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Three-Dimensional Electromagnetic High Frequency Convex Cavity Scars

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Three Dimensional Electromagnetic High Frequency Convex Cavity Scars

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Abstract

This report examines the localization of high frequency electromagnetic fields in general three-dimensional convex walled cavities along periodic paths between opposing sides of the cavity. The report examines the three-dimensional case where the mirrors at the end of the orbit have two different radii of curvature. The cases where these orbits lead to unstable localized modes are known as scars.

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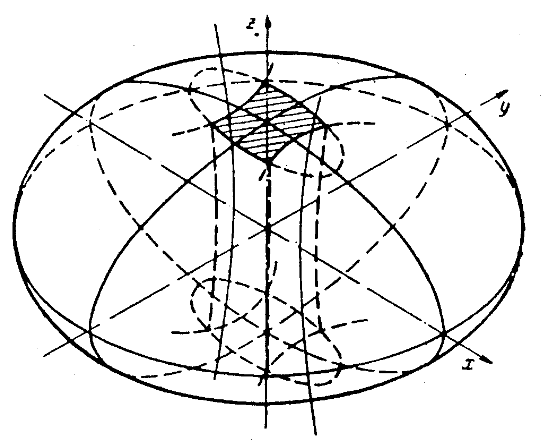

Figure 1: This figure, taken from Vaynshteyn, illustrates a bouncing ball mode beetween concave reflecting surfaces.

1 INTRODUCTION

Field behavior in high quality factor cavities typically takes on a stochastic character at high frequencies [1], particularly for geometries not supporting stable periodic ray orbits. Even in these classically unstable geometries, regions of higher modal intensity along these periodic ray orbits, known as scars [2], can exist. This report is directed at understanding the high frequency behavior of modal fields in three-dimensional cavities and the localization of the eigenfunctions about unstable periodic orbits, known as scarring, is investigated in convex walled geometry.

The random phase approach used by Antonsen [3], on convex mirror geometries in two dimensions, has been generalized [4], [5], [6], [7], [8], [9] by introducing the curved ray path formalism, used previously by Vaynshteyn [10] on stable orbits, see Figure 1. This combined approach was used recently to investigate the scars in three-dimensional axisymmetric geometry [6], [11], [12], and a previous report and journal articles detail both scalar and vector problems for convex and concave walls in three-dimensional axisymmetric geometry [13], [14], [15].

The present report explores bouncing ball modes forming scars in three-dimensional cavities with convex walls. We compare Fourier projections along the scar with projections of random plane wave representations, to illustrate how the scars modify the purely random representations.

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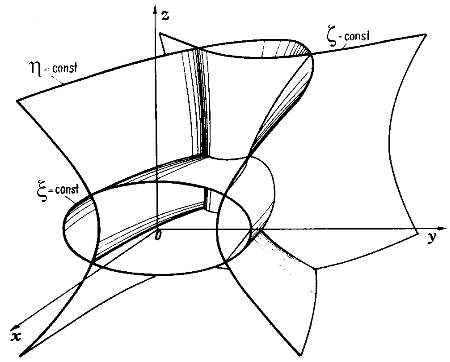

Figure 2: This figure, taken from Moon and Spencer, illustrates the types of coordinate surfaces involved in ellipsoidal coordinates.

2 CONVEX MIRRORS AND 3D BOWTIE

Vaynshteyn [10] has treatments for stable modes between concave mirrors. Here we wish to consider the generalization to unstable modes between convex mirrors. Vaynshteyn uses WKB analysis of the Lame function for the full 3D case. For three dimensions the scalar problem will be treated in ellipsoidal coordinates. Note that the vector problem is not separable in this system, but for high frequencies we will rely on the approximate correspondence between scalar and vector problems we observed in the axisymmetric case [13], [14], [15].

Another issue is stability; in the convex case both mirror axes will be unstable and the foci are located exterior to the cavity. Note in a follow-on report treating concave mirrors, although we will again be focussed on the unstable case with interior foci, it is also possible to have stability in one transverse direction and instability in the other direction (an interesting mixed case, which could relate back to 2D); alternatively, we could also have a convex mirror in one direction and a concave mirror in the other direction.

2.1 Ellipsoidal Coordinates

The ellipsoid is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

where a, b, and c are the principal semi-axes of the ellipsoid. We order the dimensions as

$$a > b > c > 0 \tag{2}$$

The system of ellipsoidal coordinates (ξ, η, ζ) shown in Figure 2 [16]corresponding to the ellipsoid is defined by the relation with Cartesian coordinates [10]

$$x^2 = \frac{\left(a^2 - \xi\right)\left(a^2 - \eta\right)\left(a^2 - \zeta\right)}{\left(b^2 - a^2\right)\left(c^2 - a^2\right)} \tag{3}$$

$$y^2 = \frac{\left(b^2 - \xi\right)\left(b^2 - \eta\right)\left(b^2 - \zeta\right)}{\left(c^2 - b^2\right)\left(a^2 - b^2\right)} \tag{4}$$

$$z^2 = \frac{\left(c^2-\xi\right)\left(c^2-\eta\right)\left(c^2-\zeta\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} \tag{5}$$

where

$$-\infty < \xi < c^2 \tag{6}$$

$$c^2 < \eta < b^2 \tag{7}$$

$$b^2 < \zeta < a^2 \tag{8}$$

Note that in Moon and Spencer [16] ellipsoidal coordinates (η, θ, λ) are defined as

$$u^1 = \eta\ ,\ \ c^2 < \eta^2 < \infty \tag{9}$$

$$u^2 = \theta\ ,\ \ b^2 < \theta^2 < c^2 \tag{10}$$

$$u^3 = \lambda\ ,\ \ 0 \leq \lambda^2 < b^2 \tag{11}$$

$$c^2 > b^2 > 0 \tag{12}$$

$$x^2 = \left(\frac{\eta\theta\lambda}{bc}\right)^2 \tag{13}$$

$$y^2 = \frac{\left(\eta^2-b^2\right)\left(\theta^2-b^2\right)\left(b^2-\lambda^2\right)}{b^2\left(c^2-b^2\right)} \tag{14}$$

$$z^2 = \frac{\left(\eta^2-c^2\right)\left(c^2-\theta^2\right)\left(c^2-\lambda^2\right)}{c^2\left(c^2-b^2\right)} \tag{15}$$

A comparison of the two systems implies that we do not need one of the three constants a, b, c in Vaynshteyn's system, or that we can set one to a fixed constant. We can take $c^2 = 0$, for example.

For fixed $\xi = \xi_0$

$$\frac{\left(b^2-a^2\right)\left(c^2-a^2\right)}{\left(a^2-\xi_0\right)}x^2 = \left(a^2-\eta\right)\left(a^2-\zeta\right) = a^4 - a^2\left(\eta+\zeta\right) + \eta\zeta \tag{16}$$

$$\frac{\left(c^2-b^2\right)\left(a^2-b^2\right)}{\left(b^2-\xi_0\right)}y^2 = \left(b^2-\eta\right)\left(b^2-\zeta\right) = b^4 - b^2\left(\eta+\zeta\right) + \eta\zeta \tag{17}$$

$$\frac{\left(a^2-c^2\right)\left(b^2-c^2\right)}{\left(c^2-\xi_0\right)}z^2 = \left(c^2-\eta\right)\left(c^2-\zeta\right) = c^4 - c^2\left(\eta+\zeta\right) + \eta\zeta \tag{18}$$

or

$$\frac{\left(c^2-a^2\right)}{\left(a^2-\xi_0\right)}x^2 + \frac{\left(c^2-b^2\right)}{\left(b^2-\xi_0\right)}y^2 = -\left(a^2+b^2\right) + \eta + \zeta \tag{19}$$

$$\frac{\left(b^2-a^2\right)}{\left(a^2-\xi_0\right)}x^2+\frac{\left(b^2-c^2\right)}{\left(c^2-\xi_0\right)}z^2=-\left(a^2+c^2\right)+\eta+\zeta \tag{20}$$

or subtracting these two

$$\frac{x^2}{\left(a^2-\xi_0\right)}+\frac{y^2}{\left(b^2-\xi_0\right)}+\frac{z^2}{\left(c^2-\xi_0\right)}=1 \tag{21}$$

defining the family of ellipsoids for $-\infty<\xi_0<c^2$. For large negative ξ_0 these become large spheres. Note that the x direction has the largest semimajor axis (since the denominator is the largest), followed by the y direction, and finally the z direction has the smallest. For $\xi_0\to c^2$ these become elliptic discs with $z=0$ and contained within the curve

$$\frac{x^2}{\left(a^2-c^2\right)}+\frac{y^2}{\left(b^2-c^2\right)}=1 \tag{22}$$

Note that the outer tip of these $z=0$ disc regions is at $x^2=a^2-c^2$ the outer focal points.

Alternatively for fixed $\eta=\eta_0$

$$\frac{\left(b^2-a^2\right)\left(c^2-a^2\right)}{\left(a^2-\eta_0\right)}x^2=\left(a^2-\xi\right)\left(a^2-\zeta\right)=a^4-a^2\left(\xi+\zeta\right)+\xi\zeta \tag{23}$$

$$\frac{\left(c^2-b^2\right)\left(a^2-b^2\right)}{\left(b^2-\eta_0\right)}y^2=\left(b^2-\xi\right)\left(b^2-\zeta\right)=b^4-b^2\left(\xi+\zeta\right)+\xi\zeta \tag{24}$$

$$\frac{\left(a^2-c^2\right)\left(b^2-c^2\right)}{\left(c^2-\eta_0\right)}z^2=\left(c^2-\xi\right)\left(c^2-\zeta\right)=c^4-c^2\left(\xi+\zeta\right)+\xi\zeta \tag{25}$$

or

$$\frac{\left(c^2-a^2\right)}{\left(a^2-\eta_0\right)}x^2+\frac{\left(c^2-b^2\right)}{\left(b^2-\eta_0\right)}y^2=-\left(a^2+b^2\right)+\xi+\zeta \tag{26}$$

$$\frac{\left(b^2-a^2\right)}{\left(a^2-\eta_0\right)}x^2+\frac{\left(b^2-c^2\right)}{\left(c^2-\eta_0\right)}z^2=-\left(a^2+c^2\right)+\xi+\zeta \tag{27}$$

or subtracting these two

$$\frac{x^2}{\left(a^2-\eta_0\right)}+\frac{y^2}{\left(b^2-\eta_0\right)}-\frac{z^2}{\left(\eta_0-c^2\right)}=1 \tag{28}$$

defining the family of hyperboloids of one sheet for $c^2<\eta_0<b^2$.

The limit $\eta_0\to c^2$ defines the disc region outside the elliptic discs defined by $z=0$ and the same elliptic curve

$$\frac{x^2}{\left(a^2-c^2\right)}+\frac{y^2}{\left(b^2-c^2\right)}=1 \tag{29}$$

The limit $\eta_0\to b^2$ defines a flat hyperbolic disc region with $y=0$ and lying within (in x) the curves satisfying

$$\frac{x^2}{\left(a^2-b^2\right)}-\frac{z^2}{\left(b^2-c^2\right)}=1 \tag{30}$$

Alternatively for fixed $\zeta = \zeta_0$

$$\frac{\left(b^2-a^2\right)\left(c^2-a^2\right)}{\left(a^2-\zeta_0\right)}x^2 = \left(a^2-\xi\right)\left(a^2-\eta\right) = a^4 - a^2\left(\xi+\eta\right)+\xi\eta \tag{31}$$

$$\frac{\left(c^2-b^2\right)\left(a^2-b^2\right)}{\left(b^2-\zeta_0\right)}y^2 = \left(b^2-\xi\right)\left(b^2-\eta\right) = b^4 - b^2\left(\xi+\eta\right)+\xi\eta \tag{32}$$

$$\frac{\left(a^2-c^2\right)\left(b^2-c^2\right)}{\left(c^2-\zeta_0\right)}z^2 = \left(c^2-\xi\right)\left(c^2-\eta\right) = c^4 - c^2\left(\xi+\eta\right)+\xi\eta \tag{33}$$

or

$$\frac{\left(c^2-a^2\right)}{\left(a^2-\zeta_0\right)}x^2 + \frac{\left(c^2-b^2\right)}{\left(b^2-\zeta_0\right)}y^2 = -\left(a^2+b^2\right)+\xi+\eta \tag{34}$$

$$\frac{\left(b^2-a^2\right)}{\left(a^2-\zeta_0\right)}x^2 + \frac{\left(b^2-c^2\right)}{\left(c^2-\zeta_0\right)}z^2 = -\left(a^2+c^2\right)+\xi+\eta \tag{35}$$

or subtracting these two

$$\frac{x^2}{\left(a^2-\zeta_0\right)} - \frac{y^2}{\left(\zeta_0-b^2\right)} - \frac{z^2}{\left(\zeta_0-c^2\right)} = 1 \tag{36}$$

defining a family of hyperboloids of two sheets for $b^2 < \zeta_0 < a^2$. Note that these elliptic hyperboloids of two sheets have a larger z dimension than a y dimension due to the fact that $\zeta_0 - b^2 < \zeta_0 - c^2$. The limit $\zeta_0 \to b^2$ defines flat disc regions with $y = 0$ and outside (in x) the same hyperbolas

$$\frac{x^2}{\left(a^2-b^2\right)} - \frac{z^2}{\left(b^2-c^2\right)} = 1 \tag{37}$$

Note that the inner tip of these $y = 0$ disc regions is at $x^2 = a^2 - b^2$ the inner focal points. The limit $\zeta_0 \to a^2$ defines planar regions with $x = 0$.

Vaynshteyn's description of the ellipsoidal system discusses the one-to-one mapping of the preceding transformations for a single octant of the space; to index the other octants we would have to use analytic continuation about the transformation branch points, for example

$$z = \sqrt{\frac{\left(c^2-\xi\right)\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}} = e^{i\theta/2}\sqrt{\frac{\widetilde{\xi}\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}} \to c^2-\xi = \widetilde{\xi}e^{i\theta} \tag{38}$$

or

$$y = \sqrt{\frac{\left(b^2-\xi\right)\left(b^2-\eta\right)\left(\zeta-b^2\right)}{\left(b^2-c^2\right)\left(a^2-b^2\right)}} = e^{i\varphi'/2}\sqrt{\frac{\left(b^2-\xi\right)\widetilde{\eta}\left(\zeta-b^2\right)}{\left(b^2-c^2\right)\left(a^2-b^2\right)}} \to b^2-\eta = \widetilde{\eta}e^{i\varphi'} \tag{39}$$

$$z = \sqrt{\frac{\left(c^2-\xi\right)\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}} = e^{i\varphi/2}\sqrt{\frac{\widetilde{\xi}\widetilde{\eta}\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}} \to \eta-c^2 = \widetilde{\eta}e^{i\varphi} \tag{40}$$

2.1.1 Unit Vectors in Ellipsoidal Coordinates

The ellipsoidal coordinate transformations [10]

$$x^2 = \frac{\left(a^2-\xi\right)\left(a^2-\eta\right)\left(a^2-\zeta\right)}{\left(a^2-b^2\right)\left(a^2-c^2\right)} \tag{41}$$

$$y^2 = \frac{\left(b^2-\xi\right)\left(b^2-\eta\right)\left(\zeta-b^2\right)}{\left(b^2-c^2\right)\left(a^2-b^2\right)} \tag{42}$$

$$z^2 = \frac{\left(c^2-\xi\right)\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} \tag{43}$$

have metric coeffi cients [10]

$$h_\xi = \frac{1}{2}\sqrt{\frac{(\eta-\xi)(\zeta-\xi)}{(a^2-\xi)(b^2-\xi)(c^2-\xi)}} \tag{44}$$

$$h_\eta = \frac{1}{2}\sqrt{\frac{(\zeta-\eta)(\eta-\xi)}{(a^2-\eta)(b^2-\eta)(\eta-c^2)}} \tag{45}$$

$$h_\zeta = \frac{1}{2}\sqrt{\frac{(\zeta-\xi)(\zeta-\eta)}{(a^2-\zeta)(\zeta-b^2)(\zeta-c^2)}} \tag{46}$$

The position vector in Cartesian coordinates is

$$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z \tag{47}$$

and the derivatives can be used to define the unit vectors and metric coeffi cients [17]

$$h_\zeta\underline{e}_\zeta = \frac{\partial\underline{r}}{\partial\zeta} = \frac{\partial x}{\partial\zeta}\underline{e}_x + \frac{\partial y}{\partial\zeta}\underline{e}_y + \frac{\partial z}{\partial\zeta}\underline{e}_z \tag{48}$$

$$h_\eta\underline{e}_\eta = \frac{\partial\underline{r}}{\partial\eta} = \frac{\partial x}{\partial\eta}\underline{e}_x + \frac{\partial y}{\partial\eta}\underline{e}_y + \frac{\partial z}{\partial\eta}\underline{e}_z \tag{49}$$

$$h_\xi\underline{e}_\xi = \frac{\partial\underline{r}}{\partial\xi} = \frac{\partial x}{\partial\xi}\underline{e}_x + \frac{\partial y}{\partial\xi}\underline{e}_y + \frac{\partial z}{\partial\xi}\underline{e}_z \tag{50}$$

The unit vectors can be found by differentiation [17]

$$\left|\frac{\partial\underline{r}}{\partial\zeta}\right|\underline{e}_\zeta = \frac{\partial\underline{r}}{\partial\zeta} = \frac{\partial x}{\partial\zeta}\underline{e}_x + \frac{\partial y}{\partial\zeta}\underline{e}_y + \frac{\partial z}{\partial\zeta}\underline{e}_z = h_\zeta\underline{e}_\zeta$$

$$= -\frac{1}{2}\sqrt{\frac{(a^2-\xi)(a^2-\eta)}{(a^2-\zeta)(a^2-b^2)(a^2-c^2)}}\underline{e}_x + \frac{1}{2}\sqrt{\frac{(b^2-\xi)(b^2-\eta)}{(a^2-b^2)(\zeta-b^2)(b^2-c^2)}}\underline{e}_y + \frac{1}{2}\sqrt{\frac{(c^2-\xi)(\eta-c^2)}{(a^2-c^2)(\zeta-c^2)(b^2-c^2)}}\underline{e}_z \tag{51}$$

$$\left|\frac{\partial\underline{r}}{\partial\eta}\right|\underline{e}_\eta = \frac{\partial\underline{r}}{\partial\eta} = \frac{\partial x}{\partial\eta}\underline{e}_x + \frac{\partial y}{\partial\eta}\underline{e}_y + \frac{\partial z}{\partial\eta}\underline{e}_z = h_\eta\underline{e}_\eta$$

$$= -\frac{1}{2}\sqrt{\frac{(a^2-\xi)(a^2-\zeta)}{(a^2-b^2)(a^2-\eta)(a^2-c^2)}}\underline{e}_x - \frac{1}{2}\sqrt{\frac{(b^2-\xi)(\zeta-b^2)}{(a^2-b^2)(b^2-\eta)(b^2-c^2)}}\underline{e}_y + \frac{1}{2}\sqrt{\frac{(c^2-\xi)(\zeta-c^2)}{(a^2-c^2)(b^2-c^2)(\eta-c^2)}}\underline{e}_z \tag{52}$$

$$\left|\frac{\partial \underline{r}}{\partial \xi}\right| \underline{e}_\xi = \frac{\partial \underline{r}}{\partial \xi} = \frac{\partial x}{\partial \xi}\underline{e}_x + \frac{\partial y}{\partial \xi}\underline{e}_y + \frac{\partial z}{\partial \xi}\underline{e}_z = h_\xi \underline{e}_\xi$$

$$= -\frac{1}{2}\sqrt{\frac{(a^2-\eta)(a^2-\zeta)}{(a^2-\xi)(a^2-b^2)(a^2-c^2)}}\underline{e}_x - \frac{1}{2}\sqrt{\frac{(b^2-\eta)(\zeta-b^2)}{(b^2-\xi)(b^2-c^2)(a^2-b^2)}}\underline{e}_y - \frac{1}{2}\sqrt{\frac{(\eta-c^2)(\zeta-c^2)}{(c^2-\xi)(a^2-c^2)(b^2-c^2)}}\underline{e}_z \tag{53}$$

Of course in this orthonormal system (ξ, η, ζ)

$$\underline{e}_\zeta \cdot \underline{e}_\zeta = \underline{e}_\xi \cdot \underline{e}_\xi = \underline{e}_\eta \cdot \underline{e}_\eta = 1 \tag{54}$$

$$\underline{e}_\zeta \cdot \underline{e}_\xi = \underline{e}_\zeta \cdot \underline{e}_\eta = \underline{e}_\eta \cdot \underline{e}_\xi = 0 \tag{55}$$

$$\underline{e}_\xi \times \underline{e}_\eta = \underline{e}_\zeta \tag{56}$$

$$\underline{e}_\eta \times \underline{e}_\zeta = \underline{e}_\xi \tag{57}$$

$$\underline{e}_\zeta \times \underline{e}_\xi = \underline{e}_\eta \tag{58}$$

2.2 Convex Mirrors With Orbit Along ζ (At Mirror)

In this present case we will take the two radii of curvature to represent convex mirrors in both tangential surface directions at the terminus of the x axis orbit. The mirrors then have fixed values of $\zeta = \zeta_0$ (hyperboloid of two sheets). The orbit center (in the transverse direction) consists of the x axis between the symmetry point 0 and the mirror center at $x = \sqrt{a^2 - \zeta_0}$; if the desired orbit length is

$$L = 2\ell = 2\sqrt{a^2 - \zeta_0} \tag{59}$$

then we have a constraint on $a^2 - \zeta_0$. The orbit is the limit $\eta = b^2$ and $\xi = c^2$ with $\zeta_0 < \zeta < a^2$ or

$$0 < x^2 = \left(a^2 - \zeta\right) < a^2 - \zeta_0 \tag{60}$$

$$y^2 = 0 = z^2 \tag{61}$$

The preceding relation on the mirror surface $\zeta = \zeta_0$ can be rewritten as

$$\frac{x^2}{(a^2-\zeta_0)} = 1 + \frac{y^2}{(\zeta_0 - b^2)} + \frac{z^2}{(\zeta_0 - c^2)} \tag{62}$$

from which we see that x^2 increases as either y^2 or z^2 increases showing that the mirror surface is convex in both tangential directions. Note that we cannot take the orbit along the y or z axes because there is no solution to the equation with the other two dimensions equal to zero for $a^2 > \zeta_0 > b^2$. We would have to consider annular orbits in these cases.

With the x direction along the orbit let us expand these relations near the point $\zeta = \zeta_0$, where $b^2 < \zeta_0 < a^2$, and $\eta \to b^2$ or $\eta = b^2 - \Delta_y^2$, and $\xi \to c^2$ or $\xi = c^2 - \Delta_z^2$. These expansions give

$$x^2 = \frac{\left(a^2 - c^2 + \Delta_z^2\right)\left(a^2 - b^2 + \Delta_y^2\right)\left(a^2 - \zeta_0\right)}{\left(b^2 - a^2\right)\left(c^2 - a^2\right)} \sim \left(a^2 - \zeta_0\right)\left[1 + \frac{\Delta_y^2}{a^2 - b^2} + \frac{\Delta_z^2}{a^2 - c^2}\right] \tag{63}$$

$$y^2 = \frac{\left(b^2 - c^2 + \Delta_z^2\right)\Delta_y^2\left(b^2 - \zeta_0\right)}{\left(c^2 - b^2\right)\left(a^2 - b^2\right)} \sim \frac{\left(\zeta_0 - b^2\right)}{\left(a^2 - b^2\right)}\Delta_y^2 \tag{64}$$

$$z^2 = \frac{\Delta_z^2\left(c^2 - b^2 + \Delta_y^2\right)\left(c^2 - \zeta_0\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} \sim \frac{\left(\zeta_0 - c^2\right)}{\left(a^2 - c^2\right)}\Delta_z^2 \tag{65}$$

Notice that near the orbit the coordinate η determines local variations in the y direction, while the coordinate ξ determines local variations in the z direction; positions x along the orbit are controlled by the coordinate ζ. Notice that at the orbit center $\zeta = a^2$ and $x^2 = 0$ with

$$y^2 = \frac{\left(b^2 - \xi\right)\left(b^2 - \eta\right)}{\left(b^2 - c^2\right)} \tag{66}$$

$$z^2 = \frac{\left(c^2 - \xi\right)\left(\eta - c^2\right)}{\left(b^2 - c^2\right)} \tag{67}$$

Now if $\xi = c^2$ and $z^2 = 0$ we find

$$y^2 = \left(b^2 - \eta\right) \ , \ c^2 < \eta < b^2 \tag{68}$$

To determine the radii of curvature we set either y or z equal to zero and examine the resulting equation compared to that of a circle

$$\left(x - x_0\right)^2 + y^2 = R_y^2 \tag{69}$$

or

$$x - x_0 = \pm R_y\sqrt{1 - y^2/R_y^2} \sim \pm R_y\left(1 - \frac{y^2/2}{R_y^2}\right) \tag{70}$$

For $z = 0$, in the above equation the circle center is located at

$$x_0 = R_y + \sqrt{a^2 - \zeta_0} = R_y + \ell \tag{71}$$

and then

$$\begin{aligned} x - x_0 &= \pm\left[\sqrt{a^2 - \zeta_0}\left(\sqrt{1 + \frac{y^2}{\zeta_0 - b^2}} - 1\right) - R_y\right] \\ &\sim \mp R_y\left(1 - \frac{\sqrt{a^2 - \zeta_0}}{R_y}\frac{y^2/2}{\zeta_0 - b^2}\right) \end{aligned} \tag{72}$$

Comparing with the previous expression for the circle we identify

$$R_y = \frac{\zeta_0 - b^2}{\sqrt{a^2 - \zeta_0}} = \frac{\zeta_0 - b^2}{\ell} \tag{73}$$

or

$$\zeta_0^2 - \left(2b^2 - R_y^2\right)\zeta_0 - \left(a^2 R_y^2 - b^4\right) = 0 \tag{74}$$

with quadratic solution

$$\zeta_0 = b^2 + \left(R_y^2/2\right)\left[\sqrt{1 + 4\left(a^2 - b^2\right)/R_y^2} - 1\right] \tag{75}$$

or from the preceding second equality, setting $\zeta_0 - b^2 = \ell R_y$ in this quadratic solution

$$\left[\left(1 + L/R_y\right)^2 - 1\right] R_y^2/4 = \left(a^2 - b^2\right) \tag{76}$$

or

$$L\left(2R_y + L\right)/4 = \ell\left(R_y + \ell\right) = \left(a^2 - b^2\right) \tag{77}$$

Similarly with $y = 0$ for the other direction

$$\left(x - x_0\right)^2 + z^2 = R_z^2 \tag{78}$$

or

$$x - x_0 = \pm R_z \sqrt{1 - z^2/R_z^2} \sim \pm R_z \left(1 - \frac{z^2/2}{R_z^2}\right) \tag{79}$$

with center at

$$x_0 = R_z + \sqrt{a^2 - \zeta_0} = R_z + \ell \tag{80}$$

yielding

$$\begin{aligned} x - x_0 &= \pm\left[\sqrt{a^2 - \zeta_0}\left(\sqrt{1 + \frac{z^2}{\zeta_0 - c^2}} - 1\right) - R_z\right] \\ &\sim \mp R_z \left(1 - \frac{\sqrt{a^2 - \zeta_0}}{R_z}\frac{z^2/2}{\zeta_0 - c^2}\right) \end{aligned} \tag{81}$$

and

$$R_z = \frac{\zeta_0 - c^2}{\sqrt{a^2 - \zeta_0}} = \frac{\zeta_0 - c^2}{\ell} \tag{82}$$

with quadratic solution

$$\zeta_0 = c^2 + \left(R_z^2/2\right)\left[\sqrt{1 + 4\left(a^2 - c^2\right)/R_z^2} - 1\right] \tag{83}$$

and using the second equality

$$\left[\left(1 + L/R_z\right)^2 - 1\right]\left(R_z^2/4\right) = \left(a^2 - c^2\right) \tag{84}$$

or

$$\left[\left(R_z+L\right)^2-R_z^2\right]/4=L\left(2R_z+L\right)/4=\ell\left(R_z+\ell\right)=\left(a^2-c^2\right) \tag{85}$$

For a consistent value of ζ_0 from the preceding two expressions

$$\left(b^2-c^2\right)+\left(R_y^2/2\right)\left[\sqrt{1+4\left(a^2-b^2\right)/R_y^2}-1\right]=\left(R_z^2/2\right)\left[\sqrt{1+4\left(a^2-c^2\right)/R_z^2}-1\right] \tag{86}$$

or

$$\left(b^2-c^2\right)=\ell\left(R_z-R_y\right)=\left(R_z-R_y\right)L/2 \tag{87}$$

which because of our desired ordering of coordinates requires $R_z \geq R_y$. In summary, to set the coordinate axes we take

$$\ell\left(R_y+\ell\right)=a^2-b^2 \tag{88}$$

$$\ell\left(R_z+\ell\right)=a^2-c^2 \tag{89}$$

$$\zeta_0=b^2+\ell R_y=c^2+\ell R_z \tag{90}$$

$$\ell^2+\zeta_0=a^2 \tag{91}$$

$$\ell\left(R_z-R_y\right)=b^2-c^2 \tag{92}$$

This final expression represents the range $y^2 \leq \left(b^2-c^2\right)=\ell\left(R_z-R_y\right)$. Note that the variation with $c^2<\eta<b^2$ when $a^2>\zeta>\zeta_0$ and $\xi=c^2$, is

$$0<x^2=\frac{\left(a^2-\eta\right)\left(a^2-\zeta\right)}{\left(a^2-b^2\right)}<\frac{\left(a^2-c^2\right)\left(a^2-\zeta_0\right)}{\left(a^2-b^2\right)}=\frac{R_z+\ell}{R_y+\ell}\ell^2 \sim \ell^2\ ,\ R_z \sim R_y \tag{93}$$

$$0<y^2=\frac{\left(b^2-\eta\right)\left(\zeta-b^2\right)}{\left(a^2-b^2\right)}<\frac{\left(b^2-c^2\right)\left(\zeta-b^2\right)}{\left(a^2-b^2\right)}=\frac{\left(R_z-R_y\right)}{\left(R_y+\ell\right)}\left(\zeta-b^2\right) \leq \left(R_z-R_y\right)\ell=b^2-c^2 \tag{94}$$

where the larger result varies over

$$\left(R_z-R_y\right)\frac{\ell R_y}{\left(R_y+\ell\right)}<y^2=\frac{\left(R_z-R_y\right)}{\left(R_y+\ell\right)}\left(\zeta-b^2\right)<\ell\left(R_z-R_y\right) \tag{95}$$

$$z^2=0 \tag{96}$$

For large and nearly equal radii of curvature this implies a variation of $0<y<\sqrt{\left(R_z-R_y\right)\ell}$. Note that there are four unknowns a, b, c, ζ_0 and three independent conditions. However, we also note that these hyperboloid surfaces of two sheets are defined by

$$\frac{x^2}{\left(a^2-\zeta_0\right)}-\frac{y^2}{\left(\zeta_0-b^2\right)}-\frac{z^2}{\left(\zeta_0-c^2\right)}=1 \tag{97}$$

and thus the actual unknowns $a^2-\zeta_0, \zeta_0-b^2, \zeta_0-c^2$ are only three in number. If we write the preceding equations in terms of these unknowns

$$a^2-\zeta_0=\ell^2 \tag{98}$$

$$\zeta_0-b^2=\ell R_y \tag{99}$$

$$\zeta_0-c^2=\ell R_z \tag{100}$$

and the equation of the hyperboloid surfaces is

$$\frac{x^2}{\ell^2}-\frac{y^2}{\ell R_y}-\frac{z^2}{\ell R_z}=1 \tag{101}$$

In this convex mirror case we will take the radii of curvature to be large in order to achieve stability exponents approaching unity from the unstable region. This argument indicates that we want to have

$$\ell R_z > \ell R_y >> \ell^2 \tag{102}$$

or

$$\zeta_0-c^2 > \zeta_0-b^2 >> a^2-\zeta_0 \tag{103}$$

Obviously if ζ_0 approaches a^2 this can hold. Also, if $\zeta_0 >> b^2 > c^2$, but $a >> b > c > 0$ with $\zeta_0 \to a^2$, it can also hold.

In the inside region ($\xi \to c^2$)

$$x^2=\frac{\left(a^2-\eta\right)}{\left(a^2-b^2\right)}\left(a^2-\zeta\right)\ ,\ c^2<\eta<b^2<\zeta<a^2 \tag{104}$$

$$y^2=\frac{\left(b^2-\eta\right)\left(\zeta-b^2\right)}{\left(a^2-b^2\right)}\ ,\ c^2<\eta<b^2<\zeta<a^2 \tag{105}$$

$$z^2=\frac{\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}\widetilde{\xi}\ ,\ \widetilde{\xi}=c^2-\xi\to 0\ ,\ c^2<\eta<b^2<\zeta<a^2 \tag{106}$$

We note that in this region $\eta \to b^2$ collapses the observation point near the scar center and is therefore an important location.

In summary for this case the orbit along x (controlled by ζ) has convex mirrors and two exterior foci.

2.2.1 Connection To Spheroidal Coordinates

Note that the limit to spheroidal coordinates is $b=c$ with geometry shown in Figure 3.

Here we expect to have unknowns a_p, b_p, ξ_{p0}, but only two conditions: radius of curvature R and orbital half length ℓ.

The prolate spheroidal coordinates $\left(\zeta_p, \varphi_p, \xi_p\right)$ are related to the cylindrical system (r,φ,z) by means of [10]

$$r=d\sinh\zeta_p\cos\xi_p \tag{107}$$

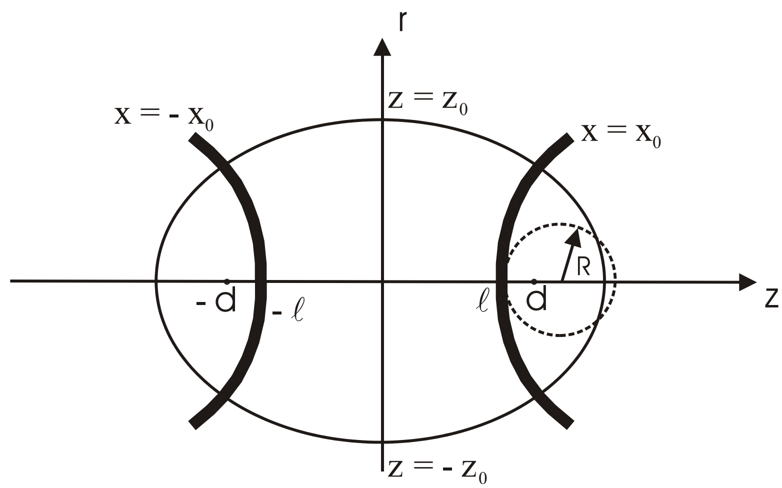

Figure 3: Spheroidal coordinate system and convex mirrors for case where the two radii of curvature are equal. The mirrors are located at $\xi = \pm\xi_0$ having locations $z = \pm\ell$ for a radius $r = 0$. The foci at $z = \pm d$ are outside the cavity region.

$$z = d\cosh\zeta_p \sin\xi_p \tag{108}$$

$$\varphi_p = \varphi \tag{109}$$

where

$$0 < \zeta_p < \infty \tag{110}$$

$$-\pi/2 < \xi_p < \pi/2\,,\; 0 < \varphi_p < 2\pi \tag{111}$$

On the mirror we can write

$$\ell = d\sin\xi_{p0} \tag{112}$$

Also about this point we can expand the coordinate relation

$$\begin{aligned} z = d\cosh\zeta_p \sin\xi_{p0} &\sim d\left(1+\zeta_p^2/2\right)\sin\xi_{p0} \sim \ell + \frac{r^2}{2d}\frac{\sin\xi_{p0}}{\cos^2\xi_{p0}} \\ &\sim \ell + R + \frac{r^2}{2d}\frac{\sin\xi_{p0}}{\cos^2\xi_{p0}} - R = z_0 + \frac{r^2}{2d}\frac{\sin\xi_{p0}}{\cos^2\xi_{p0}} - R \end{aligned} \tag{113}$$

$$r \sim d\zeta_p \cos\xi_{p0} \tag{114}$$

where R is the radius of curvature on the mirror and the center of the circle is at $z_0 = \ell + R$. For a circle of radius R and center position $r = 0$, $z = z_0$

$$r^2 + (z - z_0)^2 = R^2 \tag{115}$$

or

$$z = z_0 \pm \sqrt{R^2 - r^2} \sim z_0 \pm R\left(1 - \frac{r^2}{2R^2}\right) \tag{116}$$

Choosing the minus sign and comparing with the preceding gives

$$\frac{r^2}{2R} = \frac{r^2}{2d}\frac{\sin\xi_{p0}}{\cos^2\xi_{p0}} \tag{117}$$

or

$$R = d\cos^2\xi_{p0}/\sin\xi_{p0} = d\left(1 - \sin^2\xi_{p0}\right)/\sin\xi_{p0} \tag{118}$$

Using the length constraint (adding the length and then replacing the inverse sine by d/ℓ)

$$R + \ell = d^2/\ell \tag{119}$$

The focal distance d is chosen in terms of the mirror radius of curvature R and the half orbit length ℓ by means of

$$d = \ell\sqrt{1 + R/\ell} \tag{120}$$

The orbit extends over the range $-\xi_0 < \xi_p < \xi_0$ with $\zeta_p \to 0$ (or $-\ell < z < \ell$ and $r = 0$) and

$$\ell = d\sin\xi_0 \tag{121}$$

$$L = 2\ell \tag{122}$$

or

$$\sin\xi_0 = 1/\sqrt{1 + R/\ell} \tag{123}$$

Using the identity

$$\cos^2\xi_p + \sin^2\xi_p = 1 \tag{124}$$

we obtain the elliptical relation

$$\frac{z^2}{a_p^2} + \frac{r^2}{b_p^2} = 1 \tag{125}$$

where

$$b_p = d\sinh\zeta_p \tag{126}$$

$$a_p = d\cosh\zeta_p \tag{127}$$

$$a_p^2 - b_p^2 = d^2 = \ell\left(\ell + R\right) \tag{128}$$

Now in the ellipsoidal system the analog of the z axis would be along the ζ or x axis on the strip $\zeta = 0$ in the $x - y$ plane.

3 HELMHOLTZ EQUATION IN ELLIPSOIDAL COORDINATES

The metric coeffi cients in ellipsoidal coordinates are [10]

$$h_\xi = \frac{1}{2}\sqrt{\frac{(\xi-\eta)(\xi-\zeta)}{D(\xi)}} \tag{129}$$

$$h_\eta = \frac{1}{2}\sqrt{\frac{(\eta-\zeta)(\eta-\xi)}{D(\eta)}} \tag{130}$$

$$h_\zeta = \frac{1}{2}\sqrt{\frac{(\zeta-\xi)(\zeta-\eta)}{D(\zeta)}} \tag{131}$$

where

$$D(\theta) = \left(a^2-\theta\right)\left(b^2-\theta\right)\left(c^2-\theta\right) \tag{132}$$

Note that

$$D(\xi) > 0 \text{ for } -\infty < \xi < c^2 \tag{133}$$

$$D(\eta) < 0 \text{ for } c^2 < \eta < b^2 \tag{134}$$

$$D(\zeta) > 0 \text{ for } b^2 < \zeta < a^2 \tag{135}$$

The scalar wave equation

$$\left(\Delta + k^2\right)\Phi = \left(\nabla^2 + k^2\right)\Phi = 0 \tag{136}$$

then takes the form

$$\frac{1}{h_\xi h_\eta h_\zeta}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\zeta}{h_\xi}\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\xi h_\zeta}{h_\eta}\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{h_\xi h_\eta}{h_\zeta}\frac{\partial\Phi}{\partial\zeta}\right)\right] + k^2\Phi = 0 \tag{137}$$

or

$$\begin{aligned} &4(\eta-\zeta)\sqrt{D(\xi)}\frac{\partial}{\partial\xi}\left[\sqrt{D(\xi)}\frac{\partial\Phi}{\partial\xi}\right] - 4(\zeta-\xi)\sqrt{-D(\eta)}\frac{\partial}{\partial\eta}\left[\sqrt{-D(\eta)}\frac{\partial\Phi}{\partial\eta}\right] \\ &+4(\xi-\eta)\sqrt{D(\zeta)}\frac{\partial}{\partial\zeta}\left[\sqrt{D(\zeta)}\frac{\partial\Phi}{\partial\zeta}\right] + k^2(\xi-\eta)(\xi-\zeta)(\eta-\zeta)\Phi = 0 \end{aligned} \tag{138}$$

Let us introduce the variables [10]

$$\widetilde{\xi} = \int_0^\xi \frac{d\theta}{2\sqrt{D(\theta)}} \tag{139}$$

$$\widetilde{\eta} = \int_{c^2}^\eta \frac{d\theta}{2\sqrt{-D(\theta)}} \tag{140}$$

$$\widetilde{\zeta} = \int_{b^2}^\zeta \frac{d\theta}{2\sqrt{D(\theta)}} \tag{141}$$

and

$$\frac{\partial\widetilde{\xi}}{\partial\xi} = \frac{1}{2\sqrt{D(\xi)}} \tag{142}$$

$$\frac{\partial\widetilde{\eta}}{\partial\eta} = \frac{1}{2\sqrt{-D(\eta)}} \tag{143}$$

$$\frac{\partial\widetilde{\zeta}}{\partial\zeta} = \frac{1}{2\sqrt{D(\zeta)}} \tag{144}$$

which allow the scalar wave equation

$$(\eta-\zeta)\,2\sqrt{D(\xi)}\frac{\partial}{\partial\xi}\left[2\sqrt{D(\xi)}\frac{\partial\Phi}{\partial\xi}\right] - (\zeta-\xi)\,2\sqrt{-D(\eta)}\frac{\partial}{\partial\eta}\left[2\sqrt{-D(\eta)}\frac{\partial\Phi}{\partial\eta}\right]$$
$$+(\xi-\eta)\,2\sqrt{D(\zeta)}\frac{\partial}{\partial\zeta}\left[2\sqrt{D(\zeta)}\frac{\partial\Phi}{\partial\zeta}\right] + k^2(\xi-\eta)(\xi-\zeta)(\eta-\zeta)\,\Phi = 0 \tag{145}$$

to be written as

$$(\eta-\zeta)\frac{\partial^2\Phi}{\partial\widetilde{\xi}^2} - (\zeta-\xi)\frac{\partial^2\Phi}{\partial\widetilde{\eta}^2} + (\xi-\eta)\frac{\partial^2\Phi}{\partial\widetilde{\zeta}^2} + k^2(\xi-\eta)(\xi-\zeta)(\eta-\zeta)\,\Phi = 0 \tag{146}$$

We want to solve this equation with boundary condition on the ellipsoid surface

$$\Phi = 0 \text{ for } \zeta = \zeta_0 \tag{147}$$

or

$$\frac{\partial\Phi}{\partial\zeta} = 0 \text{ for } \zeta = \zeta_0 \tag{148}$$

Now separating variables

$$\Phi = X(\xi)\,Y(\eta)\,Z(\zeta) \tag{149}$$

$$4(\eta-\zeta)\sqrt{D(\xi)}\frac{1}{X}\frac{\partial}{\partial\xi}\left[\sqrt{D(\xi)}\frac{\partial X}{\partial\xi}\right] + 4(\zeta-\xi)\sqrt{D(\eta)}\frac{1}{Y}\frac{\partial}{\partial\eta}\left[\sqrt{D(\eta)}\frac{\partial Y}{\partial\eta}\right]$$
$$+4(\xi-\eta)\sqrt{D(\zeta)}\frac{1}{Z}\frac{\partial}{\partial\zeta}\left[\sqrt{D(\zeta)}\frac{\partial Z}{\partial\zeta}\right] + k^2(\xi-\eta)(\xi-\zeta)(\eta-\zeta) = 0 \tag{150}$$

Noting that the first term aside from the $(\eta-\zeta)$ factor is a function of ξ only, the second term aside from the $(\zeta-\xi)$ factor is a function of η only, and the third term, aside from the $(\xi-\eta)$ factor is a function of ζ only, we can write this as

$$-(\eta-\zeta)\,k^2 p(\xi) - (\zeta-\xi)\,k^2 p(\eta) - (\xi-\eta)\,k^2 p(\zeta) + k^2(\xi-\eta)(\xi-\zeta)(\eta-\zeta) = 0 \tag{151}$$

where each of these functions is a quadratic (with separation constants α and β)

$$p(\theta) = \theta^2 + \alpha\theta + \beta \tag{152}$$

in ξ, η, or ζ

$$2\sqrt{D(\xi)}\frac{1}{X}\frac{\partial}{\partial\xi}\left[2\sqrt{D(\xi)}\frac{\partial X}{\partial\xi}\right]=-k^2p(\xi) \tag{153}$$

$$2\sqrt{D(\eta)}\frac{1}{Y}\frac{\partial}{\partial\eta}\left[2\sqrt{D(\eta)}\frac{\partial Y}{\partial\eta}\right]=-k^2p(\eta) \tag{154}$$

$$2\sqrt{D(\zeta)}\frac{1}{Z}\frac{\partial}{\partial\zeta}\left[2\sqrt{D(\zeta)}\frac{\partial Z}{\partial\zeta}\right]=-k^2p(\zeta) \tag{155}$$

The quadratic terms cancel because we can write the k^2 term as

$$(\xi-\eta)(\xi-\zeta)(\eta-\zeta)=\xi^2(\eta-\zeta)+\eta^2(\zeta-\xi)+\zeta^2(\xi-\eta) \tag{156}$$

We can also show that the remaining terms cancel

$$(\eta-\zeta)(\alpha\xi+\beta)+(\zeta-\xi)(\alpha\eta+\beta)+(\xi-\eta)(\alpha\zeta+\beta)=0 \tag{157}$$

Note that these equations have the form

$$2\sqrt{D(\theta)}\frac{1}{\Theta}\frac{\partial}{\partial\theta}\left[2\sqrt{D(\theta)}\frac{\partial\Theta}{\partial\theta}\right]=-k^2p(\theta) \tag{158}$$

or

$$\frac{\partial^2\Theta}{\partial\theta^2}+\frac{1}{2}\frac{D'(\theta)}{D(\theta)}\frac{\partial\Theta}{\partial\theta}+\frac{1}{4}k^2\frac{p(\theta)}{D(\theta)}\Theta=0 \tag{159}$$

We can rewrite these as

$$\frac{d^2X}{d\widetilde{\xi}^2}+k^2p(\xi)X=0 \tag{160}$$

$$\frac{d^2Y}{d\widetilde{\eta}^2}-k^2p(\eta)Y=0 \tag{161}$$

$$\frac{d^2Z}{d\widetilde{\zeta}^2}+k^2p(\zeta)Z=0 \tag{162}$$

where $p(\theta)$ stands for the second order polynomial

$$p(\theta)=\theta^2+\alpha\theta+\beta \tag{163}$$

and α and β are separation constants. How do we determine the separation constants α and β (or θ_1 and θ_2)? In the axisymmetric problem the separation constants were related to the azimuthal number m and the wavenumber separation s. The boundary conditions on the mirror determined k_p. It would be nice if we could choose a simple solution in the direction Y (analog of $\cos(m\varphi)$ and $\sin(m\varphi)$, where we often took $m=0$). We can also write this as

$$p(\theta)=(\theta-\theta_1)(\theta-\theta_2) \tag{164}$$

$$\theta_j=-\frac{\alpha}{2}\pm\sqrt{\left(\frac{\alpha}{2}\right)^2-\beta} \tag{165}$$

These differential equations are the Lame wave equations. The functions X, Y, Z are the Lame wave functions [10].

The function Φ determines the modes inside a closed perfectly conducting surface, and therefore, apart from a constant factor, it must be real, as are the functions X, Y, and Z. From this it follows that the polynomial $p(\theta)$ must be real; the roots θ_1 and θ_2 are either real ($\beta < (\alpha/2)^2$) or are complex conjugates ($\beta > (\alpha/2)^2$). We will start with this same assumption because our total cavity is enclosed, but the scarred orbit interacts with the outer chaotic region in our case and is not enclosed by caustics. If θ_1 and θ_2 are complex conjugates, the polynomial $p(\theta)$ does not change sign, and at least one of the separation equations has a monotonic solution.

In the case of modes confined by caustic surfaces sign changes in $p(\theta)$ are desired to create the caustic boundaries [10]. However in our case with the convex mirrors we expect propagation not only in the x (or ζ) direction along the orbit, but also in the y (or η) and z (or ξ) directions as well. Hence either real or the solutions with complex conjugate values of θ_1 and θ_2 are likely to be the ones of interest. If we desired to generate modes with very little variation along the η direction (analog of m in the axisymmetric case) could we choose the separation constants appropriately to give this behavior? We might be tempted to have these modes (instead of generating a constant in η) generate the metric coeffi cient behavior as the orbit is approached $\xi \to c^2$

$$h_\eta = \frac{1}{2}\sqrt{\frac{(\eta-\zeta)(\eta-\xi)}{D(\eta)}} = \frac{1}{2}\sqrt{\frac{(\zeta-\eta)(\eta-\xi)}{(a^2-\eta)(b^2-\eta)(\eta-c^2)}} \sim \frac{1}{2}\sqrt{\frac{(\zeta-\eta)}{(a^2-\eta)(b^2-\eta)}} \tag{166}$$

since we might expect some form of generic variation over the ellipsoid (similar to cylindrical $h_\varphi = 1/\rho =$ constant in φ)? Do we expect normal chaotic behavior in this transverse direction excited by its coupling to the outer region? This is related to the question about how one would use our previous 2D solutions as approximations in a 3D cavity. If the dimensions are stretched so that there are many wavelengths in one of the transverse dimensions $k^2\left(b^2-c^2\right) >> 1$ then we might expect chaotic behavior excited from the edges of the cross-sectional disc region. Alternatively, if $k^2\left(b^2-c^2\right) = O(1)$ then the two transverse directions of the scarred orbit are similar in dimension, and we might expect the scar to be similar in character to the axisymmetric structure. Thus we could study the transitional behavior of the scar around this limit.

3.1 General Separation Equation And Choices Of Separation Constants

The preceding ordinary differential separation equations, with dependent variables $\Theta = X, Y, Z$ and independent coordinate variables $\theta = \xi, \eta, \zeta$ can be written as

$$\frac{d^2\Theta}{d\theta^2} - \frac{1}{2}\left(\frac{1}{a^2-\theta} + \frac{1}{b^2-\theta} + \frac{1}{c^2-\theta}\right)\frac{d\Theta}{d\theta} + \frac{1}{4}k^2\frac{(\theta_1-\theta)(\theta_2-\theta)}{(a^2-\theta)(b^2-\theta)(c^2-\theta)}\Theta = 0 \tag{167}$$

where again the coordinate parameters are ordered as

$$a > b > c > 0 \tag{168}$$

and the coordinates are restricted to the ranges

$$-\infty < \xi < c^2 \tag{169}$$

$$c^2 < \eta < b^2 \tag{170}$$

$$\zeta_0 < \zeta < a^2 \tag{171}$$

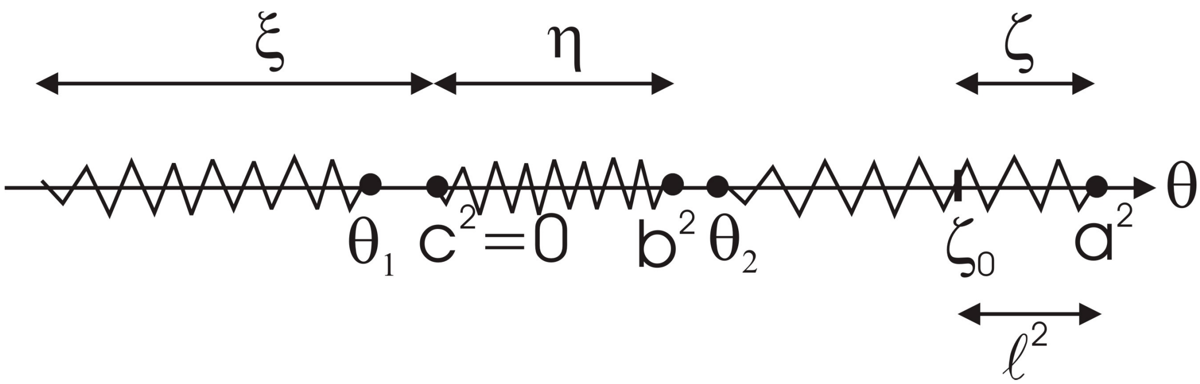

Figure 4: Elliptical system coordinate ranges and separation constant turning points, with regions of propagation illustrated by jagged lines.

We plan to order the separation constants, which are turning points, so that $\theta_2 > \theta_1$. Figure 4 illustrates the turning points and the regions of propagation (jagged lines) for the system coordinates. We also can see that for $\theta_1 < c^2$ we will see exponential behavior in ξ, whereas for $\theta_1 > c^2$ (or $\theta_2 < b^2$) we will see exponential behavior in η. Because the range of ζ is restricted to be $\zeta > \zeta_0$ we do not expect exponential behavior in ζ.

Because we expect the separation constant and turning point at θ_1 to be in the neighborhood of the singular point at c^2, and the separation constant and turning point at θ_2 to be in the neighborhood of the singular point at b^2, we will need to eventually examine the solution with combined turning and singular points for the functions $X(\xi)$ and $Y(\eta)$, whereas only the singular point limit near a^2 for $Z(\zeta)$.

3.2 General Separation Equation & Asymptotic Solutions

We discuss asymptotic formulas for the potential in each region using the combined critical point forms. Again the transformations between coordinates are

$$x^2 = \frac{\left(a^2-\xi\right)\left(a^2-\eta\right)\left(a^2-\zeta\right)}{\left(b^2-a^2\right)\left(c^2-a^2\right)} = \frac{\left(a^2-\xi\right)\left(a^2-\eta\right)\left(a^2-\zeta\right)}{\left(a^2-b^2\right)\left(a^2-c^2\right)} \tag{172}$$

$$y^2 = \frac{\left(b^2-\xi\right)\left(b^2-\eta\right)\left(b^2-\zeta\right)}{\left(c^2-b^2\right)\left(a^2-b^2\right)} = \frac{\left(b^2-\xi\right)\left(b^2-\eta\right)\left(\zeta-b^2\right)}{\left(b^2-c^2\right)\left(a^2-b^2\right)} \tag{173}$$

$$z^2 = \frac{\left(c^2-\xi\right)\left(c^2-\eta\right)\left(c^2-\zeta\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} = \frac{\left(c^2-\xi\right)\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} \tag{174}$$

where

$$-\infty < \xi < c^2 \tag{175}$$

$$c^2 < \eta < b^2 \tag{176}$$

$$\zeta_0 < \zeta < a^2 \tag{177}$$

We note that $R_z > R_y > \ell$ to make $a^2 > b^2 > c^2$. The coordinate system parameters can be found as (note that we can arbitrarily take $c^2 = 0$ if desired)

$$b^2 - c^2 = \ell \left(R_z - R_y\right) \tag{178}$$

$$a^2 - c^2 = \ell \left(\ell + R_z\right) \tag{179}$$

$$a^2 - b^2 = \ell \left(\ell + R_y\right) \tag{180}$$

The general separation equation with $\Theta = X, Y, Z$ and $\theta = \xi, \eta, \zeta$ is

$$\frac{d^2\Theta}{d\theta^2} - \frac{1}{2}\left(\frac{1}{a^2-\theta} + \frac{1}{b^2-\theta} + \frac{1}{c^2-\theta}\right)\frac{d\Theta}{d\theta} + \frac{1}{4}k^2\frac{\left(\theta_1-\theta\right)\left(\theta_2-\theta\right)}{\left(a^2-\theta\right)\left(b^2-\theta\right)\left(c^2-\theta\right)}\Theta = 0 \tag{181}$$

We will not make a specific choice of boundary conditions in this subsection but simply list the two asymptotic solutions. The Liouville [18] (also known as WKB) asymptotic solutions are

$$\Theta \sim \frac{c_0}{\sqrt[4]{p\left(\theta\right)}}\exp\left[\pm i\frac{k}{2}\int_{\theta_0}^{\theta}\sqrt{P\left(\theta'\right)}d\theta'\right] \text{ or } \frac{c_0}{\sqrt[4]{p\left(\theta\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{\theta_0}^{\theta}\sqrt{P\left(\theta'\right)}d\theta'\right]\right\} \tag{182}$$

where

$$p\left(\theta\right) = \left(\theta-\theta_1\right)\left(\theta-\theta_2\right) \tag{183}$$

$$P\left(\theta\right) = p\left(\theta\right)/D\left(\theta\right) \tag{184}$$

$$D\left(\theta\right) = \left(a^2-\theta\right)\left(b^2-\theta\right)\left(c^2-\theta\right) \tag{185}$$

The separation constants θ_1 and θ_2 are turning points of the equation. The points $a^2, b^2, c^2 = 0$ are singular points of the equation. The turning point locations in the convex walled bowtie configuration follow Figure 4. The limits of the equation at these various critical points will be given next (see also the following subsection). After that they will be used to construct the asymptotic solutions in the three coordinates along the periodic orbit. Matching along the orbit and in the outward direction will be carried out to connect the various regions and determine the complete mode behavior.

Near the critical points $\theta \to b^2, \theta_2$ we take

$$k_e^2 = k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)} \tag{186}$$

with $\widetilde{\theta} = \theta - b^2$ and

$$s = k_e\left(b^2-\theta_2\right) \tag{187}$$

or with $\widetilde{\theta} = b^2 - \theta$ and

$$s = k_e\left(\theta_2-b^2\right) \tag{188}$$

and similarly near the critical points $\theta \to c^2, \theta_1$ we take

$$k_e^2 = k^2 \frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} \tag{189}$$

with $\widetilde{\theta} = c^2 - \theta$ and

$$s = k_e \left(\theta_1 - c^2\right) \tag{190}$$

or with $\widetilde{\theta} = \theta - c^2$ and

$$s = k_e \left(c^2 - \theta_1\right) \tag{191}$$

we approximate the separation equation as

$$\frac{d^2\Theta}{d\widetilde{\theta}^2} + \frac{1}{2\widetilde{\theta}} \frac{d\Theta}{d\widetilde{\theta}} + \frac{1}{4}\left(k_e^2 + \frac{sk_e}{\widetilde{\theta}}\right)\Theta = 0 \tag{192}$$

Changing variables

$$\Theta = \widetilde{\theta}^{-1/4}\Psi \tag{193}$$

we obtain a form of the Whittaker (confluent hypergeometric) equation

$$\frac{d^2\Psi}{d\widetilde{\theta}^2} + \frac{1}{4}\left(k_e^2 + \frac{sk_e}{\widetilde{\theta}} + \frac{3}{4\widetilde{\theta}^2}\right)\Psi = 0 \tag{194}$$

where the solutions can be taken as the Whittaker (confluent hypergeometric) function forms [19], [20]

$$\Psi = W_{\pm}\left(s, 1/4, k_e\widetilde{\theta}\right) \tag{195}$$

having asymptotic forms [19]

$$W_+\left(s, 1/4, k_e\widetilde{\theta}\right) = W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right) = e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{3/4} U\left(3/4 - is/4, 3/2, -ik_e\widetilde{\theta}\right) \sim e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{is/4}, \ k_e\widetilde{\theta} \to \infty \tag{196}$$

$$U\left(3/4 - is/4, 3/2, -ik_e\widetilde{\theta}\right) = \left(-ik_e\widetilde{\theta}\right)^{-1/2}\sqrt{\pi}\left[\frac{M\left(1/4 - is/4, 1/2, -ik_e\widetilde{\theta}\right)}{\Gamma\left(3/4 - is/4\right)} - 2\frac{M\left(3/4 - is/4, 3/2, -ik_e\widetilde{\theta}\right)}{\Gamma\left(1/4 - is/4\right)}\left(-ik_e\widetilde{\theta}\right)^{1/2}\right] \tag{197}$$

$$W_+\left(s, 1/4, k_e\widetilde{\theta}\right) / \left(-ik_e\widetilde{\theta}\right)^{1/4} \sim \pi^{1/2}\left[\frac{1}{\Gamma\left(3/4 - is/4\right)} - \frac{2}{\Gamma\left(1/4 - is/4\right)}\left(-ik_e\widetilde{\theta}\right)^{1/2} + O\left(k_e\widetilde{\theta}\right)\right], \ k_e\widetilde{\theta} \to 0 \tag{198}$$

$$W_-\left(s, 1/4, k_e\widetilde{\theta}\right) = W_+\left(-s, 1/4, -k_e\widetilde{\theta}\right) \sim e^{-ik_e\widetilde{\theta}/2}\left(ik_e\widetilde{\theta}\right)^{-is/4}, \ k_e\widetilde{\theta} \to \infty \tag{199}$$

$$W_-\left(s, 1/4, k_e\widetilde{\theta}\right) / \left(ik_e\widetilde{\theta}\right)^{1/4} \sim \pi^{1/2}\left[\frac{1}{\Gamma\left(3/4 + is/4\right)} - \frac{2}{\Gamma\left(1/4 + is/4\right)}\left(ik_e\widetilde{\theta}\right)^{1/2} + O\left(k_e\widetilde{\theta}\right)\right], \ k_e\widetilde{\theta} \to 0 \tag{200}$$

We will also need derivatives with respect to $\widetilde{\theta}$. Noting that

$$W_-'\left(s,1/4,k_e\widetilde{\theta}\right)=\frac{d}{d\left(k_e\widetilde{\theta}\right)}W_-'\left(s,1/4,k_e\widetilde{\theta}\right)=-W_+'\left(-s,1/4,-k_e\widetilde{\theta}\right) \tag{201}$$

we find

$$\frac{d}{d\sqrt{\widetilde{\theta}}}\left[W_+\left(s,1/4,k_e\widetilde{\theta}\right)/\left(-ik_e\widetilde{\theta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4-is/4\right)}\left(-ik_e\right)^{1/2}+O\left(k_e\sqrt{\widetilde{\theta}}\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{202}$$

and

$$\frac{d}{d\sqrt{\widetilde{\theta}}}\left[W_-\left(s,1/4,k_e\widetilde{\theta}\right)/\left(ik_e\widetilde{\theta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4+is/4\right)}\left(ik_e\right)^{1/2}+O\left(k_e\sqrt{\widetilde{\theta}}\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{203}$$

3.3 Combined Critical (Turning & Singular) Point Limit

The actual limit of interest is when the turning point transitions through the singular point region. To accommodate this case we need to start from the original equation

$$\frac{d^2\Theta}{d\theta^2}-\frac{1}{2}\left(\frac{1}{a^2-\theta}+\frac{1}{b^2-\theta}+\frac{1}{c^2-\theta}\right)\frac{d\Theta}{d\theta}+\frac{1}{4}k^2\frac{\left(\theta_1-\theta\right)\left(\theta_2-\theta\right)}{\left(a^2-\theta\right)\left(b^2-\theta\right)\left(c^2-\theta\right)}\Theta=0 \tag{204}$$

but take the limit $\theta\to b^2,\theta_2$

$$\frac{d^2\Theta}{d\theta^2}-\frac{1}{2}\left(\frac{1}{a^2-b^2}+\frac{1}{b^2-\theta}+\frac{1}{c^2-b^2}\right)\frac{d\Theta}{d\theta}+\frac{1}{4}k^2\frac{\left(\theta_1-b^2\right)\left(\theta_2-\theta\right)}{\left(a^2-b^2\right)\left(b^2-\theta\right)\left(c^2-b^2\right)}\Theta=0 \tag{205}$$

and letting $\theta-b^2=\widetilde{\theta}$

$$\frac{d^2\Theta}{d\widetilde{\theta}^2}+\frac{1}{2}\left(-\frac{1}{a^2-b^2}+\frac{1}{\widetilde{\theta}}+\frac{1}{b^2-c^2}\right)\frac{d\Theta}{d\widetilde{\theta}}+\frac{1}{4}k^2\frac{\left(b^2-\theta_1\right)\left(b^2-\theta_2+\widetilde{\theta}\right)}{\left(a^2-b^2\right)\widetilde{\theta}\left(b^2-c^2\right)}\Theta=0 \tag{206}$$

There is a question here about whether we take $\theta\to b^2$ or $\theta\to\theta_2$ in the slowly varying factors; we have decided to use the former point, which could have some effect on the following results for k_e and s; also, we will need to follow this lead on the limits taken for the Liouville (WKB) solutions when we match. Dropping the two constant coeffi cient terms in the first derivative term gives

$$\frac{d^2\Theta}{d\widetilde{\theta}^2}+\frac{1}{2\widetilde{\theta}}\frac{d\Theta}{d\widetilde{\theta}}+\frac{1}{4}k^2\frac{\left(b^2-\theta_1\right)\left(b^2-\theta_2+\widetilde{\theta}\right)}{\left(a^2-b^2\right)\widetilde{\theta}\left(b^2-c^2\right)}\Theta=0 \tag{207}$$

If we had chosen to evaluate the slowly varying factors at $\theta\to\theta_2$

$$\frac{d^2\Theta}{d\theta^2}+\frac{1}{2}\left(-\frac{1}{a^2-\theta_2}+\frac{1}{\widetilde{\theta}}+\frac{1}{\theta_2-c^2}\right)\frac{d\Theta}{d\theta}+\frac{1}{4}k^2\frac{\left(\theta_2-\theta_1\right)\left(b^2-\theta_2+\widetilde{\theta}\right)}{\left(a^2-\theta_2\right)\widetilde{\theta}\left(\theta_2-c^2\right)}\Theta=0 \tag{208}$$

Dropping the two constant coeffi cient terms in the first derivative term factor gives

$$\frac{d^2\Theta}{d\theta^2}+\frac{1}{2\widetilde{\theta}}\frac{d\Theta}{d\theta}+\frac{1}{4}k^2\frac{(\theta_2-\theta_1)\left(b^2-\theta_2+\widetilde{\theta}\right)}{(a^2-\theta_2)\,\widetilde{\theta}\,(\theta_2-c^2)}\Theta=0 \tag{209}$$

Now transforming

$$\Theta=\widetilde{\theta}^{-1/4}\Psi \tag{210}$$

$$\frac{d^2\Psi}{d\widetilde{\theta}^2}+\left[\frac{1}{4}k^2\frac{\left(b^2-\theta_1\right)}{(a^2-b^2)\,(b^2-c^2)}+\frac{1}{4}k^2\frac{\left(b^2-\theta_1\right)\left(b^2-\theta_2\right)}{(a^2-b^2)\,\widetilde{\theta}\,(b^2-c^2)}+\frac{3}{16\widetilde{\theta}^2}\right]\Psi=0 \tag{211}$$

For the other approach (evaluating the slowly varying factors at $\theta\to\theta_2$)

$$\frac{d^2\Psi}{d\widetilde{\theta}^2}+\left[\frac{1}{4}k^2\frac{(\theta_2-\theta_1)}{(a^2-\theta_2)\,(\theta_2-c^2)}+\frac{1}{4}k^2\frac{(\theta_2-\theta_1)\left(b^2-\theta_2\right)}{(a^2-\theta_2)\,\widetilde{\theta}\,(\theta_2-c^2)}+\frac{3}{16\widetilde{\theta}^2}\right]\Psi=0 \tag{212}$$

This is a Whittaker equation [19]

$$\frac{d^2w}{dz^2}+\left(-\frac{1}{4}+\frac{\kappa}{z}+\frac{1/4-\mu^2}{z^2}\right)w=0 \tag{213}$$

A solution can be taken as the Whittaker function [19]

$$w=W_{\kappa,\mu}(z)=e^{-z/2}z^{1/2+\mu}U\left(1/2+\mu-\kappa,1+2\mu,z\right)\sim e^{-z/2}z^{\kappa}\ ,\ z\to\infty\ ,\ \left|\arg(z)\right|<3\pi/2 \tag{214}$$

$$M_{\kappa,\mu}(z)=e^{-z/2}z^{1/2+\mu}M\left(1/2+\mu-\kappa,1+2\mu,z\right) \tag{215}$$

$$\begin{aligned}U\left(1/2+\mu-\kappa,1+2\mu,z\right)&=\frac{\pi}{\sin\left(\pi\left(1+2\mu\right)\right)}\left[\frac{M\left(1/2+\mu-\kappa,1+2\mu,z\right)}{\Gamma\left(1/2-\mu-\kappa\right)\Gamma\left(1+2\mu\right)}-z^{-2\mu}\frac{M\left(1/2-\mu-\kappa,1-2\mu,z\right)}{\Gamma\left(1/2+\mu-\kappa\right)\Gamma\left(1-2\mu\right)}\right]\\&=\frac{\pi}{\sin\left(\pi\left(1+2\mu\right)\right)}\end{aligned}$$

$$\left[\frac{1}{\Gamma\left(1/2-\mu-\kappa\right)\Gamma\left(1+2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2+\mu-\kappa\right)_n z^n}{\left(1+2\mu\right)_n n!}-\frac{1}{\Gamma\left(1/2+\mu-\kappa\right)\Gamma\left(1-2\mu\right)}z^{-2\mu}\sum_{n=0}^{\infty}\frac{\left(1/2-\mu-\kappa\right)_n z^n}{\left(1-2\mu\right)_n n!}\right] \tag{216}$$

Noting the asymptotic forms [19]

$$\frac{M\left(a,b,c\right)}{\Gamma\left(b\right)}\sim\frac{e^{\pm i\pi a}z^{-a}}{\Gamma\left(b-a\right)}+\frac{e^{z}z^{a-b}}{\Gamma\left(a\right)}\ ,\quad\begin{array}{c}-\pi/2<\arg(z)\leq 3\pi/2\\-3\pi/2<\arg(z)\leq-\pi/2\end{array} \tag{217}$$

$$U\left(a,b,z\right)\sim z^{-a}\ ,\ -3\pi/2<\arg(z)<3\pi/2 \tag{218}$$

we can write

$$\frac{\Gamma(a)}{\Gamma(b)}M\left(a,b,c\right)-\frac{\Gamma(a)}{\Gamma(b-a)}e^{-i\pi a}U\left(a,b,z\right)\sim e^{z}z^{a-b}\ ,\ -3\pi/2<\arg(z)\leq-\pi/2 \tag{219}$$

$$\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1+2\mu\right)}M_{\kappa,\mu}\left(z\right)-\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1/2+\mu+\kappa\right)}e^{-i\pi\left(1/2+\mu-\kappa\right)}W_{\kappa,\mu}\left(z\right)$$

$$=e^{-z/2}z^{1/2+\mu}\left[\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1+2\mu\right)}M\left(1/2+\mu-\kappa,1+2\mu,z\right)-\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1/2+\mu+\kappa\right)}e^{-i\pi\left(1/2+\mu-\kappa\right)}U\left(1/2+\mu-\kappa,1+2\mu,z\right)\right]$$

$$\sim e^{z/2}z^{-\kappa}\ ,\quad -3\pi/2<\arg\left(z\right)\leq-\pi/2 \tag{220}$$

The series [19]

$$M\left(a,b,z\right)=\sum_{n=0}^{\infty}\frac{\left(a\right)_n z^n}{\left(b\right)_n n!} \tag{221}$$

and [19]

$$U\left(a,b,z\right)=\frac{\pi}{\sin\left(\pi b\right)}\left[\frac{M\left(a,b,z\right)}{\Gamma\left(1+a-b\right)\Gamma\left(b\right)}-z^{1-b}\frac{M\left(1+a-b,2-b,z\right)}{\Gamma\left(a\right)\Gamma\left(2-b\right)}\right]$$

$$=\frac{\pi}{\sin\left(\pi b\right)}\left[\frac{1}{\Gamma\left(1+a-b\right)\Gamma\left(b\right)}\sum_{n=0}^{\infty}\frac{\left(a\right)_n z^n}{\left(b\right)_n n!}-\frac{1}{\Gamma\left(a\right)\Gamma\left(2-b\right)}z^{1-b}\sum_{n=0}^{\infty}\frac{\left(1+a-b\right)_n z^n}{\left(2-b\right)_n n!}\right] \tag{222}$$

give

$$M_{\kappa,\mu}\left(z\right)=e^{-z/2}z^{1/2+\mu}M\left(1/2+\mu-\kappa,1+2\mu,z\right)=e^{-z/2}z^{1/2+\mu}\sum_{n=0}^{\infty}\frac{\left(1/2+\mu-\kappa\right)_n z^n}{\left(1+2\mu\right)_n n!} \tag{223}$$

$$W_{\kappa,\mu}\left(z\right)=e^{-z/2}z^{1/2+\mu}U\left(1/2+\mu-\kappa,1+2\mu,z\right)=e^{-z/2}z^{1/2+\mu}\frac{\pi}{\sin\left(\pi\left(1+2\mu\right)\right)}$$

$$\left[\frac{1}{\Gamma\left(1/2-\mu-\kappa\right)\Gamma\left(1+2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2+\mu-\kappa\right)_n z^n}{\left(1+2\mu\right)_n n!}-\frac{1}{\Gamma\left(1/2+\mu-\kappa\right)\Gamma\left(1-2\mu\right)}z^{-2\mu}\sum_{n=0}^{\infty}\frac{\left(1/2-\mu-\kappa\right)_n z^n}{\left(1-2\mu\right)_n n!}\right] \tag{224}$$

and

$$\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1+2\mu\right)}M_{\kappa,\mu}\left(z\right)-\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1/2+\mu+\kappa\right)}e^{-i\pi\left(1/2+\mu-\kappa\right)}W_{\kappa,\mu}\left(z\right)$$

$$=e^{-z/2}z^{1/2+\mu}\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1+2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2+\mu-\kappa\right)_n z^n}{\left(1+2\mu\right)_n n!}-e^{-z/2}z^{1/2+\mu}\frac{1}{\Gamma\left(1/2+\mu+\kappa\right)}\frac{\pi e^{-i\pi\left(1/2+\mu-\kappa\right)}}{\sin\pi\left(1+2\mu\right)}$$

$$\left[\frac{\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1/2-\mu-\kappa\right)\Gamma\left(1+2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2+\mu-\kappa\right)_n z^n}{\left(1+2\mu\right)_n n!}-\frac{1}{\Gamma\left(1-2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2-\mu-\kappa\right)_n z^{n-2\mu}}{\left(1-2\mu\right)_n n!}\right]$$

$$=e^{-z/2}z^{1/2+\mu}\frac{1}{\Gamma\left(1/2+\mu+\kappa\right)}\frac{e^{-i\pi\left(1/2+\mu-\kappa\right)}}{\sin\pi\left(1+2\mu\right)}\left[\frac{\pi}{\Gamma\left(1-2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2-\mu-\kappa\right)_n z^{n-2\mu}}{\left(1-2\mu\right)_n n!}\right.$$

$$+\left\{e^{i\pi(1/2+\mu-\kappa)}\sin\pi\left(1+2\mu\right)-\sin\pi\left(1/2+\mu+\kappa\right)\right\}\frac{\Gamma\left(1/2+\mu+\kappa\right)\Gamma\left(1/2+\mu-\kappa\right)}{\Gamma\left(1+2\mu\right)}\sum_{n=0}^{\infty}\frac{\left(1/2+\mu-\kappa\right)_n z^n}{\left(1+2\mu\right)_n n!}\Bigg] \tag{225}$$

It is convenient to take the solution W_+

$$\frac{d^2W_+}{d\widetilde{\theta}^2}+\frac{1}{4}\left(k_e^2+\frac{sk_e}{\widetilde{\theta}}+\frac{3}{4\widetilde{\theta}^2}\right)W_+=0 \tag{226}$$

where

$$W_+\left(s,1/4,k_e\widetilde{\theta}\right)=W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right)=e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{3/4}U\left(3/4-is/4,3/2,-ik_e\widetilde{\theta}\right)\sim e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{is/4}\ ,\ k_e\widetilde{\theta}\to\infty \tag{227}$$

Note that we can transform these confluent hypergeometric functions to the parabolic cylinder function $D_\nu(z)$ [19]

$$U\left(3/4-is/4,3/2,-ik_e\widetilde{\theta}\right)=2^{3/4-is/4}e^{-ik_e\widetilde{\theta}/2}D_{1/2+is/2}\left(\sqrt{-i2k_e\widetilde{\theta}}\right)/\sqrt{-i2k_e\widetilde{\theta}} \tag{228}$$

$$W_+\left(s,1/4,k_e\widetilde{\theta}\right)=W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right)=2^{-is/4}\left(-i2k_e\widetilde{\theta}\right)^{1/4}D_{1/2+is/2}\left(\sqrt{-i2k_e\widetilde{\theta}}\right) \tag{229}$$

Note in the axisymmetric case [13] we were using function $W_+(\tau,m,s)=W_{is/2,m/2}\left(-i\tau^2/2\right)$. The analytic continuation of the solution is

$$W_+\left(s,1/4,ze^{\pm i\pi}\right)=W_{is/4,1/4}\left(-ize^{\pm i\pi}\right)=e^{-iz/2}e^{\pm i3\pi/4}\left(-iz\right)^{3/4}U\left(3/4-is/4,3/2,-ize^{\pm i\pi}\right) \tag{230}$$

where [19]

$$U\left(a,b,-ize^{\pm i\pi}\right)=\frac{\pi}{\sin(\pi b)}e^{iz}\left[\frac{M\left(b-a,b,-iz\right)}{\Gamma\left(1+a-b\right)\Gamma\left(b\right)}-e^{\pm i\pi(1-b)-i\pi(1-b)/2}z^{1-b}\frac{M\left(1-a,2-b,-iz\right)}{\Gamma\left(a\right)\Gamma\left(2-b\right)}\right] \tag{231}$$

As a second solution we can take

$$e^{-\pi s/4}W_-\left(s,1/4,k_e\widetilde{\theta}\right)=\frac{\Gamma\left(3/4-is/4\right)}{\Gamma\left(3/2\right)}M_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right)-\frac{\Gamma\left(3/4-is/4\right)}{\Gamma\left(3/4+is/4\right)}e^{-i\pi(3/4-is/4)}W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right)$$

$$=\Gamma\left(3/4-is/4\right)e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{3/4}\left[\frac{1}{\Gamma\left(3/2\right)}M\left(3/4-is/4,3/2,-ik_e\widetilde{\theta}\right)-\frac{e^{-i\pi(3/4-is/4)}}{\Gamma\left(3/4+is/4\right)}U\left(3/4-is/4,3/2,-ik_e\widetilde{\theta}\right)\right]$$

$$\sim e^{-ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{-is/4}\ ,\ k_e\widetilde{\theta}\to\infty \tag{232}$$

which is asymptotic to W_+^* when k_e and s are real. However, we note from the invariance of the ODE when we replace $k_e\to-k_e$ and $s\to-s$, it is easier to take as a second solution

$$W_-\left(s,1/4,k_e\widetilde{\theta}\right)=W_+\left(-s,1/4,-k_e\widetilde{\theta}\right)=W_{-is/4,1/4}\left(ik_e\widetilde{\theta}\right)$$

$$=e^{-ik_e\widetilde{\theta}/2}\left(ik_e\widetilde{\theta}\right)^{3/4}U\left(3/4+is/4,3/2,ik_e\widetilde{\theta}\right)\sim e^{-ik_e\widetilde{\theta}/2}\left(ik_e\widetilde{\theta}\right)^{-is/4}\ ,\ k_e\widetilde{\theta}\to\infty \tag{233}$$

$$W_-\left(s,1/4,k_e\widetilde{\theta}\right)/\widetilde{\theta}^{1/4}\sim\pi^{1/2}\frac{1}{\Gamma\left(3/4+is/4\right)}\left(ik_e\right)^{1/4}\left[1-2\frac{\Gamma\left(3/4+is/4\right)}{\Gamma\left(1/4+is/4\right)}\left(ik_e\widetilde{\theta}\right)^{1/2}+O\left(k_e\widetilde{\theta}\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{234}$$

Note also when $s=0$

$$W_-\left(0,1/4,k_e\widetilde{\theta}\right)=W_{0,1/4}\left(ik_e\widetilde{\theta}\right)=e^{-ik_e\widetilde{\theta}/2}\left(ik_e\widetilde{\theta}\right)^{3/4}U\left(3/4,3/2,ik_e\widetilde{\theta}\right)=\frac{1}{2}\pi^{1/2}e^{-i5\pi/8}\left(ik_e\widetilde{\theta}\right)^{1/2}H^{(2)}_{1/4}\left(k_e\widetilde{\theta}/2\right) \tag{235}$$

The ODE parameters are

$$k_e^2=k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)} \tag{236}$$

$$sk_e=k^2\frac{\left(b^2-\theta_1\right)\left(b^2-\theta_2\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}=k_e^2\left(b^2-\theta_2\right) \tag{237}$$

$$s=k_e\left(b^2-\theta_2\right) \tag{238}$$

and the original variable is

$$\Theta=\widetilde{\theta}^{-1/4}W_\pm\left(s,1/4,k_e\widetilde{\theta}\right) \tag{239}$$

The turning point of the equation

$$\frac{d^2W_+}{d\widetilde{\theta}^2}+\frac{1}{4}\left(k_e^2+\frac{sk_e}{\widetilde{\theta}}+\frac{3}{4\widetilde{\theta}^2}\right)W_+=0 \tag{240}$$

is at

$$\left(k_e\widetilde{\theta}+s\right)k_e\widetilde{\theta}+\frac{3}{4}=0 \tag{241}$$

Note that for $s<0$ there is a turning point for $\widetilde{\theta}>0$, which for $k_e\widetilde{\theta}>>1$ (or if we ignore the 3/4 term) is at $\widetilde{\theta}=\theta_2-b^2$, but the asymptotic form of the local focal solution here assumes we are outside the turning point, with $\widetilde{\theta}$ greater than this location; it should therefore be fine for matching to the global solution outside this turning point.

For the other approach these become

$$\frac{d^2\Psi}{d\widetilde{\theta}^2}+\left[\frac{1}{4}k^2\frac{\left(\theta_2-\theta_1\right)}{\left(a^2-\theta_2\right)\left(\theta_2-c^2\right)}+\frac{1}{4}k^2\frac{\left(\theta_2-\theta_1\right)\left(b^2-\theta_2\right)}{\left(a^2-\theta_2\right)\widetilde{\theta}\left(\theta_2-c^2\right)}+\frac{3}{16\widetilde{\theta}^2}\right]\Psi=0 \tag{242}$$

$$k_e^2=k^2\frac{\left(\theta_2-\theta_1\right)}{\left(a^2-\theta_2\right)\left(\theta_2-c^2\right)} \tag{243}$$

$$sk_e = k^2 \frac{(\theta_2 - \theta_1)\left(b^2 - \theta_2\right)}{(a^2 - \theta_2)(\theta_2 - c^2)} = k_e^2 \left(b^2 - \theta_2\right) \tag{244}$$

$$s = k_e \left(b^2 - \theta_2\right) \tag{245}$$

and there is thus a slight change in the value of k_e^2; this does not seem to be significant. It is also useful in the limit $\theta \to b^2, \theta_2$

$$\frac{d^2\Theta}{d\theta^2} - \frac{1}{2}\left(\frac{1}{a^2 - b^2} + \frac{1}{b^2 - \theta} + \frac{1}{c^2 - b^2}\right)\frac{d\Theta}{d\theta} + \frac{1}{4}k^2 \frac{\left(\theta_1 - b^2\right)(\theta_2 - \theta)}{(a^2 - b^2)(b^2 - \theta)(c^2 - b^2)}\Theta = 0 \tag{246}$$

to let $b^2 - \theta = \widetilde{\theta}$

$$\frac{d^2\Theta}{d\widetilde{\theta}^2} + \frac{1}{2}\left(\frac{1}{a^2 - b^2} + \frac{1}{\widetilde{\theta}} - \frac{1}{b^2 - c^2}\right)\frac{d\Theta}{d\widetilde{\theta}} + \frac{1}{4}k^2 \frac{\left(b^2 - \theta_1\right)\left(\theta_2 - b^2 + \widetilde{\theta}\right)}{(a^2 - b^2)\,\widetilde{\theta}\,(b^2 - c^2)}\Theta = 0 \tag{247}$$

Dropping the two constant terms in the first derivative

$$\frac{d^2\Theta}{d\widetilde{\theta}^2} + \frac{1}{2\widetilde{\theta}}\frac{d\Theta}{d\widetilde{\theta}} + \frac{1}{4}k^2 \frac{\left(b^2 - \theta_1\right)\left(\theta_2 - b^2 + \widetilde{\theta}\right)}{(a^2 - b^2)\,\widetilde{\theta}\,(b^2 - c^2)}\Theta = 0 \tag{248}$$

or

$$\frac{d^2\Theta}{d\widetilde{\theta}^2} + \frac{1}{2\widetilde{\theta}}\frac{d\Theta}{d\widetilde{\theta}} + \frac{1}{4}\left(k_e^2 + \frac{sk_e}{\widetilde{\theta}}\right)\Theta = 0 \tag{249}$$

$$k_e^2 = k^2 \frac{\left(b^2 - \theta_1\right)}{(a^2 - b^2)(b^2 - c^2)} \tag{250}$$

$$sk_e = k^2 \frac{\left(b^2 - \theta_1\right)}{(a^2 - b^2)(b^2 - c^2)}\left(\theta_2 - b^2\right) \tag{251}$$

$$s = k_e \left(\theta_2 - b^2\right) \tag{252}$$

Now transforming

$$\Theta = \widetilde{\theta}^{-1/4}\Psi \tag{253}$$

$$\frac{d^2\Psi}{d\widetilde{\theta}^2} + \frac{1}{4}\left(k_e^2 + \frac{sk_e}{\widetilde{\theta}} + \frac{3}{4\widetilde{\theta}^2}\right)\Psi = 0 \tag{254}$$

which is again the Whittaker equation

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2}\right) w = 0 \tag{255}$$

with solutions

$$\Theta = \widetilde{\theta}^{-1/4} W_{\pm}\left(s, 1/4, k_e\widetilde{\theta}\right) \tag{256}$$

This case is another example of the invariance of the ODE, in this case under the sign reversals of $\widetilde{\theta}$ and s.

3.3.1 Evaluation Of Whittaker Function

Near the origin we use the identities

$$W_+\left(s,1/4,k_e\widetilde{\theta}\right)=W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right)=e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{3/4}U\left(3/4-is/4,3/2,-ik_e\widetilde{\theta}\right)=-\pi^{1/2}e^{ik_e\widetilde{\theta}/2}$$

$$\left(-ik_e\widetilde{\theta}\right)^{3/4}\left[\frac{2}{\Gamma\left(1/4-is/4\right)}\sum_{n=0}^{\infty}\frac{\left(3/4-is/4\right)_n\left(-ik_e\widetilde{\theta}\right)^n}{\left(3/2\right)_n n!}-\frac{1}{\Gamma\left(3/4-is/4\right)}\sum_{n=0}^{\infty}\frac{\left(1/4-is/4\right)_n\left(-ik_e\widetilde{\theta}\right)^{n-1/2}}{\left(1/2\right)_n n!}\right]$$

$$=\pi^{1/2}e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{1/4}\left[\frac{1}{\Gamma\left(3/4-is/4\right)}\sum_{n=0}^{\infty}\frac{\left(1/4-is/4\right)_n\left(-ik_e\widetilde{\theta}\right)^n}{\left(1/2\right)_n n!}-\frac{2\left(-ik_e\widetilde{\theta}\right)^{1/2}}{\Gamma\left(1/4-is/4\right)}\sum_{n=0}^{\infty}\frac{\left(3/4-is/4\right)_n\left(-ik_e\widetilde{\theta}\right)^n}{\left(3/2\right)_n n!}\right] \tag{257}$$

to find

$$W_+\left(s,1/4,k_e\widetilde{\theta}\right)/\left(-ik_e\widetilde{\theta}\right)^{1/4}\sim\pi^{1/2}\left[\frac{1-is\left(-ik_e\widetilde{\theta}\right)/2}{\Gamma\left(3/4-is/4\right)}-\frac{2-is\left(-ik_e\widetilde{\theta}\right)/3}{\Gamma\left(1/4-is/4\right)}\left(-ik_e\widetilde{\theta}\right)^{1/2}+O\left(\left(k_e\widetilde{\theta}\right)^2\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{258}$$

$$W_+\left(s,1/4,k_e\widetilde{\theta}\right)/\left(-ik_e\widetilde{\theta}\right)^{1/4}\sim\pi^{1/2}\left[\frac{1}{\Gamma\left(3/4-is/4\right)}-\frac{2}{\Gamma\left(1/4-is/4\right)}\left(-ik_e\widetilde{\theta}\right)^{1/2}+O\left(k_e\widetilde{\theta}\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{259}$$

Note also when $s=0$

$$W_+\left(0,1/4,k_e\widetilde{\theta}\right)=W_{0,1/4}\left(-ik_e\widetilde{\theta}\right)=e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{3/4}U\left(3/4,3/2,-ik_e\widetilde{\theta}\right)=\frac{1}{2}\pi^{1/2}e^{i5\pi/8}\left(-ik_e\widetilde{\theta}\right)^{1/2}H^{(1)}_{1/4}\left(k_e\widetilde{\theta}/2\right) \tag{260}$$

The asymptotic form is [19]

$$U\left(a,b,z\right)=z^{-a}\left[\sum_{n=0}^{R-1}\frac{\left(a\right)_n\left(1+a-b\right)_n}{n!}\left(-z\right)^{-n}+O\left(\left|z\right|^{-R}\right)\right]\ ,\ \ -3\pi/2<\arg\left(z\right)<3\pi/2 \tag{261}$$

$$W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right)=e^{ik_e\widetilde{\theta}/2}\left(-ik_e\widetilde{\theta}\right)^{is/4}\left[\sum_{n=0}^{R-1}\frac{\left(3/4-is/4\right)_n\left(1/4-is/4\right)_n}{n!}\left(ik_e\widetilde{\theta}\right)^{-n}+O\left(\left|k_e\widetilde{\theta}\right|^{-R}\right)\right] \tag{262}$$

An integral representation is [19]

$$\Gamma\left(a\right)U\left(a,b,z\right)=\int_0^{\infty}e^{-zt}t^{a-1}\left(1+t\right)^{b-a-1}dt$$

$$= \int_0^{e^{i\pi/2}R} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt + \int_{\pi/2}^{0} e^{-zt} t^{a-1} (1+t)^{b-a-1} ie^{i\varphi} R d\varphi \tag{263}$$

$$\Gamma(a) U(a, 3/2, -iz) \sim e^{i\pi/4} \int_0^R e^{-zt} t^{a-1} (t-i)^{1/2-a} dt + iR^{1/2} \int_{\pi/2}^{0} e^{ie^{i\varphi} zR} e^{i\varphi/2} d\varphi \tag{264}$$

$$\int_0^{\pi/2} e^{i\left(e^{i\varphi} zR + \varphi/2\right)} d\varphi = \int_0^{\pi/2} e^{(-\sin\varphi + i\cos\varphi) zR + i\varphi/2} d\varphi \sim e^{izR} \int_0^{\infty} e^{-\varphi zR} d\varphi = \frac{1}{zR} e^{izR} \tag{265}$$

$$\Gamma(a) U(a, 3/2, -iz) = e^{i\pi/4} \int_0^R e^{-zt} t^{a-1} (t-i)^{1/2-a} dt - \frac{i}{z} R^{-1/2} e^{izR} \tag{266}$$

$$\Gamma(a) U(a, 3/2, -iz) = e^{i\pi/4} \int_0^{\infty} e^{-zt} t^{a-1} (t-i)^{1/2-a} dt \tag{267}$$

which is the analytic continuation for this phase of argument. Near the lower limit with $a = 3/4 - is/4$

$$e^{i\pi/4} \int_0^{\varepsilon} e^{-zt} t^{a-1} (t-i)^{1/2-a} dt \sim e^{i\pi a/2} \int_0^{\varepsilon} (1 - zt + \cdots) t^{a-1} (1 + it(1/2-a) + \cdots) dt$$

$$\sim (i\varepsilon)^a \left[\frac{1}{a} + \frac{i(1/2-a) - z}{a+1} \varepsilon + \cdots\right] \tag{268}$$

Near the upper limit

$$e^{i\pi/4} \int_R^{\infty} e^{-zt} t^{-1/2} (1 - i/t)^{1/2-a} dt \sim e^{i\pi/4} \int_R^{\infty} e^{-zt} t^{-1/2} (1 - i(1/2-a)/t + \cdots) dt$$

$$\sim e^{i\pi/4} \left[\frac{1}{z\sqrt{R}} e^{-zR} - \frac{1}{2z} \int_R^{\infty} e^{-zt} t^{-3/2} dt\right] + e^{-i\pi/4} (1/2-a) \left[\frac{1}{zR\sqrt{R}} e^{-zR} - \frac{3/2}{z} \int_R^{\infty} e^{-zt} t^{-5/2} dt\right] + \cdots$$

$$\sim e^{i\pi/4} \left[\frac{1}{z\sqrt{R}} e^{-zR} - \frac{1/2}{z^2 R^{3/2}} e^{-zR} + \cdots\right] + e^{-i\pi/4} (1/2-a) \left[\frac{1}{zR\sqrt{R}} e^{-zR} - \frac{3/2}{z} \int_R^{\infty} e^{-zt} t^{-5/2} dt\right] + \cdots$$

$$\sim \frac{1}{z\sqrt{R}} e^{i\pi/4 - zR} \left[1 - \frac{1/(2z) + i(1/2-a)}{R}\right] + \cdots \tag{269}$$

$$W_{is/4,1/4}\left(-ik_e\widetilde{\theta}\right) = e^{ik_e\widetilde{\theta}/2} \left(-ik_e\widetilde{\theta}\right)^{3/4} U\left(3/4 - is/4, 3/2, -ik_e\widetilde{\theta}\right) \tag{270}$$

Note for the other function

$$W_-\left(s, 1/4, k_e\widetilde{\theta}\right) = W_+\left(-s, 1/4, -k_e\widetilde{\theta}\right) \tag{271}$$

and the integral representation is analytically continued in the other direction

$$\Gamma(a) U(a, b, z) = \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{b-a-1} dt$$

$$= \int_0^{e^{-i\pi/2}R} e^{-zt} t^{a-1} (1+t)^{b-a-1} \, dt + \int_{-\pi/2}^{0} e^{-zt} t^{a-1} (1+t)^{b-a-1} \, i e^{i\varphi} R d\varphi \tag{272}$$

$$\Gamma(a) U(a, 3/2, iz) \sim e^{-i\pi/4} \int_0^R e^{-zt} t^{a-1} (t+i)^{1/2-a} \, dt + i R^{1/2} \int_{-\pi/2}^{0} e^{-ie^{i\varphi} zR} e^{i\varphi/2} d\varphi$$

$$\int_0^{\pi/2} e^{-i\left(e^{-i\varphi} zR + \varphi/2\right)} d\varphi = \int_0^{\pi/2} e^{(-\sin\varphi - i\cos\varphi) zR - i\varphi/2} d\varphi \sim e^{-izR} \int_0^{\infty} e^{-\varphi z R} d\varphi = \frac{1}{zR} e^{-izR} \tag{273}$$

$$\Gamma(a) U(a, 3/2, iz) = e^{-i\pi/4} \int_0^R e^{-zt} t^{a-1} (t+i)^{1/2-a} \, dt + \frac{i}{z} R^{-1/2} e^{-izR} \tag{274}$$

$$\Gamma(a) U(a, 3/2, iz) = e^{-i\pi/4} \int_0^{\infty} e^{-zt} t^{a-1} (t+i)^{1/2-a} \, dt \tag{275}$$

which is the analytic continuation for this phase of argument. Near the lower limit with $a = 3/4 + is/4$

$$e^{-i\pi/4} \int_0^{\varepsilon} e^{-zt} t^{a-1} (t+i)^{1/2-a} \, dt \sim e^{-i\pi a/2} \int_0^{\varepsilon} (1 - zt + \cdots) t^{a-1} (1 - it(1/2-a) + \cdots) \, dt$$

$$\sim (-i\varepsilon)^a \left[\frac{1}{a} + \frac{-i(1/2-a) - z}{a+1} \varepsilon + \cdots \right] \tag{276}$$

Near the upper limit

$$e^{-i\pi/4} \int_R^{\infty} e^{-zt} t^{-1/2} (1 + i/t)^{1/2-a} \, dt \sim e^{-i\pi/4} \int_R^{\infty} e^{-zt} t^{-1/2} (1 + i(1/2-a)/t + \cdots) \, dt$$

$$\sim e^{-i\pi/4} \left[\frac{1}{z\sqrt{R}} e^{-zR} - \frac{1}{2z} \int_R^{\infty} e^{-zt} t^{-3/2} dt \right] + e^{i\pi/4} (1/2-a) \left[\frac{1}{zR\sqrt{R}} e^{-zR} - \frac{3/2}{z} \int_R^{\infty} e^{-zt} t^{-5/2} dt \right] + \cdots$$

$$\sim e^{i\pi/4} \left[\frac{1}{z\sqrt{R}} e^{-zR} - \frac{1/2}{z^2 R^{3/2}} e^{-zR} + \cdots \right] + e^{i\pi/4} (1/2-a) \left[\frac{1}{zR\sqrt{R}} e^{-zR} - \frac{3/2}{z} \int_R^{\infty} e^{-zt} t^{-5/2} dt \right] + \cdots$$

$$\sim \frac{1}{z\sqrt{R}} e^{-i\pi/4 - zR} \left[1 - \frac{1/(2z) - i(1/2-a)}{R} \right] + \cdots \tag{277}$$

$$W_{-}\left(s, 1/4, k_e \widetilde{\theta}\right) = W_{-is/4, 1/4}\left(ik_e \widetilde{\theta}\right) = e^{-ik_e \widetilde{\theta}/2} \left(ik_e \widetilde{\theta}\right)^{3/4} U\left(3/4 + is/4, 3/2, ik_e \widetilde{\theta}\right) \tag{278}$$

3.4 Singular Point Limit

The isolated singular point at $\zeta \to a^2$ (corresponding to the obit center $x^2 = 0$) leads to

$$\frac{d^2 Z}{d\zeta^2} - \frac{1}{2}\left(\frac{1}{a^2-\zeta} - \frac{1}{a^2-b^2} - \frac{1}{a^2-c^2}\right)\frac{dZ}{d\zeta} + \frac{1}{4}k^2 \frac{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}{\left(a^2-\zeta\right)\left(a^2-b^2\right)\left(a^2-c^2\right)} Z = 0 \ , \ \zeta \to a^2 \tag{279}$$

We can transform out the constant

$$Z = \exp\left[\frac{1}{4}\left(\frac{1}{a^2-b^2} + \frac{1}{a^2-c^2}\right)\left(a^2-\zeta\right)\right] O \tag{280}$$

to find

$$\frac{d^2 O}{d\zeta^2} - \frac{1}{2}\left(\frac{1}{a^2-\zeta}\right)\frac{dO}{d\zeta}$$

$$+\left[\frac{1}{4}k^2 \frac{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}{\left(a^2-\zeta\right)\left(a^2-b^2\right)\left(a^2-c^2\right)} - \frac{1}{16}\left(\frac{1}{a^2-b^2} + \frac{1}{a^2-c^2}\right)^2 - \frac{1}{8\left(a^2-\zeta\right)}\left(\frac{1}{a^2-b^2} + \frac{1}{a^2-c^2}\right)\right] O = 0 \ , \ \zeta \to a^2 \tag{281}$$

Instead in the spirit of simplicity let us drop these and approximate as

$$\frac{d^2 Z}{d\zeta^2} - \frac{1}{2}\left(\frac{1}{a^2-\zeta}\right)\frac{dZ}{d\zeta} + \frac{1}{4}k^2 \frac{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}{\left(a^2-\zeta\right)\left(a^2-b^2\right)\left(a^2-c^2\right)} Z = 0 \ , \ \zeta \to a^2 \tag{282}$$

or

$$\frac{d^2 Z}{d\tilde{\zeta}^2} + \frac{1}{2\tilde{\zeta}}\frac{dZ}{d\tilde{\zeta}} + \frac{1}{4\tilde{\zeta}} k_{e0}^2 Z = 0 \ , \ \tilde{\zeta} = a^2 - \zeta \to 0 \tag{283}$$

$$k_{e0}^2 = k^2 \frac{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}{\left(a^2-b^2\right)\left(a^2-c^2\right)} \tag{284}$$

Using the canonical equation

$$\frac{d^2 U}{du^2} + \frac{1}{2u}\frac{dU}{du} + \frac{\lambda^2}{4u} U = 0 \tag{285}$$

with solutions

$$U = u^{1/4} J_{\pm 1/2}\left(\lambda u^{1/2}\right), u^{1/4} Y_{1/2}\left(\lambda u^{1/2}\right), u^{1/4} H^{(1)}_{1/2}\left(\lambda u^{1/2}\right), u^{1/4} H^{(2)}_{1/2}\left(\lambda u^{1/2}\right) \tag{286}$$

we can write

$$Z = \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left(k_{e0}\sqrt{\tilde{\zeta}}\right)\right\} = \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left(k_{e0}\sqrt{a^2-\zeta}\right)\right\} \tag{287}$$

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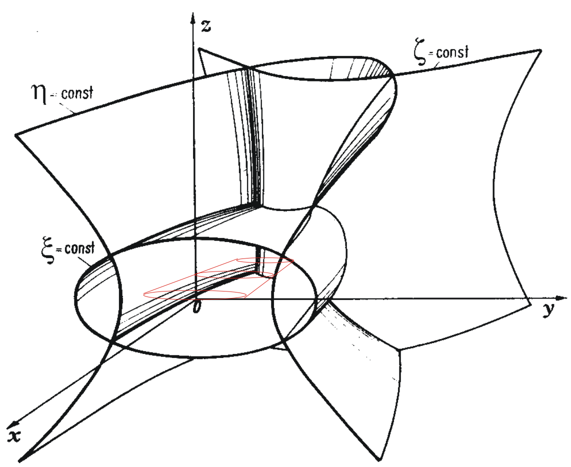

Figure 5: This figure, taken from Moon and Spencer, illustrates the types of coordinate surfaces involved in ellipsoidal coordinates. One candidate for the enclosing surface is made up of sections of the ellipse and hyperbola (orbit end cavity boundary surface) shown in red may be taken as the scar enclosing region.

4 CONSTRUCTION OF ASYMPTOTIC FIELD

We now assemble the complete asymptotic solution along the orbit of Figure 5. With the convex walls we are considering the region inside the focal points.

4.1 Radial-Like Variable

As we proceed toward the orbit (note that $\xi \to c^2$ corresponds to $z^2 \to 0$), but near the critical points $\xi \to \theta_1, c^2 = 0$, we set $\xi = c^2 - \widetilde{\xi}$ and we use the random phase reflection coeffi cient $R_x = e^{i\Phi_{0x}}$ to write the radial-like separation function as

$$X \sim \mathrm{Re}\left[c_{+x}W_{+}\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) + c_{-x}W_{-}\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right)\right] / \sqrt[4]{\widetilde{\xi}}$$

$$\sim c_{0x}\,\mathrm{Re}\left[e^{-i\Phi_{0x}/2}W_{+}\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) + e^{i\Phi_{0x}/2}W_{-}\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right)\right] / \sqrt[4]{\widetilde{\xi}} \tag{288}$$

where

$$e^{i\Phi_{0x}}c_{+x} = c_{-x} = e^{i\Phi_{0x}/2}c_{0x} \tag{289}$$

$$W_{+}\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) \sim e^{ik_{ex}\widetilde{\xi}/2}\left(-ik_{ex}\widetilde{\xi}\right)^{is_x/4}\ ,\ k_{ex}\widetilde{\xi} \to \infty \tag{290}$$

$$W_{-}\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) \sim e^{-ik_{ex}\widetilde{\xi}/2}\left(ik_{ex}\widetilde{\xi}\right)^{-is_x/4}\ ,\ k_{ex}\widetilde{\xi} \to \infty \tag{291}$$

$$k_{ex}^2 = k^2\frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} \tag{292}$$

$$s_x = k_{ex}\left(\theta_1 - c^2\right) \tag{293}$$

$$X(\xi) \sim e^{\pi s_x/8} c_{0x}\,\mathrm{Re}\left[e^{ik_{ex}\widetilde{\xi}/2+i(s_x/4)\ln\left(k_{ex}\widetilde{\xi}\right)-i\Phi_{0x}/2} + e^{i\Phi_{0x}} e^{-ik_{ex}\widetilde{\xi}/2-i(s_x/4)\ln\left(k_{ex}\widetilde{\xi}\right)+i\Phi_{0x}/2}\right] / \sqrt[4]{\widetilde{\xi}}\ ,\ k_{ex}\widetilde{\xi} \to \infty$$

$$\sim 2e^{\pi s_x/8} c_{0x} \cos\left[k_{ex}\widetilde{\xi}/2 + (s_x/4)\ln\left(k_{ex}\widetilde{\xi}\right) - \Phi_{0x}/2\right] / \sqrt[4]{\widetilde{\xi}}\ ,\ k_{ex}\widetilde{\xi} \to \infty \tag{294}$$

We note that the direction $\xi \to -\infty$, or growing $\widetilde{\xi}$, is in a direction away from the scarred orbit, so the placement here of the phase reflection coeffi cient is correct. From the preceding expressions

$$W_+\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \left(-ik_{ex}\widetilde{\xi}\right)^{1/4} \sim \pi^{1/2}\left[\frac{1}{\Gamma(3/4 - is_x/4)} - \frac{2}{\Gamma(1/4 - is_x/4)}\left(-ik_{ex}\widetilde{\xi}\right)^{1/2} + O\left(k_{ex}\widetilde{\xi}\right)\right]\ ,\ k_{ex}\widetilde{\xi} \to 0 \tag{295}$$

and

$$W_-\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \left(ik_{ex}\widetilde{\xi}\right)^{1/4} \sim \pi^{1/2}\left[\frac{1}{\Gamma(3/4 + is_x/4)} - \frac{2}{\Gamma(1/4 + is_x/4)}\left(ik_{ex}\widetilde{\xi}\right)^{1/2} + O\left(k_{ex}\widetilde{\xi}\right)\right]\ ,\ k_{ex}\widetilde{\xi} \to 0 \tag{296}$$

we find near the degenerate elliptical strip (at $z^2 = 0$)

$$X(\xi) \sim c_{0x} k_{ex}^{1/4}\,\mathrm{Re}\left[e^{-i\pi/8} e^{-i\Phi_{0x}/2} W_+\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{-ik_{ex}\widetilde{\xi}} + e^{i\pi/8} e^{i\Phi_{0x}/2} W_-\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{ik_{ex}\widetilde{\xi}}\right]$$

$$\sim c_{0x}\pi^{1/2} k_{ex}^{1/4}\,\mathrm{Re}\left[\left\{\frac{1}{\Gamma(3/4 - is_x/4)} e^{-i\Phi_{0x}/2 - i\pi/8} + \frac{1}{\Gamma(3/4 + is_x/4)} e^{i\Phi_{0x}/2 + i\pi/8}\right\}\right.$$

$$\left. -2\left\{\frac{1}{\Gamma(1/4 - is_x/4)}\left(-ik_{ex}\widetilde{\xi}\right)^{1/2} e^{-i\Phi_{0x}/2 - i\pi/8} + \frac{1}{\Gamma(1/4 + is_x/4)}\left(ik_{ex}\widetilde{\xi}\right)^{1/2} e^{i\Phi_{0x}/2 + i\pi/8}\right\} + O\left(k_{ex}\widetilde{\xi}\right)\right] \tag{297}$$

Now noting that

$$z^2 = \frac{\left(c^2 - \xi\right)\left(\eta - c^2\right)\left(\zeta - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} = \frac{\left(\eta - c^2\right)\left(\zeta - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)}\widetilde{\xi} \tag{298}$$

we see that a derivative with respect to $\sqrt{\widetilde{\xi}}$ is proportional to a derivative with respect to the distance variable z, as $\widetilde{\xi} \to 0$. Now if we choose to have evenness with respect to z as $z \to 0$ we can choose to make this derivative vanish. In the former axisymmetric problem we set the normal derivative of the potential (times the radius) equal to zero as we approached the scar center. What is the underlying reason for imposing this condition? It does represent an even type condition, but its significance is also the fact that the power flow must vanish at the scar center? In the two-dimensional problem we could argue the same thing. If this is the case, we do not necessarily have to impose this condition on Y at all, but only on X in this region; note by analogy that we did not impose a particular $\cos\varphi$ or $\sin\varphi$ choice in the axisymmetric problem. Note that in the inside region we can write

$$\lim_{\xi\to c^2}\int_{c^2}^{b^2}\frac{1}{h_\xi}\frac{\partial X}{\partial\xi}h_\eta d\eta=\sqrt{(a^2-c^2)(b^2-c^2)}\int_{c^2}^{b^2}\sqrt{\frac{(\zeta-\eta)(\zeta-c^2)}{(a^2-\eta)(b^2-\eta)(\eta-c^2)}}\sqrt{c^2-\xi}\frac{\partial X}{\partial\xi}d\eta=0 \tag{299}$$

$$h_\xi=\frac{1}{2}\sqrt{\frac{(\xi-\eta)(\xi-\zeta)}{D(\xi)}}=\frac{1}{2}\sqrt{\frac{(\eta-\xi)(\zeta-\xi)}{(a^2-\xi)(b^2-\xi)(c^2-\xi)}} \tag{300}$$

$$h_\eta=\frac{1}{2}\sqrt{\frac{(\eta-\zeta)(\eta-\xi)}{D(\eta)}}=\frac{1}{2}\sqrt{\frac{(\zeta-\eta)(\eta-\xi)}{(a^2-\eta)(b^2-\eta)(\eta-c^2)}} \tag{301}$$

$$h_\zeta=\frac{1}{2}\sqrt{\frac{(\zeta-\xi)(\zeta-\eta)}{D(\zeta)}} \tag{302}$$

$$D(\theta)=\left(a^2-\theta\right)\left(b^2-\theta\right)\left(c^2-\theta\right) \tag{303}$$

$$\frac{\partial X}{\partial\xi}\left(\xi\to c^2\right)=\lim_{\xi\to c^2}\frac{\partial\sqrt{\widetilde{\xi}}}{\partial\xi}\frac{\partial X}{\partial\sqrt{\widetilde{\xi}}}=\lim_{\xi\to c^2}\left[\frac{\partial\sqrt{c^2-\xi}}{\partial\xi}\frac{\partial X}{\partial\sqrt{\widetilde{\xi}}}\right]=\lim_{\xi\to c^2}\left[-\frac{1}{2\sqrt{c^2-\xi}}\frac{\partial X}{\partial\sqrt{\widetilde{\xi}}}\right]=\lim_{\xi\to c^2}\left[-\frac{1}{2\sqrt{\widetilde{\xi}}}\frac{\partial X}{\partial\sqrt{\widetilde{\xi}}}\right] \tag{304}$$

(In a follow-on report on concave walled cavities with interior foci, we can impose this condition on Z in the region outside the foci and in the region between foci we can impose it on both X and Z.) However, if we wish to impose evenness in y^2 for convenience we should be able to do so (like choosing $\cos\varphi$ or $\sin\varphi$ in the former axisymmetric problem).

We note from

$$\frac{d}{d\sqrt{\widetilde{\xi}}}\left[W_+\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)/\left(-ik_{ex}\widetilde{\xi}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma(1/4-is_x/4)}(-ik_{ex})^{1/2}+O\left(k_{ex}\sqrt{\widetilde{\xi}}\right)\right]\ ,\ k_{ex}\widetilde{\xi}\to 0 \tag{305}$$

$$\frac{d}{d\sqrt{\widetilde{\xi}}}\left[W_-\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)/\left(ik_{ex}\widetilde{\xi}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma(1/4+is_x/4)}(ik_{ex})^{1/2}+O\left(k_{ex}\sqrt{\widetilde{\xi}}\right)\right]\ ,\ k_{ex}\widetilde{\xi}\to 0 \tag{306}$$

that

$$\frac{d}{d\sqrt{\widetilde{\xi}}}X(\xi)\sim-2\pi^{1/2}k_{ex}^{3/4}c_{0x}\,\mathrm{Re}\left[\left\{\frac{1}{\Gamma(1/4-is_x/4)}e^{-i\Phi_{0x}/2-i3\pi/8}+\frac{1}{\Gamma(1/4+is_x/4)}e^{i\Phi_{0x}/2+i3\pi/8}\right\}+O\left(\sqrt{k_{ex}\widetilde{\xi}}\right)\right] \tag{307}$$

and if this must vanish as $\sqrt{\widetilde{\xi}}\to 0$ then

$$e^{i\pi}\frac{\Gamma(1/4+is_x/4)}{\Gamma(1/4-is_x/4)}=e^{i\Phi_{0x}+i3\pi/4} \tag{308}$$

or

$$\frac{\Gamma(1/4 + is_x/4)}{\Gamma(1/4 - is_x/4)} = e^{i(\Phi_{0x} - \pi/4)} \tag{309}$$

in which case

$$X(\xi) \sim c_{0x} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\pi/8} e^{-i\Phi_{0x}/2} W_+ \left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{-ik_{ex}\widetilde{\xi}} + e^{i\pi/8} e^{i\Phi_{0x}/2} W_- \left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{ik_{ex}\widetilde{\xi}}\right]$$

$$\sim c_{0x} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\pi/4} e^{-i(\Phi_{0x} - \pi/4)/2} W_+ \left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{-ik_{ex}\widetilde{\xi}} + e^{i\pi/4} e^{i(\Phi_{0x} - \pi/4)/2} W_- \left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{ik_{ex}\widetilde{\xi}}\right]$$

$$\sim c_{0x} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\pi/4} \sqrt{\frac{\Gamma(1/4 - is_x/4)}{\Gamma(1/4 + is_x/4)}} W_+ \left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{-ik_{ex}\widetilde{\xi}} + e^{i\pi/4} \sqrt{\frac{\Gamma(1/4 + is_x/4)}{\Gamma(1/4 - is_x/4)}} W_- \left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) / \sqrt[4]{ik_{ex}\widetilde{\xi}}\right] \tag{310}$$

Using the small argument forms of W_+

$$X(\xi) \sim c_{0x} \pi^{1/2} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\pi/4} \frac{\sqrt{\Gamma(1/4 + is_x/4)\Gamma(1/4 - is_x/4)}}{\Gamma(1/4 + is_x/4)\Gamma(3/4 - is_x/4)} + e^{i\pi/4} \frac{\sqrt{\Gamma(1/4 - is_x/4)\Gamma(1/4 + is_x/4)}}{\Gamma(3/4 + is_x/4)\Gamma(1/4 - is_x/4)} + O\left(k_{ex}\widetilde{\xi}\right)\right]$$

$$\sim c_{0x} \pi^{1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right|$$

$$\operatorname{Re}\left[e^{-i\pi/4} \frac{1}{\Gamma(1/4 + is_x/4)\Gamma(3/4 - is_x/4)} + e^{i\pi/4} \frac{1}{\Gamma(3/4 + is_x/4)\Gamma(1/4 - is_x/4)} + O\left(k_{ex}\widetilde{\xi}\right)\right]$$

$$\sim c_{0x} \pi^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[e^{-i\pi/4} \sin\pi(1/4 + is_x/4) + e^{i\pi/4} \sin\pi(1/4 - is_x/4) + O\left(k_{ex}\widetilde{\xi}\right)\right]$$

$$\sim c_{0x} (2\pi)^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right|$$

$$\operatorname{Re}\left[e^{-i\pi/4} \left\{\cosh(\pi s_x/4) + i\sinh(\pi s_x/4)\right\} + e^{i\pi/4} \left\{\cosh(\pi s_x/4) - i\sinh(\pi s_x/4)\right\} + O\left(k_{ex}\widetilde{\xi}\right)\right]$$

$$\sim c_{0x} (2\pi)^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[2\cos(\pi/4)\cosh(\pi s_x/4) + 2\sin(\pi/4)\sinh(\pi s_x/4) + O\left(k_{ex}\widetilde{\xi}\right)\right]$$

$$\sim c_{0x} \pi^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[\cosh(\pi s_x/4) + \sinh(\pi s_x/4) + O\left(k_{ex}\widetilde{\xi}\right)\right]$$

$$\sim c_{0x} \pi^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| e^{\pi s_x/4} + O\left(k_{ex}\widetilde{\xi}\right) \tag{311}$$

where we used the reflection formula $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z)$ [19]. We have thus made the connection of the random phase Φ_{0x} with the derived (connected primarily with θ_1) separation constant s_x. We note from Stirling's formula [19] we can write

$$\Gamma(1/4 - is_x/4) \sim \sqrt{2\pi}e^{-1/4+is_x/4}(1/4 - is_x/4)^{-1/4-is_x/4} \sim \sqrt{2\pi}e^{\pm i\pi/8+is_x/4}(|s_x|/4)^{-1/4-is_x/4}e^{\mp\pi s_x/8}\ ,\ s_x \to \pm\infty \tag{312}$$

$$|\Gamma(1/4 - is_x/4)| \sim \sqrt{2\pi}(|s_x|/4)^{-1/4}e^{-\pi|s_x|/8}\ ,\ s_x \to \pm\infty \tag{313}$$

4.1.1 Outward Extension

The $X(\xi)$ solution of the inside region can be extended outward in validity by matching to Liouville solutions [18]

$$X(\xi) \sim \frac{1}{\sqrt[4]{p(\xi)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{\xi_0}^{\xi}\sqrt{P(\theta)}d\theta\right]\right\} = \frac{2e^{\pi s_x/8}\sqrt{\theta_2 - c^2}c_{0x}}{\sqrt[4]{p(\xi)}}\cos\left[\frac{k}{2}\int_{\xi}^{\min\left(c^2,\theta_1\right)}\sqrt{P(\theta)}d\theta + k\delta_0\right] \tag{314}$$

$$p(\theta) = (\theta - \theta_1)(\theta - \theta_2) \tag{315}$$

$$P(\theta) = p(\theta)/D(\theta) \tag{316}$$

$$D(\theta) = \left(a^2 - \theta\right)\left(b^2 - \theta\right)\left(c^2 - \theta\right) \tag{317}$$

which are in turn matched to outward solutions with a random phase reflection coefficient. If we evaluate this global solution in the limit as $c^2 - \xi = \widetilde{\xi}$ small but large compared to $\theta_1 - c^2$

$$X(\xi) \sim \frac{2e^{\pi s_x/8}\sqrt{(\theta_2 - c^2)}c_{0x}}{\sqrt[4]{(\theta_1 - \xi)(\theta_2 - \xi)}}\cos\left[\frac{k}{2}\int_{\xi}^{\min\left(c^2,\theta_1\right)}\sqrt{\frac{(\theta_1 - \theta)(\theta_2 - \theta)}{(a^2 - \theta)(b^2 - \theta)(c^2 - \theta)}}d\theta + k\delta_0\right] \tag{318}$$

$$\begin{aligned}
X(\xi) &\sim \frac{2e^{\pi s_x/8}\sqrt[4]{(\theta_2 - c^2)}c_{0x}}{\sqrt[4]{(\theta_1 - c^2 + c^2 - \xi)(\theta_2 - c^2)}}\cos\left[\frac{k}{2}\int_{\xi}^{\min\left(c^2,\theta_1\right)}\sqrt{\frac{(\theta_1 - \theta)(\theta_2 - \theta)}{(a^2 - \theta)(b^2 - \theta)(c^2 - \theta)}}d\theta + k\delta_0\right] \\
&\sim \frac{2e^{\pi s_x/8}c_{0x}}{\sqrt[4]{\widetilde{\xi}}}\cos\left[\frac{k}{2}\int_{\xi}^{\min\left(c^2,\theta_1\right)}\sqrt{\frac{(\theta_1 - c^2 + c^2 - \theta)(\theta_2 - c^2)}{(a^2 - c^2)(b^2 - c^2)(c^2 - \theta)}}d\theta + k\delta_0\right] \\
&\sim \frac{2e^{\pi s_x/8}c_{0x}}{\sqrt[4]{\widetilde{\xi}}}\cos\left[\frac{k}{2}\sqrt{\frac{(\theta_2 - c^2)}{(a^2 - c^2)(b^2 - c^2)}}\int_{c^2-\min(c^2,\theta_1)}^{\widetilde{\xi}}\sqrt{\frac{\left(\theta_1 - c^2 + \widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} + k\delta_0\right] \\
&\sim \frac{2e^{\pi s_x/8}c_{0x}}{\sqrt[4]{\widetilde{\xi}}}\cos\left[\frac{k}{2}\sqrt{\frac{(\theta_2 - c^2)}{(a^2 - c^2)(b^2 - c^2)}}\int_{\max(0,c^2-\theta_1)}^{\widetilde{\xi}}\sqrt{\frac{\left(\theta_1 - c^2 + \widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} + k\delta_0\right]
\end{aligned} \tag{319}$$

Noting

$$u = \frac{\left|\theta_1 - c^2\right|}{\widetilde{\theta}} \rightarrow d\widetilde{\theta} = -\frac{\left|\theta_1 - c^2\right|}{u^2} du \tag{320}$$

$$\int_{\max(0,c^2-\theta_1)}^{\widetilde{\xi}} \sqrt{\frac{\left(\theta_1 - c^2 + \widetilde{\theta}\right)}{\widetilde{\theta}}} d\widetilde{\theta} = \left|\theta_1 - c^2\right| \int_{\left|\theta_1 - c^2\right|/\widetilde{\xi}}^{(\infty,1)} \sqrt{1 \pm u} \frac{du}{u^2} \text{ , where } \pm \quad \rightarrow \operatorname{sgn}\left(\theta_1 - c^2\right)$$

$$= \left|\theta_1 - c^2\right| \left[\frac{\sqrt{1 \pm \left|\theta_1 - c^2\right|/\widetilde{\xi}}}{\left|\theta_1 - c^2\right|/\widetilde{\xi}} \pm \frac{1}{2} \int_{\left|\theta_1 - c^2\right|/\widetilde{\xi}}^{(\infty,1)} \frac{du}{u\sqrt{1 \pm u}} \right]$$

$$= \left|\theta_1 - c^2\right| \left[\frac{\sqrt{1 \pm \left|\theta_1 - c^2\right|/\widetilde{\xi}}}{\left|\theta_1 - c^2\right|/\widetilde{\xi}} \pm \frac{1}{2} \ln \left| \frac{\sqrt{1 \pm \left|\theta_1 - c^2\right|/\widetilde{\xi}} + 1}{\sqrt{1 \pm \left|\theta_1 - c^2\right|/\widetilde{\xi}} - 1} \right| \right] \tag{321}$$

where we used

$$\int \frac{du}{u\sqrt{1 \pm u}} = \ln \left| \frac{\sqrt{1 \pm u} - 1}{\sqrt{1 \pm u} + 1} \right| \tag{322}$$

Expanding for $\left(c^2 - \theta_1\right) << \widetilde{\xi}$

$$\int_{\max(0,c^2-\theta_1)}^{\widetilde{\xi}} \sqrt{\frac{\left(\theta_1 - c^2 + \widetilde{\theta}\right)}{\widetilde{\theta}}} d\widetilde{\theta} \sim \widetilde{\xi} + \frac{1}{2}\left(\theta_1 - c^2\right) + \frac{1}{2}\left(\theta_1 - c^2\right) \ln \left| \frac{4\widetilde{\xi}}{\left(\theta_1 - c^2\right)} \right| \tag{323}$$

$$X(\xi) \sim \frac{2e^{\pi s_x/8} c_{0x}}{\sqrt[4]{\widetilde{\xi}}} \cos \left[\frac{k}{2} \sqrt{\frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)}} \left\{ \widetilde{\xi} + \frac{1}{2}\left(\theta_1 - c^2\right) + \frac{1}{2}\left(\theta_1 - c^2\right) \ln \left| \frac{4\widetilde{\xi}}{\left(\theta_1 - c^2\right)} \right| \right\} + k\delta_0 \right]$$

$$\sim \frac{2e^{\pi s_x/8} c_{0x}}{\sqrt[4]{\widetilde{\xi}}} \cos \left[\frac{1}{2} k_{ex} \widetilde{\xi} + \frac{1}{4} s_x + \frac{1}{4} s_x \ln \left| \frac{4k_{ex}\widetilde{\xi}}{s_x} \right| + k\delta_0 \right] \tag{324}$$

$$k_{ex}^2 = k^2 \frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} \tag{325}$$

$$s_x = k_{ex}\left(\theta_1 - c^2\right) \tag{326}$$

Matching to the local solution

$$X(\xi) \sim 2e^{\pi s_x/8} c_{0x} \cos \left[k_{ex}\widetilde{\xi}/2 + \left(s_x/4\right) \ln \left(k_{ex}\widetilde{\xi}\right) - \Phi_{0x}/2 \right] / \sqrt[4]{\widetilde{\xi}} \text{ , } k_{ex}\widetilde{\xi} \rightarrow \infty \tag{327}$$

gives

$$k\delta_0 = \left(s_x/4\right) \ln \left|s_x/4\right| - \Phi_{0x}/2 \tag{328}$$

4.2 Azimuth-Like Variable

For the azimuth-like coordinate separation function we use the Liouville representation [18] as the global solution

$$Y(\eta) \sim \frac{1}{\sqrt[4]{p(\eta)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_{c^2}^{\eta} \sqrt{P(\theta)} d\theta \right] \right\} = \frac{c_{0y}}{\sqrt[4]{p(\eta)}} \cos \left[\frac{k}{2} \int_{\max(c^2,\theta_1)}^{\eta} \sqrt{P(\theta)} d\theta + k\delta_{c0} \right] \tag{329}$$

where

$$p(\theta) = (\theta - \theta_1)(\theta_2 - \theta) \tag{330}$$

$$P(\theta) = p(\theta) / D(\theta) \tag{331}$$

$$D(\theta) = \left(a^2 - \theta\right)\left(b^2 - \theta\right)\left(\theta - c^2\right) \tag{332}$$

and the shift δ_{c0}, which is presently unknown, will become apparent later when matching to the local solutions at the critical points. Note that the choice of cosine in the azimuth-like solution is not essential because the phase shift $k\delta_{c0}$ can change this to sine. The local solutions below that match to this global solution are chosen to be consistent with this cosine choice, but the phases $e^{i\Phi_{1cy}}$ and $e^{i\Phi_{1by}}$ would likely change with the sine choice.

4.2.1 Inner Critical Points

Near the critical points $\eta \to c^2, \theta_1$ with $\eta - c^2 = \widetilde{\eta}$ the local solution is

$$\begin{aligned} Y &\sim \mathrm{Re}\left[c_{+cy} W_+ \left(s_y, 1/4, k_{ey}\widetilde{\eta}\right) + c_{-cy} W_- \left(s_y, 1/4, k_{ey}\widetilde{\eta}\right)\right] / \sqrt[4]{\widetilde{\eta}} \\ &\sim \mathrm{Re}\left[c_{+cy} W_+ \left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) + c_{-cy} W_- \left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right)\right] / \sqrt[4]{\widetilde{\eta}} \end{aligned} \tag{333}$$

where (here we also give the association with the prior subsection radial-like parameters)

$$k_{ey}^2 = k^2 \frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} = k_{ex}^2 \tag{334}$$

$$s_y = k_{ey}\left(c^2 - \theta_1\right) = k_{ex}\left(c^2 - \theta_1\right) = -s_x \tag{335}$$

Note here that $\widetilde{\eta} = \eta - c^2 \to \infty$ corresponds to η moving away from c^2 toward b^2. Because $\sqrt{\eta - c^2} = \sqrt{\widetilde{\eta}}$, or $\sqrt{b^2 - \eta} = \sqrt{\widetilde{\eta}}$, represents a linear spatial coordinate with limit, $z^2 = 0$, or $y^2 = 0$,

$$y^2 = \frac{\left(b^2 - \xi\right)\left(\zeta - b^2\right)}{\left(b^2 - c^2\right)\left(a^2 - b^2\right)}\left(b^2 - \eta\right) = \frac{\left(b^2 - \xi\right)\left(\zeta - b^2\right)}{\left(b^2 - c^2\right)\left(a^2 - b^2\right)}\widetilde{\eta} \tag{336}$$

$$z^2 = \frac{\left(c^2 - \xi\right)\left(\zeta - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)}\left(\eta - c^2\right) = \frac{\left(c^2 - \xi\right)\left(\zeta - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)}\widetilde{\eta} \tag{337}$$

and if we want to enforce, say, even boundary conditions, we would take

$$\frac{dY}{d\sqrt{\widetilde{\eta}}}\left(\widetilde{\eta}=\eta-c^2\rightarrow 0\right)=0 \text{ or} \frac{dY}{d\sqrt{\widetilde{\eta}}}\left(\widetilde{\eta}=b^2-\eta\rightarrow 0\right)=0 \tag{338}$$

In this region for $b^2<\zeta_0<\zeta<a^2$ we can only drive $y^2\rightarrow 0$ toward the symmetry plane by taking $\eta\rightarrow b^2$. Thus in this inside region, for even boundary conditions, it seems appropriate to impose the symmetry condition on this limit. (In a follow-on report on concave walled cavities with interior foci, in the outside region it seems appropriate to impose the symmetry condition on the limit $\eta\rightarrow c^2$. In the region between foci we do not need to impose this symmetry condition since in that region it is already imposed on X and Z.) Then using

$$Y\sim k_{ey}^{1/4}\operatorname{Re}\left[e^{-i\pi/8}c_{+cy}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ey}\widetilde{\eta}}+e^{i\pi/8}c_{-cy}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{ik_{ey}\widetilde{\eta}}\right] \tag{339}$$

$$\frac{d}{d\sqrt{\widetilde{\eta}}}\left[W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(-ik_{ey}\widetilde{\eta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\right)^{1/2}+O\left(k_{ey}\sqrt{\widetilde{\eta}}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\rightarrow 0 \tag{340}$$

$$\frac{d}{d\sqrt{\widetilde{\eta}}}\left[W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(ik_{ey}\widetilde{\eta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\right)^{1/2}+O\left(k_{ey}\sqrt{\widetilde{\eta}}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\rightarrow 0 \tag{341}$$

$$\frac{dY}{d\sqrt{\widetilde{\eta}}}\left(\eta\rightarrow c^2\right)\sim -2\pi^{1/2}k_{ey}^{3/4}\operatorname{Re}\left[c_{+cy}e^{-i3\pi/8}\frac{1}{\Gamma\left(1/4-is_y/4\right)}+c_{-cy}e^{i3\pi/8}\frac{1}{\Gamma\left(1/4+is_y/4\right)}\right]=0 \tag{342}$$

Setting (where c_{0cy} is taken as real)

$$e^{i\Phi_{1cy}}c_{+cy}=c_{-cy}=e^{i\Phi_{1cy}/2}c_{0cy} \tag{343}$$

we see that

$$\frac{dY}{d\sqrt{\widetilde{\eta}}}\left(\eta\rightarrow c^2\right)\sim c_{0cy}\operatorname{Re}\left[e^{-i\Phi_{1cy}/2}e^{-i3\pi/8}\frac{1}{\Gamma\left(1/4-is_y/4\right)}+e^{i\Phi_{1cy}/2}e^{i3\pi/8}\frac{1}{\Gamma\left(1/4+is_y/4\right)}\right]=0 \tag{344}$$

gives

$$e^{i\Phi_{1cy}+i3\pi/4}=e^{i\pi}\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)} \tag{345}$$

$$e^{i(\Phi_{1cy}-\pi/4)}=\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}=\frac{\Gamma\left(1/4-is_x/4\right)}{\Gamma\left(1/4+is_x/4\right)}=e^{-i(\Phi_{0x}-\pi/4)} \tag{346}$$

$$\Phi_{1cy}=-\Phi_{0x}+\pi/2 \tag{347}$$

Then

$$Y\sim\operatorname{Re}\left[c_{+cy}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+c_{-cy}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}$$

$$\sim c_{0cy}\,\mathrm{Re}\left[e^{-i\Phi_{1cy}/2}W_{+}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+e^{i\Phi_{1cy}/2}W_{-}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}} \tag{348}$$

and using

$$W_{+}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\sim e^{ik_{ey}\widetilde{\eta}/2}\left(-ik_{ey}\widetilde{\eta}\right)^{is_y/4}\ ,\ k_{ey}\widetilde{\eta}\to\infty \tag{349}$$

$$W_{+}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(-ik_{ey}\widetilde{\eta}\right)^{1/4}\sim\pi^{1/2}\left[\frac{1}{\Gamma\left(3/4-is_y/4\right)}-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}+O\left(k_{ey}\widetilde{\eta}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\to 0 \tag{350}$$

$$W_{-}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)=W_{+}\left(-s_y,1/4,-k_{ey}\widetilde{\eta}\right)\sim e^{-ik_{ey}\widetilde{\eta}/2}\left(ik_{ey}\widetilde{\eta}\right)^{-is_y/4}\ ,\ k_{ey}\widetilde{\eta}\to\infty \tag{351}$$

$$W_{-}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(ik_{ey}\widetilde{\eta}\right)^{1/4}\sim\pi^{1/2}\left[\frac{1}{\Gamma\left(3/4+is_y/4\right)}-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}+O\left(k_{ey}\widetilde{\eta}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\to 0 \tag{352}$$

the asymptotic forms are

$$Y\sim c_{0cy}\,\mathrm{Re}\left[e^{-i\Phi_{1cy}/2}e^{ik_{ey}\widetilde{\eta}/2}\left(-ik_{ey}\widetilde{\eta}\right)^{is_y/4}+e^{i\Phi_{1cy}/2}e^{-ik_{ey}\widetilde{\eta}/2}\left(ik_{ey}\widetilde{\eta}\right)^{-is_y/4}\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ k_{ey}\widetilde{\eta}\to\infty$$

$$\sim e^{\pi s_y/8}c_{0cy}\,\mathrm{Re}\left[e^{ik_{ey}\widetilde{\eta}/2+i(s_y/4)\ln(k_{ey}\widetilde{\eta})-i\Phi_{1cy}/2}+e^{-ik_{ey}\widetilde{\eta}/2-i(s_y/4)\ln(k_{ey}\widetilde{\eta})+i\Phi_{1cy}/2}\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ k_{ey}\widetilde{\eta}\to\infty$$

$$\sim 2e^{\pi s_y/8}c_{0cy}\cos\left[k_{ey}\widetilde{\eta}/2+(s_y/4)\ln\left(k_{ey}\widetilde{\eta}\right)-\Phi_{1cy}/2\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ k_{ey}\widetilde{\eta}\to\infty \tag{353}$$

and

$$Y\sim k_{ey}^{1/4}\,\mathrm{Re}\left[e^{-i\pi/8}c_{+cy}W_{+}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ey}\widetilde{\eta}}+e^{i\pi/8}c_{-cy}W_{-}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{ik_{ey}\widetilde{\eta}}\right]$$

$$\sim c_{0cy}k_{ey}^{1/4}\,\mathrm{Re}\left[e^{-i\Phi_{1cy}/2-i\pi/8}W_{+}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ey}\widetilde{\eta}}+e^{i\Phi_{1cy}/2+i\pi/8}W_{-}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{ik_{ey}\widetilde{\eta}}\right]$$

$$\sim c_{0cy}k_{ey}^{1/4}\,\mathrm{Re}\left[e^{-i(\Phi_{1cy}-\pi/4)/2-i\pi/4}W_{+}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ey}\widetilde{\eta}}+e^{i(\Phi_{1cy}-\pi/4)/2+i\pi/4}W_{-}\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{ik_{ey}\widetilde{\eta}}\right]$$

$$\sim c_{0cy}\pi^{1/2}k_{ey}^{1/4}\,\mathrm{Re}\left[e^{-i\pi/4}\sqrt{\frac{\Gamma\left(1/4-is_y/4\right)}{\Gamma\left(1/4+is_y/4\right)}}\left\{\frac{1}{\Gamma\left(3/4-is_y/4\right)}-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}\right.$$
$$\left.+e^{i\pi/4}\sqrt{\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}}\left\{\frac{1}{\Gamma\left(3/4+is_y/4\right)}-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\pi^{1/2}k_{ey}^{1/4}\,\mathrm{Re}\left[e^{-i\pi/4}\left\{\frac{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}{\Gamma\left(3/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}-\frac{2}{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}\right.$$

$$\left.+e^{i\pi/4}\left\{\frac{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}{\Gamma\left(3/4+is_y/4\right)\Gamma\left(1/4-is_y/4\right)}-\frac{2}{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\pi^{-1/2}k_{ey}^{1/4}\left|\Gamma\left(1/4-is_y/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\sin\pi\left(1/4+is_y/4\right)-\frac{2\pi e^{-i\pi/2}}{\left|\Gamma\left(1/4-is_y/4\right)\right|^2}\left(k_{ey}\widetilde{\eta}\right)^{1/2}\right.$$

$$\left.+e^{i\pi/4}\sin\pi\left(1/4-is_y/4\right)-\frac{2\pi e^{i\pi/2}}{\left|\Gamma\left(1/4-is_y/4\right)\right|^2}\left(k_{ey}\widetilde{\eta}\right)^{1/2}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\pi^{-1/2}k_{ey}^{1/4}\left|\Gamma\left(1/4-is_y/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\sin\pi\left(1/4+is_y/4\right)+e^{i\pi/4}\sin\pi\left(1/4-is_y/4\right)+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\left(2\pi\right)^{-1/2}k_{ey}^{1/4}\left|\Gamma\left(1/4-is_y/4\right)\right|$$

$$\mathrm{Re}\left[e^{-i\pi/4}\left\{\cosh\left(\pi s_y/4\right)+i\sinh\left(\pi s_y/4\right)\right\}+e^{i\pi/4}\left\{\cosh\left(\pi s_y/4\right)-i\sinh\left(\pi s_y/4\right)\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\left(2\pi\right)^{-1/2}k_{ey}^{1/4}\left|\Gamma\left(1/4-is_y/4\right)\right|\mathrm{Re}\left[2\cos\left(\pi/4\right)\left\{\cosh\left(\pi s_y/4\right)+\sinh\left(\pi s_y/4\right)\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\pi^{-1/2}k_{ey}^{1/4}\left|\Gamma\left(1/4-is_y/4\right)\right|\left[\cosh\left(\pi s_y/4\right)+\sinh\left(\pi s_y/4\right)+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0cy}\pi^{-1/2}k_{ey}^{1/4}\left|\Gamma\left(1/4-is_y/4\right)\right|\left[e^{\pi s_y/4}+O\left(k_{ey}\widetilde{\eta}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\rightarrow 0 \tag{354}$$

Now to match to the global form

$$Y\left(\eta\right)\sim\frac{c_{0y}}{\sqrt[4]{\left(\eta-\theta_1\right)\left(\theta_2-\eta\right)}}\cos\left[\frac{k}{2}\int_{\max\left(c^2,\theta_1\right)}^{\eta}\sqrt{\frac{\left(\theta-\theta_1\right)\left(\theta_2-\theta\right)}{\left(a^2-\theta\right)\left(b^2-\theta\right)\left(\theta-c^2\right)}}d\theta+k\delta_{c0}\right] \tag{355}$$

we let $\widetilde{\eta}=\eta-c^2$

$$Y\left(\eta\right)\sim\frac{c_{0y}}{\sqrt[4]{\left(\widetilde{\eta}+c^2-\theta_1\right)\left(\theta_2-c^2\right)}}\cos\left[\frac{k}{2}\sqrt{\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}}\int_{\max\left(0,\theta_1-c^2\right)}^{\widetilde{\eta}}\sqrt{1+\frac{c^2-\theta_1}{\widetilde{\theta}}}d\widetilde{\theta}+k\delta_{c0}\right]$$

$$\sim\frac{c_{0y}}{\sqrt[4]{\widetilde{\eta}\left(\theta_2-c^2\right)}}\cos\left[\frac{k}{2}\sqrt{\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}}\int_{\max\left(0,\theta_1-c^2\right)}^{\widetilde{\eta}}\sqrt{1+\frac{c^2-\theta_1}{\widetilde{\theta}}}d\widetilde{\theta}+k\delta_{c0}\right] \tag{356}$$

where we have assumed that $\widetilde{\eta} >> c^2 - \theta_1$. Letting

$$u = \frac{\left|c^2 - \theta_1\right|}{\widetilde{\theta}} \tag{357}$$

$$d\widetilde{\theta} = -\frac{\left|c^2 - \theta_1\right|}{u^2} du \tag{358}$$

$$\int_{\max(0,\theta_1 - c^2)}^{\widetilde{\eta}} \sqrt{1 + \frac{c^2 - \theta_1}{\widetilde{\theta}}} d\widetilde{\theta} = \left|c^2 - \theta_1\right| \int_{\left|c^2 - \theta_1\right|/\widetilde{\eta}}^{(\infty,1)} \sqrt{1 \pm u} \frac{du}{u^2} \text{ , where } \pm \ \ \to \operatorname{sgn}\left(c^2 - \theta_1\right)$$

$$= \left|c^2 - \theta_1\right| \left[\frac{\sqrt{1 \pm \left|c^2 - \theta_1\right|/\widetilde{\eta}}}{\left|c^2 - \theta_1\right|/\widetilde{\eta}} \pm \frac{1}{2} \int_{\left|c^2 - \theta_1\right|/\widetilde{\eta}}^{(\infty,1)} \frac{du}{u\sqrt{1 \pm u}} \right]$$

$$= \left|c^2 - \theta_1\right| \left[\frac{\sqrt{1 \pm \left|c^2 - \theta_1\right|/\widetilde{\eta}}}{\left|c^2 - \theta_1\right|/\widetilde{\eta}} \pm \frac{1}{2} \ln \left| \frac{\sqrt{1 \pm \left|c^2 - \theta_1\right|/\widetilde{\eta}} + 1}{\sqrt{1 \pm \left|c^2 - \theta_1\right|/\widetilde{\eta}} - 1} \right| \right]$$

$$= \widetilde{\eta}\sqrt{1 + \left(c^2 - \theta_1\right)/\widetilde{\eta}} + \frac{1}{2}\left(c^2 - \theta_1\right) \ln \left| \frac{\sqrt{1 + \left(c^2 - \theta_1\right)/\widetilde{\eta}} + 1}{\sqrt{1 + \left(c^2 - \theta_1\right)/\widetilde{\eta}} - 1} \right| \tag{359}$$

where we used

$$\int \frac{du}{u\sqrt{1 \pm u}} = \ln \left| \frac{\sqrt{1 \pm u} - 1}{\sqrt{1 \pm u} + 1} \right| \tag{360}$$

Expanding for $\left(c^2 - \theta_1\right) << \widetilde{\eta}$

$$\int_{\max(0,\theta_1 - c^2)}^{\widetilde{\eta}} \sqrt{1 + \frac{c^2 - \theta_1}{\widetilde{\theta}}} d\widetilde{\theta} \sim \widetilde{\eta} + \frac{1}{2}\left(c^2 - \theta_1\right) + \frac{1}{2}\left(c^2 - \theta_1\right) \ln \left| \frac{4\widetilde{\eta}}{c^2 - \theta_1} \right| \tag{361}$$

Using

$$k_{ex}^2 = k^2 \frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} = k_{ey}^2 \tag{362}$$

$$-s_x = k_{ex}\left(c^2 - \theta_1\right) = k_{ey}\left(c^2 - \theta_1\right) = s_y \tag{363}$$

gives

$$Y(\eta) \sim \frac{c_{0y}}{\sqrt[4]{\widetilde{\eta}\left(\theta_2 - c^2\right)}} \cos \left[\frac{k_{ey}}{2}\widetilde{\eta} + \frac{1}{4}k_{ey}\left(c^2 - \theta_1\right) + \frac{1}{4}k_{ey}\left(c^2 - \theta_1\right) \ln \left| \frac{4\widetilde{\eta}}{c^2 - \theta_1} \right| + k\delta_{c0} \right]$$

$$\sim \frac{c_{0y}}{\sqrt[4]{\widetilde{\eta}\left(\theta_2 - c^2\right)}} \cos \left[k_{ey}\widetilde{\eta}/2 + \frac{s_y}{4} + \frac{s_y}{4} \ln \left|4k_{ey}\widetilde{\eta}/s_y\right| + k\delta_{c0} \right] \tag{364}$$

Comparing to the local solution

$$Y \sim 2e^{\pi s_y/8} c_{0cy} \cos\left[k_{ey}\widetilde{\eta}/2 + \left(s_y/4\right)\ln\left(k_{ey}\widetilde{\eta}\right) - \Phi_{1cy}/2\right]/\sqrt[4]{\widetilde{\eta}} \ , \ \ k_{ey}\widetilde{\eta} \to \infty \tag{365}$$

gives

$$\frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\left(\theta_2-c^2\right)}}=2e^{\pi s_y/8}c_{0cy}=2e^{-\pi s_x/8}c_{0cy} \tag{366}$$

$$\frac{s_y}{4}\ln\left|4e/s_y\right|+k\delta_{c0}-n_c\pi=-\Phi_{1cy}/2=-\frac{s_x}{4}\ln\left|4e/s_x\right|+k\delta_{c0}-n_c\pi=\Phi_{0x}/2-\pi/4 \tag{367}$$

4.2.2 Outer Critical Points

Similarly, the local solution near the critical points $\eta\to b^2,\theta_2$, with $b^2-\eta=\widetilde{\eta}$, is

$$\begin{aligned}Y &\sim \operatorname{Re}\left[c_{+by}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+c_{-by}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\\ &\sim \operatorname{Re}\left[c_{+by}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+c_{-by}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\end{aligned} \tag{368}$$

where

$$k_{ey}^2=k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}=k_{ez}^2 \tag{369}$$

$$s_y=k_{ey}\left(\theta_2-b^2\right)=k_{ez}\left(\theta_2-b^2\right)=-s_z \tag{370}$$

This connection to the axial-like coordinate parameters k_{ez}^2 and s_z will become clear in a follow-on report when we deal with the inner focus, but we will nevertheless use these notations for these quantities here in the convex walled case. Note here that $\widetilde{\eta}\to\infty$ corresponds to η moving from b^2 toward c^2. Then using

$$Y\sim k_{ey}^{1/4}\operatorname{Re}\left[e^{-i\pi/8}c_{+by}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ey}\widetilde{\eta}}+e^{i\pi/8}c_{-by}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{ik_{ey}\widetilde{\eta}}\right] \tag{371}$$

$$\frac{d}{d\sqrt{\widetilde{\eta}}}\left[W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(-ik_{ey}\widetilde{\eta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\right)^{1/2}+O\left(k_{ey}\sqrt{\widetilde{\eta}}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\to 0 \tag{372}$$

$$\frac{d}{d\sqrt{\widetilde{\eta}}}\left[W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(ik_{ey}\widetilde{\eta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\right)^{1/2}+O\left(k_{ey}\sqrt{\widetilde{\eta}}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\to 0 \tag{373}$$

$$\frac{dY}{d\sqrt{\widetilde{\eta}}}\left(\eta\to b^2\right)\sim-2\pi^{1/2}k_{ey}^{3/4}\operatorname{Re}\left[c_{+by}e^{-i3\pi/8}\frac{1}{\Gamma\left(1/4-is_y/4\right)}+c_{-by}e^{i3\pi/8}\frac{1}{\Gamma\left(1/4+is_y/4\right)}\right]=0 \tag{374}$$

Setting (where c_{0by} is taken as real)

$$e^{i\Phi_{1by}}c_{+by}=c_{-by}=e^{i\Phi_{1by}/2}c_{0by} \tag{375}$$

$$\frac{dY}{d\sqrt{\widetilde{\eta}}}\left(\eta\to b^2\right)\sim-2\pi^{1/2}k_{ey}^{3/4}c_{0by}\operatorname{Re}\left[e^{-i\Phi_{1by}/2-i3\pi/8}\frac{1}{\Gamma\left(1/4-is_y/4\right)}+e^{i\Phi_{1by}/2+i3\pi/8}\frac{1}{\Gamma\left(1/4+is_y/4\right)}\right]=0 \tag{376}$$

gives (where in the next subsection we define Φ_{1z})

$$e^{i\Phi_{1by}+i3\pi/4}=e^{i\pi}\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)} \tag{377}$$

$$e^{i(\Phi_{1by}-\pi/4)}=\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}=\frac{\Gamma\left(1/4-is_z/4\right)}{\Gamma\left(1/4+is_z/4\right)}=e^{-i(\Phi_{1z}-\pi/4)} \tag{378}$$

$$\Phi_{1by}=-\Phi_{1z}+\pi/2 \tag{379}$$

This connection to the axial-like coordinate phase Φ_{1z} will become clear in a follow-on report when we deal with the inner focus, but we will nevertheless use this notation for the phase here in the convex wall case. Then

$$\begin{aligned}
Y &\sim \mathrm{Re}\left[c_{+by}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+c_{-by}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}} \\
&\sim \mathrm{Re}\left[c_{+by}\left\{W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+e^{i\Phi_{1by}}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right\}\right]/\sqrt[4]{\widetilde{\eta}} \\
&\sim c_{0by}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+e^{i\Phi_{1by}/2}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}} \\
&\sim \mathrm{Re}\left[c_{+by}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+c_{-by}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}} \\
&\sim \mathrm{Re}\left[c_{+by}\left\{W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+e^{i\Phi_{1by}}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right\}\right]/\sqrt[4]{\widetilde{\eta}} \\
&\sim c_{0by}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+e^{i\Phi_{1by}/2}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}} \\
&\sim c_{0by}\,\mathrm{Re}\left[e^{i\Phi_{1z}/2-i\pi/4}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+e^{-i\Phi_{1z}/2+i\pi/4}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}
\end{aligned} \tag{380}$$

and using

$$W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\sim e^{ik_{ey}\widetilde{\eta}/2}\left(-ik_{ey}\widetilde{\eta}\right)^{is_y/4}\ \ ,\ \ k_{ey}\widetilde{\eta}\to\infty \tag{381}$$

$$W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(-ik_{ey}\widetilde{\eta}\right)^{1/4}\sim\pi^{1/2}\left[\frac{1}{\Gamma\left(3/4-is_y/4\right)}-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}+O\left(k_{ey}\widetilde{\eta}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\to 0 \tag{382}$$

$$W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)=W_+\left(-s_y,1/4,-k_{ey}\widetilde{\eta}\right)\sim e^{-ik_{ey}\widetilde{\eta}/2}\left(ik_{ey}\widetilde{\eta}\right)^{-is_y/4}\ \ ,\ \ k_{ey}\widetilde{\eta}\to\infty \tag{383}$$

$$W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\left(ik_{ey}\widetilde{\eta}\right)^{1/4}\sim\pi^{1/2}\left[\frac{1}{\Gamma\left(3/4+is_y/4\right)}-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}+O\left(k_{ey}\widetilde{\eta}\right)\right]\ ,\ k_{ey}\widetilde{\eta}\to 0 \tag{384}$$

the asymptotic forms are

$$Y\sim\mathrm{Re}\left[c_{+by}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+c_{-by}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}$$

$$\sim c_{0by}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+e^{i\Phi_{1by}/2}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}$$

$$\sim c_{0by}e^{\pi s_y/8}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2}e^{ik_{ey}\widetilde{\eta}/2}e^{i(s_y/4)\ln(k_{ey}\widetilde{\eta})}+e^{i\Phi_{1by}/2}e^{-ik_{ey}\widetilde{\eta}/2}e^{-i(s_y/4)\ln(k_{ey}\widetilde{\eta})}\right]/\sqrt[4]{\widetilde{\eta}}$$

$$\sim 2c_{0by}e^{\pi s_y/8}\cos\left[k_{ey}\widetilde{\eta}/2+(s_y/4)\ln\left(k_{ey}\widetilde{\eta}\right)-\Phi_{1by}/2\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \ k_{ey}\widetilde{\eta}\to\infty$$

$$\sim c_{0by}\,\mathrm{Re}\left[e^{i\Phi_{1z}/2-i\pi/4}e^{ik_{ez}\widetilde{\eta}/2}\left(-ik_{ez}\widetilde{\eta}\right)^{-is_z/4}+e^{-i\Phi_{1z}/2+i\pi/4}e^{-ik_{ez}\widetilde{\eta}/2}\left(ik_{ez}\widetilde{\eta}\right)^{is_z/4}\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \ k_{ez}\widetilde{\eta}\to\infty$$

$$\sim e^{-\pi s_z/8}c_{0by}\,\mathrm{Re}\left[e^{i\Phi_{1z}/2-i\pi/4}e^{ik_{ez}\widetilde{\eta}/2}e^{-i(s_z/4)\ln(k_{ez}\widetilde{\eta})}+e^{-i\Phi_{1z}/2+i\pi/4}e^{-ik_{ez}\widetilde{\eta}/2}e^{i(s_z/4)\ln(k_{ez}\widetilde{\eta})}\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ k_{ez}\widetilde{\eta}\to\infty$$

$$\sim 2e^{-\pi s_z/8}c_{0by}\cos\left[k_{ez}\widetilde{\eta}/2-(s_z/4)\ln\left(k_{ez}\widetilde{\eta}\right)+\Phi_{1z}/2-\pi/4\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \ k_{ez}\widetilde{\eta}\to\infty \tag{385}$$

and

$$Y\sim\mathrm{Re}\left[c_{+by}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+c_{-by}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}$$

$$\sim c_{0by}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)+e^{i\Phi_{1by}/2}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}$$

$$\sim c_{0by}k_{ey}^{1/2}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2-i\pi/8}W_+\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ey}\widetilde{\eta}}+e^{i\Phi_{1by}/2+i\pi/8}W_-\left(s_y,1/4,k_{ey}\widetilde{\eta}\right)/\sqrt[4]{ik_{ey}\widetilde{\eta}}\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ey}^{1/2}\,\mathrm{Re}\left[e^{-i(\Phi_{1by}-\pi/4)/2-i\pi/4}\left\{\frac{1}{\Gamma\left(3/4-is_y/4\right)}-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}\right.$$
$$\left.+e^{i(\Phi_{1by}-\pi/4)/2+i\pi/4}\left\{\frac{1}{\Gamma\left(3/4+is_y/4\right)}-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ey}^{1/2}\,\mathrm{Re}\left[e^{-i\pi/4}\sqrt{\frac{\Gamma\left(1/4-is_y/4\right)}{\Gamma\left(1/4+is_y/4\right)}}\left\{\frac{1}{\Gamma\left(3/4-is_y/4\right)}-\frac{2}{\Gamma\left(1/4-is_y/4\right)}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}\right.$$
$$\left.+e^{i\pi/4}\sqrt{\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}}\left\{\frac{1}{\Gamma\left(3/4+is_y/4\right)}-\frac{2}{\Gamma\left(1/4+is_y/4\right)}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ey}^{1/2}\,\mathrm{Re}\left[e^{-i\pi/4}\left\{\frac{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}{\Gamma\left(3/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}-\frac{2}{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}\right.$$

$$\left.+e^{i\pi/4}\left\{\frac{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}{\Gamma\left(3/4+is_y/4\right)\Gamma\left(1/4-is_y/4\right)}-\frac{2}{\sqrt{\Gamma\left(1/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ey}^{1/2}\left|\Gamma\left(1/4+is_y/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\left\{\frac{1}{\Gamma\left(3/4-is_y/4\right)\Gamma\left(1/4+is_y/4\right)}-\frac{2}{\left|\Gamma\left(1/4+is_y/4\right)\right|^2}\left(-ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}\right.$$

$$\left.+e^{i\pi/4}\left\{\frac{1}{\Gamma\left(3/4+is_y/4\right)\Gamma\left(1/4-is_y/4\right)}-\frac{2}{\left|\Gamma\left(1/4+is_y/4\right)\right|^2}\left(ik_{ey}\widetilde{\eta}\right)^{1/2}\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{-1/2}k_{ey}^{1/2}\left|\Gamma\left(1/4+is_y/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\sin\pi\left(1/4+is_y/4\right)-\frac{2\pi e^{-i\pi/2}}{\left|\Gamma\left(1/4+is_y/4\right)\right|^2}\left(k_{ey}\widetilde{\eta}\right)^{1/2}\right.$$

$$\left.+e^{i\pi/4}\sin\pi\left(1/4-is_y/4\right)-\frac{2\pi e^{i\pi/2}}{\left|\Gamma\left(1/4+is_y/4\right)\right|^2}\left(k_{ey}\widetilde{\eta}\right)^{1/2}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{-1/2}k_{ey}^{1/2}\left|\Gamma\left(1/4+is_y/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\sin\pi\left(1/4+is_y/4\right)+e^{i\pi/4}\sin\pi\left(1/4-is_y/4\right)+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\left(2\pi\right)^{-1/2}k_{ey}^{1/2}\left|\Gamma\left(1/4+is_y/4\right)\right|$$

$$\mathrm{Re}\left[e^{-i\pi/4}\left\{\cosh\left(\pi s_y/4\right)+i\sinh\left(\pi s_y/4\right)\right\}+e^{i\pi/4}\left\{\cosh\left(\pi s_y/4\right)-i\sinh\left(\pi s_y/4\right)\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\left(2\pi\right)^{-1/2}k_{ey}^{1/2}\left|\Gamma\left(1/4+is_y/4\right)\right|\left[2\cos\left(\pi/4\right)\left\{\cosh\left(\pi s_y/4\right)+\sinh\left(\pi s_y/4\right)\right\}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{-1/2}k_{ey}^{1/2}\left|\Gamma\left(1/4+is_y/4\right)\right|\left[e^{\pi s_y/4}+O\left(k_{ey}\widetilde{\eta}\right)\right]$$

$$\sim c_{0by}\pi^{-1/2}k_{ez}^{1/2}\left|\Gamma\left(1/4-is_z/4\right)\right|\left[e^{-\pi s_z/4}+O\left(k_{ey}\widetilde{\eta}\right)\right] \tag{386}$$

Now we match this to the limit of the global solution

$$Y\left(\eta\right)\sim\frac{c_{0y}}{\sqrt[4]{p\left(\eta\right)}}\cos\left[\frac{k}{2}\int_{\max\left(c^2,\theta_1\right)}^{\eta}\sqrt{P\left(\theta\right)}d\theta+k\delta_{c0}\right] \tag{387}$$

where

$$p\left(\theta\right)=\left(\theta-\theta_1\right)\left(\theta_2-\theta\right) \tag{388}$$

$$P\left(\theta\right)=p\left(\theta\right)/D\left(\theta\right) \tag{389}$$

$$D\left(\theta\right)=\left(a^2-\theta\right)\left(b^2-\theta\right)\left(\theta-c^2\right) \tag{390}$$

Expanding this Liouville WKB solution near the upper integration point with $b^2-\eta=\widetilde{\eta}$

$$Y(\eta) \sim \frac{c_{0y}}{\sqrt[4]{p(\eta)}} \cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)}d\theta + \frac{k}{2}\int_{\min(b^2,\theta_2)}^{\eta} \sqrt{P(\theta)}d\theta + k\delta_{c0}\right]$$

$$\sim \frac{c_{0y}}{\sqrt[4]{(\eta-\theta_1)(\theta_2-\eta)}} \cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)}d\theta + \frac{k}{2}\int_{\min(b^2,\theta_2)}^{\eta} \sqrt{\frac{(\theta-\theta_1)(\theta_2-\theta)}{(a^2-\theta)(b^2-\theta)(\theta-c^2)}}d\theta + k\delta_{c0}\right]$$

$$\sim \frac{c_{0y}}{\sqrt[4]{(b^2-\theta_1-\widetilde{\eta})(\theta_2-b^2+\widetilde{\eta})}}$$

$$\cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)}d\theta - \frac{k}{2}\int_{\max(0,b^2-\theta_2)}^{\widetilde{\eta}} \sqrt{\frac{\left(b^2-\theta_1-\widetilde{\theta}\right)}{\left(a^2-b^2-\widetilde{\theta}\right)\left(b^2-c^2-\widetilde{\theta}\right)}}\sqrt{\frac{\left(\theta_2-b^2+\widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} + k\delta_{c0}\right] \;,\; b^2-\theta=\widetilde{\theta}$$

$$\sim \frac{c_{0y}}{\sqrt[4]{(b^2-\theta_1)\,\widetilde{\eta}}} \cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)}d\theta - \frac{k}{2}\sqrt{\frac{(b^2-\theta_1)}{(a^2-b^2)(b^2-c^2)}}\int_{\max(0,b^2-\theta_2)}^{\widetilde{\eta}} \sqrt{\frac{\left(\theta_2-b^2+\widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} + k\delta_{c0}\right] \;,\; b^2-\theta=\widetilde{\theta}$$

$$\sim \frac{c_{0y}}{\sqrt[4]{(b^2-\theta_1)\,\widetilde{\eta}}} \cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)}d\theta - \frac{k_{ey}}{2}\int_{\max(0,b^2-\theta_2)}^{\widetilde{\eta}} \sqrt{\frac{\left(\theta_2-b^2+\widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} + k\delta_{c0}\right] \tag{391}$$

where we have assumed that $\widetilde{\eta} >> b^2-\theta_2$. Letting

$$u = \frac{\left|\theta_2-b^2\right|}{\widetilde{\theta}} \to d\widetilde{\theta} = -\frac{\left|\theta_2-b^2\right|}{u^2}du \tag{392}$$

$$\int_{\max(0,b^2-\theta_2)}^{\widetilde{\eta}} \sqrt{\frac{\left(\theta_2-b^2+\widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} = \left|\theta_2-b^2\right|\int_{\left|\theta_2-b^2\right|/\widetilde{\eta}}^{(\infty,1)} \sqrt{1\pm u}\frac{du}{u^2} \;,\; \text{where } \pm \;\; \to \operatorname{sgn}\left(\theta_2-b^2\right)$$

$$= \left|\theta_2-b^2\right|\left[\frac{\sqrt{1\pm\left|\theta_2-b^2\right|/\widetilde{\eta}}}{\left|\theta_2-b^2\right|/\widetilde{\eta}} \pm \frac{1}{2}\int_{\left|\theta_2-b^2\right|/\widetilde{\eta}}^{(\infty,1)} \frac{du}{u\sqrt{1\pm u}}\right]$$

$$= \left|\theta_2-b^2\right|\left[\frac{\sqrt{1\pm\left|\theta_2-b^2\right|/\widetilde{\eta}}}{\left|\theta_2-b^2\right|/\widetilde{\eta}} \pm \frac{1}{2}\ln\left|\frac{\sqrt{1\pm\left|\theta_2-b^2\right|/\widetilde{\eta}}+1}{\sqrt{1\pm\left|\theta_2-b^2\right|/\widetilde{\eta}}-1}\right|\right] \tag{393}$$

where we again used

$$\int \frac{du}{u\sqrt{1\pm u}} = \ln\left|\frac{\sqrt{1\pm u}-1}{\sqrt{1\pm u}+1}\right| \tag{394}$$

Expanding for $\left(b^2-\theta_2\right) << \widetilde{\eta}$

$$\int_{\max(0,b^2-\theta_2)}^{\widetilde{\eta}} \sqrt{\frac{\left(\theta_2-b^2+\widetilde{\theta}\right)}{\widetilde{\theta}}}d\widetilde{\theta} \sim \widetilde{\eta}+\frac{1}{2}\left(\theta_2-b^2\right)+\frac{1}{2}\left(\theta_2-b^2\right)\ln\left|\frac{4\widetilde{\eta}}{\left(\theta_2-b^2\right)}\right| \tag{395}$$

Using

$$k_{ey}^2 = k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)} = k_{ez}^2 \tag{396}$$

$$s_y = k_{ey}\left(\theta_2-b^2\right) = k_{ez}\left(\theta_2-b^2\right) = -s_z \tag{397}$$

gives

$$Y\left(\eta\right) \sim \frac{c_{0y}}{\sqrt[4]{\left(b^2-\theta_1\right)\widetilde{\eta}}}\cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta - k_{ey}\widetilde{\eta}/2-\frac{s_y}{4}-\frac{s_y}{4}\ln\left|\frac{4k_{ey}\widetilde{\eta}}{s_y}\right|+k\delta_{c0}\right] \tag{398}$$

Comparing to the local solution

$$Y \sim 2c_{0by}e^{\pi s_y/8}\cos\left[k_{ey}\widetilde{\eta}/2+\left(s_y/4\right)\ln\left(k_{ey}\widetilde{\eta}\right)-\Phi_{1by}/2\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \ k_{ey}\widetilde{\eta}\to\infty \tag{399}$$

gives

$$\frac{c_{0y}\left(-1\right)^{n_b}}{\sqrt[4]{\left(b^2-\theta_1\right)}} = 2c_{0by}e^{\pi s_y/8} = 2c_{0by}e^{-\pi s_z/8} \tag{400}$$

and

$$\begin{aligned}\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta-\frac{s_y}{4}\ln\left|\frac{4e}{s_y}\right|+k\delta_{c0}-n_b\pi &= \Phi_{1by}/2 = -\left(\Phi_{1z}-\pi/2\right)/2 \\ &= \frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta+\frac{s_z}{4}\ln\left|\frac{4e}{-s_z}\right|+k\delta_{c0}-n_b\pi\end{aligned} \tag{401}$$

Note that in the aligned limit

$$\frac{k}{2}\int_{c^2}^{b^2}\sqrt{P\left(\theta\right)}d\theta = \frac{k}{2}\int_{c^2}^{b^2}\frac{d\theta}{\sqrt{a^2-\theta}} = k\left(\sqrt{a^2-c^2}-\sqrt{a^2-b^2}\right) \tag{402}$$

4.3 Axial-Like Variable

For the axial-like coordinate, taking odd or even symmetry at the orbit center $x=0$ the Liouville global solution is [18]

$$Z\left(\zeta\right) \sim \frac{c_{0z}}{\sqrt[4]{p\left(\zeta\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{\zeta}^{a^2}\sqrt{P\left(\theta\right)}d\theta\right]\right\} \tag{403}$$

$$p\left(\theta\right) = \left(\theta-\theta_1\right)\left(\theta-\theta_2\right) \tag{404}$$

$$P(\theta) = p(\theta)/D(\theta) \tag{405}$$

$$D(\theta) = \left(a^2 - \theta\right)\left(\theta - b^2\right)\left(\theta - c^2\right) \tag{406}$$

4.3.1 Center Singular Point

Near $x = 0$ we have $\zeta \to a^2$ which has the singular point asymptotic solution (the multiplicative coeffi cient of this local singular form has been chosen so it will match to the Liouville global solution)

$$Z \sim \frac{c_{0z}}{\sqrt[4]{(a^2-\theta_1)(a^2-\theta_2)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left(k_{e0}\sqrt{\widetilde{\zeta}}\right)\right\} \ , \ \widetilde{\zeta} = a^2 - \zeta \tag{407}$$

where

$$k_{e0}^2 = k^2 \frac{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}{\left(a^2-b^2\right)\left(a^2-c^2\right)} \tag{408}$$

Note that the preceding Liouville phase in this limit becomes

$$\frac{k}{2}\int_{\zeta}^{a^2} \sqrt{P(\theta)}d\theta \sim \frac{k_{e0}}{2}\int_{\zeta}^{a^2} \frac{d\theta}{\sqrt{a^2-\theta}} = k_{e0}\sqrt{a^2-\zeta} \tag{409}$$

and thus

$$Z(\zeta) \sim \frac{c_{0z}}{\sqrt[4]{p(a^2)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left(k_{e0}\sqrt{a^2-\zeta}\right)\right\} \tag{410}$$

matches to the local singular solution.

4.3.2 Cavity Boundary Condition

As we proceed outward from the symmetry point at the ray path center toward the inner focus in the bowtie cavity we encounter the conductive wall with the required vanishing of the potential

$$0 = Z(\zeta_0) = \frac{c_{0z}}{\sqrt[4]{p(\zeta_0)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2}\int_{\zeta_0}^{a^2} \sqrt{P(\theta)}d\theta\right]\right\} \tag{411}$$

which requires

$$k_p\ell = \left\{ \begin{array}{c} p\pi \\ (p-1/2)\,\pi \end{array} \right\} = \frac{k}{2}\int_{\zeta_0}^{a^2} \sqrt{P(\theta)}d\theta \tag{412}$$

Note in the aligned case $\theta_2 = b^2$ and $\theta_1 = c^2$ this phase condition simplifies to

$$\frac{k}{2}\int_{\zeta_0}^{a^2} \frac{d\theta}{\sqrt{a^2-\theta}} = k\sqrt{a^2-\zeta_0} = k\ell \tag{413}$$

4.4 Summary Of Solutions And Matching Conditions

From the preceding subsections the solutions and matching results are now listed

$$\Phi\left(\xi,\eta,\zeta\right)=X\left(\xi\right)Y\left(\eta\right)Z\left(\zeta\right) \tag{414}$$

In the convex walled cavity the region of interest is inside the foci where we summarize

$$X\left(\xi\right)\sim c_{0x}\,\mathrm{Re}\left[e^{-i\Phi_{0x}/2}W_{+}\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)+e^{i\Phi_{0x}/2}W_{-}\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)\right]/\sqrt[4]{\widetilde{\xi}}\ ,\ \xi=c^2-\widetilde{\xi} \tag{415}$$

$$k_{ex}^2=k^2\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} \tag{416}$$

$$s_x=k_{ex}\left(\theta_1-c^2\right) \tag{417}$$

$$\frac{\Gamma\left(1/4+is_x/4\right)}{\Gamma\left(1/4-is_x/4\right)}=e^{i\left(\Phi_{0x}-\pi/4\right)} \tag{418}$$

$$Y\left(\eta\right)\sim\frac{1}{\sqrt[4]{p\left(\eta\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\eta}\sqrt{P\left(\theta\right)}d\theta\right]\right\}=\frac{c_{0y}}{\sqrt[4]{p\left(\eta\right)}}\cos\left[\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\eta}\sqrt{P\left(\theta\right)}d\theta+k\delta_{c0}\right] \tag{419}$$

$$p\left(\theta\right)=\left(\theta-\theta_1\right)\left(\theta_2-\theta\right) \tag{420}$$

$$P\left(\theta\right)=p\left(\theta\right)/D\left(\theta\right) \tag{421}$$

$$D\left(\theta\right)=\left(a^2-\theta\right)\left(b^2-\theta\right)\left(\theta-c^2\right) \tag{422}$$

$$Y\left(\eta\right)\sim c_{0cy}\,\mathrm{Re}\left[e^{-i\Phi_{1cy}/2}W_{+}\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)+e^{i\Phi_{1cy}/2}W_{-}\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \eta-c^2=\widetilde{\eta}$$

$$\sim c_{0cy}\,\mathrm{Re}\left[e^{i\Phi_{0x}/2-i\pi/4}W_{+}\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)+e^{-i\Phi_{0x}/2+i\pi/4}W_{-}\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \eta-c^2=\widetilde{\eta} \tag{423}$$

$$Y\left(\eta\right)\sim 2e^{\pi s_y/8}c_{0cy}\cos\left[k_{ey}\widetilde{\eta}/2+\left(s_y/4\right)\ln\left(k_{ey}\widetilde{\eta}\right)-\Phi_{1cy}/2\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ k_{ey}\widetilde{\eta}\to\infty \tag{424}$$

$$k_{ey}^2=k^2\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}=k_{ex}^2 \tag{425}$$

$$s_y=k_{ey}\left(c^2-\theta_1\right)=k_{ex}\left(c^2-\theta_1\right)=-s_x \tag{426}$$

$$e^{i\left(\Phi_{1cy}-\pi/4\right)}=\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}=\frac{\Gamma\left(1/4-is_x/4\right)}{\Gamma\left(1/4+is_x/4\right)}=e^{-i\left(\Phi_{0x}-\pi/4\right)} \tag{427}$$

$$\frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\left(\theta_2-c^2\right)}}=2e^{\pi s_y/8}c_{0cy}=2e^{-\pi s_x/8}c_{0cy} \tag{428}$$

$$\frac{s_y}{4}\ln\left|4e/s_y\right|+k\delta_{c0}-n_c\pi=-\Phi_{1cy}/2=-\frac{s_x}{4}\ln\left|4e/s_x\right|+k\delta_{c0}-n_c\pi=\Phi_{0x}/2-\pi/4 \tag{429}$$

$$Y\left(\eta\right)\sim c_{0by}\operatorname{Re}\left[e^{-i\Phi_{1by}/2}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+e^{i\Phi_{1by}/2}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ b^2-\eta=\widetilde{\eta}$$

$$\sim c_{0by}\operatorname{Re}\left[e^{i\Phi_{0z}/2-i\pi/4}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+e^{-i\Phi_{0z}/2+i\pi/4}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ b^2-\eta=\widetilde{\eta} \tag{430}$$

$$Y\left(\eta\right)\sim 2c_{0by}e^{\pi s_y/8}\cos\left[k_{ey}\widetilde{\eta}/2+\left(s_y/4\right)\ln\left(k_{ey}\widetilde{\eta}\right)-\Phi_{1by}/2\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ k_{ey}\widetilde{\eta}\to\infty \tag{431}$$

$$k_{ey}^2=k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}=k_{ez}^2 \tag{432}$$

$$s_y=k_{ey}\left(\theta_2-b^2\right)=k_{ez}\left(\theta_2-b^2\right)=-s_z \tag{433}$$

$$e^{i\left(\Phi_{1by}-\pi/4\right)}=\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}=\frac{\Gamma\left(1/4-is_z/4\right)}{\Gamma\left(1/4+is_z/4\right)}=e^{-i\left(\Phi_{1z}-\pi/4\right)} \tag{434}$$

$$\frac{c_{0y}\left(-1\right)^{n_b}}{\sqrt[4]{\left(b^2-\theta_1\right)}}=2c_{0by}e^{\pi s_y/8}=2c_{0by}e^{-\pi s_z/8} \tag{435}$$

$$\frac{k}{2}\int_{\max\left(c^2,\theta_1\right)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta-\frac{s_y}{4}\ln\left|\frac{4e}{s_y}\right|+k\delta_{c0}-n_b\pi=\Phi_{1by}/2=-\left(\Phi_{1z}-\pi/2\right)/2$$

$$=\frac{k}{2}\int_{\max\left(c^2,\theta_1\right)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta+\frac{s_z}{4}\ln\left|4e/s_z\right|+k\delta_{c0}-n_b\pi \tag{436}$$

$$Z\left(\zeta\right)\sim\frac{c_{0z}}{\sqrt[4]{p\left(\zeta\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{\zeta}^{a^2}\sqrt{P\left(\theta\right)}d\theta\right]\right\} \tag{437}$$

$$p\left(\theta\right)=\left(\theta-\theta_1\right)\left(\theta-\theta_2\right) \tag{438}$$

$$P\left(\theta\right)=p\left(\theta\right)/D\left(\theta\right) \tag{439}$$

$$D\left(\theta\right)=\left(a^2-\theta\right)\left(\theta-b^2\right)\left(\theta-c^2\right) \tag{440}$$

$$Z\left(\zeta\right)\sim\frac{c_{0z}}{\sqrt[4]{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left(k_{e0}\sqrt{\widetilde{\zeta}}\right)\right\}\ ,\ \widetilde{\zeta}=a^2-\zeta \tag{441}$$

$$k_{e0}^2=k^2\frac{\left(a^2-\theta_1\right)\left(a^2-\theta_2\right)}{\left(a^2-b^2\right)\left(a^2-c^2\right)} \tag{442}$$

$$0 = Z(\zeta_0) = \frac{c_{0z}}{\sqrt[4]{p(\zeta_0)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_{\zeta_0}^{a^2} \sqrt{P(\theta)} d\theta \right] \right\} \tag{443}$$

$$k_p \ell = \left\{ \begin{array}{c} p\pi \\ (p - 1/2)\pi \end{array} \right\} = \frac{k}{2} \int_{\zeta_0}^{a^2} \sqrt{P(\theta)} d\theta \tag{444}$$

4.4.1 Phase Matching Condition

Eliminating $k\delta_{c0}$ in the phase conditions

$$\frac{s_y}{4} \ln |4e/s_y| + k\delta_{c0} - n_c\pi = -\Phi_{1cy}/2 = -\frac{s_x}{4} \ln |4e/s_x| + k\delta_{c0} - n_c\pi = \Phi_{0x}/2 - \pi/4 \tag{445}$$

and

$$\frac{k}{2} \int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)} d\theta - \frac{s_y}{4} \ln \left| \frac{4e}{s_y} \right| + k\delta_{c0} - n_b\pi = \Phi_{1by}/2 \tag{446}$$

$$-(\Phi_{1z} - \pi/2)/2 = \frac{k}{2} \int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)} d\theta + \frac{s_z}{4} \ln |4e/s_z| + k\delta_{c0} - n_b\pi \tag{447}$$

gives

$$(n_b - n_c)\pi = \frac{k}{2} \int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)} d\theta + (\Phi_{0x} + \Phi_{1z} - \pi/2)/2 + \frac{s_x}{4} \ln |4e/s_x| + \frac{s_z}{4} \ln |4e/s_z| - \pi/4 \tag{448}$$

where

$$\arg\Gamma(1/4 + is_x/4) = \ln \sqrt{\frac{\Gamma(1/4 + is_x/4)}{\Gamma(1/4 - is_x/4)}} = (\Phi_{0x} - \pi/4)/2 \tag{449}$$

$$\arg\Gamma(1/4 + is_z/4) = \ln \sqrt{\frac{\Gamma(1/4 + is_z/4)}{\Gamma(1/4 - is_z/4)}} = (\Phi_{1z} - \pi/4)/2 \tag{450}$$

We can then write this phase matching condition as

$$\frac{k}{2} \int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)} \sqrt{P(\theta)} d\theta + \arg\Gamma(1/4 + is_z/4) + \frac{s_z}{4} \ln |4e/s_z| + \arg\Gamma(1/4 + is_x/4) + \frac{s_x}{4} \ln |4e/s_x| = (n_b - n_c)\pi + \pi/4 \tag{451}$$

This condition arises because of matching of $Y(\eta)$ and hence we would anticipate that these conditions arise from quantization around the orbit. The first term of this equation contains the total phase shift over half the interval; the two evanescent intervals are missing but these do not have a phase advance. This phase condition thus restricts choices of the connection between s_x and s_z (and θ_1 and θ_2) to a discrete set of solutions for various choices of the integer $n_b - n_c$. For higher frequencies the increase in the value of the wavenumber k in the first term indicates that this discrete set becomes more dense.

Note in the aligned limit $s_x = 0 = s_z$ ($\theta_1 = c^2$, $\theta_2 = b^2$) and $k = k_p$ this phase matching condition becomes

$$\frac{k}{2}\int_{c^2}^{b^2}\frac{d\theta}{\sqrt{a^2-\theta}}=k_p\left(\sqrt{a^2-c^2}-\sqrt{a^2-b^2}\right)=(n_b-n_c)\,\pi+\pi/4 \tag{452}$$

Alternatively, if we approximate the integral here for large a^2

$$\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta\approx\frac{k}{2\sqrt{a^2-\left(b^2+c^2\right)/2}}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{\frac{\left(\theta-\theta_1\right)\left(\theta_2-\theta\right)}{\left(b^2-\theta\right)\left(\theta-c^2\right)}}d\theta \tag{453}$$

For large shifts we may need to retain the exact forms of k_{ez} and k_{ex}, but otherwise

$$\frac{k}{2}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{P\left(\theta\right)}d\theta\approx\frac{k}{2\sqrt{a^2-\left(b^2+c^2\right)/2}}\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\sqrt{\frac{\left(\theta-\theta_1\right)\left(\theta_2-\theta\right)}{\left(b^2-\theta\right)\left(\theta-c^2\right)}}d\theta$$

$$\approx\frac{k}{2\sqrt{a^2-\left(b^2+c^2\right)/2}}\int_{\max(0,\theta_1-c^2)}^{\min\left(b^2-c^2,\theta_2-b^2+b^2-c^2\right)}\sqrt{\frac{\left(\widetilde{\theta}+c^2-\theta_1\right)\left(\theta_2-b^2+b^2-c^2-\widetilde{\theta}\right)}{\left(b^2-c^2-\widetilde{\theta}\right)\widetilde{\theta}}}d\widetilde{\theta}\ ,\ \widetilde{\theta}=\theta-c^2$$

$$\approx\frac{k}{\sqrt{2}\sqrt{\left(a^2-b^2\right)+\left(a^2-c^2\right)}}\int_{\max\left(0,s_x\frac{1}{k}\sqrt{a^2-c^2}\right)}^{\min\left(b^2-c^2,-s_z\frac{1}{k}\sqrt{a^2-b^2}+b^2-c^2\right)}\sqrt{\frac{\left(\widetilde{\theta}-s_x\frac{1}{k}\sqrt{a^2-c^2}\right)\left(-s_z\frac{1}{k}\sqrt{a^2-b^2}+b^2-c^2-\widetilde{\theta}\right)}{\left(b^2-c^2-\widetilde{\theta}\right)\widetilde{\theta}}}d\widetilde{\theta} \tag{454}$$

Suppose instead of trying to identify the different values of s_x and s_z in the model, we suppress these values and focus on only the difference $k-k_p$. In other words we make a plot as a function of only this parameter. The procedure we have in mind is to scan over the s_x and s_z separation constants (each analytic construction being individually normalized) determining the $k-k_p$ constant for each, and plot the mean of an observable for each of the $k-k_p$ values. We carry out this procedure with success, but there are several questions to keep in mind if there are any issues: 1) Are the separation constants real?, 2) Is the two-dimensional space of these separation constants of limited extent for a limit on $k-k_p$ because of the resonance condition connection (we hope this is true or the normalization weight might become very small)?, 3) Do we need to select regions of s_x, s_z space with some criterion (equal areas?) to then develop the statistics for the observable versus $k-k_p$, or can we base the statistics on the observable with the values of $k-k_p$ alone (averaging the observable over the values falling within a $k-k_p$ bin) and not worry about how the underlying s_x, s_z values were chosen?

4.4.2 Resonant Condition Phase Integral

The quantization condition along the orbit is

$$k_p\ell=\left\{\begin{array}{c}p\pi\\ \left(p-1/2\right)\pi\end{array}\right\}=\frac{k}{2}\int_{\zeta_0}^{a^2}\sqrt{P\left(\theta\right)}d\theta=\frac{k}{2}\int_{\zeta_0}^{a^2}\sqrt{\frac{\left(\theta-\theta_1\right)\left(\theta-\theta_2\right)}{\left(a^2-\theta\right)\left(\theta-b^2\right)\left(\theta-c^2\right)}}d\theta \tag{455}$$

We can write the phase integral along the path approximately as

$$\frac{k}{2}\int_{\zeta_0}^{a^2}\sqrt{P\left(\theta\right)}d\theta=\frac{k}{2}\int_{\zeta_0}^{a^2}\sqrt{\frac{\left(\theta-\theta_1\right)\left(\theta-\theta_2\right)}{\left(a^2-\theta\right)\left(\theta-b^2\right)\left(\theta-c^2\right)}}d\theta=\frac{k}{2}\int_{\zeta_0}^{a^2}\sqrt{\frac{\left(\theta-c^2+c^2-\theta_1\right)\left(\theta-b^2+b^2-\theta_2\right)}{\left(a^2-\theta\right)\left(\theta-b^2\right)\left(\theta-c^2\right)}}d\theta$$

$$= \frac{k}{2}\int_0^{a^2-\zeta_0}\sqrt{\frac{\left(a^2-c^2-\theta'+c^2-\theta_1\right)\left(a^2-b^2-\theta'+b^2-\theta_2\right)}{\left(a^2-c^2-\theta'\right)\theta'\left(a^2-b^2-\theta'\right)}}d\theta' = \frac{k}{2}\int_0^{a^2-\zeta_0}\sqrt{\frac{\left(d_z^2-\theta'+c^2-\theta_1\right)\left(d_y^2-\theta'+b^2-\theta_2\right)}{\left(d_z^2-\theta'\right)\theta'\left(d_y^2-\theta'\right)}}d\theta'$$

$$= \frac{k}{2}\int_0^{a^2-\zeta_0}\sqrt{\frac{\left(1-\theta'/d_z^2+\frac{c^2-\theta_1}{d_z^2}\right)\left(1-\theta'/d_y^2+\frac{b^2-\theta_2}{d_y^2}\right)}{\left(1-\theta'/d_z^2\right)\theta'\left(1-\theta'/d_y^2\right)}}d\theta' \sim \frac{k}{2}\int_0^{\ell^2}\sqrt{\frac{\left\{1-\theta'/d_z^2-s_x/\left(kd_z\right)\right\}\left\{1-\theta'/d_y^2+s_z/\left(kd_y\right)\right\}}{\left(1-\theta'/d_z^2\right)\theta'\left(1-\theta'/d_y^2\right)}}d\theta' \quad (456)$$

$$\theta' = a^2-\theta \quad (457)$$

If we assume that

$$a^2-\zeta_0 = \ell^2 << a^2-b^2 = d_y^2 < a^2-c^2 = d_z^2 \quad (458)$$

$$\frac{k}{2}\int_{\zeta_0}^{a^2}\sqrt{P\left(\theta\right)}d\theta \sim \frac{k}{2}\left\{1+\frac{1}{2}s_z/\left(kd_y\right)-\frac{1}{2}s_x/\left(kd_z\right)\right\}\int_0^{\ell^2}\frac{d\theta'}{\sqrt{\theta'}} = k\ell\left\{1+\frac{1}{2}s_z/\left(kd_y\right)-\frac{1}{2}s_x/\left(kd_z\right)\right\} \quad (459)$$

and then

$$k_p\ell = k\ell\left\{1+\frac{1}{2}s_z/\left(kd_y\right)-\frac{1}{2}s_x/\left(kd_z\right)\right\} \quad (460)$$

where

$$k_p\ell = \left\{\begin{array}{c} p\pi \\ \left(p-1/2\right)\pi \end{array}\right\} \quad (461)$$

and

$$a^2-c^2 = d_z^2 \quad (462)$$

$$a^2-b^2 = d_y^2 \quad (463)$$

$$d_z > d_y \quad (464)$$

$$k_{ex}^2 = k^2\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} \sim \frac{k^2}{a^2-c^2} = \frac{k^2}{d_z^2} \quad (465)$$

$$k_{ez}^2 = k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)} \sim \frac{k^2}{a^2-b^2} = \frac{k^2}{d_y^2} \quad (466)$$

$$s_z = k_{ez}\left(b^2-\theta_2\right) \sim \frac{k}{d_y}\left(b^2-\theta_2\right) \quad (467)$$

$$s_x = k_{ex}\left(\theta_1-c^2\right) \sim \frac{k}{d_z}\left(\theta_1-c^2\right) \quad (468)$$

This result implies that

$$k\ell = \frac{\ell}{2}\left(s_x/\sqrt{a^2-c^2} - s_z/\sqrt{a^2-b^2}\right) + k_p\ell \tag{469}$$

or

$$2\left(k-k_p\right)\ell = \ell\left(s_x/d_z - s_z/d_y\right) \tag{470}$$

which corresponds to lines of constant frequency separation $k - k_p$ on a two-dimensional plot with s_x and s_z axes.

In the section below on the Fourier projection along the ray path orbit, we will generate a two-dimensional probability plot $F(s_x, s_z)$ as a function of s_x and s_z. The preceding relation will then be used to integrate out the dependence on s_x and s_z to end up with a simple plot as a function of $k - k_p$

$$F\left(2\Delta k\ell\right) = \int_{-\infty}^{\infty} F\left(2\Delta k\ell d_z/\ell + s_z d_z/d_y, s_z\right) ds_z \tag{471}$$

4.4.3 Field Along Scar

Here we will use the limits of the Whittaker functions to write

$$W_+\left(s, 1/4, k_e\widetilde{\theta}\right)/\left(-ik_e\widetilde{\theta}\right)^{1/4} \sim \pi^{1/2}\left[\frac{1}{\Gamma\left(3/4-is/4\right)} - \frac{2}{\Gamma\left(1/4-is/4\right)}\left(-ik_e\widetilde{\theta}\right)^{1/2} + O\left(k_e\widetilde{\theta}\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{472}$$

$$W_-\left(s, 1/4, k_e\widetilde{\theta}\right)/\left(ik_e\widetilde{\theta}\right)^{1/4} \sim \pi^{1/2}\left[\frac{1}{\Gamma\left(3/4+is/4\right)} - \frac{2}{\Gamma\left(1/4+is/4\right)}\left(ik_e\widetilde{\theta}\right)^{1/2} + O\left(k_e\widetilde{\theta}\right)\right]\ ,\ k_e\widetilde{\theta}\to 0 \tag{473}$$

and determine the limits of the separation functions. The radial-like function is then

$$X\left(\xi\right) \sim c_{0x}\,\mathrm{Re}\left[e^{-i\Phi_{0x}/2}W_+\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right) + e^{i\Phi_{0x}/2}W_-\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right)\right]/\sqrt[4]{\widetilde{\xi}}\ ,\ \widetilde{\xi} = c^2 - \xi$$

$$\sim c_{0x}k_{ex}^{1/4}\,\mathrm{Re}\left[e^{-i\Phi_{0x}/2-i\pi/8}W_+\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right)/\sqrt[4]{-ik_{ex}\widetilde{\xi}} + e^{i\Phi_{0x}/2+i\pi/8}W_-\left(s_x, 1/4, k_{ex}\widetilde{\xi}\right)/\sqrt[4]{ik_{ex}\widetilde{\xi}}\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\pi^{1/2}\,\mathrm{Re}\left[e^{-i\Phi_{0x}/2-i\pi/8}\frac{1}{\Gamma\left(3/4-is_x/4\right)} + e^{i\Phi_{0x}/2+i\pi/8}\frac{1}{\Gamma\left(3/4+is_x/4\right)}\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\pi^{1/2}\,\mathrm{Re}\left[e^{-i(\Phi_{0x}-\pi/4)/2-i\pi/4}\frac{1}{\Gamma\left(3/4-is_x/4\right)} + e^{i(\Phi_{0x}-\pi/4)/2+i\pi/4}\frac{1}{\Gamma\left(3/4+is_x/4\right)}\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\pi^{1/2}\,\mathrm{Re}\left[e^{-i\pi/4}\frac{1}{\Gamma\left(3/4-is_x/4\right)}\sqrt{\frac{\Gamma\left(1/4-is_x/4\right)}{\Gamma\left(1/4+is_x/4\right)}} + e^{i\pi/4}\frac{1}{\Gamma\left(3/4+is_x/4\right)}\sqrt{\frac{\Gamma\left(1/4+is_x/4\right)}{\Gamma\left(1/4-is_x/4\right)}}\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\pi^{1/2}\,\mathrm{Re}\left[e^{-i\pi/4}\frac{\sqrt{\Gamma\left(1/4-is_x/4\right)\Gamma\left(1/4+is_x/4\right)}}{\Gamma\left(3/4-is_x/4\right)\Gamma\left(1/4+is_x/4\right)} + e^{i\pi/4}\frac{\sqrt{\Gamma\left(1/4+is_x/4\right)\Gamma\left(1/4-is_x/4\right)}}{\Gamma\left(3/4+is_x/4\right)\Gamma\left(1/4-is_x/4\right)}\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\pi^{-1/2}\left|\Gamma\left(1/4-is_x/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\sin\pi\left(1/4+is_x/4\right)+e^{i\pi/4}\sin\pi\left(1/4-is_x/4\right)\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\left(2\pi\right)^{-1/2}\left|\Gamma\left(1/4-is_x/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\left\{\cosh\left(\pi s_x/4\right)+i\sinh\left(\pi s_x/4\right)\right\}+e^{i\pi/4}\left\{\cosh\left(\pi s_x/4\right)-i\sinh\left(\pi s_x/4\right)\right\}\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\left(2\pi\right)^{-1/2}\left|\Gamma\left(1/4-is_x/4\right)\right|\mathrm{Re}\left[2\cos\left(\pi/4\right)\cosh\left(\pi s_x/4\right)+2\sin\left(\pi/4\right)\sinh\left(\pi s_x/4\right)\right]$$

$$\sim c_{0x}k_{ex}^{1/4}\pi^{-1/2}\left|\Gamma\left(1/4-is_x/4\right)\right|\mathrm{Re}\left[\cosh\left(\pi s_x/4\right)+\sinh\left(\pi s_x/4\right)\right] \tag{474}$$

$$X\left(\xi\to c^2\right)\sim c_{0x}k_{ex}^{1/4}\pi^{-1/2}\left|\Gamma\left(1/4-is_x/4\right)\right|e^{\pi s_x/4} \tag{475}$$

where we used the reflection formula

$$\Gamma\left(z\right)\Gamma\left(1-z\right)=\pi\csc\left(\pi z\right) \tag{476}$$

and

$$\frac{\Gamma\left(1/4+is_x/4\right)}{\Gamma\left(1/4-is_x/4\right)}=e^{i\left(\Phi_{0x}-\pi/4\right)} \tag{477}$$

$$k_{ex}^2=k^2\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)} \tag{478}$$

$$s_x=k_{ex}\left(\theta_1-c^2\right) \tag{479}$$

Similarly the azimuth-like function is

$$Y\left(\eta\right)\sim c_{0by}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)+e^{i\Phi_{1by}/2}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)\right]/\sqrt[4]{\widetilde{\eta}}\ ,\ \widetilde{\eta}=b^2-\eta$$

$$\sim c_{0by}k_{ez}^{1/4}\,\mathrm{Re}\left[e^{-i\Phi_{1by}/2-i\pi/8}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ez}\widetilde{\eta}}+e^{i\Phi_{1by}/2+i\pi/8}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]$$

$$\sim c_{0by}k_{ez}^{1/4}\,\mathrm{Re}\left[e^{-i\left(\Phi_{1by}-\pi/4\right)/2-i\pi/4}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ez}\widetilde{\eta}}+e^{i\left(\Phi_{1by}-\pi/4\right)/2+i\pi/4}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]$$

$$\sim c_{0by}k_{ez}^{1/4}\,\mathrm{Re}\left[e^{i\left(\Phi_{1z}-\pi/4\right)/2-i\pi/4}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ez}\widetilde{\eta}}+e^{-i\left(\Phi_{1z}-\pi/4\right)/2+i\pi/4}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ez}^{1/4}\,\mathrm{Re}\left[e^{i\left(\Phi_{1z}-\pi/4\right)/2-i\pi/4}\frac{1}{\Gamma\left(3/4+is_z/4\right)}+e^{-i\left(\Phi_{1z}-\pi/4\right)/2+i\pi/4}\frac{1}{\Gamma\left(3/4-is_z/4\right)}\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ez}^{1/4}\,\mathrm{Re}\left[e^{-i\pi/4}\frac{1}{\Gamma\left(3/4+is_z/4\right)}\sqrt{\frac{\Gamma\left(1/4+is_z/4\right)}{\Gamma\left(1/4-is_z/4\right)}}+e^{i\pi/4}\frac{1}{\Gamma\left(3/4-is_z/4\right)}\sqrt{\frac{\Gamma\left(1/4-is_z/4\right)}{\Gamma\left(1/4+is_z/4\right)}}\right]$$

$$\sim c_{0by}\pi^{1/2}k_{ez}^{1/4}\,\mathrm{Re}\left[e^{-i\pi/4}\frac{\sqrt{\Gamma\left(1/4+is_z/4\right)\Gamma\left(1/4-is_z/4\right)}}{\Gamma\left(3/4+is_z/4\right)\Gamma\left(1/4-is_z/4\right)}+e^{i\pi/4}\frac{\sqrt{\Gamma\left(1/4+is_z/4\right)\Gamma\left(1/4-is_z/4\right)}}{\Gamma\left(3/4-is_z/4\right)\Gamma\left(1/4+is_z/4\right)}\right]$$

$$\sim c_{0by}\pi^{-1/2}k_{ez}^{1/4}\left|\Gamma\left(1/4-is_z/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\sin\pi\left(1/4-is_z/4\right)+e^{i\pi/4}\sin\pi\left(1/4+is_z/4\right)\right]$$

$$\sim c_{0by}\left(2\pi\right)^{-1/2}k_{ez}^{1/4}\left|\Gamma\left(1/4-is_z/4\right)\right|\mathrm{Re}\left[e^{-i\pi/4}\left\{\cosh\left(\pi s_z/4\right)-i\sinh\left(\pi s_z/4\right)\right\}+e^{i\pi/4}\left\{\cosh\left(\pi s_z/4\right)+i\sinh\left(\pi s_z/4\right)\right\}\right]$$

$$\sim c_{0by}\left(2\pi\right)^{-1/2}k_{ez}^{1/4}\left|\Gamma\left(1/4-is_z/4\right)\right|\mathrm{Re}\left[2\cos\left(\pi/4\right)\cosh\left(\pi s_z/4\right)-2\sin\left(\pi/4\right)\sinh\left(\pi s_z/4\right)\right]$$

$$\sim c_{0by}\pi^{-1/2}k_{ez}^{1/4}\left|\Gamma\left(1/4-is_z/4\right)\right|\mathrm{Re}\left[\cosh\left(\pi s_z/4\right)-\sinh\left(\pi s_z/4\right)\right] \tag{480}$$

$$Y\left(\eta\to b^2\right)\sim c_{0by}\pi^{-1/2}k_{ez}^{1/4}\left|\Gamma\left(1/4-is_z/4\right)\right|e^{-\pi s_z/4}$$

$$\sim\pi^{-1/2}k_{ez}^{1/4}c_{0by}\left|\Gamma\left(1/4+is_y/4\right)\right|e^{\pi s_y/4} \tag{481}$$

where

$$e^{i\left(\Phi_{1by}-\pi/4\right)}=\frac{\Gamma\left(1/4+is_y/4\right)}{\Gamma\left(1/4-is_y/4\right)}=\frac{\Gamma\left(1/4-is_z/4\right)}{\Gamma\left(1/4+is_z/4\right)}=e^{-i\left(\Phi_{1z}-\pi/4\right)} \tag{482}$$

$$k_{ey}^2=k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}=k_{ez}^2 \tag{483}$$

$$s_y=k_{ey}\left(\theta_2-b^2\right)=k_{ez}\left(\theta_2-b^2\right)=-s_z \tag{484}$$

$$\frac{c_{0y}\left(-1\right)^{n_b}}{\sqrt[4]{\left(b^2-\theta_1\right)}}=2c_{0by}e^{\pi s_y/8}=2c_{0by}e^{-\pi s_z/8} \tag{485}$$

Then

$$Y\left(\eta\to b^2\right)\sim\frac{1}{2}\pi^{-1/2}k_{ez}^{1/4}\frac{c_{0y}\left(-1\right)^{n_b}}{\sqrt[4]{\left(b^2-\theta_1\right)}}\left|\Gamma\left(1/4+is_y/4\right)\right|e^{\pi s_y/8}$$

$$\sim\frac{1}{2}\pi^{-1/2}k_{ez}^{1/4}\frac{c_{0y}\left(-1\right)^{n_b}}{\sqrt[4]{\left(b^2-\theta_1\right)}}\left|\Gamma\left(1/4-is_z/4\right)\right|e^{-\pi s_z/8} \tag{486}$$

Noting from Stirling's formula that

$$\Gamma\left(1/4+is_y/4\right)\sim\sqrt{2\pi}e^{-1/4-is_y/4}\left(1/4+is_y/4\right)^{-1/4+is_y/4}\sim\sqrt{2\pi}e^{\mp i\pi/8-is_y/4}\left(\left|s_y\right|/4\right)^{-1/4+is_y/4}e^{\mp\pi s_y/8}\ ,\ s_y\to\pm\infty \tag{487}$$

this becomes

$$Y\left(\eta\to b^2\right)\sim 2^{-1/2}k_{ez}^{1/4}\frac{c_{0y}\left(-1\right)^{n_b}}{\sqrt[4]{\left(b^2-\theta_1\right)}}\left(\left|s_y\right|/4\right)^{-1/4}e^{\pi s_y/8\mp\pi s_y/8}\ ,\ s_y\to\pm\infty \tag{488}$$

where we see that there is exponential decay for $s_y << -1$ but no exponential growth for $s_y >> 1$. The negative region $s_y < 0$ occurs when $\theta_2 < b^2$ intruding on the interval occupied by η. It thus appears like there can be exponential decay behavior from the end regions of the integration range in η (note that the interior global region in η has scale c_{0y} without exponential decay or growth). There is no exponential decay in ζ for the bowtie since $b^2 << \zeta_0 < \zeta < a^2$. Thus we can observe exponentially small values at the scar center in $\eta \to b^2$ or $y \to 0$ (on the strip center for $s_y = -s_z << -1$). At the strip edges

$$Y(\eta) \sim c_{0cy} \operatorname{Re}\left[e^{-i\Phi_{1cy}/2} W_+\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) + e^{i\Phi_{1cy}/2} W_-\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right)\right] / \sqrt[4]{\widetilde{\eta}}\ ,\ \eta - c^2 = \widetilde{\eta}$$

$$\sim c_{0cy} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\Phi_{1cy}/2 - i\pi/8} W_+\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{-ik_{ex}\widetilde{\eta}} + e^{i\Phi_{1cy}/2 + i\pi/8} W_-\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{ik_{ex}\widetilde{\eta}}\right]$$

$$\sim c_{0cy} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i(\Phi_{1cy} - \pi/4)/2 - i\pi/4} W_+\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{-ik_{ex}\widetilde{\eta}} + e^{i(\Phi_{1cy} - \pi/4)/2 + i\pi/4} W_-\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{ik_{ex}\widetilde{\eta}}\right]$$

$$\sim c_{0cy} k_{ex}^{1/4} \operatorname{Re}\left[e^{i(\Phi_{0x} - \pi/4)/2 - i\pi/4} W_+\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{-ik_{ex}\widetilde{\eta}} + e^{-i(\Phi_{0x} - \pi/4)/2 + i\pi/4} W_-\left(-s_x, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{ik_{ex}\widetilde{\eta}}\right]$$

$$\sim c_{0cy} \pi^{1/2} k_{ex}^{1/4} \operatorname{Re}\left[e^{i(\Phi_{0x} - \pi/4)/2 - i\pi/4} \frac{1}{\Gamma(3/4 + is_x/4)} + e^{-i(\Phi_{0x} - \pi/4)/2 + i\pi/4} \frac{1}{\Gamma(3/4 - is_x/4)}\right]$$

$$\sim c_{0cy} \pi^{1/2} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\pi/4} \frac{1}{\Gamma(3/4 + is_x/4)} \sqrt{\frac{\Gamma(1/4 + is_x/4)}{\Gamma(1/4 - is_x/4)}} + e^{i\pi/4} \frac{1}{\Gamma(3/4 - is_x/4)} \sqrt{\frac{\Gamma(1/4 - is_x/4)}{\Gamma(1/4 + is_x/4)}}\right]$$

$$\sim c_{0cy} \pi^{1/2} k_{ex}^{1/4} \operatorname{Re}\left[e^{-i\pi/4} \frac{\sqrt{\Gamma(1/4 + is_x/4)\,\Gamma(1/4 - is_x/4)}}{\Gamma(3/4 + is_x/4)\,\Gamma(1/4 - is_x/4)} + e^{i\pi/4} \frac{\sqrt{\Gamma(1/4 + is_x/4)\,\Gamma(1/4 - is_x/4)}}{\Gamma(3/4 - is_x/4)\,\Gamma(1/4 + is_x/4)}\right]$$

$$\sim c_{0cy} \pi^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[e^{-i\pi/4} \sin\pi(1/4 - is_x/4) + e^{i\pi/4} \sin\pi(1/4 + is_x/4)\right]$$

$$\sim c_{0cy} (2\pi)^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[e^{-i\pi/4}\left\{\cosh(\pi s_x/4) - i\sinh(\pi s_x/4)\right\} + e^{i\pi/4}\left\{\cosh(\pi s_x/4) + i\sinh(\pi s_x/4)\right\}\right]$$

$$\sim c_{0cy} (2\pi)^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[2\cos(\pi/4)\cosh(\pi s_x/4) - 2\sin(\pi/4)\sinh(\pi s_x/4)\right]$$

$$\sim c_{0cy} \pi^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \operatorname{Re}\left[\cosh(\pi s_x/4) - \sinh(\pi s_x/4)\right] \tag{489}$$

$$Y\left(\eta \to c^2\right) \sim c_{0cy} \pi^{-1/2} k_{ex}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| e^{-\pi s_x/4}$$

$$\sim \pi^{-1/2} k_{ex}^{1/4} c_{0cy} \left|\Gamma(1/4 + is_y/4)\right| e^{\pi s_y/4} \tag{490}$$

where

$$e^{i(\Phi_{1cy}-\pi/4)} = \frac{\Gamma(1/4 + is_y/4)}{\Gamma(1/4 - is_y/4)} = \frac{\Gamma(1/4 - is_x/4)}{\Gamma(1/4 + is_x/4)} = e^{-i(\Phi_{0x}-\pi/4)} \tag{491}$$

$$k_{ey}^2 = k^2 \frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} = k_{ex}^2 \tag{492}$$

$$s_y = k_{ey}\left(c^2 - \theta_1\right) = k_{ex}\left(c^2 - \theta_1\right) = -s_x \tag{493}$$

$$\frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\left(\theta_2 - c^2\right)}} = 2e^{\pi s_y/8} c_{0cy} = 2e^{-\pi s_x/8} c_{0cy} \tag{494}$$

Then

$$\begin{aligned} Y\left(\eta \to c^2\right) &\sim \frac{1}{2}\pi^{-1/2} k_{ex}^{1/4} \frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\left(\theta_2 - c^2\right)}} \left|\Gamma\left(1/4 + is_y/4\right)\right| e^{\pi s_y/8} \\ &\sim \frac{1}{2}\pi^{-1/2} k_{ex}^{1/4} \frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\left(\theta_2 - c^2\right)}} \left|\Gamma\left(1/4 - is_x/4\right)\right| e^{-\pi s_x/8} \end{aligned} \tag{495}$$

From Stirlings̀ formula

$$\Gamma\left(1/4 + is_y/4\right) \sim \sqrt{2\pi} e^{-1/4 - is_y/4} \left(1/4 + is_y/4\right)^{-1/4 + is_y/4} \sim \sqrt{2\pi} e^{\mp i\pi/8 - is_y/4} \left(\left|s_y\right|/4\right)^{-1/4 + is_y/4} e^{\mp\pi s_y/8}\ ,\ s_y \to \pm\infty \tag{496}$$

$$Y\left(\eta \to c^2\right) \sim \frac{1}{2}\pi^{-1/2} k_{ex}^{1/4} \frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\left(\theta_2 - c^2\right)}} \sqrt{2\pi} \left(\left|s_y\right|/4\right)^{-1/4} e^{\pi s_y/8 \mp \pi s_y/8}\ ,\ s_y \to \pm\infty \tag{497}$$

and therefore we can also observe exponential decay at the edges versus the interior global region (for $s_y = -s_x << -1$).

Using the axial-like function

$$Z\left(\zeta\right) \sim \frac{c_{0z}}{\sqrt[4]{\left(\zeta - \theta_1\right)\left(\zeta - \theta_2\right)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2}\int_{\zeta}^{a^2} \sqrt{\frac{\left(\theta - \theta_1\right)\left(\theta - \theta_2\right)}{\left(a^2 - \theta\right)\left(\theta - b^2\right)\left(\theta - c^2\right)}} d\theta\right]\right\} \tag{498}$$

in the inside region, the global solution along the scar ray path at the orbit center is

$$\begin{aligned} \Phi\left(c^2, b^2, \zeta\right) = X\left(c^2\right) Y\left(b^2\right) Z\left(\zeta\right) = \frac{\left(-1\right)^{n_b}}{2\pi} \frac{c_{0x} c_{0y} c_{0z}}{\sqrt[4]{\left(b^2 - \theta_1\right)}} k_{ex}^{1/4} k_{ez}^{1/4} \left|\Gamma\left(1/4 - is_x/4\right)\right| \left|\Gamma\left(1/4 - is_z/4\right)\right| e^{\pi s_x/4 - \pi s_z/8} \\ \frac{1}{\sqrt[4]{\left(\zeta - \theta_1\right)\left(\zeta - \theta_2\right)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2}\int_{\zeta}^{a^2} \sqrt{\frac{\left(\theta - \theta_1\right)\left(\theta - \theta_2\right)}{\left(a^2 - \theta\right)\left(\theta - b^2\right)\left(\theta - c^2\right)}} d\theta\right]\right\} \end{aligned} \tag{499}$$

5 ENERGY THEOREM NORMALIZATION

The overall normalizing constant $c_0 = c_{0x}c_{0y}c_{0z}$ must be determined for the preceding scar asymptotic forms near the orbit. This is done using an energy theorem [21], [22].

5.1 Acoustic Energy Normalization

A physical scalar wave problem of interest is acoustics. The equations of motion are [23]

$$\rho \frac{\partial}{\partial t} \underline{u} = -\nabla P \tag{500}$$

$$\kappa \frac{\partial}{\partial t} P = -\nabla \cdot \underline{u} \tag{501}$$

where $\underline{u}$ is the particle velocity, P is the pressure, κ is the compressibility, and ρ is the density. Thus eliminating the velocity gives the scalar wave equation

$$\rho \frac{\partial}{\partial t}\left(\kappa \frac{\partial}{\partial t} P\right) = \nabla^2 P \tag{502}$$

Let us suppress time harmonic dependence $e^{-i\omega t}$

$$i\omega\rho\underline{u} = \nabla P \tag{503}$$

$$i\omega\kappa P = \nabla \cdot \underline{u} \tag{504}$$

and eliminating the velocity (with P taking the place of the scalar quantity)

$$\nabla^2 P + \omega^2 \kappa\rho P = \left(\nabla^2 + k^2\right) P = 0 \tag{505}$$

Now examine the quantity

$$\begin{aligned}
\nabla \cdot \left(\frac{\partial P}{\partial \omega}\underline{u}^* + P^* \frac{\partial \underline{u}}{\partial \omega}\right) &= \frac{\partial P}{\partial \omega}\nabla \cdot \underline{u}^* + \frac{\partial \nabla P}{\partial \omega} \cdot \underline{u}^* + \nabla P^* \cdot \frac{\partial \underline{u}}{\partial \omega} + P^* \frac{\partial \nabla \cdot \underline{u}}{\partial \omega} \\
= -\frac{\partial P}{\partial \omega} i\omega\kappa P^* &+ i\rho\left(\underline{u} + \omega \frac{\partial \underline{u}}{\partial \omega}\right) \cdot \underline{u}^* - i\omega\rho\underline{u}^* \cdot \frac{\partial \underline{u}}{\partial \omega} + i\kappa P^* \left(\omega \frac{\partial P}{\partial \omega} + P\right) \\
&= i\left(\rho\underline{u} \cdot \underline{u}^* + \kappa P^* P\right) = i\left(\frac{1}{\omega^2\rho}\left|\nabla P\right|^2 + \kappa\left|P\right|^2\right)
\end{aligned} \tag{506}$$

or

$$\begin{aligned}
\nabla \cdot \left[\frac{\partial P}{\partial \omega}\left(\frac{1}{\omega}\nabla P^*\right) - P^* \frac{\partial}{\partial \omega}\left(\frac{1}{\omega}\nabla P\right)\right] &= \frac{1}{\omega^2}\left(\left|\nabla P\right|^2 + \omega^2\rho\kappa\left|P\right|^2\right) \\
&= \frac{1}{\omega^2}\left(\left|\nabla P\right|^2 + k^2\left|P\right|^2\right)
\end{aligned} \tag{507}$$

where the wavenumber is

$$k^2 = \omega^2 \rho\kappa \tag{508}$$

Now using

$$|\nabla P|^2 = \nabla P \cdot \nabla P^* = \nabla \cdot (P^* \nabla P) - P^* \nabla^2 P = \nabla \cdot (P^* \nabla P) + k^2 |P|^2 \tag{509}$$

for either the soft outer boundary with $P = 0$, or the hard outer boundary with $\partial P/\partial n = 0$, gives the relation between the volume integral over the cavity and the surface integral over the scar

$$\oint_{S_{scar}} \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \nabla P^* \right) - P^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \nabla P \right) \right] \cdot \underline{n} dS = 2 \frac{k^2}{\omega^2} \int_V |P|^2 \, dV$$

$$= \oint_{S_{scar}} \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P^*}{\partial n} \right) - P^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P}{\partial n} \right) \right] dS \tag{510}$$

where here n points into the scar region. Now if the pressure is taken to be real

$$\oint_{S_{scar}} \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P}{\partial n} \right) - P \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P}{\partial n} \right) \right] dS$$

$$= 2 \frac{k^2}{\omega^2} \int_V |P|^2 \, dV \tag{511}$$

In addition if we take

$$\frac{\partial P}{\partial n} = 0 \ , \ \text{on } S_{scar} \tag{512}$$

then

$$- \oint_{S_{scar}} \left[P \frac{\partial^2 P}{\partial \omega \partial n} \right] dS = 2 \frac{k^2}{\omega} \int_V |P|^2 \, dV \tag{513}$$

where here again n points into the scar region.

Changing from the preceding acoustic case with scalar P to scalar u the normalization condition is taken as

$$\oint_{S_{scar}} \left[\frac{\partial u}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u^*}{\partial n} \right) - u^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) \right] dS$$

$$= 2 \frac{k^2}{\omega^2} \int_V |u|^2 \, dV \tag{514}$$

or for a real function

$$\oint_{S_{scar}} \left[\frac{\partial u}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) - u \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) \right] dS = 2 \frac{k^2}{\omega^2} \int_V |u|^2 \, dV \tag{515}$$

If we set

$$\frac{\partial u}{\partial n} = 0 \ , \ \text{on } S_{scar} \tag{516}$$

we have

$$2\frac{k^2}{\omega}\int_V |u|^2\,dV = -\int_{S_{scar}} u^2\frac{\partial}{\partial\omega}\left(\frac{\partial u}{\partial n}/u\right)dS = -\int_{S_{scar}} u\frac{\partial^2 u}{\partial\omega\partial n}dS \tag{517}$$

5.1.1 Electromagnetic Energy Theorem Normalization

The electromagnetic energy theorem [21] has been used to normalize the scar constructions [4], [13]. If we take the result in the two-dimensional problem [4] derived from the electromagnetic energy theorem [21], [22] (with either $u = 0$ or $\partial u/\partial n = 0$ on the outer boundary, this gives the relation between the volume integral over the cavity and the surface integral over the scar), and change the line integral to a surface and the surface integral to a volume as an approximation we obtain

$$\int_{S_{scar}}\left[\frac{\partial u}{\partial\omega}\nabla u^* - u^*\frac{\partial}{\partial\omega}\nabla u\right]\cdot\underline{n}dS = \int_{S_{scar}}\left[\frac{\partial u}{\partial\omega}\frac{\partial u^*}{\partial n} - u^*\frac{\partial^2 u}{\partial\omega\partial n}\right]dS = 2\frac{k^2}{\omega}\int_V |u|^2\,dV \tag{518}$$

which is the same as the acoustic case above. In the electromagnetic case we can take $u = \Phi = E_j$ where E_j is a transverse component along the ray path orbit.

5.2 Scalar Normalization

We begin here by taking the solution to be a scalar and consider the normalization as in the acoustic case (with either $u = 0$ or $\partial u/\partial n = 0$ on the outer boundary). The normalization condition is thus taken as

$$\oint_{S_{scar}}\left[\frac{\partial u}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial u^*}{\partial n}\right) - u^*\frac{\partial}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial u}{\partial n}\right)\right]dS$$

$$= 2\frac{k^2}{\omega^2}\int_V |u|^2\,dV \tag{519}$$

or for a real function (where $\underline{n}$ points into the scarred region)

$$\oint_{S_{scar}}\left[\frac{\partial u}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial u}{\partial n}\right) - u\frac{\partial}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial u}{\partial n}\right)\right]dS = 2\frac{k^2}{\omega^2}\int_V |u|^2\,dV \tag{520}$$

If we set

$$\frac{\partial u}{\partial n} = 0\ ,\ \text{on } S_{scar} \tag{521}$$

and take the normalization in the three dimensional problem to be

$$\int_V |u|^2\,dV = 1 \tag{522}$$

we find

$$2\frac{k^2}{\omega} = -\int_{S_{scar}} u^2\frac{\partial}{\partial\omega}\left(\frac{\partial u}{\partial n}/u\right)dS = -\int_{S_{scar}} u\frac{\partial^2 u}{\partial\omega\partial n}dS \tag{523}$$

We will focus on the case where $R_y, R_z >> \ell$ and a is very much larger than b or c. Furthermore, we will assume that b is near but not equal to c in the ellipsoidal coordinate system. In this case the scar orbit will be taken to be centered on the region where $\xi \to c^2$ inside the foci.

In the inside region we are integrating over the limit of an ellipse approaching a strip area as shown in Figure 5; the normal limit to the strip is $\xi \to c^2$. This strip integration is the three-dimensional generalization of the two-dimensional normalization [4], where unit width strip integrations were in-effect used; these also degenerate to line integrals when the two foci degenerate into a single focus, as in the three-dimensional axisymmetric case [13]. Let us apply the energy theorem over the orbit shown in the figure with

$$u = \Phi = X\left(\xi\right) Y\left(\eta\right) Z\left(\zeta\right) \tag{524}$$

The normal derivative ∂n with respect to a direction into the scarred region is proportional to $\partial\sqrt{\widetilde{\xi}}$, arising from $h_\xi \partial\xi$; for the inside region $\widetilde{\xi} = c^2 - \xi$, and thus ∂n has the same sign as $\partial\xi$ but the opposite sign as $\partial\widetilde{\xi}$, (also the reason for concentrating the ω derivative on the factor involved in the normal derivative of X, is that this normal derivative vanishes without the ω derivative due to symmetry-power conditions). If the normal derivative does not vanish (for example, if we are off the strip symmetry location) we must revert back to the original form.

Because we are using symmetry we only carry out one quarter of the azimuthal-like integrations and multiply the result by 4; in the axial direction we multiply by 2 since we are only carrying this out for the positive half of the orbit

$$2\frac{k^2}{\omega} \sim -\oint_{S_{scar}} u\left(\frac{\partial^2 u}{\partial\omega\partial n}\right) dS \sim -2\int_{\zeta_0}^{a^2}\left[4\int_{c^2}^{b^2}\left(\frac{\partial^2 u}{\partial\omega h_\xi \partial\xi}\right)_{\xi\to c^2} h_\eta d\eta\right] u h_\zeta d\zeta \tag{525}$$

or using

$$\left(\frac{\partial^2 u}{\partial\omega h_\xi \partial\xi}\right)_{\xi\to c^2} \sim Y\left(\eta\right) Z\left(\zeta\right)\left(\frac{\partial^2 X}{\partial\omega h_\xi \partial\xi}\right)_{\xi\to c^2} \tag{526}$$

we can write

$$2\frac{k^2}{\omega} \sim -2\int_{\zeta_0}^{a^2}\left[4\int_{c^2}^{b^2} Y^2\left(\eta\right) Z^2\left(\zeta\right)\left(X\frac{\partial^2 X}{\partial\omega h_\xi \partial\xi}\right)_{\xi\to c^2} h_\eta d\eta\right] h_\zeta d\zeta \tag{527}$$

The ellipsoidal metric coeffi cients are given by [10]

$$h_\xi = \frac{1}{2}\sqrt{\frac{\left(\xi-\eta\right)\left(\xi-\zeta\right)}{D\left(\xi\right)}} = \frac{1}{2}\sqrt{\frac{\left(\eta-\xi\right)\left(\zeta-\xi\right)}{\left(a^2-\xi\right)\left(b^2-\xi\right)\left(c^2-\xi\right)}} \tag{528}$$

$$h_\eta = \frac{1}{2}\sqrt{\frac{\left(\eta-\zeta\right)\left(\eta-\xi\right)}{D\left(\eta\right)}} = \frac{1}{2}\sqrt{\frac{\left(\zeta-\eta\right)\left(\eta-\xi\right)}{\left(a^2-\eta\right)\left(b^2-\eta\right)\left(\eta-c^2\right)}} \tag{529}$$

$$h_\zeta = \frac{1}{2}\sqrt{\frac{\left(\zeta-\xi\right)\left(\zeta-\eta\right)}{D\left(\zeta\right)}} = \frac{1}{2}\sqrt{\frac{\left(\zeta-\xi\right)\left(\zeta-\eta\right)}{\left(a^2-\zeta\right)\left(\zeta-b^2\right)\left(\zeta-c^2\right)}} \tag{530}$$

$$D\left(\theta\right) = \left(a^2-\theta\right)\left(b^2-\theta\right)\left(c^2-\theta\right) \tag{531}$$

The normal metric coeffi cient can then be written as

$$h_\xi \sim \frac{1}{2}\sqrt{\frac{\left(\eta-c^2\right)\left(\zeta-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)\widetilde{\xi}}}\ ,\ c^2-\xi = \widetilde{\xi} \tag{532}$$

with the scar normal

$$\partial n \sim h_\xi \partial \xi \sim -\sqrt{\frac{(\eta - c^2)(\zeta - c^2)}{(a^2 - c^2)(b^2 - c^2)}} \partial \sqrt{\xi} \tag{533}$$

giving

$$\left(\frac{\partial^2 u}{\partial \omega h_\xi \partial \xi}\right)_{\xi \to c^2} \sim Y(\eta) Z(\zeta) \left(\frac{\partial^2 X}{\partial \omega h_\xi \partial \xi}\right)_{\xi \to c^2} \sim -Y(\eta) Z(\zeta) \sqrt{\frac{(a^2 - c^2)(b^2 - c^2)}{(\eta - c^2)(\zeta - c^2)}} \left(\frac{\partial^2 X}{\partial \omega \partial \sqrt{\xi}}\right)_{\xi \to c^2}$$

$$\sim -\sqrt{\frac{(a^2 - c^2)(b^2 - c^2)}{(\eta - c^2)(\zeta - c^2)}} \left(\frac{\partial^2 u}{\partial \omega \partial \sqrt{\xi}}\right)_{\xi \to c^2} \tag{534}$$

The energy theorem can then be written as

$$\frac{k^2}{\omega \sqrt{(a^2 - c^2)(b^2 - c^2)}} \sim \int_{\zeta_0}^{a^2} \left[4 \int_{c^2}^{b^2} \frac{1}{\sqrt{(\eta - c^2)(\zeta - c^2)}} \left(\frac{\partial^2 u}{\partial \omega \partial \sqrt{\xi}}\right)_{\xi \to c^2} h_\eta d\eta\right] u h_\zeta d\zeta \tag{535}$$

or

$$\frac{k^2}{\omega \sqrt{(a^2 - c^2)(b^2 - c^2)}} \sim \int_{\zeta_0}^{a^2} \left[\int_{c^2}^{b^2} \left(u \frac{\partial^2 u}{\partial \omega \partial \sqrt{\xi}}\right)_{\xi \to c^2} \frac{(\zeta - \eta)\, d\eta}{\sqrt{(a^2 - \eta)(b^2 - \eta)(\eta - c^2)}}\right] \frac{d\zeta}{\sqrt{(a^2 - \zeta)(\zeta - b^2)(\zeta - c^2)}} \tag{536}$$

or

$$\frac{k^2}{\omega \sqrt{(a^2 - c^2)(b^2 - c^2)}} \sim \int_{\zeta_0}^{a^2} \left[\int_{c^2}^{b^2} \left(X \frac{\partial^2 X}{\partial \omega \partial \sqrt{\xi}}\right)_{\xi \to c^2} \frac{Y^2(\eta)(\zeta - \eta)\, d\eta}{\sqrt{(a^2 - \eta)(b^2 - \eta)(\eta - c^2)}}\right] \frac{Z^2(\zeta)\, d\zeta}{\sqrt{(a^2 - \zeta)(\zeta - b^2)(\zeta - c^2)}} \tag{537}$$

Noting for high frequencies that the global solutions

$$Y(\eta) \sim \frac{c_{0y}}{\sqrt[4]{(\eta - \theta_1)(\theta_2 - \eta)}} \cos\left[\frac{k}{2} \int_{\max(c^2, \theta_1)}^{\eta} \sqrt{\frac{(\theta - \theta_1)(\theta_2 - \theta)}{(a^2 - \theta)(b^2 - \theta)(\theta - c^2)}} d\theta + k\delta_{c0}\right] \tag{538}$$

$$Z(\zeta) \sim \frac{c_{0z}}{\sqrt[4]{(\zeta - \theta_1)(\zeta - \theta_2)}} \left\{\begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_{\zeta}^{a^2} \sqrt{\frac{(\theta - \theta_1)(\theta - \theta_2)}{(a^2 - \theta)(\theta - b^2)(\theta - c^2)}} d\theta\right]\right\} \tag{539}$$

are rapidly varying we can average over these to find

$$\left\langle Z^2(\zeta) \right\rangle \sim \frac{c_{0z}^2/2}{\sqrt{(\zeta - \theta_1)(\zeta - \theta_2)}} \tag{540}$$

$$\left\langle Y^2(\eta)\right\rangle \sim \frac{c_{0y}^2/2}{\sqrt{(\eta-\theta_1)(\theta_2-\eta)}} \ , \ \ \min\left(b^2,\theta_2\right) > \eta > \max\left(c^2,\theta_1\right) \tag{541}$$

where if θ_1 or θ_2 intrude into the region $c^2 < \eta < b^2$, the solution can exhibit exponential decay near the end points $\theta_2 < \eta < b^2$ or $c^2 < \eta < \theta_1$, and in these local regions the averages may not apply; hence, we use the local solutions in these regions. The energy theorem can then be written as

$$\frac{k^2}{\omega\sqrt{(a^2-c^2)(b^2-c^2)}} \sim \left(X\frac{\partial^2 X}{\partial\omega\partial\sqrt{\widetilde{\xi}}}\right)_{\xi\to c^2}$$

$$\int_{\zeta_0}^{a^2}\left[\int_{\max(c^2,\theta_1)}^{\min\left(b^2,\theta_2\right)}\left\{\frac{\left\langle Y^2(\eta)\right\rangle(\zeta-\eta)}{\sqrt{(a^2-\eta)(b^2-\eta)(\eta-c^2)}}-\frac{\left\langle Y^2\left(\eta\to b^2\right)\right\rangle\left(\zeta-b^2\right)}{\sqrt{(a^2-b^2)(b^2-\eta)(b^2-c^2)}}-\frac{\left\langle Y^2\left(\eta\to c^2\right)\right\rangle\left(\zeta-c^2\right)}{\sqrt{(a^2-c^2)(b^2-c^2)(\eta-c^2)}}\right\}d\eta\right.$$

$$\left.+\frac{\left(\zeta-b^2\right)}{\sqrt{(a^2-b^2)(b^2-c^2)}}\int_{\max(c^2,\theta_1)}^{b^2} Y_{b^2}^2(\eta)\,\frac{d\eta}{\sqrt{b^2-\eta}}+\frac{\left(\zeta-c^2\right)}{\sqrt{(a^2-c^2)(b^2-c^2)}}\int_{c^2}^{\min\left(b^2,\theta_2\right)} Y_{c^2}^2(\eta)\,\frac{d\eta}{\sqrt{\eta-c^2}}\right]$$

$$\frac{\left\langle Z^2(\zeta)\right\rangle d\zeta}{\sqrt{(a^2-\zeta)(\zeta-b^2)(\zeta-c^2)}} \tag{542}$$

where in the subtracted terms representing the average global solutions near the end points, with $\left\langle Y^2\left(\eta\to b^2\right)\right\rangle \to \left(c_{0y}^2/2\right)/\sqrt{(\theta_2-\eta)(b^2-\theta_1)}$ and $\left\langle Y^2\left(\eta\to c^2\right)\right\rangle \to \left(c_{0y}^2/2\right)/\sqrt{(\theta_2-c^2)(\eta-\theta_1)}$, we have taken limits of factors which are not varying near b^2, and near c^2, respectively. These subtracted average solutions result in improved convergence near the integration end points, but are approximately cancelled away from the integration end points by the local integrations on the next line, where the local solutions are being denoted as

$$Y_{b^2}(\eta) \sim \frac{k_{ez}^{1/4}c_{0y}(-1)^{n_b}}{2\sqrt[4]{(b^2-\theta_1)}}e^{\pi s_z/8}$$

$$\mathrm{Re}\left[e^{i(\Phi_{1z}-\pi/4)/2-i\pi/4}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ez}\widetilde{\eta}}+e^{-i(\Phi_{1z}-\pi/4)/2+i\pi/4}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]$$

$$\to \frac{c_{0y}(-1)^{n_b}}{\sqrt[4]{\widetilde{\eta}}\sqrt[4]{(b^2-\theta_1)}}\cos\left[\Phi_{1z}/2-\pi/4+k_{ez}\widetilde{\eta}/2-(s_z/4)\ln\left(k_{ez}\widetilde{\eta}\right)\right] \ , \ \ k_{ez}\widetilde{\eta} >> 1 \tag{543}$$

$$\frac{\Gamma(1/4+is_z/4)}{\Gamma(1/4-is_z/4)} = e^{i(\Phi_{1z}-\pi/4)} \tag{544}$$

$$Y_{c^2}(\eta) \sim \frac{k_{ex}^{1/4}c_{0y}(-1)^{n_c}}{2\sqrt[4]{(\theta_2-c^2)}}e^{\pi s_x/8}$$

$$\mathrm{Re}\left[e^{i(\Phi_{0x}-\pi/4)/2-i\pi/4}W_+\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ex}\widetilde{\eta}}+e^{-i(\Phi_{0x}-\pi/4)/2+i\pi/4}W_-\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]$$

$$\rightarrow \frac{c_{0y}\left(-1\right)^{n_c}}{\sqrt[4]{\widetilde{\eta}}\sqrt[4]{\left(\theta_2-c^2\right)}}\cos\left[\Phi_{0x}/2-\pi/4+k_{ex}\widetilde{\eta}/2-\left(s_x/2\right)\ln\left(k_{ex}\widetilde{\eta}\right)\right]\ ,\ k_{ex}\widetilde{\eta}>>1 \tag{545}$$

$$\frac{\Gamma\left(1/4+is_x/4\right)}{\Gamma\left(1/4-is_x/4\right)}=e^{i\left(\Phi_{0x}-\pi/4\right)} \tag{546}$$

Substituting the global solutions

$$\frac{k^2}{\omega\sqrt{\left(a^2-c^2\right)\left(b^2-c^2\right)}}\sim\frac{1}{4}c_{0y}^2c_{0z}^2\left(X\frac{\partial^2X}{\partial\omega\partial\sqrt{\widetilde{\xi}}}\right)_{\xi\rightarrow c^2}$$

$$\int_{\zeta_0}^{a^2}\left[\int_{\max\left(c^2,\theta_1\right)}^{\min\left(b^2,\theta_2\right)}\left\{\frac{\left(\zeta-\eta\right)}{\sqrt{\left(a^2-\eta\right)\left(b^2-\eta\right)\left(\theta_2-\eta\right)\left(\eta-c^2\right)\left(\eta-\theta_1\right)}}\right.\right.$$

$$\left.-\frac{\left(\zeta-b^2\right)}{\sqrt{\left(a^2-b^2\right)\left(b^2-\eta\right)\left(\theta_2-\eta\right)\left(b^2-c^2\right)\left(b^2-\theta_1\right)}}-\frac{\left(\zeta-c^2\right)}{\sqrt{\left(a^2-c^2\right)\left(b^2-c^2\right)\left(\theta_2-c^2\right)\left(\eta-c^2\right)\left(\eta-\theta_1\right)}}\right\}d\eta$$

$$\left.+\frac{\left(\zeta-b^2\right)2/c_{0y}^2}{\sqrt{\left(a^2-b^2\right)\left(b^2-c^2\right)}}\int_{\max\left(c^2,\theta_1\right)}^{b^2}Y_{b^2}^2\left(\eta\right)\frac{d\eta}{\sqrt{b^2-\eta}}+\frac{\left(\zeta-c^2\right)2/c_{0y}^2}{\sqrt{\left(a^2-c^2\right)\left(b^2-c^2\right)}}\int_{c^2}^{\min\left(b^2,\theta_2\right)}Y_{c^2}^2\left(\eta\right)\frac{d\eta}{\sqrt{\eta-c^2}}\right]$$

$$\frac{d\zeta}{\sqrt{\left(a^2-\zeta\right)\left(\zeta-b^2\right)\left(\zeta-\theta_2\right)\left(\zeta-c^2\right)\left(\zeta-\theta_1\right)}} \tag{547}$$

Using the derivatives of the Whittaker functions

$$\frac{d}{d\sqrt{\widetilde{\theta}}}\left[W_+\left(s,1/4,k_e\widetilde{\theta}\right)/\left(-ik_e\widetilde{\theta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4-is/4\right)}\left(-ik_e\right)^{1/2}+O\left(k_e\sqrt{\widetilde{\theta}}\right)\right]\ ,\ k_e\widetilde{\theta}\rightarrow 0 \tag{548}$$

$$\frac{d}{d\sqrt{\widetilde{\theta}}}\left[W_-\left(s,1/4,k_e\widetilde{\theta}\right)/\left(ik_e\widetilde{\theta}\right)^{1/4}\right]\sim\pi^{1/2}\left[-\frac{2}{\Gamma\left(1/4+is/4\right)}\left(ik_e\right)^{1/2}+O\left(k_e\sqrt{\widetilde{\theta}}\right)\right]\ ,\ k_e\widetilde{\theta}\rightarrow 0 \tag{549}$$

the function

$$X\left(\xi\right)\sim c_{0x}\,\mathrm{Re}\left[e^{-i\Phi_{0x}/2}W_+\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)+e^{i\Phi_{0x}/2}W_-\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)\right]/\sqrt[4]{\widetilde{\xi}}\ ,\ \xi=c^2-\widetilde{\xi} \tag{550}$$

then has frequency-normal derivatives

$$\left(\frac{\partial^2X}{\partial\omega\partial\sqrt{\widetilde{\xi}}}\right)_{\xi\rightarrow c^2}\sim\frac{1}{2}k_{ex}^{1/4}c_{0x}\frac{\partial\Phi_{0x}}{\partial\omega}$$

$$\mathrm{Re}\left[-ie^{-i\Phi_{0x}/2-i\pi/8}\frac{d}{d\sqrt{\widetilde{\xi}}}\left\{W_+\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)/\sqrt[4]{-ik_{ex}\widetilde{\xi}}\right\}+ie^{i\Phi_{0x}/2+i\pi/8}\frac{d}{d\sqrt{\widetilde{\xi}}}\left\{W_-\left(s_x,1/4,k_{ex}\widetilde{\xi}\right)/\sqrt[4]{ik_{ex}\widetilde{\xi}}\right\}\right]\Bigg|_{\widetilde{\xi}\rightarrow 0}$$

$$\sim \sqrt{\pi} k_{ex}^{3/4} c_{0x} \frac{\partial \Phi_{0x}}{\partial \omega} \left\{ i e^{-i\Phi_{0x}/2 - i3\pi/8} \frac{1}{\Gamma\left(1/4 - is_x/4\right)} - i e^{i\Phi_{0x}/2 + i3\pi/8} \frac{1}{\Gamma\left(1/4 + is_x/4\right)} \right\}$$

$$\sim \sqrt{\pi} k_{ex}^{3/4} c_{0x} \frac{\partial \Phi_{0x}}{\partial \omega} \left\{ e^{-i\Phi_{0x}/2 + i\pi/8} \frac{1}{\Gamma\left(1/4 - is_x/4\right)} + e^{i\Phi_{0x}/2 - i\pi/8} \frac{1}{\Gamma\left(1/4 + is_x/4\right)} \right\}$$

$$\sim \sqrt{\pi} k_{ex}^{3/4} c_{0x} \frac{\partial \Phi_{0x}}{\partial \omega} \left\{ e^{-i(\Phi_{0x} - \pi/4)/2} \frac{1}{\Gamma\left(1/4 - is_x/4\right)} + e^{i(\Phi_{0x} - \pi/4)/2} \frac{1}{\Gamma\left(1/4 + is_x/4\right)} \right\}$$

$$\sim \sqrt{\pi} k_{ex}^{3/4} c_{0x} \frac{\partial \Phi_{0x}}{\partial \omega} \left\{ \frac{1}{\Gamma\left(1/4 - is_x/4\right)} \sqrt{\frac{\Gamma\left(1/4 - is_x/4\right)}{\Gamma\left(1/4 + is_x/4\right)}} + \frac{1}{\Gamma\left(1/4 + is_x/4\right)} \sqrt{\frac{\Gamma\left(1/4 + is_x/4\right)}{\Gamma\left(1/4 - is_x/4\right)}} \right\}$$

$$\sim \frac{2\sqrt{\pi} k_{ex}^{3/4} c_{0x}}{\left|\Gamma\left(1/4 - is_x/4\right)\right|} \frac{\partial \Phi_{0x}}{\partial \omega} \tag{551}$$

where we assumed that the frequency derivative is dominated by the derivative of the reflection phase [3], and we previously have shown that

$$X\left(\widetilde{\xi} \to 0\right) \sim c_{0x} k_{ex}^{1/4} \pi^{-1/2} \left|\Gamma\left(1/4 - is_x/4\right)\right| e^{\pi s_x/4} \ , \ \xi = c^2 - \widetilde{\xi} \tag{552}$$

Then

$$\left(X \frac{\partial^2 X}{\partial \omega \partial \sqrt{\widetilde{\xi}}} \right)_{\xi \to c^2} \sim 2 c_{0x}^2 k_{ex} e^{\pi s_x/4} \frac{\partial \Phi_{0x}}{\partial \omega} \tag{553}$$

where

$$k_{ex}^2 = k^2 \frac{\left(\theta_2 - c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} \tag{554}$$

$$s_x = k_{ex} \left(\theta_1 - c^2\right) \tag{555}$$

To simplify the energy result we take the limit of alignment $\theta_2 \to b^2$ and $\theta_1 \to c^2$ (except in this normal derivative function)

$$\frac{k^2}{\omega \sqrt{\left(a^2 - c^2\right)\left(b^2 - c^2\right)}} \approx \frac{1}{4} c_{0y}^2 c_{0z}^2 \left(X \frac{\partial^2 X}{\partial \omega \partial \sqrt{\widetilde{\xi}}} \right)_{\xi \to c^2}$$

$$\int_{\zeta_0}^{a^2} \left[\int_{c^2}^{b^2} \left\{ \frac{(\zeta - \eta)}{\sqrt{\left(a^2 - \eta\right)\left(b^2 - \eta\right)^2 \left(\eta - c^2\right)^2}} - \frac{\left(\zeta - b^2\right)}{\sqrt{\left(a^2 - b^2\right)\left(b^2 - \eta\right)^2 \left(b^2 - c^2\right)^2}} - \frac{\left(\zeta - c^2\right)}{\sqrt{\left(a^2 - c^2\right)\left(b^2 - c^2\right)^2 \left(\eta - c^2\right)^2}} \right\} d\eta \right.$$

$$\left. + \frac{\left(\zeta - b^2\right) 2/c_{0y}^2}{\sqrt{\left(a^2 - b^2\right)\left(b^2 - c^2\right)}} \left\{ \int_{c^2}^{b^2} Y_{b^2}^2(\eta) \frac{d\eta}{\sqrt{\widetilde{\eta}}} \right\}_{\widetilde{\eta} = b^2 - \eta} + \frac{\left(\zeta - c^2\right) 2/c_{0y}^2}{\sqrt{\left(a^2 - c^2\right)\left(b^2 - c^2\right)}} \left\{ \int_{c^2}^{b^2} Y_{c^2}^2(\eta) \frac{d\eta}{\sqrt{\widetilde{\eta}}} \right\}_{\widetilde{\eta} = \eta - c^2} \right]$$

$$\frac{d\zeta}{\sqrt{\left(a^2-\zeta\right)\left(\zeta-b^2\right)^2\left(\zeta-c^2\right)^2}} \tag{556}$$

From our previous identities for the Whittaker functions we can write the aligned limit as

$$\mathrm{Re}\left[e^{-i\pi/4}W_+\left(0,1/4,k_e\widetilde{\theta}\right)/\sqrt[4]{-ik_e\widetilde{\theta}}+e^{i\pi/4}W_-\left(0,1/4,k_e\widetilde{\theta}\right)/\sqrt[4]{ik_e\widetilde{\theta}}\right]$$

$$=\frac{\pi^{1/2}}{2}\,\mathrm{Re}\left[e^{-i\pi/8+i\pi/2}H^{(1)}_{1/4}\left(k_e\widetilde{\theta}/2\right)\sqrt[4]{-ik_e\widetilde{\theta}}+e^{i\pi/8-i\pi/2}H^{(2)}_{1/4}\left(k_e\widetilde{\theta}/2\right)\sqrt[4]{ik_e\widetilde{\theta}}\right]$$

$$=\frac{\pi^{1/2}}{2}\,\mathrm{Re}\left[e^{i\pi/4}H^{(1)}_{1/4}\left(k_e\widetilde{\theta}/2\right)+e^{-i\pi/4}H^{(2)}_{1/4}\left(k_e\widetilde{\theta}/2\right)\right]\sqrt[4]{k_e\widetilde{\theta}}$$

$$=\frac{\pi^{1/2}}{2}\,\mathrm{Re}\left[e^{i\pi/4}\left\{J_{1/4}\left(k_e\widetilde{\theta}/2\right)+iY_{1/4}\left(k_e\widetilde{\theta}/2\right)\right\}+e^{-i\pi/4}\left\{J_{1/4}\left(k_e\widetilde{\theta}/2\right)-iY_{1/4}\left(k_e\widetilde{\theta}/2\right)\right\}\right]\sqrt[4]{k_e\widetilde{\theta}}$$

$$=\pi^{1/2}\left[\cos\left(\pi/4\right)J_{1/4}\left(k_e\widetilde{\theta}/2\right)-\sin\left(\pi/4\right)Y_{1/4}\left(k_e\widetilde{\theta}/2\right)\right]\sqrt[4]{k_e\widetilde{\theta}}$$

$$=\pi^{1/2}J_{-1/4}\left(k_e\widetilde{\theta}/2\right)\sqrt[4]{k_e\widetilde{\theta}} \tag{557}$$

where $J_\nu\left(z\right)$ is the Bessel function of order ν. Note that we will not retain the exponentials in the coeffi cients in the following, because these exponentials cancel an exponential arising from the misaligned form of the Whittaker functions for large argument (see preceding summary of solutions), so the interior part of the η interval does not exhibit the decay possible at the end points. Therefore we use the aligned limits of $Y\left(\eta\right)$ at the end points to facilitate convergence of the integral in the energy theorem

$$Y\left(\eta\right)\sim\frac{k_{ez}^{1/4}c_{0y}\left(-1\right)^{n_b}}{2\sqrt[4]{\left(b^2-\theta_1\right)}}e^{\pi s_z/8}$$

$$\mathrm{Re}\left[\sqrt{\frac{\Gamma\left(1/4+is_z/4\right)}{\Gamma\left(1/4-is_z/4\right)}}e^{-i\pi/4}W_+\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ez}\widetilde{\eta}}+\sqrt{\frac{\Gamma\left(1/4-is_z/4\right)}{\Gamma\left(1/4+is_z/4\right)}}e^{i\pi/4}W_-\left(-s_z,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]$$

$$\to\frac{k_{ez}^{1/4}c_{0y}\left(-1\right)^{n_b}}{2\sqrt[4]{\left(b^2-c^2\right)}}\,\mathrm{Re}\left[e^{-i\pi/4}W_+\left(0,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ez}\widetilde{\eta}}+e^{i\pi/4}W_-\left(0,1/4,k_{ez}\widetilde{\eta}\right)/\sqrt[4]{ik_{ez}\widetilde{\eta}}\right]\ ,\ b^2-\eta=\widetilde{\eta}\ ,\ s_z\to0$$

$$\sim\frac{\pi^{1/2}k_{ez}^{1/4}c_{0y}\left(-1\right)^{n_b}/2}{\sqrt[4]{\left(b^2-c^2\right)}}J_{-1/4}\left(k_{ez}\widetilde{\eta}/2\right)\sqrt[4]{k_{ez}\widetilde{\eta}}\ ,\ b^2-\eta=\widetilde{\eta}\ ,\ s_z\to0 \tag{558}$$

$$Y\left(\eta\right)\sim\frac{k_{ex}^{1/4}c_{0y}\left(-1\right)^{n_c}}{2\sqrt[4]{\left(\theta_2-c^2\right)}}e^{\pi s_x/8}$$

$$\mathrm{Re}\left[\sqrt{\frac{\Gamma\left(1/4+is_x/4\right)}{\Gamma\left(1/4-is_x/4\right)}}e^{-i\pi/4}W_+\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)/\sqrt[4]{-ik_{ex}\widetilde{\eta}}+\sqrt{\frac{\Gamma\left(1/4-is_x/4\right)}{\Gamma\left(1/4+is_x/4\right)}}e^{i\pi/4}W_-\left(-s_x,1/4,k_{ex}\widetilde{\eta}\right)/\sqrt[4]{ik_{ex}\widetilde{\eta}}\right]$$

$$\to \frac{k_{ex}^{1/4} c_{0y} \left(-1\right)^{n_c}}{2\sqrt[4]{(b^2-c^2)}} \operatorname{Re}\left[e^{-i\pi/4} W_+ \left(0, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{-ik_{ex}\widetilde{\eta}} + e^{i\pi/4} W_- \left(0, 1/4, k_{ex}\widetilde{\eta}\right) / \sqrt[4]{ik_{ex}\widetilde{\eta}}\right] \ , \ \eta - c^2 = \widetilde{\eta} \ , \ s_x \to 0$$

$$\sim \frac{\pi^{1/2} k_{ex}^{1/4} c_{0y} \left(-1\right)^{n_c} /2}{\sqrt[4]{(b^2-c^2)}} J_{-1/4}\left(k_{ex}\widetilde{\eta}/2\right) \sqrt[4]{k_{ex}\widetilde{\eta}} \ , \ \eta - c^2 = \widetilde{\eta} \ , \ s_x \to 0 \tag{559}$$

Inserting these aligned forms then gives the energy theorem

$$\frac{k^2}{\omega}\sqrt{\frac{(b^2-c^2)}{(a^2-c^2)}} \sim \frac{1}{4} c_{0y}^2 c_{0z}^2 \left(X \frac{\partial^2 X}{\partial\omega\partial\sqrt{\xi}}\right)_{\xi \to c^2}$$

$$\int_{\zeta_0}^{a^2} \left[\int_0^{b^2-c^2} \left\{\frac{(\zeta - b^2 + \widetilde{\eta})}{\sqrt{(a^2-b^2+\widetilde{\eta})}} - \frac{(\zeta - b^2)}{\sqrt{(a^2-b^2)}} + \frac{(\zeta - c^2 - \widetilde{\eta})}{\sqrt{(a^2-c^2-\widetilde{\eta})}} - \frac{(\zeta - c^2)}{\sqrt{(a^2-c^2)}}\right\} \frac{d\widetilde{\eta}}{\widetilde{\eta}}\right.$$

$$\left. + \frac{(\zeta - b^2)}{\sqrt{(a^2-b^2)}} \int_0^{k_{ez}(b^2-c^2)/2} \pi J_{-1/4}^2 (u)\, du + \frac{(\zeta - c^2)}{\sqrt{(a^2-c^2)}} \int_0^{k_{ex}(b^2-c^2)/2} \pi J_{-1/4}^2 (u)\, du\right] \frac{d\zeta}{(\zeta - b^2)(\zeta - c^2)\sqrt{(a^2-\zeta)}} \tag{560}$$

where we note that

$$\pi J_{-1/4}^2 (u) \sim \frac{2}{u}\left[\cos\left(u - \pi/8\right) + O\left(u^{-1}\right)\sin\left(u - \pi/8\right)\right]^2 \ , \ u >> 1 \tag{561}$$

To carry these out analytically, suppose we consider the integral [24], [19]

$$\int_0^\infty \left[\pi J_{-1/4}^2 (u) - \frac{1}{\sqrt{u^2+1}}\right] u^{-\lambda} du = \int_0^\infty \pi J_{-1/4}^2 (u)\, u^{-\lambda} du - \frac{1}{2}\int_0^\infty \frac{v^{-\lambda/2-1/2}}{\sqrt{v+1}} dv$$

$$= \frac{\pi\Gamma(\lambda)\,\Gamma\left(\frac{1/2-\lambda}{2}\right)}{2^\lambda \Gamma^2\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{1/2+\lambda}{2}\right)} - \frac{\Gamma(\lambda/2)\,\Gamma(1/2-\lambda/2)}{2\Gamma(1/2)}$$

$$= \frac{\Gamma(\lambda/2)\,\Gamma(1/2-\lambda/2)}{2\Gamma(1/2)}\left[\frac{\pi\Gamma\left(\frac{1/2-\lambda}{2}\right)}{\Gamma(1/2-\lambda/2)\,\Gamma(\lambda/2+1/2)\,\Gamma\left(\frac{1/2+\lambda}{2}\right)} - 1\right]$$

$$= \frac{\Gamma(1+\lambda/2)\,\Gamma(1/2-\lambda/2)}{\lambda\Gamma(1/2)}\left[\frac{\Gamma(1/4-\lambda/2)}{\Gamma(1/4+\lambda/2)}\cos(\pi\lambda/2) - 1\right]$$

$$\sim \frac{1 + \{\psi(1) + \psi(1/2)\}(\lambda/2)}{\lambda}\left[\{1 - \psi(1/4)\lambda\}\left\{1 - \pi^2\lambda^2/8\right\} - 1\right]$$

$$\sim -\psi(1/4)\left[1 + \{\psi(1) + \psi(1/2)\}(\lambda/2)\right] \to -\psi(1/4) \tag{562}$$

where $\Gamma(z)$ is the gamma function and $\psi(z)$ is the digamma function [19], so that

$$\int_0^\infty \left[\pi J_{-1/4}^2 (u) - \frac{1}{\sqrt{u^2+1}}\right] du = \gamma' + 3\ln 2 + \pi/2 \tag{563}$$

where we note that $\psi\left(1\right)=-\gamma'$, $\psi\left(1/2\right)=-\gamma'-2\ln 2$, Euler's constant $\gamma'\approx 0.577215664$ and

$$\psi\left(1/2\right)=\frac{1}{2}\psi\left(1/4\right)+\frac{1}{2}\psi\left(3/4\right)+\ln 2 \tag{564}$$

where from the reflection formula

$$\psi\left(3/4\right)=\psi\left(1/4\right)+\pi\cot\left(\pi/4\right)=\psi\left(1/4\right)+\pi \tag{565}$$

$$\psi\left(1/4\right)=-\gamma'-3\ln 2-\pi/2 \tag{566}$$

and from the duplication formula

$$\Gamma\left(\lambda\right)=\frac{1}{2}\frac{2^{\lambda}\Gamma\left(\lambda/2\right)\Gamma\left(\lambda/2+1/2\right)}{\Gamma\left(1/2\right)} \tag{567}$$

finally giving

$$\int_0^{\infty}J_{-1/4}^2\left(x\right)x^{-\lambda}dx=\frac{\Gamma\left(\lambda\right)\Gamma\left(\frac{1/2-\lambda}{2}\right)}{2^{\lambda}\Gamma^2\left(\frac{\lambda+1}{2}\right)\Gamma\left(\frac{1/2+\lambda}{2}\right)}\ ,\ 1/2>\mathrm{Re}\left(\lambda\right)>0 \tag{568}$$

We also note from the integral representation of the beta function $B\left(p,q\right)$ [19]

$$\int_0^{\infty}\frac{v^{-\lambda/2-1/2}}{\sqrt{v+1}}dv=B\left(1/2-\lambda/2,\lambda/2\right)=\frac{\Gamma\left(\lambda/2\right)\Gamma\left(1/2-\lambda/2\right)}{\Gamma\left(1/2\right)} \tag{569}$$

From these we can write

$$\int_0^{R}\pi J_{-1/4}^2\left(u\right)du\sim\int_0^{\infty}\left[\pi J_{-1/4}^2\left(u\right)-\frac{1}{\sqrt{u^2+1}}\right]du+\int_0^{R}\frac{du}{\sqrt{1+u^2}}+O\left(R^{-1}\right)=\gamma'+3\ln 2+\pi/2+\mathrm{Arcsinh}\left(R\right)$$

$$\sim\gamma'+3\ln 2+\pi/2+\ln\left(R+\sqrt{1+R^2}\right)\sim\gamma'+\pi/2+\ln\left(16R\right) \tag{570}$$

where we used

$$\mathrm{Arcsinh}\left(R\right)=\ln\left(R+\sqrt{1+R^2}\right)=\ln\left(R+R\sqrt{1+1/R^2}\right)\sim\ln\left(2R\right)+1/\left(2R\right)^2 \tag{571}$$

and the order symbol represents the leading correction $-\cos\left(2R+\pi/4\right)/\left(2R\right)$. The indefinite integral

$$\int\left\{\frac{\left(\zeta-b^2+\widetilde{\eta}\right)}{\sqrt{\left(a^2-b^2+\widetilde{\eta}\right)}}-\frac{\left(\zeta-b^2\right)}{\sqrt{\left(a^2-b^2\right)}}+\frac{\left(\zeta-c^2-\widetilde{\eta}\right)}{\sqrt{\left(a^2-c^2-\widetilde{\eta}\right)}}-\frac{\left(\zeta-c^2\right)}{\sqrt{\left(a^2-c^2\right)}}\right\}\frac{d\widetilde{\eta}}{\widetilde{\eta}}$$

$$=\frac{\left(\zeta-b^2\right)}{\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{\left(a^2-b^2+\widetilde{\eta}\right)}-\sqrt{a^2-b^2}}{\sqrt{\left(a^2-b^2+\widetilde{\eta}\right)}+\sqrt{a^2-b^2}}\right|+2\sqrt{\left(a^2-b^2+\widetilde{\eta}\right)}-\frac{\left(\zeta-b^2\right)}{\sqrt{\left(a^2-b^2\right)}}\ln\left(\widetilde{\eta}\right)$$

$$+\frac{\left(\zeta-c^2\right)}{\sqrt{a^2-c^2}}\ln\left|\frac{\sqrt{\left(a^2-c^2+\widetilde{\eta}\right)}-\sqrt{a^2-c^2}}{\sqrt{\left(a^2-c^2+\widetilde{\eta}\right)}+\sqrt{a^2-c^2}}\right|+2\sqrt{\left(a^2-c^2-\widetilde{\eta}\right)}-\frac{\left(\zeta-c^2\right)}{\sqrt{\left(a^2-c^2\right)}}\ln\left(\widetilde{\eta}\right) \tag{572}$$

then yields

$$\int_0^{b^2-c^2}\left\{\frac{\left(\zeta-b^2+\widetilde{\eta}\right)}{\sqrt{(a^2-b^2+\widetilde{\eta})}}-\frac{\left(\zeta-b^2\right)}{\sqrt{(a^2-b^2)}}+\frac{\left(\zeta-c^2-\widetilde{\eta}\right)}{\sqrt{(a^2-c^2-\widetilde{\eta})}}-\frac{\left(\zeta-c^2\right)}{\sqrt{(a^2-c^2)}}\right\}\frac{d\widetilde{\eta}}{\widetilde{\eta}}$$

$$=\frac{\left(\zeta-b^2\right)}{\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{(a^2-c^2)}-\sqrt{a^2-b^2}}{\sqrt{(a^2-c^2)}+\sqrt{a^2-b^2}}\right|+2\sqrt{(a^2-c^2)}-\frac{\left(\zeta-b^2\right)}{\sqrt{(a^2-b^2)}}\ln\left(b^2-c^2\right)$$

$$+\frac{\left(\zeta-c^2\right)}{\sqrt{a^2-c^2}}\ln\left|\frac{\sqrt{(a^2-b^2)}-\sqrt{a^2-c^2}}{\sqrt{(a^2-b^2)}+\sqrt{a^2-c^2}}\right|+2\sqrt{(a^2-b^2)}-\frac{\left(\zeta-c^2\right)}{\sqrt{(a^2-c^2)}}\ln\left(b^2-c^2\right)$$

$$+\frac{\left(\zeta-b^2\right)}{\sqrt{a^2-b^2}}\ln\left|4\left(a^2-b^2\right)\right|-2\sqrt{(a^2-b^2)}$$

$$+\frac{\left(\zeta-c^2\right)}{\sqrt{a^2-c^2}}\ln\left|4\left(a^2-c^2\right)\right|-2\sqrt{(a^2-c^2)}$$

$$=\frac{\left(\zeta-b^2\right)}{\sqrt{a^2-b^2}}\ln\left|4\frac{\sqrt{(a^2-c^2)}-\sqrt{a^2-b^2}}{\sqrt{(a^2-c^2)}+\sqrt{a^2-b^2}}\frac{\left(a^2-b^2\right)}{(b^2-c^2)}\right|$$

$$+\frac{\left(\zeta-c^2\right)}{\sqrt{a^2-c^2}}\ln\left|4\frac{\sqrt{(a^2-b^2)}-\sqrt{a^2-c^2}}{\sqrt{(a^2-b^2)}+\sqrt{a^2-c^2}}\frac{\left(a^2-c^2\right)}{(b^2-c^2)}\right| \tag{573}$$

Using these the normalization condition becomes

$$\frac{k^2}{\omega}\sqrt{\frac{(b^2-c^2)}{(a^2-c^2)}}\sim\frac{1}{4}c_{0y}^2c_{0z}^2\left(X\frac{\partial^2 X}{\partial\omega\partial\sqrt{\widetilde{\xi}}}\right)_{\xi\to c^2}$$

$$\int_{\zeta_0}^{a^2}\left[\left\{\frac{1}{(\zeta-c^2)\sqrt{a^2-b^2}}+\frac{1}{(\zeta-b^2)\sqrt{a^2-c^2}}\right\}\ln\left|32e^{\gamma'+\pi/2}\frac{\sqrt{a^2-c^2}-\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}+\sqrt{a^2-b^2}}\right|\right.$$

$$\left.+\frac{\ln\left(k_{ez}\left(a^2-b^2\right)\right)}{(\zeta-c^2)\sqrt{(a^2-b^2)}}+\frac{\ln\left(k_{ex}\left(a^2-c^2\right)\right)}{(\zeta-b^2)\sqrt{(a^2-c^2)}}\right]\frac{d\zeta}{\sqrt{(a^2-\zeta)}} \tag{574}$$

Noting that

$$\int_{\zeta_0}^{a^2}\frac{d\zeta}{(\zeta-c^2)\sqrt{(a^2-\zeta)}}=\int_{\zeta_0-c^2}^{a^2-c^2}\frac{du}{u\sqrt{(a^2-c^2-u)}}=\frac{1}{\sqrt{a^2-c^2}}\ln\left|\frac{\sqrt{(a^2-c^2-u)}-\sqrt{a^2-c^2}}{\sqrt{(a^2-c^2-u)}+\sqrt{a^2-c^2}}\right|_{\zeta_0-c^2}^{a^2-c^2}$$

$$=\frac{1}{\sqrt{a^2-c^2}}\ln\left|\frac{\sqrt{a^2-c^2}+\sqrt{(a^2-\zeta_0)}}{\sqrt{a^2-c^2}-\sqrt{(a^2-\zeta_0)}}\right| \tag{575}$$

$$\int_{\zeta_0}^{a^2}\frac{d\zeta}{(\zeta-b^2)\sqrt{(a^2-\zeta)}}=\int_{\zeta_0-b^2}^{a^2-b^2}\frac{du}{u\sqrt{(a^2-b^2-u)}}=\frac{1}{\sqrt{a^2-b^2}}\ln\left|\frac{\sqrt{(a^2-b^2-u)}-\sqrt{a^2-b^2}}{\sqrt{(a^2-b^2-u)}+\sqrt{a^2-b^2}}\right|_{\zeta_0-b^2}^{a^2-b^2}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \ln\left|\frac{\sqrt{a^2-b^2}+\sqrt{(a^2-\zeta_0)}}{\sqrt{a^2-b^2}-\sqrt{(a^2-\zeta_0)}}\right| \tag{576}$$

finally gives

$$\frac{k^2}{\omega}\sqrt{(a^2-b^2)(b^2-c^2)} \sim \frac{1}{4}c_{0y}^2 c_{0z}^2 \left(X\frac{\partial^2 X}{\partial\omega\partial\sqrt{\tilde{\xi}}}\right)_{\xi\to c^2}$$
$$\left[\ln\left|32k_{ez}\left(a^2-b^2\right)e^{\gamma'+\pi/2}\frac{\sqrt{a^2-c^2}-\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}+\sqrt{a^2-b^2}}\right|\ln\left|\frac{\sqrt{a^2-c^2}+\sqrt{(a^2-\zeta_0)}}{\sqrt{a^2-c^2}-\sqrt{(a^2-\zeta_0)}}\right|\right.$$
$$\left.+\ln\left|32k_{ex}\left(a^2-c^2\right)e^{\gamma'+\pi/2}\frac{\sqrt{a^2-c^2}-\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}+\sqrt{a^2-b^2}}\right|\ln\left|\frac{\sqrt{a^2-b^2}+\sqrt{(a^2-\zeta_0)}}{\sqrt{a^2-b^2}-\sqrt{(a^2-\zeta_0)}}\right|\right] \tag{577}$$

Note in the bowtie geometry we have $\ell < R_y < R_z$ and often for near stable situations $\ell << R_y < R_z$ and then $a^2 > \zeta_0 >> b^2 > c^2$. Inserting the ray orbit half length ℓ and the radii of curvature

$$b^2 - c^2 = \ell\left(R_z - R_y\right) \tag{578}$$

$$a^2 - \zeta_0 = \ell^2 \tag{579}$$

$$a^2 - b^2 = \ell\left(R_y + \ell\right) \tag{580}$$

$$a^2 - c^2 = \ell\left(R_z + \ell\right) \tag{581}$$

we then obtain

$$\frac{k^2}{\omega}\ell\sqrt{(R_y+\ell)(R_z-R_y)} \sim \frac{1}{4}c_{0y}^2 c_{0z}^2 \left(X\frac{\partial^2 X}{\partial\omega\partial\sqrt{\tilde{\xi}}}\right)_{\xi\to c^2}$$
$$\left[\ln\left|32k\sqrt{\ell(R_y+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right.$$
$$\left.+\ln\left|32k\sqrt{\ell(R_z+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right] \tag{582}$$

or inserting the normal derivative factor

$$\frac{k\ell}{\omega}\sqrt{\ell(R_y+\ell)(R_z-R_y)(R_z+\ell)} \sim \frac{1}{4}c_{0x}^2 c_{0y}^2 c_{0z}^2 e^{\pi s_x/4}\frac{\partial\Phi_{0x}}{\partial\omega}2$$
$$\left[\ln\left|32k\sqrt{\ell(R_y+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right.$$
$$\left.+\ln\left|32k\sqrt{\ell(R_z+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right] \tag{583}$$

5.3 Phase Derivative & Normalization

The frequency derivative of the phase can be written as

$$\frac{\partial \Phi_{0x}}{\partial \omega} = \sqrt{\mu_0 \varepsilon_0}\frac{\partial \Phi_{0x}}{\partial k} = 2k\sqrt{\mu_0 \varepsilon_0}\frac{\partial \Phi_{0x}}{\partial k^2} \tag{584}$$

In a three-dimensional cavity of volume V the scalar field mean modal spacing is [25], [13], [14]

$$\Delta k \sim 2\pi^2/\left(k^2 V\right) \to \Delta k^2 = 2k\Delta k \sim 4\pi^2/\left(kV\right) \tag{585}$$

If we impose three axes of symmetry

$$\Delta k^2 \to 8\Delta k^2 \sim 32\pi^2/\left(kV\right) \tag{586}$$

Then using a phase change $\Delta\Phi_{0x} = 2\pi$ between modes we can write the inverse phase derivative as [3], [13], [14]

$$\left(\frac{\partial \Phi_{0x}}{\partial k^2}\right)^{-1} = \frac{16\pi}{kV} v_x^2 \ , \ f\left(v_x\right) = \frac{1}{\sqrt{2\pi}} e^{-v_x^2/2} \tag{587}$$

where $f\left(v_x\right)$ is a unit variance normally distributed density function of the random variable v_x. Then the energy theorem becomes

$$\begin{aligned} c_{0x}^2 c_{0y}^2 c_{0z}^2 e^{\pi s_x/4} \sim v_x^2 \frac{16\pi\ell}{k^2 V} &\sqrt{\ell\left(R_y+\ell\right)\left(R_z-R_y\right)\left(R_z+\ell\right)} \\ \left[\ln\left|32k\sqrt{\ell\left(R_y+\ell\right)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right. \\ \left. +\ln\left|32k\sqrt{\ell\left(R_z+\ell\right)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right]^{-1} \end{aligned} \tag{588}$$

We see from the dimensional relations that an approach to a two-dimensional situation results if $R_z >> R_y >> \ell$ and then $a^2 > \zeta_0 > b^2 >> c^2$. For this limit with $R_z >> R_y >> \ell$ we see that the y dimension has a very large range. The variation of the y coordinate is governed largely by the variation in η. In past treatments of the two-dimensional case, where we chose to have no variation in the long dimension this can thus be simulated by taking $Y^2\left(\eta\right)$ as a constant.

5.3.1 Vector Case

The vector mean modal spacing in a three-dimensional cavity of volume V is half of the scalar case [25], [13], [14]

$$\Delta k \sim \pi^2/\left(k^2 V\right) \to \Delta k^2 = 2k\Delta k \sim 2\pi^2/\left(kV\right) \tag{589}$$

If we impose three axes of symmetry

$$\Delta k^2 \to 8\Delta k^2 \sim 16\pi^2/\left(kV\right) \tag{590}$$

Then using a phase change $\Delta\Phi_{0x} = 2\pi$ between modes we can write the inverse phase derivative as

$$\left(\frac{\partial\Phi_{0x}}{\partial k^2}\right)^{-1} = \frac{8\pi}{kV}v_x^2\ ,\ f\left(v_x\right) = \frac{1}{\sqrt{2\pi}}e^{-v_x^2/2} \tag{591}$$

This implies that the normalization condition

$$\begin{aligned}
c_{0x}^2 c_{0y}^2 c_{0z}^2 e^{\pi s_x/4} &\sim v_x^2 \frac{8\pi\ell}{k^2V}\sqrt{\ell\left(R_y+\ell\right)\left(R_z-R_y\right)\left(R_z+\ell\right)} \\
&\left[\ln\left|32k\sqrt{\ell\left(R_y+\ell\right)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right. \\
&\left.+\ln\left|32k\sqrt{\ell\left(R_z+\ell\right)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right]^{-1}
\end{aligned} \tag{592}$$

leads to half the scalar level (for the squared amplitude).

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6 PROJECTION ALONG SCAR

We now examine the trigonometric projection along the scarred orbit [3], [4]

$$V_p = \int_{-\ell}^{\ell} u(x) \left\{ \begin{array}{c} \sin(k_p x) \\ \cos(k_p x) \end{array} \right\} dx = 2\int_0^{\ell} u(x) \left\{ \begin{array}{c} \sin(k_p x) \\ \cos(k_p x) \end{array} \right\} dx \tag{593}$$

where

$$k_p \ell = \left\{ \begin{array}{c} p\pi \\ (p-1/2)\,\pi \end{array} \right\}, \; p = 1, 2, ... \tag{594}$$

Our previous scalar field along the scar center was

$$u(x) = \Phi\left(c^2, b^2, \zeta\right) = X\left(c^2\right) Y\left(b^2\right) Z(\zeta) = \frac{(-1)^{n_b}}{2\pi} \frac{c_{0x} c_{0y} c_{0z}}{\sqrt[4]{(b^2-\theta_1)}} k_{ex}^{1/4} k_{ez}^{1/4} \left|\Gamma(1/4 - is_x/4)\right| \left|\Gamma(1/4 - is_z/4)\right| e^{\pi s_x/4 - \pi s_z/8}$$

$$\frac{1}{\sqrt[4]{(\zeta-\theta_1)(\zeta-\theta_2)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_{\zeta}^{a^2} \sqrt{\frac{(\theta-\theta_1)(\theta-\theta_2)}{(a^2-\theta)(\theta-b^2)(\theta-c^2)}} d\theta \right] \right\} \tag{595}$$

$$x^2 = \frac{\left(a^2-c^2\right)\left(a^2-b^2\right)\left(a^2-\zeta\right)}{(a^2-b^2)(a^2-c^2)} = a^2 - \zeta \tag{596}$$

$$y^2 = \frac{\left(b^2-c^2\right)\left(b^2-b^2\right)\left(\zeta-b^2\right)}{(b^2-c^2)(a^2-b^2)} = 0 \tag{597}$$

$$z^2 = \frac{\left(c^2-c^2\right)\left(b^2-c^2\right)\left(\zeta-c^2\right)}{(a^2-c^2)(b^2-c^2)} = 0 \tag{598}$$

We transform to

$$a^2 - \zeta = \widetilde{\zeta} = x^2 \tag{599}$$

$$a^2 - \theta = \widetilde{\theta} \tag{600}$$

$$\begin{aligned} &\frac{1}{\sqrt[4]{(\zeta-\theta_1)(\zeta-\theta_2)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_{\zeta}^{a^2} \sqrt{\frac{(\theta-\theta_1)(\theta-\theta_2)}{(a^2-\theta)(\theta-b^2)(\theta-c^2)}} d\theta \right] \right\} \\ &= \frac{1}{\sqrt[4]{(a^2-\theta_1-x^2)(a^2-\theta_2-x^2)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_0^{a^2-\zeta} \sqrt{\frac{\left(a^2-\theta_1-\widetilde{\theta}\right)\left(a^2-\theta_2-\widetilde{\theta}\right)}{\widetilde{\theta}\left(a^2-b^2-\widetilde{\theta}\right)\left(a^2-c^2-\widetilde{\theta}\right)}} d\widetilde{\theta} \right] \right\} \\ &= \frac{1}{\sqrt[4]{(a^2-\theta_1-x^2)(a^2-\theta_2-x^2)}} \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left[\frac{k}{2} \int_0^{x^2} \sqrt{\frac{\left(a^2-\theta_1-\widetilde{\theta}\right)\left(a^2-\theta_2-\widetilde{\theta}\right)}{\widetilde{\theta}\left(a^2-b^2-\widetilde{\theta}\right)\left(a^2-c^2-\widetilde{\theta}\right)}} d\widetilde{\theta} \right] \right\} \end{aligned} \tag{601}$$

$$b^2 - c^2 = \ell\,(R_z - R_y) \tag{602}$$

$$a^2-\zeta_0=\ell^2 \tag{603}$$

$$a^2-b^2=\ell\left(R_y+\ell\right) \tag{604}$$

$$a^2-c^2=\ell\left(R_z+\ell\right) \tag{605}$$

For the case where $a^2>\zeta>\zeta_0>>b^2>c^2$ following from $R_z>R_y>>\ell$, or $a^2-\zeta=x^2<a^2-\zeta_0=\ell<<a^2-b^2=\ell\left(R_y+\ell\right)<a^2-c^2=\ell\left(R_z+\ell\right)$, we can approximate the final integral as

$$\frac{1}{\sqrt[4]{\left(\zeta-\theta_1\right)\left(\zeta-\theta_2\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{\zeta}^{a^2}\sqrt{\frac{\left(\theta-\theta_1\right)\left(\theta-\theta_2\right)}{\left(a^2-\theta\right)\left(\theta-b^2\right)\left(\theta-c^2\right)}}d\theta\right]\right\}$$

$$\approx\frac{1}{\sqrt[4]{\left(a^2-c^2\right)\left(a^2-b^2\right)}}\left\{\begin{array}{c}\sin\\ \cos\end{array}\left[\frac{k}{2}\int_{0}^{x^2}\frac{d\widetilde{\theta}}{\sqrt{\widetilde{\theta}}}\right]\right\}=\frac{1}{\sqrt[4]{\left(a^2-c^2\right)\left(a^2-b^2\right)}}\left\{\begin{array}{c}\sin\left(kx\right)\\ \cos\left(kx\right)\end{array}\right\} \tag{606}$$

Then we can write

$$u\left(x\right)=\Phi\left(c^2,b^2,\zeta\right)\approx\frac{\left(-1\right)^{n_b}}{2\pi}\frac{c_{0x}c_{0y}c_{0z}}{\sqrt[4]{\left(b^2-c^2\right)}}k_{ex}^{1/4}k_{ez}^{1/4}\left|\Gamma\left(1/4-is_x/4\right)\right|\left|\Gamma\left(1/4-is_z/4\right)\right|e^{\pi s_x/4-\pi s_z/8}$$

$$\frac{1}{\sqrt[4]{\left(a^2-c^2\right)\left(a^2-b^2\right)}}\left\{\begin{array}{c}\sin\left(kx\right)\\ \cos\left(kx\right)\end{array}\right\}$$

$$\approx\frac{\left(-1\right)^{n_b}}{2\pi}\frac{\sqrt{k}c_{0x}c_{0y}c_{0z}}{\sqrt[4]{\left(b^2-c^2\right)\left(a^2-c^2\right)\left(a^2-b^2\right)\sqrt{\left(a^2-c^2\right)\left(a^2-b^2\right)}}}\left|\Gamma\left(1/4-is_x/4\right)\right|\left|\Gamma\left(1/4-is_z/4\right)\right|e^{\pi s_x/4-\pi s_z/8}\left\{\begin{array}{c}\sin\left(kx\right)\\ \cos\left(kx\right)\end{array}\right\} \tag{607}$$

where we used

$$k_{ex}^2=k^2\frac{\left(\theta_2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}$$

$$=k^2\frac{\left(\theta_2-b^2+b^2-c^2\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}=k^2\frac{\left(b^2-c^2-s_z/k_{ez}\right)}{\left(a^2-c^2\right)\left(b^2-c^2\right)}=\frac{k^2}{\left(a^2-c^2\right)}-\frac{s_zk^2}{\left(a^2-c^2\right)k_{ez}\left(b^2-c^2\right)}\sim\frac{k^2}{\left(a^2-c^2\right)} \tag{608}$$

$$k_{ez}^2=k^2\frac{\left(b^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}$$

$$=k^2\frac{\left(b^2-c^2+c^2-\theta_1\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}=k^2\frac{\left(b^2-c^2-s_x/k_{ex}\right)}{\left(a^2-b^2\right)\left(b^2-c^2\right)}=\frac{k^2}{\left(a^2-b^2\right)}-\frac{s_xk^2}{\left(a^2-b^2\right)k_{ex}\left(b^2-c^2\right)}\sim\frac{k^2}{\left(a^2-b^2\right)} \tag{609}$$

Inserting the radii and ray path length

$$u\left(x\right)=\Phi\left(c^2,b^2,\zeta\right)\approx\frac{\left(-1\right)^{n_b}}{2\pi}\frac{\sqrt{k}c_{0x}c_{0y}c_{0z}}{\sqrt[4]{\ell\left(R_z-R_y\right)\ell\left(R_z+\ell\right)\ell\left(R_y+\ell\right)\sqrt{\ell\left(R_z+\ell\right)\ell\left(R_y+\ell\right)}}}$$

$$|\Gamma(1/4 - is_x/4)|\,|\Gamma(1/4 - is_z/4)|\,e^{\pi s_x/4-\pi s_z/8}\left\{\begin{array}{c}\sin(kx)\\ \cos(kx)\end{array}\right\} \tag{610}$$

and scalar normalization

$$c_{0x}^2 c_{0y}^2 c_{0z}^2 e^{\pi s_x/4} \sim v_x^2\frac{16\pi\ell}{k^2 V}\sqrt{\ell(R_y+\ell)(R_z-R_y)(R_z+\ell)}$$

$$\left[\ln\left|32k\sqrt{\ell(R_y+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right.$$

$$\left.+\ln\left|32k\sqrt{\ell(R_z+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right]^{-1} \tag{611}$$

to give

$$u(x) = \Phi\left(c^2, b^2, \zeta\right) = \frac{A_0}{\sqrt[4]{\ell(R_z+\ell)\,\ell(R_y+\ell)}}\left\{\begin{array}{c}\sin(kx)\\ \cos(kx)\end{array}\right\} \tag{612}$$

with

$$A_0 \approx \frac{(-1)^{n_b}}{2\pi}\frac{\sqrt{k}c_{0x}c_{0y}c_{0z}}{\sqrt[4]{\ell(R_z-R_y)\sqrt{\ell(R_z+\ell)\,\ell(R_y+\ell)}}}|\Gamma(1/4 - is_x/4)|\,|\Gamma(1/4 - is_z/4)|\,e^{\pi s_x/4-\pi s_z/8}$$

$$\approx v_x\frac{2(-1)^{n_b}}{\sqrt{\pi k V}}\sqrt[4]{\ell\sqrt{(R_z+\ell)(R_y+\ell)}}\,|\Gamma(1/4 - is_x/4)|\,|\Gamma(1/4 - is_z/4)|\,e^{\pi s_x/8-\pi s_z/8}$$

$$\left[\ln\left|32k\sqrt{\ell(R_y+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right.$$

$$\left.+\ln\left|32k\sqrt{\ell(R_z+\ell)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right]^{-1/2} \tag{613}$$

Then

$$V_p = 2\int_0^\ell u(x)\left\{\begin{array}{c}\sin(k_p x)\\ \cos(k_p x)\end{array}\right\}dx = 2A_0\int_0^\ell\left\{\begin{array}{c}\sin(kx)\\ \cos(kx)\end{array}\right\}\left\{\begin{array}{c}\sin(k_p x)\\ \cos(k_p x)\end{array}\right\}dx \tag{614}$$

or

$$V_p = \frac{A_0}{\sqrt[4]{\ell(R_z+\ell)\,\ell(R_y+\ell)}}\int_0^\ell\left[\cos((k-k_p)x)\mp\cos((k+k_p)x)\right]dx$$

$$= \frac{A_0}{\sqrt[4]{\ell(R_z+\ell)\,\ell(R_y+\ell)}}\left[\frac{\sin((k-k_p)\ell)}{(k-k_p)}\mp\frac{\sin((k+k_p)\ell)}{(k+k_p)}\right]\approx\frac{A_0\ell}{\sqrt[4]{\ell(R_z+\ell)\,\ell(R_y+\ell)}}\frac{\sin((k-k_p)\ell)}{(k-k_p)\ell} \tag{615}$$

where at high frequencies we neglect the sum term. Then squaring this and taking the mean

$$\left\langle V_p^2\right\rangle = \left\langle \frac{A_0^2 \ell}{\sqrt{(R_z+\ell)(R_y+\ell)}} \right\rangle \left[\frac{\sin\left((k-k_p)\ell\right)}{(k-k_p)\ell} \right]^2 \tag{616}$$

where we used $\left\langle v_x^2\right\rangle = 1$. Then defining $G_1(s_x, s_z)$ by means of [3], [4], [13]

$$\left\langle kLV_p^2\right\rangle \sim L^2 G_1(s_x, s_z)/V \tag{617}$$

gives

$$\begin{aligned} G_1(s_x, s_z) = {} & \frac{2/\pi}{\sqrt{\sqrt{(R_y+\ell)(R_z+\ell)}/\ell}} \left|\Gamma(1/4 - is_z/4)\right|^2 \left|\Gamma(1/4 - is_x/4)\right|^2 e^{\pi s_x/4 - \pi s_z/4} \left[\frac{\sin\left((k-k_p)\ell\right)}{(k-k_p)\ell} \right]^2 \\ & \left[\ln\left| 32k\sqrt{\ell(R_y+\ell)} e^{\gamma' + \pi/2} \frac{\sqrt{R_z+\ell} - \sqrt{R_y+\ell}}{\sqrt{R_z+\ell} + \sqrt{R_y+\ell}} \right| \ln\left| \frac{\sqrt{R_z+\ell} + \sqrt{\ell}}{\sqrt{R_z+\ell} - \sqrt{\ell}} \right| \right. \\ & \left. + \ln\left| 32k\sqrt{\ell(R_z+\ell)} e^{\gamma' + \pi/2} \frac{\sqrt{R_z+\ell} - \sqrt{R_y+\ell}}{\sqrt{R_z+\ell} + \sqrt{R_y+\ell}} \right| \ln\left| \frac{\sqrt{R_y+\ell} + \sqrt{\ell}}{\sqrt{R_y+\ell} - \sqrt{\ell}} \right| \right]^{-1} \end{aligned} \tag{618}$$

In previous work [7], [14] we have often approximated the $\sin(kx)$ or $\cos(kx)$ factor as the Fourier series terms $\sin(k_p x)$ or $\cos(k_p x)$ as the radii of curvature become large, denoted byV_{pp}. This is certainly also true of the transition region, corresponding to the larger values of $G_1(s_x, s_z)$ near $k \to k_p$ which corresponds to taking the factor $\sin\left((k-k_p)\ell\right)/\left((k-k_p)\ell\right) \to 1$

$$\left\langle kLV_{pp}^2\right\rangle \approx \left\langle kLV_p^2\right\rangle \sim L^2 G_1(s_x, s_z)/V \tag{619}$$

where

$$G_1(s_x, s_z) = \frac{G_1(0,0)}{\Gamma^4(1/4)} \left|\Gamma(1/4 - is_z/4)\right|^2 \left|\Gamma(1/4 - is_x/4)\right|^2 e^{\pi s_x/4 - \pi s_z/4} \tag{620}$$

and

$$\begin{aligned} G_1(0,0) = {} & \frac{2/\pi}{\sqrt{\sqrt{(R_y+\ell)(R_z+\ell)}/\ell}} \Gamma^4(1/4) \\ & \left[\ln\left| 32k\sqrt{\ell(R_y+\ell)} e^{\gamma' + \pi/2} \frac{\sqrt{R_z+\ell} - \sqrt{R_y+\ell}}{\sqrt{R_z+\ell} + \sqrt{R_y+\ell}} \right| \ln\left| \frac{\sqrt{R_z+\ell} + \sqrt{\ell}}{\sqrt{R_z+\ell} - \sqrt{\ell}} \right| \right. \\ & \left. + \ln\left| 32k\sqrt{\ell(R_z+\ell)} e^{\gamma' + \pi/2} \frac{\sqrt{R_z+\ell} - \sqrt{R_y+\ell}}{\sqrt{R_z+\ell} + \sqrt{R_y+\ell}} \right| \ln\left| \frac{\sqrt{R_y+\ell} + \sqrt{\ell}}{\sqrt{R_y+\ell} - \sqrt{\ell}} \right| \right]^{-1} \end{aligned} \tag{621}$$

We note that this result has taken symmetry in all three directions so we expect a factor of eight enhancement over a general scalar case.

We can also use the gamma function reflection formula to write

$$\Gamma(1/4 - is_x/4)\,\Gamma(3/4 + is_x/4) = \pi/\sin\pi(1/4 - is_x/4) = \pi\sqrt{2}/\left[\cosh(\pi s_x/4) - i\sinh(\pi s_x/4)\right] \tag{622}$$

$$\left|\Gamma\left(1/4-is_x/4\right)\Gamma\left(3/4+is_x/4\right)\right|^2=2\pi^2/\left[\cosh^2\left(\pi s_x/4\right)+\sinh^2\left(\pi s_x/4\right)\right]=4\pi^2/\left(e^{\pi s_x/2}+e^{-\pi s_x/2}\right) \quad (623)$$

$$\Gamma\left(1/4\right)\Gamma\left(3/4\right)=\pi/\sin\left(\pi/4\right)=\pi\sqrt{2} \quad (624)$$

$$\Gamma^2\left(1/4\right)\Gamma^2\left(3/4\right)=2\pi^2 \quad (625)$$

and

$$G_1\left(s_x,s_z\right)/\left[\frac{\Gamma^2\left(3/4\right)}{\Gamma^2\left(1/4\right)}G_1\left(0,0\right)\right]=\frac{\left|\Gamma\left(1/4-is_z/4\right)\right|^2}{\left|\Gamma\left(3/4-is_x/4\right)\right|^2}\frac{2e^{\pi s_x/4-\pi s_z/4}}{e^{\pi s_x/2}+e^{-\pi s_x/2}} \quad (626)$$

where

$$\frac{\Gamma^2\left(3/4\right)}{\Gamma^2\left(1/4\right)}G_1\left(0,0\right)=\frac{4\pi}{\sqrt{\sqrt{\left(R_y+\ell\right)\left(R_z+\ell\right)}/\ell}}$$

$$\left[\ln\left|32k\sqrt{\ell\left(R_y+\ell\right)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_z+\ell}+\sqrt{\ell}}{\sqrt{R_z+\ell}-\sqrt{\ell}}\right|\right.$$

$$\left.+\ln\left|32k\sqrt{\ell\left(R_z+\ell\right)}e^{\gamma'+\pi/2}\frac{\sqrt{R_z+\ell}-\sqrt{R_y+\ell}}{\sqrt{R_z+\ell}+\sqrt{R_y+\ell}}\right|\ln\left|\frac{\sqrt{R_y+\ell}+\sqrt{\ell}}{\sqrt{R_y+\ell}-\sqrt{\ell}}\right|\right]^{-1} \quad (627)$$

Figure 6 shows an example of the behavior of this function on a contour plot for parameters $L=2\ell=2$ m, $R_y=10$ m, $R_z=12$ m, $kL=67.367$, where $\zeta_0=12$ m^2, $a^2=13$ m^2, $b^2=2$ m^2, $c^2=0$ m^2, $d_z/d_y\approx 1.0871146$, $d_z/\ell=a/\ell\approx 3.60555$. Figure 7 shows a cut for $s_z=0$.

$$b^2-c^2=\ell\left(R_z-R_y\right) \quad (628)$$

$$a^2-\zeta_0=\ell^2 \quad (629)$$

$$a^2-b^2=\ell\left(R_y+\ell\right)=d_y^2 \quad (630)$$

$$a^2-c^2=\ell\left(R_z+\ell\right)=d_z^2 \quad (631)$$

The value chosen for kL was chosen to compare to a numerical simulation run with these dimensions over the frequency range $1-2$ GHz. Using the asymptotic formula for the scalar field mean spacing

$$\Delta k\sim 2\pi^2/\left(k^2V\right) \quad (632)$$

the number of modes over the wave number range $k_1<k<k_2$ is

$$N=\int_{k_1}^{k_2}\frac{dN}{dk}dk=\int_{k_1}^{k_2}\left(1/\Delta k\right)dk=\frac{V}{2\pi^2}\int_{k_1}^{k_2}k^2dk=\frac{V}{6\pi^2}\left(k_2^3-k_1^3\right) \quad (633)$$

and therefore the mean wavenumber over this same range is

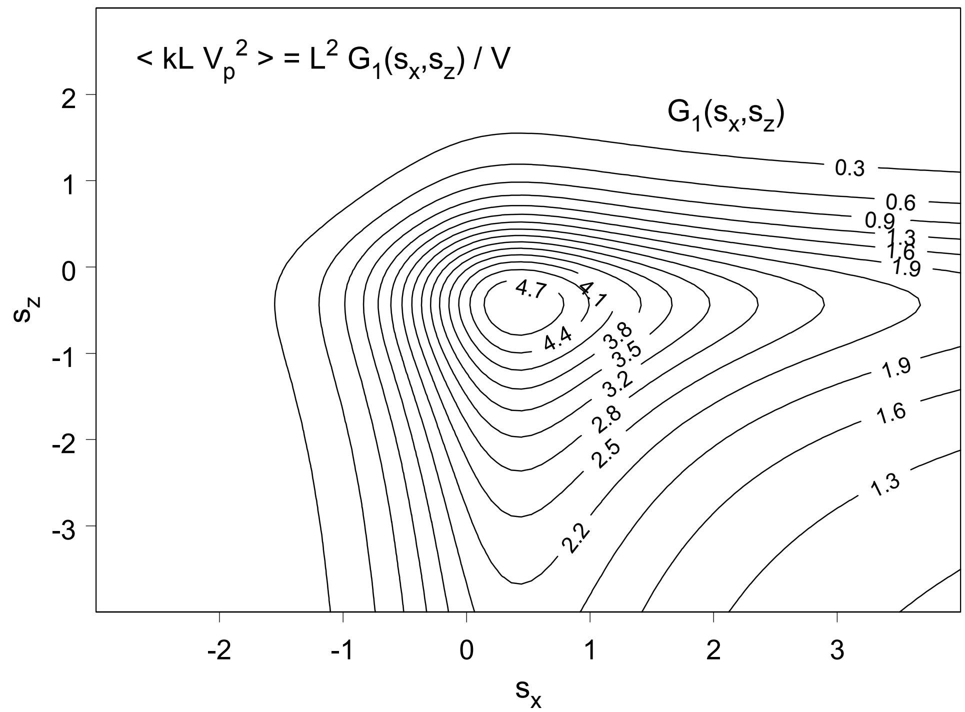

Figure 6: Contour plot of $G_1(s_x, s_z)$ for example where $R_z = 12$ m, $R_y = 10$ m, $L = 2\ell = 2$ m, and $kL = 67.367$.

$$\langle k \rangle = \frac{1}{N}\int_{k_1}^{k_2} (k/\Delta k)\, dk = \frac{V}{N 2\pi^2}\int_{k_1}^{k_2} k^3 dk = \frac{V}{N 8\pi^2}\left(k_2^4 - k_1^4\right) = \frac{3}{4}\frac{k_2^4 - k_1^4}{k_2^3 - k_1^3} = \frac{3}{4}\frac{(k_2 + k_1)\left(k_2^2 + k_1^2\right)}{\left(k_2^2 + k_2 k_1 + k_1^2\right)} \quad (634)$$

We can also write (using $k_1\ell \approx 20.958622$ at 1 GHz)

$$\langle k\ell \rangle = \frac{3}{4}\frac{(k_2 + k_1)\left(k_2^2 + k_1^2\right)}{\left(k_2^2 + k_2 k_1 + k_1^2\right)}\ell \approx \frac{3}{4}\frac{(2+1)(4+1)}{4+2+1}(20.958622) \approx \frac{45}{28}(20.958622) \rightarrow 33.6835 \;\; (f = 1.60714 \text{ GHz}) \quad (635)$$

where the final result corresponds to an average $\langle k \rangle$ being at $f = 1.60714$ GHz. We can then write

$$\langle kL \rangle = \frac{3}{4}\frac{(k_2 + k_1)\left(k_2^2 + k_1^2\right)}{\left(k_2^2 + k_2 k_1 + k_1^2\right)}L \approx \frac{45}{14}(20.958622) \approx 67.367 \quad (636)$$

Note that this average choice will not change for the vector spacing $\Delta k \sim \pi^2/\left(k^2 V\right)$ or including symmetries.

From the prior result in the section on the "Resonant Condition phase integral"

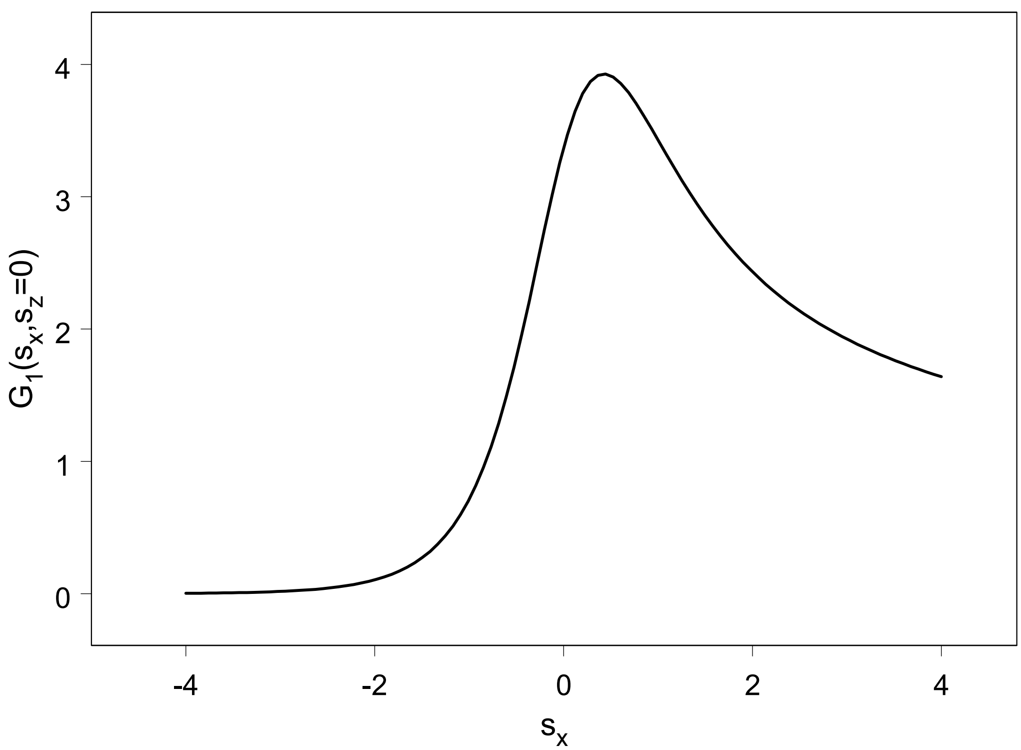

Figure 7: Cut of function $G_1\left(s_x, s_z\right)$ for $s_z = 0$, for example where $R_z = 12$ m, $R_y = 10$ m, $L = 2\ell = 2$ m, and $kL = 67.367$.

$$2\left(k-k_{p}\right)\ell=\ell\left(s_{x}/d_{z}-s_{z}/d_{y}\right) \tag{637}$$

we can eliminate the separation constants and determine a result dependent on $k-k_p$. To do this we need to integrate out the probability along these constant lines of frequency separation to end up with a single plot as a function of $k-k_p$

$$F\left(2\Delta k\ell\right)=\int_{-\infty}^{\infty}F\left(2\Delta k\ell d_{z}/\ell+s_{z}d_{z}/d_{y},s_{z}\right)ds_{z} \tag{638}$$

Note that the key point here in this integration is that we have already selected the eigenvalue spacing based on the three-dimensional asymptotic scalar eigenvalue density (using eight-fold symmetry). This introduces a scaling in the eigenfunction amplitude accounting for the three-dimensional spacing (not representing the density for fixed s_x or s_z, as in the fixed m for the axisymmetric problem) and thus we do not have to introduce another density function in this averaging. Thus with

$$u=2\left(k-k_{p}\right)\ell\left(d_{z}/\ell\right) \tag{639}$$

we can write

$$G_{1}\left(u\right)=\int_{-\infty}^{\infty}G_{1}\left(u+s_{z}d_{z}/d_{y},s_{z}\right)ds_{z} \tag{640}$$

and

$$G_{1}\left(u\right)/\left[\frac{\Gamma^{2}\left(3/4\right)}{\Gamma^{2}\left(1/4\right)}G_{1}\left(0,0\right)\right]=2\int_{-\infty}^{\infty}\frac{\left|\Gamma\left(1/4-is_{z}/4\right)\right|^{2}}{\left|\Gamma\left(3/4-i\left(u+s_{z}d_{z}/d_{y}\right)/4\right)\right|^{2}}\frac{e^{\pi\left(u+s_{z}d_{z}/d_{y}\right)/4-\pi s_{z}/4}}{e^{\pi\left(u+s_{z}d_{z}/d_{y}\right)/2}+e^{-\pi\left(u+s_{z}d_{z}/d_{y}\right)/2}}ds_{z} \tag{641}$$

where

$$\left|\Gamma\left(3/4-is_{x}/4\right)\right|^{2}\sim2\pi\left(\left|s_{x}\right|/4\right)^{1/2}e^{-\pi\left|s_{x}\right|/4}\ ,\ s_{x}\rightarrow\pm\infty \tag{642}$$

$$\left|\Gamma\left(1/4-is_{z}/4\right)\right|^{2}\sim2\pi\left(\left|s_{z}\right|/4\right)^{-1/2}e^{-\pi\left|s_{z}\right|/4}\ ,\ s_{z}\rightarrow\mp\infty \tag{643}$$

$$\frac{\left|\Gamma\left(1/4-is_{z}/4\right)\right|^{2}}{\left|\Gamma\left(3/4-i\left(u+s_{z}d_{z}/d_{y}\right)/4\right)\right|^{2}}\frac{e^{\pi\left(u+s_{z}d_{z}/d_{y}\right)/4-\pi s_{z}/4}}{e^{\pi\left(u+s_{z}d_{z}/d_{y}\right)/2}+e^{-\pi s_{x}/2}}\sim\frac{e^{-\pi\left(\left|s_{z}\right|+s_{z}\right)/4}}{\sqrt{\left|u+s_{z}d_{z}/d_{y}\right|\left|s_{z}\right|}}e^{\pi\left(u+s_{z}d_{z}/d_{y}-\left|u+s_{z}d_{z}/d_{y}\right|\right)/4} \tag{644}$$

The convergence is thus exponential in both directions: $e^{-\pi s_z/2}$, $s_z\rightarrow+\infty$ and $e^{\pi s_z(d_z/d_y)/2}$, $s_z\rightarrow-\infty$.

We could plot this result as a function of

$$\lambda=2\left(k-k_{p}\right)L=u/\left(d_{z}/L\right) \tag{645}$$

but instead from former work in two-dimensional and three-dimensional axisymmetric cases we have chosen to plot the result as a function of s instead of u or λ, where

$$s=\left(k-k_{p}\right)\ell/\ln\sqrt{\frac{a+\ell}{a-\ell}}=\left(k-k_{p}\right)\ell/\ln\sqrt{\frac{\sqrt{\ell\left(R_{z}+\ell\right)}+\ell}{\sqrt{\ell\left(R_{z}+\ell\right)}-\ell}}=\left(k-k_{p}\right)L/\ln\left(\frac{d_{z}+\ell}{d_{z}-\ell}\right) \tag{646}$$

$$a^2 - c^2 = a^2 = \ell \left(R_z + \ell \right) = d_z^2 \tag{647}$$

and we have arbitrarily decided to use the larger radius of curvature R_z instead of R_y (in previous work there was only a single radius of curvature). This can be written as

$$s = 2\left(k - k_p\right)\ell/\ln\left(\frac{a+\ell}{a-\ell}\right) = 2\left(k - k_p\right)\ell/\ln\left(\frac{\sqrt{\ell\left(R_z+\ell\right)}+\ell}{\sqrt{\ell\left(R_z+\ell\right)}-\ell}\right) = \left(k - k_p\right)L/\ln\left(\frac{\sqrt{R_z/\ell+1}+1}{\sqrt{R_z/\ell+1}-1}\right) \tag{648}$$

and related to $u = 2\left(k - k_p\right)\ell\left(d_z/\ell\right)$ by means of

$$s = u/\left[\left(d_z/\ell\right)\ln\left(\frac{\sqrt{R_z/\ell+1}+1}{\sqrt{R_z/\ell+1}-1}\right)\right] \approx u\left(\ell/d_z\right)1.175556 \approx u0.4869 \tag{649}$$

where the final two results use this example geometry. The stability exponent with respect to this radius can be written as [26]

$$\Lambda_+ = \left(\frac{d_z+\ell}{d_z-\ell}\right)^2 \approx 3.124 \tag{650}$$

and then s becomes

$$s = 2\left(k - k_p\right)L/\ln\left(\Lambda_+\right) = \lambda/\ln\left(\Lambda_+\right) = u/\left[\left(d_z/L\right)\ln\left(\Lambda_+\right)\right] \tag{651}$$

Figure 8 shows this scalar scar theory result $G_1(s)/8$ as the solid curve and the dashed curve in this figure shows the scalar random plane wave result $G(\lambda)$, discussed in the next subsection, for comparison.

Figure 9 shows the surface mesh used to model one eighth of the three-dimensional convex surface lantern (bowtie) geometry and Figure 10 shows an example of the electric field. Figure 11 shows a histogram from the vector electromagnetic simulation (using the method of moments code EIGER) for $G_1(s)/4$ (for the vector problem this quantity corresponds to $G_1(s)/8$ in the scalar problem) versus $s = 2\left(k - k_p\right)L/\ln\left(\Lambda_+\right)$, $\Lambda_+ = \left(\frac{d_z+\ell}{d_z-\ell}\right)^2$ [27]. The transition near $s = 0$ is quite similar to the scar result (solid curve) in Figure 8 and is quite different from the random plane wave result (dashed curve).

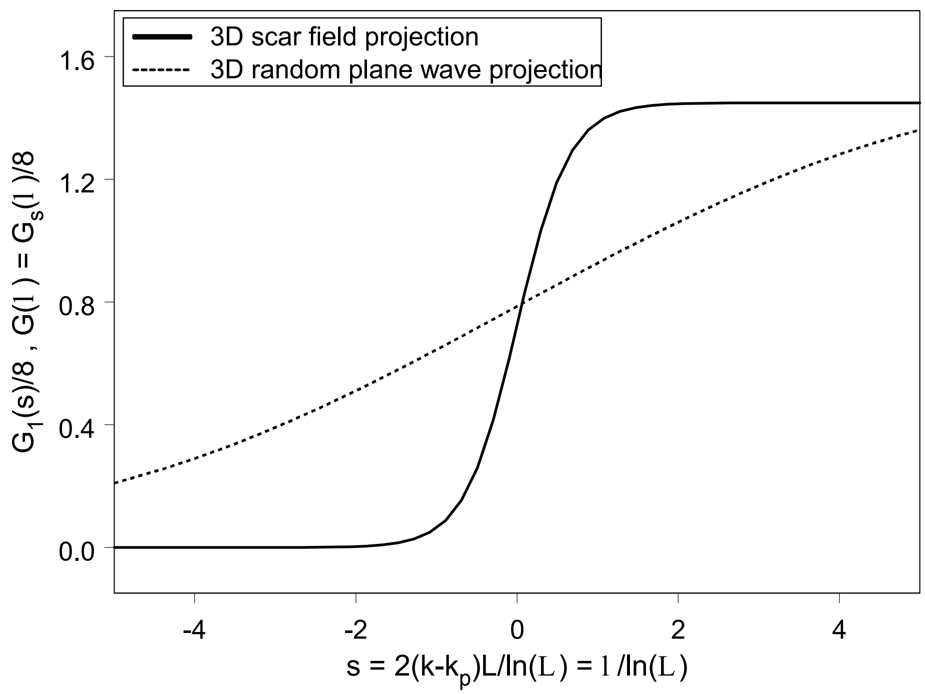

Figure 8: Function $G_1(s)/8$ compared to random plane wave result $G(\lambda) = G_s(\lambda)/8$, for example where $R_z = 12$ m, $R_y = 10$ m, $L = 2\ell = 2$ m, and $kL = 67.367$.

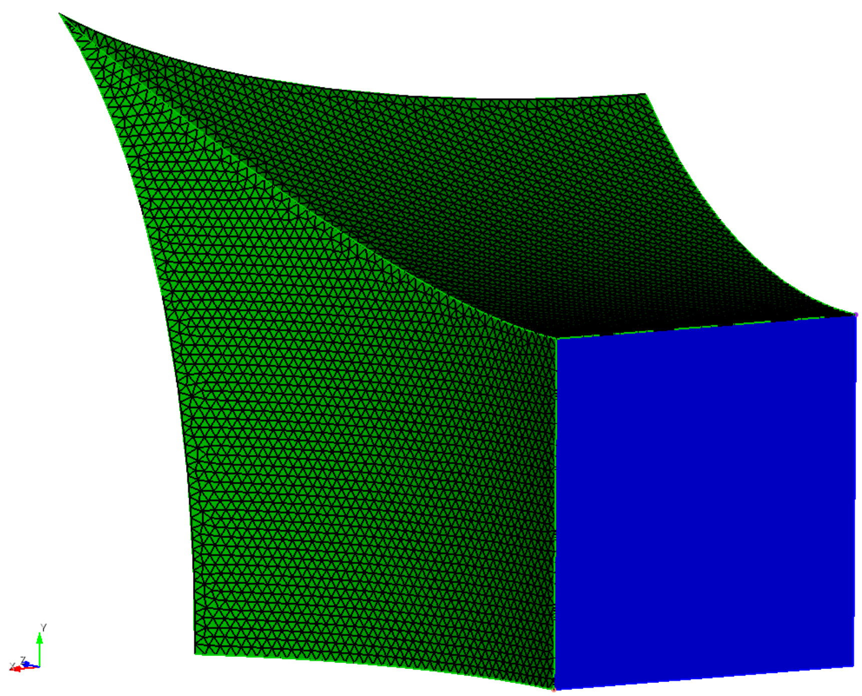

Figure 9: One eighth of lantern cavity structure, where $R_y = 10$ m, $R_z = 12$ m, and $L = 2\ell = 2$ m.

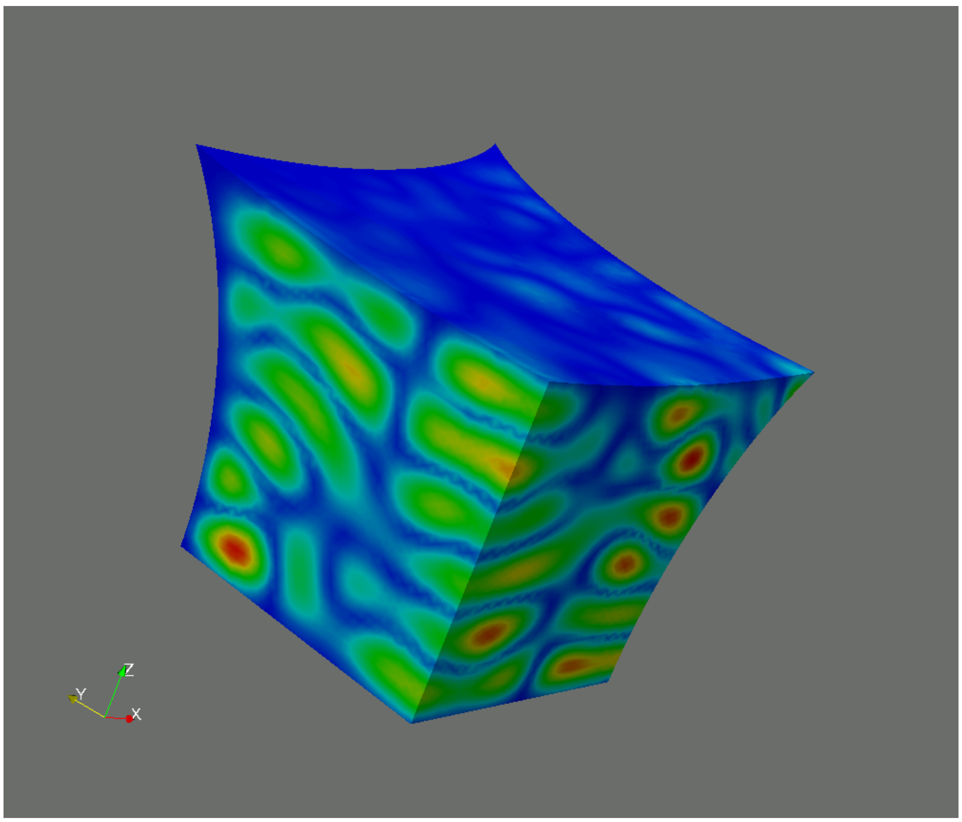

Figure 10: An example of electric field in three-dimensional geometry.

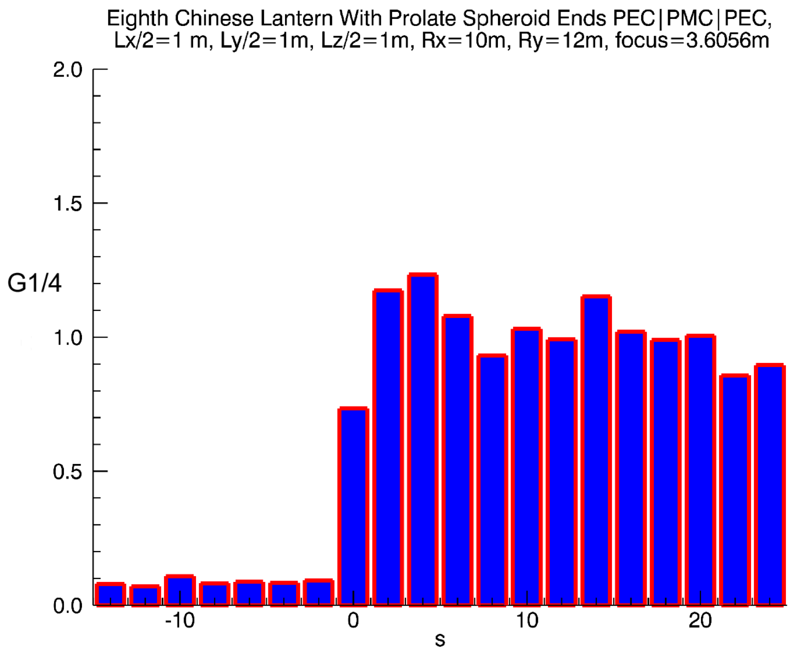

Figure 11: Histogram from numerical simulation of convex lantern (bowtie) geometry for $G_1(s)/4$ in the vector case. This simulation used one eighth geometry with symmetry planes PEC|PMC|PEC. In the simulation the ray path is along z with the radii of curvature noted in the x and y directions; the axes must be permuted to correspond to our $R_y = 10$ m, $R_z = 12$ m, orbit half length $L/2 = 1$ m in the x direction, and the focal point $d_z = 3.6056$ m.

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7 3D RANDOM PLANE WAVE PROJECTION

Let us take the random plane wave description in three-dimensions and consider the trigonometric projections.

7.1 Scalar Case

The axisymmetric report [13] gives the scalar 3D random plane wave representation

$$u_r = \lim_{N\to\infty} \sqrt{2/\left(VN\right)}\, \mathrm{Re}\left[\sum_{j=1}^{N} a_j e^{i\alpha_j + i\underline{k}_j\cdot\underline{r}}\right] \tag{652}$$

where a_j are real random numbers with $\left\langle a_j^2\right\rangle = 1$, $\left|\underline{k}_j\right| = k$ are random wave vectors uniformly distributed in angles the sphere with 4π solid angle, and the random phases α_j are uniformly distributed on a 2π interval. This has been normalized so that mean square is

$$\left\langle u_r^2\right\rangle = 1/V \tag{653}$$

The 3D trigonometric projection is

$$V_{pr} = \int_{-\ell}^{\ell} \cos\left(k_p x\right) u_r\left(0,x\right) dx \tag{654}$$

We define

$$\left\langle kLV_p^2\right\rangle_r = L^2 G\left(\lambda\right)/V \tag{655}$$

with

$$\lambda = 2\left(k-k_p\right) L \tag{656}$$

giving

$$G\left(\lambda\right) = \frac{\pi}{4} - \frac{\sin^2\left(\lambda/4\right)}{\lambda/2} + \frac{1}{2}\,\mathrm{Si}\left(\lambda/2\right) = \frac{1}{2}\left[\mathrm{Si}\left(\lambda/2\right) - \frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \frac{\pi}{2}\right] \tag{657}$$

The asymptotic limits are

$$G\left(\lambda\right) \to 0\ ,\ \ \lambda \to -\infty \tag{658}$$

$$G\left(0\right) = \pi/4 \tag{659}$$

$$G\left(\infty\right) = \pi/2 \approx 1.570796 \tag{660}$$

and

$$s = 2\left(k-k_p\right) L/\ln\left(\Lambda_+\right) = \lambda/\ln\left(\Lambda_+\right) \tag{661}$$

with [26]

$$\Lambda_+ = \left(\frac{d_z+\ell}{d_z-\ell}\right)^2 \tag{662}$$

If we were to impose symmetries in all three dimensions we could define

$$G_s(\lambda) = 8G(\lambda) \tag{663}$$

Then we would have

$$G_s(\infty) = 4\pi \approx 12.56637 \tag{664}$$

The prior $G_1(s \to \infty) \approx 11.6$ result (eight times the saturation level 1.45 in Figure 8), using the scalar eigenvalue spacing, is reasonably close to this result for $G_s(\infty)$.

7.2 Vector Case

The vector case has random plane wave representation [13]

$$\underline{E}_r = \lim_{N\to\infty} \sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j \left(\cos\varphi_{pj}\underline{e}_j + \sin\varphi_{pj}\underline{e}'_j\right) e^{i\alpha_j + i\underline{k}_j\cdot\underline{r}}\right] \tag{665}$$

where the polarization angles φ_{pj} are random numbers uniformly distributed over a 2π interval, and the unit vector $\underline{e}_j$ is perpendicular to $\underline{k}_j$ with.

$$\underline{e}'_j = \left(\underline{k}_j \times \underline{e}_j\right)/k \tag{666}$$

The normalization gives

$$\langle \underline{E}_r \cdot \underline{E}_r \rangle = 1/V \tag{667}$$

The 3D trigonometric projection is taken as

$$V_{pr} = \int_{-\ell}^{\ell} \cos(k_p x)\, E_v(x,0,0)\, dx = \int_{-\ell}^{\ell} \cos(k_p x)\, \underline{e}_v \cdot \underline{E}_r(x,0,0)\, dx \tag{668}$$

where the unit vector $\underline{e}_v$ is taken to be perpendicular to x. Again defining

$$\left\langle kLV_p^2 \right\rangle_r = L^2 G(\lambda)/V \tag{669}$$

$$\lambda = 2(k - k_p) L \tag{670}$$

gives

$$G(\lambda) = \frac{1}{4}\left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \frac{\pi}{2} + \mathrm{Si}(\lambda/2)\right] \tag{671}$$

which is exactly half of the scalar result. Therefore the asymptotic results in this case are

$$G(\lambda) \to 0\ ,\ \ \lambda \to -\infty \tag{672}$$

$$G(0) = \pi/8 \tag{673}$$

$$G(\infty) = \pi/4 \approx 0.785398 \tag{674}$$

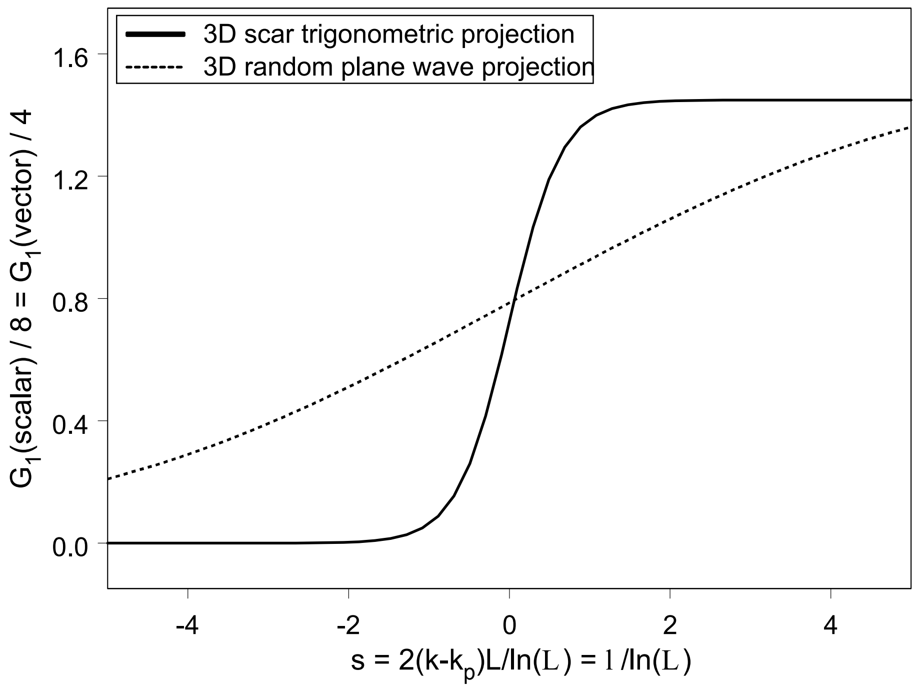

Figure 12: Scalar function $G_1(s)/8$ or vector function $G_1(s)/4$ compared to scalar random plane wave result $G(\lambda) = G_s(\lambda)/8$ or vector random plane wave result $2G(\lambda) = G_s(\lambda)/4$, for example where $R_z = 12$ m, $R_y = 10$ m, $L = 2\ell = 2$ m, and $kL = 67.367$.

If we were to impose symmetries in all three dimensions we could again define

$$G_s(\lambda) = 8G(\lambda) \tag{675}$$

Then we would have

$$G_s(\infty) = 2\pi \approx 6.2831853 \tag{676}$$

The prior scalar result would go to $G_1(s \to \infty) \approx 5.8$ (four times the saturation level 1.45 in Figure 8 when using the vector eigenvalue spacing) is close to this result for $G_s(\infty)$. Figure 12 again shows this scar transition versus the random plane wave result. Figure 11 showed the histogram from the electromagnetic simulation [27] and Figure 13 shows the histogram when the first two symmetry planes are permuted (which rotates the polarization state) [27].

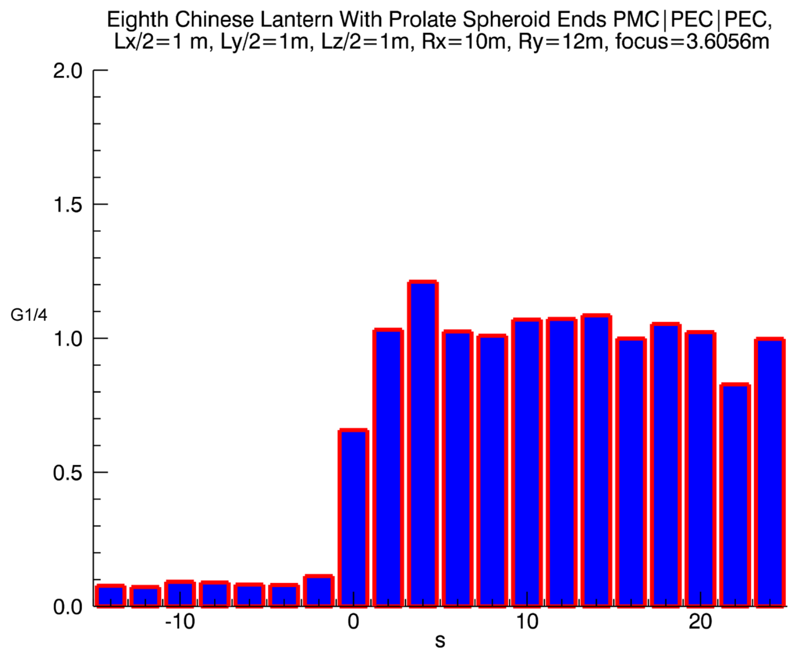

Figure 13: Histogram from numerical simulation of convex lantern (bowtie) geometry for $G_1(s)/4$ in the vector case. This simulation used one eighth geometry with symmetry planes PMC|PEC|PEC. In the simulation the ray path is along z with the radii of curvature noted in the x and y directions; the axes must be permuted to correspond to our $R_y = 10$ m, $R_z = 12$ m, orbit half length $L/2 = 1$ m in the x direction, and the focal point $d_z = 3.6056$ m.

8 CONCLUSIONS

This report constructed high frequency asymptotic field solutions along an unstable periodic ray orbit in a three-dimensional cavity at high frequencies. In this case the mirrors in general have two distinct radii of curvature, which in this report are assumed to be convex. The modal solutions are normalized by using the energy theorem. Random plane wave chaotic field representations are also constructed. Fourier projections of the field along the periodic ray orbit are taken and compared to the projection using the random plane wave form of the field. These comparisons illustrate orbital field behavior transitions (in the Fourier projection) when the eigenmode frequencies pass through values where phase variations along the orbit length align with the required boundary conditions on the terminating mirrors. With the foci located exterior to the cavity we do not observe significant field enhancements at single locations along the orbit, as in the two-dimension case [7].

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