du$$

$$=-2h^3V_1\left\{C_e^o+2\ln\left(h\right)\right\}\left\{-\frac{1}{kh}\cos\left(kh\right)+\frac{1}{k^2h^2}\sin\left(kh\right)-\frac{1}{3}\sin\left(kh\right)\right\}$$

$$+2V_1\left[\left\{-\frac{h}{k}\left(u-h\right)\cos\left(k\left(u-h\right)\right)-\frac{h^2}{k}\cos\left(kh\right)+\frac{h}{k^2}\sin\left(k\left(u-h\right)\right)\right.\right.$$

$$\left.\left.-\frac{1}{3}\left(u-h\right)^3\sin\left(kh\right)-\left(\frac{1}{3}h^2-\frac{1}{k^2}\right)h\sin\left(kh\right)\right\}\ln\left(u\right)\right]_0^{2h}$$

$$-2V_1\int_0^{2h}\left\{-\frac{h}{k}\left(u-h\right)\cos\left(k\left(u-h\right)\right)-\frac{h^2}{k}\cos\left(kh\right)+\frac{h}{k^2}\sin\left(k\left(u-h\right)\right)\right.$$

$$\left.-\frac{1}{3}\left(u-h\right)^3\sin\left(kh\right)-\left(\frac{1}{3}h^2-\frac{1}{k^2}\right)h\sin\left(kh\right)\right\}\frac{du}{u}$$

$$+2V_1\left[-\left(h/2\right)\left\{\frac{1}{k}\left(u-h\right)\sin\left(k\left(u-h\right)\right)+\frac{1}{k^2}\cos\left(k\left(u-h\right)\right)\right\}\mathrm{Si}\left(2ku\right)\right.$$

$$\left.-\left(h/2\right)\left\{-\frac{1}{k}\left(u-h\right)\cos\left(k\left(u-h\right)\right)+\frac{1}{k^2}\sin\left(k\left(u-h\right)\right)\right\}\mathrm{Cin}\left(2ku\right)+\sin\left(kh\right)\frac{1}{3}\left(u-h\right)^3\mathrm{Cin}\left(ku\right)\right]_0^{2h}$$

$$-2V_1\int_0^{2h}\left[-\left(h/2\right)\left\{\frac{1}{k}\left(u-h\right)\sin\left(k\left(u-h\right)\right)+\frac{1}{k^2}\cos\left(k\left(u-h\right)\right)\right\}\frac{\sin\left(2ku\right)}{u}\right.$$

$$- (h/2) \left\{ -\frac{1}{k} (u-h) \cos(k(u-h)) + \frac{1}{k^2} \sin(k(u-h)) \right\} \frac{1-\cos(2ku)}{u} + \sin(kh) \frac{1}{3} (u-h)^3 \frac{1-\cos(ku)}{u} \Bigg] du$$

$$+\frac{2}{k} V_1 \sin(kh) \int_0^{2h} (u-h) \sin(ku)\, du$$

$$+i2V_1 \Bigg[(h/2) \left\{ -\frac{1}{k} (u-h) \cos(k(u-h)) + \frac{1}{k^2} \sin(k(u-h)) \right\} \mathrm{Si}(2ku)$$

$$- (h/2) \left\{ \frac{1}{k} (u-h) \sin(k(u-h)) + \frac{1}{k^2} \cos(k(u-h)) \right\} \mathrm{Cin}(2ku) - \frac{1}{3} (u-h)^3 \sin(kh)\, \mathrm{Si}(ku) \Bigg]_0^{2h}$$

$$-i2V_1 \int_0^{2h} \Bigg[(h/2) \left\{ -\frac{1}{k} (u-h) \cos(k(u-h)) + \frac{1}{k^2} \sin(k(u-h)) \right\} \frac{\sin(2ku)}{u}$$

$$- (h/2) \left\{ \frac{1}{k} (u-h) \sin(k(u-h)) + \frac{1}{k^2} \cos(k(u-h)) \right\} \frac{1-\cos(2ku)}{u} - \frac{1}{3} (u-h)^3 \sin(kh) \frac{\sin(ku)}{u} \Bigg] du$$

$$+i\frac{2}{k} V_1 \sin(kh) \int_0^{2h} (u-h) \{1-\cos(ku)\}\, du$$

$$= -2h^3 V_1 \{C_e^o + 2\ln(h)\} \left\{ -\frac{1}{kh} \cos(kh) + \frac{1}{k^2h^2} \sin(kh) - \frac{1}{3} \sin(kh) \right\}$$

$$+2V_1 \left[\left\{ -\frac{h^2}{k} \cos(kh) - \frac{h^2}{k} \cos(kh) + \frac{h}{k^2} \sin(kh) - \frac{1}{3} h^3 \sin(kh) - \left(\frac{1}{3} h^2 - \frac{1}{k^2} \right) h \sin(kh) \right\} \ln(2h) \right]$$

$$-2V_1 \int_0^{2h} \left\{ -\frac{h}{k} \cos(k(u-h)) - \frac{1}{3} \left(u^2 - 3hu + 3h^2 \right) \sin(kh) \right\} du$$

$$-2V_1 \int_0^{2h} \left\{ -\frac{h^2}{k} \cos(kh) \frac{1-\cos(ku)}{u} + \frac{h^2}{k} \sin(kh) \frac{\sin(ku)}{u} + \frac{h}{k^2} \cos(kh) \frac{\sin(ku)}{u} + \frac{h}{k^2} \sin(kh) \frac{1-\cos(ku)}{u} \right\} du$$

$$+2V_1 \left[-(h/2) \left\{ \frac{h}{k} \sin(kh) + \frac{1}{k^2} \cos(kh) \right\} \mathrm{Si}(4kh) - (h/2) \left\{ -\frac{h}{k} \cos(kh) + \frac{1}{k^2} \sin(kh) \right\} \mathrm{Cin}(4kh) + \sin(kh) \frac{1}{3} h^3 \mathrm{Cin}(2kh) \right]$$

$$-2V_1 \int_0^{2h} \Bigg[-(h/2) \left\{ \frac{1}{k} (u-h) \sin(k(u-h)) + \frac{1}{k^2} \cos(k(u-h)) \right\} \frac{\sin(2ku)}{u}$$

$$- (h/2) \left\{ -\frac{1}{k} (u-h) \cos(k(u-h)) + \frac{1}{k^2} \sin(k(u-h)) \right\} \frac{1-\cos(2ku)}{u} + \sin(kh) \frac{1}{3} (u-h)^3 \frac{1-\cos(ku)}{u} \Bigg] du$$

$$+\frac{2}{k}V_1\sin(kh)\left\{-\frac{1}{k}(u-h)\cos(ku)+\frac{1}{k^2}\sin(ku)\right\}_0^{2h}$$

$$+i2V_1\left[(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Si}(4kh)-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Cin}(4kh)-\frac{1}{3}h^3\sin(kh)\,\mathrm{Si}(2kh)\right]$$

$$-i2V_1\int_0^{2h}\left[(h/2)\left\{-\frac{1}{k}(u-h)\cos(k(u-h))+\frac{1}{k^2}\sin(k(u-h))\right\}\frac{\sin(2ku)}{u}\right.$$

$$\left.-(h/2)\left\{\frac{1}{k}(u-h)\sin(k(u-h))+\frac{1}{k^2}\cos(k(u-h))\right\}\frac{1-\cos(2ku)}{u}-\frac{1}{3}(u-h)^3\sin(kh)\frac{\sin(ku)}{u}\right]du$$

$$+i\frac{2}{k}V_1\sin(kh)\left\{\frac{1}{2}(u-h)^2-\frac{1}{k}(u-h)\sin(ku)-\frac{1}{k^2}\cos(ku)\right\}_0^{2h}$$

$$=-2h^3V_1\left\{C_e^o+2\ln(h)\right\}\left\{-\frac{1}{kh}\cos(kh)+\frac{1}{k^2h^2}\sin(kh)-\frac{1}{3}\sin(kh)\right\}$$

$$+2V_1\left[\left\{-\frac{h^2}{k}\cos(kh)-\frac{h^2}{k}\cos(kh)+\frac{h}{k^2}\sin(kh)-\frac{1}{3}h^3\sin(kh)-\left(\frac{1}{3}h^2-\frac{1}{k^2}\right)h\sin(kh)\right\}\ln(2h)\right]$$

$$-2V_1\left\{-\frac{h}{k^2}\sin(k(u-h))-\frac{1}{3}\left(\frac{1}{3}u^3-\frac{3}{2}hu^2+3h^2u\right)\sin(kh)\right\}_0^{2h}$$

$$-2V_1\left\{-\frac{h^2}{k}\cos(kh)\,\mathrm{Cin}(2kh)+\frac{h^2}{k}\sin(kh)\,\mathrm{Si}(2kh)+\frac{h}{k^2}\cos(kh)\,\mathrm{Si}(2kh)+\frac{h}{k^2}\sin(kh)\,\mathrm{Cin}(2kh)\right\}$$

$$+2V_1\left[-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Si}(4kh)-(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Cin}(4kh)+\sin(kh)\frac{1}{3}h^3\mathrm{Cin}(2kh)\right]$$

$$+\frac{h}{2}V_1\int_0^{2h}\left\{\frac{1}{k}(u-h)\cos(k(u+h))-\frac{1}{k}(u-h)\cos(k(3u-h))+\frac{1}{k^2}\sin(k(u+h))+\frac{1}{k^2}\sin(k(3u-h))\right\}\frac{du}{u}$$

$$+\frac{h}{2}V_1\int_0^{2h}\left\{-\frac{2}{k}(u-h)\cos(k(u-h))+\frac{2}{k^2}\sin(k(u-h))+\frac{1}{k}(u-h)\cos(k(u+h))\right.$$

$$\left.+\frac{1}{k}(u-h)\cos(k(3u-h))+\frac{1}{k^2}\sin(k(u+h))-\frac{1}{k^2}\sin(k(3u-h))\right\}\frac{du}{u}$$

$$-2V_1\sin(kh)\frac{1}{3}\int_0^{2h}\left[\left(u^2-3hu+3h^2\right)-\left(u^2-3hu+3h^2\right)\cos(ku)\right]du+2V_1\sin(kh)\frac{1}{3}h^3\int_0^{2h}\frac{1-\cos(ku)}{u}du$$

$$+\frac{2}{k}V_1\sin(kh)\left\{-\frac{h}{k}\cos(2kh)-\frac{h}{k}+\frac{1}{k^2}\sin(2kh)\right\}$$

$$+i2V_1\left[(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Si}(4kh)-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Cin}(4kh)-\frac{1}{3}h^3\sin(kh)\,\mathrm{Si}(2kh)\right]$$

$$-i\frac{h}{2}V_1\int_0^{2h}\left\{-\frac{1}{k}(u-h)\sin(k(u+h))-\frac{1}{k}(u-h)\sin(k(3u-h))+\frac{1}{k^2}\cos(k(u+h))-\frac{1}{k^2}\cos(k(3u-h))\right\}\frac{du}{u}$$

$$+i\frac{h}{2}V_1\int_0^{2h}\left\{\frac{2}{k}(u-h)\sin(k(u-h))+\frac{2}{k^2}\cos(k(u-h))+\frac{1}{k}(u-h)\sin(k(u+h))\right.$$

$$\left.-\frac{1}{k}(u-h)\sin(k(3u-h))-\frac{1}{k^2}\cos(k(u+h))-\frac{1}{k^2}\cos(k(3u-h))\right\}\frac{du}{u}$$

$$+i\frac{2}{3}V_1\sin(kh)\int_0^{2h}\left(u^3-3hu^2+3h^2u-h^3\right)\frac{\sin(ku)}{u}du$$

$$+i\frac{2}{k}V_1\sin(kh)\left\{\frac{1}{k^2}-\frac{h}{k}\sin(2kh)-\frac{1}{k^2}\cos(2kh)\right\}$$

$$=-2h^3V_1\left\{C_e^o+2\ln(h)\right\}\left\{-\frac{1}{kh}\cos(kh)+\frac{1}{k^2h^2}\sin(kh)-\frac{1}{3}\sin(kh)\right\}$$

$$+2V_1\left[\left\{-\frac{h^2}{k}\cos(kh)-\frac{h^2}{k}\cos(kh)+\frac{h}{k^2}\sin(kh)-\frac{1}{3}h^3\sin(kh)-\left(\frac{1}{3}h^2-\frac{1}{k^2}\right)h\sin(kh)\right\}\ln(2h)\right]$$

$$-2V_1\left\{-\frac{2h}{k^2}\sin(kh)-\frac{1}{3}\left(\frac{8}{3}h^3-6h^3+6h^3\right)\sin(kh)\right\}$$

$$-2V_1\left\{\frac{h^2}{k}\sin(kh)\,\mathrm{Si}(2kh)-\frac{h^2}{k}\cos(kh)\,\mathrm{Cin}(2kh)+\frac{h}{k^2}\cos(kh)\,\mathrm{Si}(2kh)+\frac{h}{k^2}\sin(kh)\,\mathrm{Cin}(2kh)\right\}$$

$$+2V_1\left[-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Si}(4kh)-(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Cin}(4kh)+\sin(kh)\frac{1}{3}h^3\mathrm{Cin}(2kh)\right]$$

$$+\frac{h}{2k^2}V_1\left\{\sin(k(u+h))-\frac{1}{3}\sin(k(3u-h))\right\}_0^{2h}$$

$$+\frac{h}{2k}V_1\int_0^{2h}\left\{-h\cos(ku)\cos(kh)+h\cos(3ku)\cos(kh)+h\sin(ku)\sin(kh)+h\sin(3ku)\sin(kh)\right\}\frac{du}{u}$$

$$+\frac{h}{2k}V_1\int_0^{2h}\left\{\frac{1}{k}\sin(ku)\cos(kh)+\frac{1}{k}\sin(3ku)\cos(kh)+\frac{1}{k}\sin(kh)\cos(ku)-\frac{1}{k}\sin(kh)\cos(3ku)\right\}\frac{du}{u}$$

$$+\frac{h}{2k}V_1\left\{-\frac{2}{k}\sin\left(k\left(u-h\right)\right)+\frac{1}{k}\sin\left(k\left(u+h\right)\right)+\frac{1}{3k}\sin\left(k\left(3u-h\right)\right)\right\}_0^{2h}$$

$$+\frac{h}{2k}V_1\int_0^{2h}\left\{h\cos\left(ku\right)\cos\left(kh\right)-h\cos\left(3ku\right)\cos\left(kh\right)-\frac{1}{k}\cos\left(ku\right)\sin\left(kh\right)+\frac{1}{k}\cos\left(3ku\right)\sin\left(kh\right)\right\}\frac{du}{u}$$

$$+\frac{h}{2k}V_1\int_0^{2h}\left\{3h\sin\left(ku\right)\sin\left(kh\right)-h\sin\left(3ku\right)\sin\left(kh\right)+\frac{3}{k}\sin\left(ku\right)\cos\left(kh\right)-\frac{1}{k}\sin\left(3ku\right)\cos\left(kh\right)\right\}\frac{du}{u}$$

$$+\frac{2}{3}V_1\sin\left(kh\right)\left\{\frac{1}{k}u^2\sin\left(ku\right)+\frac{2}{k^2}u\cos\left(ku\right)-\frac{2}{k^3}\sin\left(ku\right)-\frac{3}{k}hu\sin\left(ku\right)-\frac{3}{k^2}h\cos\left(ku\right)+\frac{3}{k}h^2\sin\left(ku\right)\right\}_0^{2h}$$

$$-\frac{2}{3}V_1\sin\left(kh\right)\frac{8}{3}h^3+2V_1\sin\left(kh\right)\frac{1}{3}h^3\mathrm{Cin}\left(2kh\right)$$

$$+\frac{2}{k}V_1\sin\left(kh\right)\left\{-\frac{h}{k}\cos\left(2kh\right)-\frac{h}{k}+\frac{1}{k^2}\sin\left(2kh\right)\right\}$$

$$+i2V_1\left[\left(h/2\right)\left\{-\frac{h}{k}\cos\left(kh\right)+\frac{1}{k^2}\sin\left(kh\right)\right\}\mathrm{Si}\left(4kh\right)-\left(h/2\right)\left\{\frac{h}{k}\sin\left(kh\right)+\frac{1}{k^2}\cos\left(kh\right)\right\}\mathrm{Cin}\left(4kh\right)-\frac{1}{3}h^3\sin\left(kh\right)\mathrm{Si}\left(2kh\right)\right]$$

$$-i\frac{h}{2k^2}V_1\left\{\cos\left(k\left(u+h\right)\right)+\frac{1}{3}\cos\left(k\left(3u-h\right)\right)\right\}_0^{2h}$$

$$+i\frac{h}{2}V_1\left\{-\frac{2}{k^2}\cos\left(k\left(u-h\right)\right)-\frac{1}{k^2}\cos\left(k\left(u+h\right)\right)+\frac{1}{3k^2}\cos\left(k\left(3u-h\right)\right)\right\}_0^{2h}$$

$$-i\frac{h}{2}V_1\int_0^{2h}\left\{\frac{h}{k}\sin\left(k\left(u+h\right)\right)+\frac{h}{k}\sin\left(k\left(3u-h\right)\right)+\frac{1}{k^2}\cos\left(k\left(u+h\right)\right)-\frac{1}{k^2}\cos\left(k\left(3u-h\right)\right)\right\}\frac{du}{u}$$

$$\begin{aligned}+i\frac{h}{2}V_1\int_0^{2h}&\left\{-\frac{2}{k}h\sin\left(k\left(u-h\right)\right)+\frac{2}{k^2}\cos\left(k\left(u-h\right)\right)-\frac{1}{k}h\sin\left(k\left(u+h\right)\right)\right.\\&\left.+\frac{h}{k}\sin\left(k\left(3u-h\right)\right)-\frac{1}{k^2}\cos\left(k\left(u+h\right)\right)-\frac{1}{k^2}\cos\left(k\left(3u-h\right)\right)\right\}\frac{du}{u}\end{aligned}$$

$$+i\frac{2}{3}V_1\sin\left(kh\right)\left\{-\frac{1}{k}u^2\cos\left(ku\right)+\frac{2}{k^2}u\sin\left(ku\right)+\frac{2}{k^3}\cos\left(ku\right)+\frac{3}{k}hu\cos\left(ku\right)-\frac{3}{k^2}h\sin\left(ku\right)-\frac{3}{k}h^2\cos\left(ku\right)\right\}_0^{2h}$$

$$-i\frac{2}{3}h^3V_1\sin\left(kh\right)\mathrm{Si}\left(2kh\right)$$

$$+i\frac{2}{k}V_1\sin\left(kh\right)\left\{\frac{1}{k^2}-\frac{h}{k}\sin\left(2kh\right)-\frac{1}{k^2}\cos\left(2kh\right)\right\}$$

$$= -2h^3 V_1 \left\{C_e^o + 2\ln(h)\right\} \left\{-\frac{1}{kh}\cos(kh) + \frac{1}{k^2h^2}\sin(kh) - \frac{1}{3}\sin(kh)\right\}$$

$$+2V_1 \left[\left\{-\frac{h^2}{k}\cos(kh) - \frac{h^2}{k}\cos(kh) + \frac{h}{k^2}\sin(kh) - \frac{1}{3}h^3\sin(kh) - \left(\frac{1}{3}h^2 - \frac{1}{k^2}\right)h\sin(kh)\right\}\ln(2h)\right]$$

$$-2V_1 \left\{\frac{h^2}{k}\sin(kh)\,\mathrm{Si}(2kh) - \frac{h^2}{k}\cos(kh)\,\mathrm{Cin}(2kh) + \frac{h}{k^2}\cos(kh)\,\mathrm{Si}(2kh) + \frac{h}{k^2}\sin(kh)\,\mathrm{Cin}(2kh)\right\}$$

$$-2V_1 \left\{-\frac{2h}{k^2}\sin(kh) - \frac{1}{3}\left(\frac{8}{3}h^3 - 6h^3 + 6h^3\right)\sin(kh)\right\}$$

$$+2V_1 \left[-(h/2)\left\{\frac{h}{k}\sin(kh) + \frac{1}{k^2}\cos(kh)\right\}\mathrm{Si}(4kh) - (h/2)\left\{-\frac{h}{k}\cos(kh) + \frac{1}{k^2}\sin(kh)\right\}\mathrm{Cin}(4kh) + \sin(kh)\,\frac{1}{3}h^3\mathrm{Cin}(2kh)\right]$$

$$+\frac{h}{2k^2}V_1 \left\{\sin(3kh) - \sin(kh) - \frac{1}{3}\sin(5kh) - \frac{1}{3}\sin(kh)\right\}$$

$$+\frac{h}{2k}V_1 \left\{h\cos(kh)\,\mathrm{Cin}(2kh) - h\cos(kh)\,\mathrm{Cin}(6kh) + h\sin(kh)\,\mathrm{Si}(2kh) + h\sin(kh)\,\mathrm{Si}(6kh)\right\}$$

$$+\frac{h}{2k}V_1 \left\{\frac{1}{k}\cos(kh)\,\mathrm{Si}(2kh) + \frac{1}{k}\cos(kh)\,\mathrm{Si}(6kh) - \frac{1}{k}\sin(kh)\,\mathrm{Cin}(2kh) + \frac{1}{k}\sin(kh)\,\mathrm{Cin}(6kh)\right\}$$

$$+\frac{h}{2k}V_1 \left\{-\frac{4}{k}\sin(kh) + \frac{1}{k}\sin(3kh) - \frac{1}{k}\sin(kh) + \frac{1}{3k}\sin(5kh) + \frac{1}{3k}\sin(kh)\right\}$$

$$+\frac{h}{2k}V_1 \left\{-h\cos(kh)\,\mathrm{Cin}(2kh) + h\cos(kh)\,\mathrm{Cin}(6kh) + \frac{1}{k}\sin(kh)\,\mathrm{Cin}(2kh) - \frac{1}{k}\sin(kh)\,\mathrm{Cin}(6kh)\right\}$$

$$+\frac{h}{2k}V_1 \left\{3h\sin(kh)\,\mathrm{Si}(2kh) - h\sin(kh)\,\mathrm{Si}(6kh) + \frac{3}{k}\cos(kh)\,\mathrm{Si}(2kh) - \frac{1}{k}\cos(kh)\,\mathrm{Si}(6kh)\right\}$$

$$+\frac{2}{3}V_1\sin(kh) \left\{\frac{4}{k}h^2\sin(2kh) + \frac{4}{k^2}h\cos(2kh) - \frac{2}{k^3}\sin(2kh) - \frac{6}{k}h^2\sin(2kh) - \frac{3}{k^2}h\cos(2kh) + \frac{3}{k^2}h + \frac{3}{k}h^2\sin(2kh)\right\}$$

$$-\frac{2}{3}V_1\sin(kh)\,\frac{8}{3}h^3 + 2V_1\sin(kh)\,\frac{1}{3}h^3\mathrm{Cin}(2kh)$$

$$+\frac{2}{k}V_1\sin(kh)\left\{-\frac{h}{k}\cos(2kh) - \frac{h}{k} + \frac{1}{k^2}\sin(2kh)\right\}$$

$$+i2V_1\left[(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Si}(4kh)\right.$$

$$\left.-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Cin}(4kh)-\frac{1}{3}h^3\sin(kh)\,\mathrm{Si}(2kh)\right]$$

$$-i\frac{h}{2k^2}V_1\left\{\cos(3kh)-\cos(kh)+\frac{1}{3}\cos(5kh)-\frac{1}{3}\cos(kh)\right\}$$

$$+i\frac{h}{2}V_1\left\{-\frac{1}{k^2}\cos(3kh)+\frac{1}{k^2}\cos(kh)+\frac{1}{3k^2}\cos(5kh)-\frac{1}{3k^2}\cos(kh)\right\}$$

$$-i\frac{h}{2}V_1\int_0^{2h}\left\{\frac{h}{k}\cos(kh)\sin(ku)+\frac{h}{k}\sin(3ku)\cos(kh)-\frac{1}{k^2}\sin(ku)\sin(kh)-\frac{1}{k^2}\sin(3ku)\sin(kh)\right\}\frac{du}{u}$$

$$-i\frac{h}{2}V_1\int_0^{2h}\left\{\frac{h}{k}\cos(ku)\sin(kh)-\frac{h}{k}\cos(3ku)\sin(kh)+\frac{1}{k^2}\cos(ku)\cos(kh)-\frac{1}{k^2}\cos(3ku)\cos(kh)\right\}\frac{du}{u}$$

$$+i\frac{h}{2}V_1\int_0^{2h}\left\{-\frac{3}{k}h\sin(ku)\cos(kh)+\frac{3}{k^2}\sin(ku)\sin(kh)+\frac{h}{k}\sin(3ku)\cos(kh)-\frac{1}{k^2}\sin(3ku)\sin(kh)\right\}\frac{du}{u}$$

$$+i\frac{h}{2}V_1\int_0^{2h}\left\{\frac{h}{k}\cos(ku)\sin(kh)-\frac{h}{k}\cos(3ku)\sin(kh)+\frac{1}{k^2}\cos(ku)\cos(kh)-\frac{1}{k^2}\cos(3ku)\cos(kh)\right\}\frac{du}{u}$$

$$+i\frac{2}{3}V_1\sin(kh)\left\{-\frac{4}{k}h^2\cos(2kh)+\frac{4}{k^2}h\sin(2kh)+\frac{2}{k^3}\cos(2kh)-\frac{2}{k^3}+\frac{6}{k}h^2\cos(2kh)\right.$$

$$\left.-\frac{3}{k^2}h\sin(2kh)-\frac{3}{k}h^2\cos(2kh)+\frac{3}{k}h^2\right\}$$

$$-i\frac{2}{3}h^3V_1\sin(kh)\,\mathrm{Si}(2kh)+i\frac{2}{k}V_1\sin(kh)\left\{\frac{1}{k^2}-\frac{h}{k}\sin(2kh)-\frac{1}{k^2}\cos(2kh)\right\}$$

$$=-2h^3V_1\left\{C_e^o+2\ln(h)\right\}\left\{-\frac{1}{kh}\cos(kh)+\frac{1}{k^2h^2}\sin(kh)-\frac{1}{3}\sin(kh)\right\}$$

$$+2V_1\left[\left\{-\frac{h^2}{k}\cos(kh)-\frac{h^2}{k}\cos(kh)+\frac{h}{k^2}\sin(kh)-\frac{1}{3}h^3\sin(kh)-\left(\frac{1}{3}h^2-\frac{1}{k^2}\right)h\sin(kh)\right\}\ln(2h)\right]$$

$$-2V_1\left\{\frac{h^2}{k}\sin(kh)\,\mathrm{Si}(2kh)-\frac{h^2}{k}\cos(kh)\,\mathrm{Cin}(2kh)+\frac{h}{k^2}\cos(kh)\,\mathrm{Si}(2kh)+\frac{h}{k^2}\sin(kh)\,\mathrm{Cin}(2kh)\right\}$$

$$-2V_1\left\{-\frac{2h}{k^2}\sin(kh)-\frac{1}{3}\left(\frac{8}{3}h^3-6h^3+6h^3\right)\sin(kh)\right\}$$

$$+2V_1\left[-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Si}(4kh)\right.$$

$$\left.-(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Cin}(4kh)+\sin(kh)\frac{1}{3}h^3\mathrm{Cin}(2kh)\right]$$

$$+\frac{h}{2k^2}V_1\left\{\sin(3kh)-\sin(kh)-\frac{1}{3}\sin(5kh)-\frac{1}{3}\sin(kh)\right\}$$

$$+\frac{h}{2k}V_1\left\{h\cos(kh)\mathrm{Cin}(2kh)-h\cos(kh)\mathrm{Cin}(6kh)+h\sin(kh)\mathrm{Si}(2kh)+h\sin(kh)\mathrm{Si}(6kh)\right\}$$

$$+\frac{h}{2k}V_1\left\{\frac{1}{k}\cos(kh)\mathrm{Si}(2kh)+\frac{1}{k}\cos(kh)\mathrm{Si}(6kh)-\frac{1}{k}\sin(kh)\mathrm{Cin}(2kh)+\frac{1}{k}\sin(kh)\mathrm{Cin}(6kh)\right\}$$

$$+\frac{h}{2k}V_1\left\{-\frac{4}{k}\sin(kh)+\frac{1}{k}\sin(3kh)-\frac{1}{k}\sin(kh)+\frac{1}{3k}\sin(5kh)+\frac{1}{3k}\sin(kh)\right\}$$

$$+\frac{h}{2k}V_1\left\{-h\cos(kh)\mathrm{Cin}(2kh)+h\cos(kh)\mathrm{Cin}(6kh)+\frac{1}{k}\sin(kh)\mathrm{Cin}(2kh)-\frac{1}{k}\sin(kh)\mathrm{Cin}(6kh)\right\}$$

$$+\frac{h}{2k}V_1\left\{3h\sin(kh)\mathrm{Si}(2kh)-h\sin(kh)\mathrm{Si}(6kh)+\frac{3}{k}\cos(kh)\mathrm{Si}(2kh)-\frac{1}{k}\cos(kh)\mathrm{Si}(6kh)\right\}$$

$$+\frac{2}{3}V_1\sin(kh)\left\{\frac{4}{k}h^2\sin(2kh)+\frac{4}{k^2}h\cos(2kh)-\frac{2}{k^3}\sin(2kh)-\frac{6}{k}h^2\sin(2kh)-\frac{3}{k^2}h\cos(2kh)+\frac{3}{k^2}h+\frac{3}{k}h^2\sin(2kh)\right\}$$

$$-\frac{2}{3}V_1\sin(kh)\frac{8}{3}h^3+2V_1\sin(kh)\frac{1}{3}h^3\mathrm{Cin}(2kh)$$

$$+\frac{2}{k}V_1\sin(kh)\left\{-\frac{h}{k}\cos(2kh)-\frac{h}{k}+\frac{1}{k^2}\sin(2kh)\right\}$$

$$+i2V_1\left[(h/2)\left\{-\frac{h}{k}\cos(kh)+\frac{1}{k^2}\sin(kh)\right\}\mathrm{Si}(4kh)-(h/2)\left\{\frac{h}{k}\sin(kh)+\frac{1}{k^2}\cos(kh)\right\}\mathrm{Cin}(4kh)-\frac{1}{3}h^3\sin(kh)\mathrm{Si}(2kh)\right]$$

$$-i\frac{h}{2k^2}V_1\left\{\cos(3kh)-\cos(kh)+\frac{1}{3}\cos(5kh)-\frac{1}{3}\cos(kh)\right\}$$

$$+i\frac{h}{2}V_1\left\{-\frac{1}{k^2}\cos(3kh)+\frac{1}{k^2}\cos(kh)+\frac{1}{3k^2}\cos(5kh)-\frac{1}{3k^2}\cos(kh)\right\}$$

$$-i\frac{h}{2}V_1\left\{\frac{h}{k}\cos(kh)\mathrm{Si}(2kh)+\frac{h}{k}\cos(kh)\mathrm{Si}(6kh)-\frac{1}{k^2}\sin(kh)\mathrm{Si}(2kh)-\frac{1}{k^2}\sin(kh)\mathrm{Si}(6kh)\right\}$$

$$-i\frac{h}{2}V_1\left\{-\frac{h}{k}\sin(kh)\operatorname{Cin}(2kh)+\frac{h}{k}\sin(kh)\operatorname{Cin}(6kh)-\frac{1}{k^2}\cos(kh)\operatorname{Cin}(2kh)+\frac{1}{k^2}\cos(kh)\operatorname{Cin}(6kh)\right\}$$

$$+i\frac{h}{2}V_1\left\{-\frac{3}{k}h\cos(kh)\operatorname{Si}(2kh)+\frac{3}{k^2}\sin(kh)\operatorname{Si}(2kh)+\frac{h}{k}\cos(kh)\operatorname{Si}(6kh)-\frac{1}{k^2}\sin(kh)\operatorname{Si}(6kh)\right\}$$

$$+i\frac{h}{2}V_1\left\{-\frac{h}{k}\sin(kh)\operatorname{Cin}(2kh)+\frac{h}{k}\sin(kh)\operatorname{Cin}(6kh)-\frac{1}{k^2}\cos(kh)\operatorname{Cin}(2kh)+\frac{1}{k^2}\cos(kh)\operatorname{Cin}(6kh)\right\}$$

$$+i\frac{2}{3}V_1\sin(kh)\left\{-\frac{4}{k}h^2\cos(2kh)+\frac{4}{k^2}h\sin(2kh)+\frac{2}{k^3}\cos(2kh)-\frac{2}{k^3}\right.$$

$$\left.+\frac{6}{k}h^2\cos(2kh)-\frac{3}{k^2}h\sin(2kh)-\frac{3}{k}h^2\cos(2kh)+\frac{3}{k}h^2\right\}$$

$$-i\frac{2}{3}h^3V_1\sin(kh)\operatorname{Si}(2kh)+i\frac{2}{k}V_1\sin(kh)\left\{\frac{1}{k^2}-\frac{h}{k}\sin(2kh)-\frac{1}{k^2}\cos(2kh)\right\}$$

Thus we find

$$\int_{-h}^{h}sF_1(s)\,ds=-2h^3V_1\left\{C_e^o-2\ln(2)\right\}\left\{-\frac{1}{kh}\cos(kh)+\frac{1}{k^2h^2}\sin(kh)-\frac{1}{3}\sin(kh)\right\}$$

$$+\frac{4}{3k}V_1\sin(kh)\left\{\left(\frac{1}{k^2}+h^2\right)\sin(kh)\cos(kh)-\frac{h}{k}\sin^2(kh)+\frac{2}{k}h\right\}$$

$$+V_1\left[-\frac{h}{k}\left\{h\sin(kh)+\frac{1}{k}\cos(kh)\right\}\operatorname{Si}(4kh)-\frac{h}{k}\left\{-h\cos(kh)+\frac{1}{k}\sin(kh)\right\}\left\{\operatorname{Cin}(4kh)+2\operatorname{Cin}(2kh)\right\}\right.$$

$$\left.+\frac{4}{3}h^3\sin(kh)\operatorname{Cin}(2kh)\right]$$

$$+iV_1\left[\frac{h}{k}\left\{\frac{1}{k}\sin(kh)-h\cos(kh)\right\}\left\{2\operatorname{Si}(2kh)+\operatorname{Si}(4kh)\right\}-\frac{h}{k}\left\{h\sin(kh)+\frac{1}{k}\cos(kh)\right\}\operatorname{Cin}(4kh)-\frac{4}{3}h^3\sin(kh)\operatorname{Si}(2kh)\right]$$

$$+i\frac{4}{3k}V_1\sin(kh)\left\{\left(h^2+\frac{1}{k^2}\right)\sin^2(kh)+\frac{h}{k}\sin(kh)\cos(kh)+h^2\right\}$$

Then

$$\pi\eta_0hY_{rad}^o\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=\frac{i}{kh}\left\{kh\cos(kh)-\sin(kh)\right\}\frac{2}{hV_1}F_1(h)+i\frac{k}{h^2V_1}\sin(kh)\int_{-h}^{h}sF_1(s)\,ds$$

$$\frac{2}{hV_1}F_1(h) = \left[-\cos(kh)\,\mathrm{Si}(4kh) + \sin(kh)\frac{2}{kh}\sin(2kh) - \sin(kh)\,\mathrm{Cin}(4kh) + 2\sin(kh)\,\mathrm{Cin}(2kh)\right]$$

$$+i\left[\sin(kh)\,\mathrm{Si}(4kh) - \cos(kh)\,\mathrm{Cin}(4kh) - 2\sin(kh)\,\mathrm{Si}(2kh) + 2\sin(kh)\frac{1}{kh}\{1-\cos(2kh)\}\right]$$

$$\pi\eta_0 hY^o_{rad}\left\{1 + 3\cos(kh)\frac{\sin(kh)}{kh} - 4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$= -i2\sin(kh)\left\{-\cos(kh) + \frac{1}{kh}\sin(kh) - \frac{1}{3}kh\sin(kh)\right\}\{C^o_e - 2\ln(2)\}$$

$$+i\frac{4}{3}\sin^2(kh)\left\{\left(\frac{1}{k^2h^2}+1\right)\sin(kh)\cos(kh) - \frac{1}{kh}\sin^2(kh) + \frac{2}{kh}\right\}$$

$$+i\left\{\cos(kh) - \frac{1}{kh}\sin(kh)\right\}\left[-\cos(kh)\,\mathrm{Si}(4kh) + \sin(kh)\frac{2}{kh}\sin(2kh) - \sin(kh)\,\mathrm{Cin}(4kh) + 2\sin(kh)\,\mathrm{Cin}(2kh)\right]$$

$$+i\sin(kh)\left[-\left\{\sin(kh) + \frac{1}{kh}\cos(kh)\right\}\mathrm{Si}(4kh) + \left\{\cos(kh) - \frac{1}{kh}\sin(kh)\right\}\{\mathrm{Cin}(4kh) + 2\mathrm{Cin}(2kh)\}\right.$$

$$\left.+\frac{4}{3}kh\sin(kh)\,\mathrm{Cin}(2kh)\right]$$

$$-\left\{\cos(kh) - \frac{1}{kh}\sin(kh)\right\}\left[\sin(kh)\,\mathrm{Si}(4kh) - \cos(kh)\,\mathrm{Cin}(4kh) - 2\sin(kh)\,\mathrm{Si}(2kh) + 2\sin(kh)\frac{1}{kh}\{1-\cos(2kh)\}\right]$$

$$-\sin(kh)\left[\left\{\frac{1}{kh}\sin(kh) - \cos(kh)\right\}\{2\,\mathrm{Si}(2kh) + \mathrm{Si}(4kh)\} - \left\{\sin(kh) + \frac{1}{kh}\cos(kh)\right\}\mathrm{Cin}(4kh) - \frac{4}{3}kh\sin(kh)\,\mathrm{Si}(2kh)\right]$$

$$-\frac{4}{3}\sin^2(kh)\left\{\left(1+\frac{1}{k^2h^2}\right)\sin^2(kh) + \frac{1}{kh}\sin(kh)\cos(kh) + 1\right\}$$

or

$$\pi\eta_0 hY^o_{rad}\left\{1 + 3\cos(kh)\frac{\sin(kh)}{kh} - 4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$= i2\sin(kh)\left\{\cos(kh) - \frac{1}{kh}\sin(kh) + \frac{1}{3}kh\sin(kh)\right\}\{C^o_e - 2\ln(2)\}$$

$$+i\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh) - \frac{1}{kh}\sin(kh) + \frac{1}{2}kh\sin(kh)\right\}\sin^2(kh)\cos(kh) + i\frac{4}{3}\frac{1}{kh}\sin^2(kh)$$

$$+i\left[-\operatorname{Si}(4kh)+\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}4\operatorname{Cin}(2kh)\right]$$

$$+\operatorname{Cin}(4kh)+\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}4\operatorname{Si}(2kh)$$

$$-\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh)-\frac{1}{kh}\sin(kh)\right\}\sin^3(kh)-\frac{4}{3}\sin^2(kh)\left\{\sin^2(kh)+1\right\}$$

Therefore we can write

$$\pi\eta_0 hG^o_{rad}\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=\operatorname{Cin}(4kh)+\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}4\operatorname{Si}(2kh)$$

$$-\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh)-\frac{1}{kh}\sin(kh)+\frac{1}{2}kh\sin(kh)\right\}\sin^3(kh)-\frac{4}{3}\sin^2(kh)$$

$$\pi\eta_0 hB^o_{rad}\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=-2\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}\left\{C^o_e-2\ln(2)\right\}$$

$$-\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh)-\frac{1}{kh}\sin(kh)+\frac{1}{2}kh\sin(kh)\right\}\sin^2(kh)\cos(kh)-\frac{4}{3}\frac{1}{kh}\sin^2(kh)$$

$$+\operatorname{Si}(4kh)-\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}4\operatorname{Cin}(2kh)$$

As a check we compared these with numerical integration of the integrals in

$$\pi\eta_0 hG^o_{rad}\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=-\frac{1}{kh}\left\{kh\cos(kh)-\sin(kh)\right\}\frac{2}{hV_1}\operatorname{Im}\left\{F_1(h)\right\}-\frac{k}{h^2V_1}\sin(kh)\operatorname{Im}\left\{\int_{-h}^{h}sF_1(s)\,ds\right\}$$

$$\pi\eta_0 hB^o_{rad}\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=-\frac{1}{kh}\left\{kh\cos(kh)-\sin(kh)\right\}\frac{2}{hV_1}\operatorname{Re}\left\{F_1(h)\right\}-\frac{k}{h^2V_1}\sin(kh)\operatorname{Re}\left\{\int_{-h}^{h}sF_1(s)\,ds\right\}$$

If we take the low frequency limit of the right hand side of B^o_{rad}, using the expansions

$$\operatorname{Si}(x)=\int_0^x\frac{\sin u}{u}du\sim\int_0^x\left[1-u^2/6+u^4/120-u^5/5040\right]du=x\left(1-x^2/18+x^4/600-x^6/35280\right)$$

$$\mathrm{Cin}\,(x)=\int_0^x \frac{1-\cos u}{u}du \sim \int_0^x \left[u/2-u^3/24+u^5/720-u^7/40320\right]du = x^2/4-x^4/96+x^6/4320-x^8/322560$$

$$=\left(x^2/4\right)\left(1-x^2/24+x^4/1080-x^6/80640\right)$$

we find

$$\sim -2kh\left\{1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4-\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4\right)+\frac{1}{3}k^2h^2\left(1-\frac{1}{6}k^2h^2\right)\right\}\{C_e^o-2\ln(2)\}$$

$$-\frac{8}{3}\left\{2\left(1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4\right)-\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4\right)+\frac{1}{2}k^2h^2\left(1-\frac{1}{6}k^2h^2\right)\right\}$$

$$kh\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4\right)\left(1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4\right)$$

$$-\frac{4}{3}kh\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4\right)$$

$$+4kh-\frac{32}{9}k^3h^3+\frac{128}{75}k^5h^5-kh\left(1-\frac{1}{6}k^2h^2\right)\left\{\left(1-\frac{1}{2}k^2h^2\right)-\left(1-\frac{1}{3}k^2h^2\right)\left(1-\frac{1}{6}k^2h^2\right)\right\}4\left(k^2h^2-\frac{1}{6}k^4h^4+\frac{2}{135}k^6h^6\right)$$

$$\sim\left[\frac{1}{3}\{C_e^o-2\ln(2)\}+\frac{64}{5}-\frac{2}{9}49\right]\frac{2}{15}k^5h^5=\left[\{C_e^o-2\ln(2)\}+3\frac{64}{5}-\frac{2}{3}49\right]\frac{2}{45}k^5h^5$$

$$\sim\left[\{C_e^o-2\ln(2)\}+\frac{86}{15}\right]\frac{2}{45}k^5h^5$$

The second line was simplified as

$$-\frac{8}{3}kh\left\{1-\frac{1}{3}k^2h^2-\frac{1}{120}k^4h^4\right\}\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4\right)\left(1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4\right)-\frac{4}{3}kh\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4\right)$$

$$=-\frac{8}{3}kh\left\{1-\frac{1}{3}k^2h^2-\frac{1}{120}k^4h^4\right\}\left(1-\frac{5}{6}k^2h^2+\frac{91}{360}k^4h^4\right)-\frac{4}{3}kh\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4\right)$$

$$=-\frac{8}{3}kh\left\{1-\frac{7}{6}k^2h^2+\frac{47}{90}k^4h^4\right\}-\frac{4}{3}kh\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4\right)$$

$$=-\frac{4}{3}kh\left\{3-\frac{8}{3}k^2h^2+\frac{49}{45}k^4h^4\right\}=-kh\left\{4-\frac{32}{9}k^2h^2+\frac{4}{3}\frac{49}{45}k^4h^4\right\}$$

If we take

$$C_e^o=2\ln(2)-86/15=2\left[\ln(2)-43/15\right]$$

we get $O\left(k^7h^7\right)$ and

$$\pi\eta_0 h B^o_{rad}\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$=2\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}86/15$$

$$-\frac{8}{3}\frac{1}{kh}\left\{2\cos(kh)-\frac{1}{kh}\sin(kh)+\frac{1}{2}kh\sin(kh)\right\}\sin^2(kh)\cos(kh)-\frac{4}{3}\frac{1}{kh}\sin^2(kh)$$

$$+\mathrm{Si}(4kh)-\sin(kh)\left\{\cos(kh)-\left(\frac{1}{kh}-\frac{1}{3}kh\right)\sin(kh)\right\}4\mathrm{Cin}(2kh)$$

If we take the low frequency limit of the right hand side of G^o_{rad}

$$\sim 4k^2h^2\left(1-\frac{2}{3}k^2h^2+\frac{32}{135}k^4h^4-\frac{16}{315}k^6h^6\right)$$

$$+8k^2h^2\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)\left\{1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4-\frac{1}{720}k^6h^6\right.$$

$$\left.-\left(1-\frac{1}{3}k^2h^2\right)\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)\right\}\left(1-\frac{2}{9}k^2h^2+\frac{2}{75}k^4h^4-\frac{4}{2205}k^6h^6\right)$$

$$-\frac{8}{3}k^2h^2\left\{2\left(1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4-\frac{1}{720}k^6h^6\right)-\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)\right.$$

$$\left.+\frac{1}{2}k^2h^2\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)\right\}\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)^3$$

$$-\frac{4}{3}k^2h^2\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)^2$$

$$\sim 4k^2h^2\left(1-\frac{2}{3}k^2h^2+\frac{32}{135}k^4h^4-\frac{16}{315}k^6h^6\right)$$

$$+8k^2h^2\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)\left\{1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4-\frac{1}{720}k^6h^6-\left(1-\frac{1}{2}k^2h^2+\frac{23}{360}k^4h^4-\frac{1}{336}k^6h^6\right)\right\}$$

$$\left(1-\frac{2}{9}k^2h^2+\frac{2}{75}k^4h^4-\frac{4}{2205}k^6h^6\right)$$

$$-\frac{8}{3}k^2h^2\left\{1-\frac{1}{3}k^2h^2-\frac{1}{120}k^4h^4+\frac{1}{630}k^6h^6\right\}\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4-\frac{1}{315}k^6h^6\right)\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)$$

$$-\frac{4}{3}k^2h^2\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4-\frac{1}{315}k^6h^6\right)$$

$$\sim 4k^2h^2\left(1-\frac{2}{3}k^2h^2+\frac{32}{135}k^4h^4-\frac{16}{315}k^6h^6\right)$$

$$-\frac{8}{45}k^6h^6\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)\left\{1-\frac{1}{14}k^2h^2\right\}\left(1-\frac{2}{9}k^2h^2+\frac{2}{75}k^4h^4-\frac{4}{2205}k^6h^6\right)$$

$$-\frac{8}{3}k^2h^2\left\{1-\frac{1}{3}k^2h^2-\frac{1}{120}k^4h^4+\frac{1}{630}k^6h^6\right\}\left(1-\frac{1}{2}k^2h^2+\frac{13}{120}k^4h^4-\frac{41}{3024}k^6h^6\right)$$

$$-\frac{4}{3}k^2h^2\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4-\frac{1}{315}k^6h^6\right)$$

$$\sim 4k^2h^2\left(1-\frac{2}{3}k^2h^2+\frac{32}{135}k^4h^4-\frac{16}{315}k^6h^6\right)-\frac{8}{45}k^6h^6\left(1-\frac{29}{63}k^2h^2\right)$$

$$-\frac{8}{3}k^2h^2\left\{1-\frac{1}{3}k^2h^2-\frac{1}{120}k^4h^4+\frac{1}{630}k^6h^6\right\}\left(1-\frac{1}{2}k^2h^2+\frac{13}{120}k^4h^4-\frac{41}{3024}k^6h^6\right)$$

$$-\frac{4}{3}k^2h^2\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4-\frac{1}{315}k^6h^6\right)$$

$$\sim 4k^2h^2\left(1-\frac{2}{3}k^2h^2+\frac{26}{135}k^4h^4-\frac{202}{2835}k^6h^6\right)$$

$$-\frac{8}{3}k^2h^2\left\{1-\frac{5}{6}k^2h^2+\frac{4}{15}k^4h^4-\frac{83}{1890}k^6h^6\right\}$$

$$-\frac{4}{3}k^2h^2\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4-\frac{1}{315}k^6h^6\right)$$

$$\sim 4k^2h^2\left(-\frac{2}{3}k^2h^2+\frac{26}{135}k^4h^4-\frac{404}{27\cdot 2\cdot 3\cdot 5\cdot 7}k^6h^6\right)$$

$$+4k^2h^2\left\{\frac{2}{3}k^2h^2-\frac{26}{135}k^4h^4+\frac{267}{27\cdot 2\cdot 3\cdot 5\cdot 7}k^6h^6\right\}$$

However, a numerical check in this limit indicates a smaller value

$$\pi\eta_0 hG^o_{rad}D^o_0 \approx \frac{4\cdot 4}{450\cdot 135}k^{10}h^{10}$$

Next the coefficient has the low frequency behavior

$$D^o_0=\left\{1+3\cos(kh)\frac{\sin(kh)}{kh}-4\left(1-\frac{1}{6}k^2h^2\right)\frac{\sin^2(kh)}{k^2h^2}\right\}$$

$$\sim 1+3\left(1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4-\frac{1}{720}k^6h^6\right)\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)$$

$$-4\left(1-\frac{1}{6}k^2h^2\right)\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)^2$$

$$\sim 1+3\left(1-\frac{1}{2}k^2h^2+\frac{1}{24}k^4h^4-\frac{1}{720}k^6h^6\right)\left(1-\frac{1}{6}k^2h^2+\frac{1}{120}k^4h^4-\frac{1}{5040}k^6h^6\right)$$

$$-4\left(1-\frac{1}{6}k^2h^2\right)\left(1-\frac{1}{3}k^2h^2+\left(\frac{1}{36}+\frac{1}{60}\right)k^4h^4-2\left(\frac{1}{5040}+\frac{1}{720}\right)k^6h^6\right)$$

$$\sim 1+3\left(1-\left(\frac{1}{2}+\frac{1}{6}\right)k^2h^2+\left(\frac{1}{12}+\frac{1}{24}+\frac{1}{120}\right)k^4h^4-\left(\frac{1}{5040}+\frac{1}{720}+\frac{1}{240}+\frac{1}{144}\right)k^6h^6\right)$$

$$-4\left(1-\frac{1}{6}k^2h^2\right)\left(1-\frac{1}{3}k^2h^2+\frac{2}{45}k^4h^4-\frac{1}{315}k^6h^6\right)$$

$$\sim 1+3\left(1-\frac{2}{3}k^2h^2+\frac{2}{15}k^4h^4-\frac{4}{315}k^6h^6\right)-4\left(1-\left(\frac{1}{3}+\frac{1}{6}\right)k^2h^2+\left(\frac{2}{45}+\frac{1}{18}\right)k^4h^4-\left(\frac{1}{315}+\frac{1}{135}\right)k^6h^6\right)$$

$$\sim 1+3\left(1-\frac{2}{3}k^2h^2+\frac{2}{15}k^4h^4-\frac{4}{315}k^6h^6\right)-4\left(1-\frac{1}{2}k^2h^2+\frac{1}{10}k^4h^4-\frac{2}{189}k^6h^6\right)$$

$$\sim 4-2k^2h^2+\frac{2}{5}k^4h^4-\frac{4}{105}k^6h^6-\left(4-2k^2h^2+\frac{2}{5}k^4h^4-\frac{8}{189}k^6h^6\right)=\left(\frac{8}{189}-\frac{4}{105}\right)k^6h^6$$

$$\sim \frac{40-36}{945}k^6h^6=\frac{4}{945}k^6h^6$$

This page left blank

References

[1] L. K. Warne, R. E. Jorgenson, J. T. Williams, L. I. Basilio, W. L. Langston, R. S. Coats, S. Campione, and K. C. Chen, "A Bound On Electromagnetic Penetration Through A Slot Aperture With Backing Cavity," Sandia National Laboratories Report, SAND2016-9029, September 2016.

[2] L. K. Warne, S. Campione, L. San Martin, W. L. Langston, A. Pack, and B. Zinser, "Penetration Bounds For Slot With Backing Rectangular Cavity", Sandia Report in preparation.

[3] S. Campione, L. K. Warne, W. L. Langston, R. A. Pfeiffer, N. Martin, J. T. Williams, R. K. Guitierrez, I. C. Reines, J. G. Huerta, and V. Q. Dang, "Penetration through Slots in Cylindrical Cavities Operating at Fundamental Cavity Modes," IEEE Transactions on Electromagnetic Compatibility, 62(5), pp1980-1988, (2020).

[4] S. Campione, L. K. Warne, W. L. Langston, R. K. Guitierrez, J. W. Hicks, I. C. Reines, R. A. Pfeiffer, J. J. Himbele, and J. T. Williams, "Penetration through slots in cylindrical cavities with cavity modes overlapping with the first slot resonance," Electromagnetics, 41(2), pp. 98-109, (2021).

[5] S. Campione, L. K. Warne, I. C. Reines, R. K. Guitierrez, and J. T. Williams, "Penetration through Slots in Cylindrical Cavities Operating at Fundamental Cavity Modes in the Presence of Electromagnetic Absorbers," Progress in Electromagnetics Research, M96, pp. 119-127, (2020).

[6] S. Campione, L. K. Warne, C. Grimms, J. T. Williams, R. K. Gutierrez, R. S. Coats, and L. I. Basilio, "Perturbation theory to model shielding effectiveness of cavities loaded with electromagnetic dampeners," Electronics Letters, 55(11), 644-646, (2019).

[7] J. R. Wait, "Electromagnetic Field Analysis for a Coaxial Cable with Periodic Slots," IEEE Transactions on Electromagnetic Compatibility, Vol. EMC-19, No. 1, Feb. 1977, pp. 7-13.

[8] J. A. Stratton, **Electromagnetic Theory**, New York: McGraw-Hill Book Co., 1941, p. 372.

[9] L. K. Warne and K. C. Chen, "A Simple Transmission Line Model for Narrow Slots Having Depth and Losses," IEEE Transactions on Electromagnetic Compatibility, Vol. 34, No. 3, August 1992, pp. 173-182.

[10] L. K. Warne, W. A. Johnson, B. F. Zinser, W. L. Langston, R. S. Coats, I. C. Reines, J. T. Williams, L. I. Basilio, and K. C. Chen, "Narrow Slot Algorithm," Sandia National Laboratories Report, SAND2020-3979, April 2020.

[11] K. S. H. Lee and F. C. Yang, "Trends and Bounds in RF Coupling to a Wire Inside a Slotted Cavity," IEEE Transactions On Electromagnetic Compatibility, Vol. 34, No. 3, Aug. 1992.

[12] S. Campione and L. K. Warne, "Penetration through slots in overmoded cavities," IEEE Transactions on Electromagnetic Compatibility, DOI:10.1109/TEMC.2021.3067005 (2021).

[13] M. Abramowitz and I. A. Stegun (editors), **Handbook of Mathematical Functions**, New York: Dover Pub. Inc., pp. 231, 358, 360.

[14] L. K. Warne, H. G. Hudson, W. A. Johnson, R. E. Jorgenson, S. L. Stronach, and K. S. H. Lee, "Input Impedance of Antennas in High Frequency Cavities," Sandia National Laboratories Report, SAND2000-3112, December 2000.

[15] L. K. Warne, R. E. Jorgenson, and H. G. Hudson, "Electromagnetic Field Limits Set By The V-Curve," Sandia National Laboratories Report, SAND2014-5038, July 2014.

[16] G. N. Watson, **A Treatise On The Theory Of Bessel Functions**,Cambridge: The University Press, 1966, pp. 135, 137.

[17] K. S. H. Lee, **EMP Interaction: Principles, Techniques, and Reference Data**, New York: Hemisphere Pub. Corp., 1986.

[18] E. C. Jordan and K. G. Balmain, **Electromagnetic Waves And Radiating Systems**, Englewood Cliffs, NJ: Prentice-Hall, Inc., 1968, pp. 544, 584.

[19] L. K. Warne, K. S. H. Lee, H. G. Hudson, W. A. Johnson, R. E. Jorgenson, and S. L. Stronach, "Statistical properties of Linear Antenna Impedance in an Electrically Large Cavity," IEEE Transactions on Antennas and Propagation, Vol. 51, No. 5, May 2003.

DISTRIBUTION

Email—Internal

Name	Org.	Sandia Email Address
K. O. Merewether	02900	MS0382
K. C. Chen	00531	MS0492
W. L. Langston	01352	MS1152
S. Campione	01352	MS1152
J. D. Kotulski	01352	MS1152
A. R. Pack	01352	MS1152
R. A. Pfeiffer	01352	MS1152
L. San Martin	01352	MS1152
I. Timmons	01352	MS1152
J. W. Wallace	01352	MS1152
L. K. Warne	01352	MS1152
Technical Library	01911	sanddocs@sandia.gov

Hardcopy—Internal

Number of Copies	Name	Org.	Mailstop
2	S. Campione	01352	MS1152
2	A. R. Pack	01352	MS1152
2	L. San Martin	01352	MS1152
5	L. K. Warne	01352	MS1152

This page left blank

This page left blank

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia LLC, a wholly owned subsidiary of Honeywell International Inc. for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.