$(2, 2)$ **Scattering and the celestial torus**

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ABSTRACT: Analytic continuation from Minkowski space to $(2, 2)$ split signature spacetime has proven to be a powerful tool for the study of scattering amplitudes. Here we show that, under this continuation, null infinity becomes the product of a null interval with a celestial torus (replacing the celestial sphere) and has only one connected component. Spacelike and timelike infinity are time-periodic quotients of AdS$_3$. These three components of infinity combine to an S^3 represented as a toric fibration over the interval. Privileged scattering states of scalars organize into $\mathrm{SL}(2, \mathbb{R})_L \times \mathrm{SL}(2, \mathbb{R})_R$ conformal primary wave functions and their descendants with real integral or half-integral conformal weights, giving the normally continuous scattering problem a discrete character.

KEYWORDS: Scattering Amplitudes, Space-Time Symmetries

Contents

1   Introduction

Scattering amplitudes in quantum field theory are often defined by analytic continuation from $(4, 0)$ Euclidean signature to $(3, 1)$ Lorentzian signature. This provides an efficient prescription for the Feynman-diagram singularities encountered in perturbation theory. Moreover, positivity properties in Euclidean space enable powerful non-perturbative instanton and axiomatic analyses. Euclidean methods have also proven effective in quantum gravity.

In recent years, however, analytic continuation from Minkowski space to a split $(2, 2)$ signature spacetime — which we shall refer to as Klein space[1] $\mathbb{K}^{2,2}$ — has emerged as a complementary and surprisingly effective tool in quantum field theory. An awkward feature of Euclidean space is that particles cannot be on-shell. Amplitudes are therefore represented as analytic continuations of sums of off-shell processes, which can both become inordinately complicated and obscure the underlying physics. Dramatic simplifications have been found in some on-shell descriptions in Klein space [1–8]. The group-theoretic reduction of the 4D $(2, 2)$ Lorentz group to the product of two 1D conformal groups, the associated reality of the self duality condition [9], and the non-degeneracy of massless three-point scattering also lead to significant simplifications.

In quantum gravity in asymptotically flat spacetimes, there are yet further reasons to consider Klein space. The paucity of generally covariant bulk observables — and more generally the holographic principle — suggests that any theory of quantum gravity should be defined by boundary observables. In Euclidean space, the conformal boundary is just a point. It seems challenging to formulate a holographic dual which encodes the richness of asymptotically flat quantum gravity by observables at a zero-dimensional point. Here we

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[1]After the mathematician Felix Klein, who pioneered the study of these spaces in the Erlangen Program.

find that, in contrast, Klein space has a rich conformal boundary at infinity, providing a suitable potential home for a holographic dual.

In section 2 we show that the conformal boundary at null infinity in Klein space, denoted $\mathcal{I}$, is the product of a null interval with the Lorentzian signature celestial torus. Both spatial and timelike infinity i^0 and i' are the product of a disk with a circle and are endowed with the conformal metric of $AdS_3/\mathbb{Z}$. Here the $\mathbb{Z}$-quotient makes the familiar AdS_3 cylinder periodic. The gluing of the toroidal boundaries of these $AdS_3/\mathbb{Z}$ geometries to the celestial tori at the two ends of $\mathcal{I}$ trivializes different cycles of the latter, giving a toric representation of the $\mathcal{I} \cup i^0 \cup i'$ infinity as S^3. Since $\mathcal{I}$ has only one connected component, observables are given by an $\mathcal{S}$-vector rather than an $\mathcal{S}$-matrix. The fact that the continuation from Minkowski to Klein space leads to the replacement of the sphere with a torus will perhaps prove useful for sharpening the concept of a celestial conformal field theory.

Section 3 reviews the $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ symmetry of Klein space. Expressions are given for L_n, $\bar{L}_n$, $n = -1, 0, 1$ in a natural basis where $L_0 \pm \bar{L}_0$ generate the compact space and time directions of the celestial torus, as well as for the finite group action on the celestial torus. The group action preserves the $AdS_3/\mathbb{Z}$ hypersurfaces which are a fixed distance from the origin and foliate Klein space.

Section 4 considers conformal basis wave functions for massless scalars. Single-valuedness on the celestial torus requires that the L_0 and $\bar{L}_0$ eigenvalues are either both integer or both half-integer. "L-primary" solutions are found corresponding to highest-weight states annihilated by L_1 and $\bar{L}_1$. More general solutions are then obtained by taking descendants. Convolutions of these wave functions with the bulk field operator create states which have an interpretation as $L_0, \bar{L}_0$ eigenstates of the 1+1D celestial CFT living on a spatial circle of the celestial torus. The fact that the time direction of the torus is periodic is not a problem because $L_0 + \bar{L}_0$ is quantized. We also find lowest-weight solutions annihilated by L_{-1} and $\bar{L}_{-1}$, as well as mixed solutions annihilated by $L_{\pm 1}, \bar{L}_{\mp 1}$.

A striking feature of this construction is that the solutions are labelled by three integers: the conformal weights and the levels of the left and right descendants, giving L-primary scattering a discrete character. This contrasts with dynamics on the celestial sphere in Minkowski space, where the conformal basis solutions are labelled by three continuous parameters: a position on the sphere and a continuous complex conformal dimension. The discrete character of celestial scattering in Klein space resonates with several other recent developments. Spacetime translations shift conformal weights by a half-integer [10], so the set of all L-primaries and their descendants associated to a given spacetime field form a representation of the Poincaré group.[2] In gauge theory and gravity, the infinite hierarchy of soft currents appears at negative integer weight, while the positive integer weights appear related to Goldstone bosons [13–16]. Poles at negative even integer conformal weights in celestial scattering amplitudes were recently shown [17] to encode the coefficents in the Wilsonian effective action. These poles characterize much or all of the theory and may be naturally probed by scattering L-primaries.

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[2]Unlike the continuous complex highest weight representations discussed in [11, 12] which are restricted to have the real part of the conformal weight equal to unity and cannot be put in representations of translations.

In section 5 we construct, as Mellin transforms of plane waves, modes corresponding to particles which emerge at a fixed point on the celestial torus. These correspond to "H-primary" operators which are primary with respect to elements $H_1, \bar{H}_1$ leaving fixed the point at which the particles emerge. Scattering of such particles takes the form of a correlation function on the celestial torus. We show that L-primary wave functions can be expressed as weighted integrals over the torus of H-primary wave functions with quantized weights. This is a version of the celestial state-operator correspondence. Hence L-primary scattering amplitudes are weighted celestial integrals of Mellin transforms of plane wave scattering amplitudes. We close with a few comments in section 6.

2 Null, spacelike and timelike infinity

In this section we conformally compactify $\mathbb{K}^{2,2}$ and derive the conformal geometry of null infinity $\mathcal{I}$, spatial infinity i^0 and timelike infinity i'.[3] The flat metric on $\mathbb{K}^{2,2}$ is

$$ds_4^2 = dz d\bar{z} - dw d\bar{w}. \tag{2.1}$$

In polar coordinates $z = r e^{i\phi}$ and $w = q e^{i\psi}$, this becomes

$$ds^2 = -dq^2 - q^2 d\psi^2 + dr^2 + r^2 d\phi^2. \tag{2.2}$$

Now define $q - r = \tan U$, $q + r = \tan V$, giving

$$ds^2 = \frac{1}{\cos^2 U \cos^2 V} \left(-dU dV - \frac{1}{4}\sin^2(V + U) d\psi^2 + \frac{1}{4}\sin^2(V - U) d\phi^2 \right). \tag{2.3}$$

The coordinate ranges are the solid triangle $-\frac{\pi}{2} < U < \frac{\pi}{2}$ and $|U| < V < \frac{\pi}{2}$, as depicted in figure 1. Null infinity $\mathcal{I}$ is at $V = \pi/2$ where the factor out front blows up. Spacelike (timelike) infinity i^0 (i') is the boundary at $U = -\frac{\pi}{2}$ ($U = \frac{\pi}{2}$).

Note that, unlike the case of $\mathbb{M}^{3,1}$, null infinity has only one connected component. This means we cannot define an $\mathcal{S}$-matrix. Instead we have only an $\mathcal{S}$-vector in the sense of [19]. It is an amplitude for a collection of incoming particles on $\mathcal{I}$ to scatter into nothing — which they must as there is nowhere to go! This $\mathcal{S}$-vector together with a suitable analytic continuation procedure can in principle be used to define an $\mathcal{S}$-matrix in $\mathbb{M}^{3,1}$.

$\mathcal{I}$ is parameterized by the null coordinate $-\frac{\pi}{2} < U < \frac{\pi}{2}$ and the periodic coordinates ψ and ϕ. Taking $V \to \frac{\pi}{2}$ while rescaling (2.3) by $\cos^2 V$ one finds the conformal metric on $\mathcal{I}$ to be the square, Lorentzian torus

$$ds_{\mathcal{I}}^2 = -d\psi^2 + d\phi^2, \quad \psi \sim \psi + 2\pi, \ \phi \sim \phi + 2\pi. \tag{2.4}$$

Hence $\mathcal{I}$ is the product of the celestial torus with a null interval.

Now we turn to i^0, i'. Since the boundary of a boundary is nothing, we must be able to glue these to $\mathcal{I}$ to get an S^3 which is the topological boundary of $\mathbb{K}^{2,2}$. S^3 is topologically represented in toric geometry as a torus fibration over the interval in which one of the two torus cycles shrinks to zero at one end of the interval, and the other at the other end. Then

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[3]An alternate conformal compactification of $\mathbb{K}^{2,2}$ has been studied in [18].
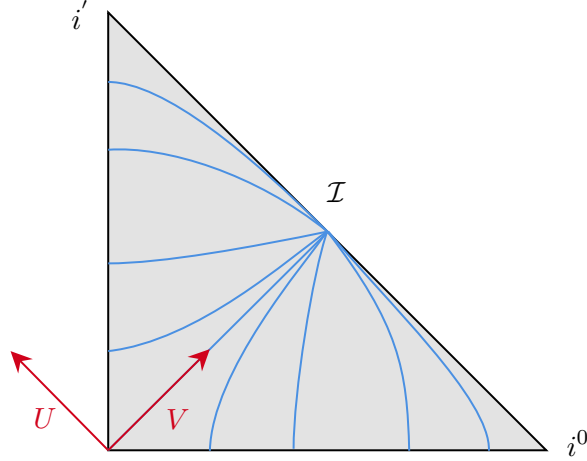
Figure 1. Toric Penrose diagram for signature $(2,2)$ Klein space. $45°$ lines are null as usual. A Lorentzian torus is fibered over every point in the diagram. The spacelike cycle of the torus degenerates along the timelike line $U = V$, while the timelike cycle degenerates along the spacelike line $U = -V$. Neither cycle degenerates at null infinity $\mathcal{I}$ which is the interval $-\frac{\pi}{2} < U < \frac{\pi}{2}$, $V = \frac{\pi}{2}$. Spacelike infinity i^0 is at $(U, V) = (-\frac{\pi}{2}, \frac{\pi}{2})$ and has the conformal geometry of signature $(1,2)$ AdS$_3/\mathbb{Z}$. Timelike infinity i' is at $(U, V) = (\frac{\pi}{2}, \frac{\pi}{2})$ and has the conformal geometry of signature $(2,1)$ AdS$_3/\mathbb{Z}$. The blue lines are lines of constant $w\bar{w} - z\bar{z}$ with $\tau = 0$ at $U = 0$.

there are no non-contractible cycles. In order to complete $\mathcal{I}$ to S^3 in this manner, the i^0, i' "caps" must both be topologically the product of a disk and a circle. We now show that this is indeed the case and moreover that the conformal geometry on each cap is AdS$_3/\mathbb{Z}$.

Following procedures which are standard for $\mathbb{M}^{3,1}$ [20–24], we resolve i' by taking the $\tau \to \infty$ limit of the signature $(2,1)$ surface

$$z\bar{z} - w\bar{w} = -\tau^2. \tag{2.5}$$

We denote the two regions of $\mathbb{K}^{2,2}$ with positive or negative $z\bar{z} - w\bar{w}$ by $\mathbb{K}^{2,2\pm}$. Coordinates covering the region $\mathbb{K}^{2,2-}$, which contains i', are

$$
\begin{aligned}
z &= \tau e^{i\phi} \sinh \rho, \\
w &= \tau e^{i\psi} \cosh \rho.
\end{aligned}
\tag{2.6}
$$

The inverse relations are

$$
\begin{aligned}
\tau &= \sqrt{w\bar{w} - z\bar{z}}, &\quad \tanh \rho &= \sqrt{\frac{z\bar{z}}{w\bar{w}}}, \\
e^{i\phi} &= \sqrt{z/\bar{z}}, &\quad e^{i\psi} &= \sqrt{w/\bar{w}}.
\end{aligned}
\tag{2.7}
$$

The Klein space metric in these coordinates is

$$ds_4^2 = -d\tau^2 + \tau^2 ds_3^2, \tag{2.8}$$

where

$$ds_3^2 = -\cosh^2 \rho \, d\psi^2 + \sinh^2 \rho \, d\phi^2 + d\rho^2 \tag{2.9}$$

is the conformal geometry of i'. We recognize it as the standard metric on $\mathrm{AdS}_3/\mathbb{Z}$, where the $\mathbb{Z}$ acts as the time-like quotient $\psi \to \psi + 2\pi$.

A similar construction for i^0 begins with $(\tilde\tau, \tilde\rho, \phi, \psi)$ covering $\mathbb{K}^{2,2+}$ with $z\bar{z} - w\bar{w} = +\tilde\tau^2$:

$$
\begin{aligned}
z &= \tilde\tau e^{i\phi} \cosh\tilde\rho, \\
w &= \tilde\tau e^{i\psi} \sinh\tilde\rho.
\end{aligned}
\tag{2.10}
$$

The inverse relations are

$$
\begin{aligned}
\tilde\tau = \sqrt{z\bar{z} - w\bar{w}}, &\quad \tanh\tilde\rho = \sqrt{\frac{w\bar{w}}{z\bar{z}}}, \\
e^{i\phi} = \sqrt{z/\bar{z}}, &\quad e^{i\psi} = \sqrt{w/\bar{w}}.
\end{aligned}
\tag{2.11}
$$

One finds

$$
ds_4^2 = d\tilde\tau^2 - \tilde\tau^2 ds_3^2,
\tag{2.12}
$$
$$
ds_3^2 = -\cosh^2\tilde\rho\, d\phi^2 + \sinh^2\tilde\rho\, d\psi^2 + d\tilde\rho^2.
\tag{2.13}
$$

We see that the non-contractible loop in the $\mathrm{AdS}_3/\mathbb{Z}$ factor is now ϕ instead of ψ and spacelike instead of timelike in the $\mathbb{K}^{2,2\pm}$ embedding space. Hence gluing the conformal geometries of the two $\mathrm{AdS}_3/\mathbb{Z}$ caps to $\mathcal{I}$ trivializes both cycles of the celestial torus and the full topology of infinity is S^3.

3 Symmetries

The "Lorentz group" of $\mathbb{K}^{2,2}$ is $\mathrm{SO}(2,2) \cong \frac{\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R}{\mathbb{Z}_2}$, where the $\mathbb{Z}_2$ is generated by $-1_L \times -1_R$. The spin group is the double cover $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$. The symmetry is generated on Klein space by (real combinations of) the six Killing vector fields

$$
\begin{aligned}
L_1 &= \bar{z}\partial_w + \bar{w}\partial_z, & \bar{L}_1 &= z\partial_w + \bar{w}\partial_{\bar{z}}, \\
L_0 &= \frac{1}{2}\left(z\partial_z + w\partial_w - \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}\right), & \bar{L}_0 &= \frac{1}{2}\left(-z\partial_z + w\partial_w + \bar{z}\partial_{\bar{z}} - \bar{w}\partial_{\bar{w}}\right), \\
L_{-1} &= -z\partial_{\bar{w}} - w\partial_{\bar{z}}, & \bar{L}_{-1} &= -\bar{z}\partial_{\bar{w}} - w\partial_z.
\end{aligned}
\tag{3.1}
$$

In $\mathbb{K}^{2,2-}$ we may also write

$$
\begin{aligned}
L_1 &= \frac{1}{2}e^{-i\psi - i\phi}\left(\partial_\rho - i\tanh\rho\,\partial_\psi - i\coth\rho\,\partial_\phi\right), \\
L_0 &= -\frac{i}{2}\left(\partial_\psi + \partial_\phi\right), \\
L_{-1} &= \frac{1}{2}e^{i\psi + i\phi}\left(-\partial_\rho - i\tanh\rho\,\partial_\psi - i\coth\rho\,\partial_\phi\right), \\
\bar{L}_1 &= \frac{1}{2}e^{-i\psi + i\phi}\left(\partial_\rho - i\tanh\rho\,\partial_\psi + i\coth\rho\,\partial_\phi\right), \\
\bar{L}_0 &= -\frac{i}{2}\left(\partial_\psi - \partial_\phi\right), \\
\bar{L}_{-1} &= \frac{1}{2}e^{i\psi - i\phi}\left(-\partial_\rho - i\tanh\rho\,\partial_\psi + i\coth\rho\,\partial_\phi\right),
\end{aligned}
\tag{3.2}
$$

while on $\mathbb{K}^{2,2+}$

$$L_1 = \frac{1}{2} e^{-i\psi - i\phi} \left(\partial_{\tilde{\rho}} - i \coth \tilde{\rho}\, \partial_\psi - i \tanh \tilde{\rho}\, \partial_\phi \right),$$

$$L_0 = -\frac{i}{2} \left(\partial_\psi + \partial_\phi \right),$$

$$L_{-1} = \frac{1}{2} e^{i\psi + i\phi} \left(-\partial_{\tilde{\rho}} - i \coth \tilde{\rho}\, \partial_\psi - i \tanh \tilde{\rho}\, \partial_\phi \right),$$

$$\bar{L}_1 = \frac{1}{2} e^{-i\psi + i\phi} \left(\partial_{\tilde{\rho}} - i \coth \tilde{\rho}\, \partial_\psi + i \tanh \tilde{\rho}\, \partial_\phi \right), \tag{3.3}$$

$$\bar{L}_0 = -\frac{i}{2} \left(\partial_\psi - \partial_\phi \right),$$

$$\bar{L}_{-1} = \frac{1}{2} e^{i\psi - i\phi} \left(-\partial_{\tilde{\rho}} - i \coth \tilde{\rho}\, \partial_\psi + i \tanh \tilde{\rho}\, \partial_\phi \right).$$

In either case on the boundary at $\rho \to \infty$ (or $\tilde{\rho} \to \infty$) these reduce to the familiar circle action

$$L_n = -\frac{i}{2} e^{-in(\psi+\phi)} \left(\partial_\psi + \partial_\phi \right), \quad \bar{L}_n = -\frac{i}{2} e^{-in(\psi-\phi)} \left(\partial_\psi - \partial_\phi \right), \tag{3.4}$$

for $n = -1, 0, 1$. The L_n obey (for all ρ or $\tilde{\rho}$) the $\mathrm{SL}(2,\mathbb{R})$ Lie bracket algebra

$$[L_n, L_m] = (n - m) L_{m+n}, \tag{3.5}$$

and similarly for the $\bar{L}_n$.

AdS$_3$/$\mathbb{Z}$ is the $\mathrm{SL}(2,\mathbb{R})$ group manifold which admits an $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ group action. The generators above leave fixed the AdS$_3$/$\mathbb{Z}$ hypersurfaces of constant $w\bar{w} - z\bar{z}$. $L_0 - \bar{L}_0$ generates AdS rotations and $L_0 + \bar{L}_0$ generates AdS global time translations for $w\bar{w} - z\bar{z} = \tau^2$, while for $z\bar{z} - w\bar{w} = \tilde{\tau}^2$ it is the other way around. In either case because of the mod $\mathbb{Z}$ quotient, the eigenvalues of $L_0 \pm \bar{L}_0$ are separately quantized. This is standard in $\mathrm{SL}(2,\mathbb{R})$ representation theory, but differs from familiar string theory applications in which one works on the simply-connected universal cover AdS$_3$ of AdS$_3$/$\mathbb{Z}$, and only $L_0 - \bar{L}_0$ is quantized.

SO(2, 2) acts faithfully on the celestial torus. We define the null angles

$$x^\pm \equiv \psi \pm \phi. \tag{3.6}$$

While ψ, ϕ naturally parametrize the cycles of the celestial torus, the symmetry group acts more simply on $x^\pm$. In particular $\mathrm{SL}(2,\mathbb{R})_L$ acts only on x^+, while $\mathrm{SL}(2,\mathbb{R})_R$ acts only on x^-. The price for working with $x^\pm$ is that their periodicity properties are not independent. Rather one has

$$(x^+, x^-) \sim (x^+ + 2\pi, x^- + 2\pi) \sim (x^+ + 2\pi, x^- - 2\pi). \tag{3.7}$$

Finite elements of $\mathrm{SL}(2,\mathbb{R})_L$ act as Möbius transformations on $\tan \frac{x^+}{2}$ by sending $x^+ \to x^{+\prime}$ such that

$$\tan \frac{x^{+\prime}}{2} = \frac{a \tan \frac{x^+}{2} + b}{c \tan \frac{x^+}{2} + d} \tag{3.8}$$

with $ad - bc = 1$. Note that $\tan \frac{x^+}{2} = \tan \frac{x^+ + 2\pi}{2}$ despite the fact that (x^+, x^-) and $(x^+ + 2\pi, x^-)$ are distinct points, so $\tan \frac{x^+}{2}$ is not a good coordinate on the whole torus.

4 Primary scalar states

In this section we construct the highest- and lowest-weight conformal primary wave functions for a massless scalar.

Solving the massless scalar wave equation $\Box\Phi = 0$ in a conformal basis reduces to finding representations of $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ as functions on the $SL(2,\mathbb{R})$ group manifold $AdS_3/\mathbb{Z}$. On $\mathbb{K}^{2,2-}$ the equation separates as $\Phi(\tau,\psi,\phi,\rho) = \Phi_1(\tau)\Phi_3(\psi,\phi,\rho)$ and can be written

$$\frac{1}{\tau}(\partial_\tau \tau^3 \partial_\tau)\Phi_1 = K\Phi_1, \tag{4.1}$$

$$\nabla_3^2 \Phi_3 = K\Phi_3, \tag{4.2}$$

where the separation constant K can be anything at this point. The first equation (4.1) has two power law solutions which depend on K. (4.2) is the wave equation for a scalar of mass $m^2 = K$ on $AdS_3/\mathbb{Z}$. In a standard basis, $L_0 + \bar{L}_0$ generates time translations, while $L_0 - \bar{L}_0$ generates space rotations. Both must be integers, implying that L_0 and $\bar{L}_0$ are either both integers or both half-integers.[4] (4.2) can be rewritten in terms of either the $SL(2,\mathbb{R})_L$ or $SL(2,\mathbb{R})_R$ Casimirs on $AdS_3/\mathbb{Z}$

$$(4\bar{L}_0^2 - 2\bar{L}_{-1}\bar{L}_1 - 2\bar{L}_1\bar{L}_{-1})\Phi_3 = (4L_0^2 - 2L_{-1}L_1 - 2L_1L_{-1})\Phi_3 = K\Phi_3. \tag{4.3}$$

Here we consider conformal primary solutions in a basis of $(L_0, \bar{L}_0)$ eigenstates. These obey the eigenvalue condition

$$L_0\Phi_3 = h\Phi_3, \quad \bar{L}_0\Phi_3 = \bar{h}\Phi_3 \tag{4.4}$$

for integer or half-integer $(h, \bar{h})$, as well as the highest-weight condition

$$L_1\Phi_3 = \bar{L}_1\Phi_3 = 0. \tag{4.5}$$

We refer to these as "L-primary", to distinguish them from operator-type primaries discussed in the next section. Commuting L_1 and $\bar{L}_1$ to the right where they annihilate Φ_3, the wave equation reduces to (4.5) together with

$$K = 4h(h-1) = 4\bar{h}(\bar{h}-1) \tag{4.6}$$

for some integer or half-integer $(h, \bar{h})$ eigenvalues of $L_0, \bar{L}_0$. (4.1) then has two solutions

$$\Phi_1 = \tau^{-2h}, \quad \widetilde{\Phi}_1 = \tau^{2h-2}, \tag{4.7}$$

which are indirectly related by the shadow transform. Moreover the highest-weight conditions (4.5) imply

$$h = \bar{h} \tag{4.8}$$

together with

$$\partial_\rho \Phi_3 + 2h\tanh\rho\,\Phi_3 = 0. \tag{4.9}$$

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[4]In the familiar case of AdS_3, ψ is not periodically identified and $L_0 + \bar{L}_0$ is not quantized.

This is solved by

$$\Phi_3 \propto \frac{e^{2ih\psi}}{\cosh^{2h}\rho}. \tag{4.10}$$

Putting this together we have a pair of conformal primary solutions in $\mathbb{K}^{2,2-}$ for every half-integer value of h,

$$\Phi_h^{++} = \frac{e^{2ih\psi}}{\cosh^{2h}\rho}\tau^{-2h}, \qquad \widetilde{\Phi}_h^{++} = \frac{e^{2ih\psi}}{\cosh^{2h}\rho}\tau^{2h-2}. \tag{4.11}$$

We can construct descendant solutions by acting with $L_{-1}, \bar{L}_{-1}$ on Φ_h^{++}. However we still have to match this to a solution on $\mathbb{K}^{2,2+}$. For this purpose it is easiest to work in terms of the $(z, \bar{z}, w, \bar{w})$ coordinates. Then we find

$$\Phi_h^{++} = \bar{w}^{-2h}, \quad \widetilde{\Phi}_h^{++} = \bar{w}^{-2h}(w\bar{w} - z\bar{z})^{2h-1}. \tag{4.12}$$

The equation of motion

$$\Box\Phi = 4(\partial_z\partial_{\bar{z}} - \partial_w\partial_{\bar{w}})\Phi = 0 \tag{4.13}$$

has $\partial_{\bar{w}}^{2h}\delta^{(2)}(w)$ sources at $w = 0$ for positive h, which may be important or need regulation in some applications. The singularity along the light cone of the origin $z\bar{z} = w\bar{w}$ can be regulated with a $\pm i\varepsilon$ prescription, a choice of which may be necessary for example to define scattering amplitudes. Similar regulators are likely needed in solutions below but will not be analyzed herein. Near $\mathcal{I}$ at $V = \frac{\pi}{2}$ one finds

$$\Phi_h^{++} \to (\pi - 2V)^{2h}e^{2ih\psi}, \quad \widetilde{\Phi}_h^{++} \to (\pi - 2V)e^{2ih\psi}(2\tan U)^{2h-1}. \tag{4.14}$$

We could also consider lowest-weight solutions obeying

$$L_{-1}\Phi = \bar{L}_{-1}\Phi = 0. \tag{4.15}$$

Inspection of (3.1) immediately reveals that the complex conjugates

$$\Phi_h^{--} = (\Phi_h^{++})^* = w^{-2h}, \quad \widetilde{\Phi}_h^{--} = (\widetilde{\Phi}_h^{++})^* = w^{-2h}(w\bar{w} - z\bar{z})^{2h-1} \tag{4.16}$$

obey (4.15) and

$$L_0\Phi_h^{--} = \bar{L}_0\Phi_h^{--} = -h\Phi_h^{--}, \quad L_0\widetilde{\Phi}_h^{--} = \bar{L}_0\widetilde{\Phi}_h^{--} = -h\widetilde{\Phi}_h^{--}. \tag{4.17}$$

There are further mixed solutions obeying

$$L_1\Phi = \bar{L}_{-1}\Phi = 0. \tag{4.18}$$

Again from (3.1) we see that under the exchange $z \leftrightarrow w$, we have

$$L_n \leftrightarrow L_n, \quad \bar{L}_n \leftrightarrow -\bar{L}_{-n}. \tag{4.19}$$

It follows that

$$\Phi_h^{+-} = \bar{z}^{-2h}, \quad \widetilde{\Phi}_h^{+-} = \bar{z}^{-2h}(w\bar{w} - z\bar{z})^{2h-1} \tag{4.20}$$

obey (4.18) and

$$L_0\Phi_h^{+-} = -\bar{L}_0\Phi_h^{+-} = h\Phi_h^{+-}, \quad L_0\widetilde{\Phi}_h^{+-} = -\bar{L}_0\widetilde{\Phi}_h^{+-} = h\widetilde{\Phi}_h^{+-}. \qquad (4.21)$$

Finally the other class of mixed solutions

$$L_{-1}\Phi = \bar{L}_1\Phi = 0 \qquad (4.22)$$

is given by

$$\Phi_h^{-+} = z^{-2h}, \quad \widetilde{\Phi}_h^{-+} = z^{-2h}(w\bar{w} - z\bar{z})^{2h-1}. \qquad (4.23)$$

The full $\mathrm{SL}(2,\mathbb{R})_L\times\mathrm{SL}(2,\mathbb{R})_R$ multiplets can for all cases be obtained by suitable actions of $L_{\pm 1}, \bar{L}_{\pm 1}$.

5 Primary scalar operators

In the previous section we constructed wave functions whose convolutions with bulk field operators create states in the $(1,1)$ celestial CFT on the Lorentzian torus. These are L-primary with respect to the standard $\mathrm{SL}(2,\mathbb{R})_L\times\mathrm{SL}(2,\mathbb{R})_R$ action, diagonalizing both time translations and space rotations.

The $(1,1)$ CFT also contains local operators acting at points on the torus

$$\mathcal{O}_{h,\bar{h}}(\hat{x}^+, \hat{x}^-), \quad \hat{x}^\pm = \hat{\psi} \pm \hat{\phi}, \qquad (5.1)$$

which are H-primary rather than L-primary [25]. They are annihilated by the raising operators in the basis that diagonalizes boosts towards $(\hat{x}^+, \hat{x}^-)$. This basis is

$$H_0^{\hat{x}} = \frac{1}{2}\left(e^{i\hat{x}^+}L_1 - e^{-i\hat{x}^+}L_{-1}\right), \quad H_{\pm 1}^{\hat{x}} = iL_0 \mp \frac{i}{2}\left(e^{i\hat{x}^+}L_1 + e^{-i\hat{x}^+}L_{-1}\right),$$
$$\bar{H}_0^{\hat{x}} = \frac{1}{2}\left(e^{i\hat{x}^-}\bar{L}_1 - e^{-i\hat{x}^-}\bar{L}_{-1}\right), \quad \bar{H}_{\pm 1}^{\hat{x}} = i\bar{L}_0 \mp \frac{i}{2}\left(e^{i\hat{x}^-}\bar{L}_1 + e^{-i\hat{x}^-}\bar{L}_{-1}\right). \qquad (5.2)$$

These obey the commutation relations

$$[H_n, H_m] = (n-m)H_{n+m}, \quad [\bar{H}_n, \bar{H}_m] = (n-m)\bar{H}_{n+m}. \qquad (5.3)$$

Analogous primary operators were constructed in Minkowski space, where they live on the sphere rather than the torus, as Mellin transforms of momentum space field operators in [11]. The construction is easily continued to Klein space. Let us write in $(z, \bar{z}, w, \bar{w})$ coordinates

$$p = \omega\hat{p}(\hat{x}) = \omega(e^{i\hat{\phi}}, e^{-i\hat{\phi}}, e^{i\hat{\psi}}, e^{-i\hat{\psi}}), \qquad (5.4)$$

$$X = (re^{i\phi}, re^{-i\phi}, qe^{i\psi}, qe^{-i\psi}), \qquad (5.5)$$

so that $p^2 = 0$ and

$$\hat{p}(\hat{x})\cdot X = r\cos(\hat{\phi}-\phi) - q\cos(\hat{\psi}-\psi)$$
$$= (r-q)\cos\frac{\hat{x}^+ - x^+}{2}\cos\frac{\hat{x}^- - x^-}{2} + (r+q)\sin\frac{\hat{x}^+ - x^+}{2}\sin\frac{\hat{x}^- - x^-}{2}, \qquad (5.6)$$

where $\hat{x} = (\hat{x}^+, \hat{x}^-)$. As usual the Mellin transform gives[5]

$$\varphi_h(X; \hat{x}) = \int_0^\infty d\omega \omega^{2h-1} e^{i\omega \hat{p} \cdot X} = \frac{e^{-\pi i h} \Gamma(2h)}{(\hat{p} \cdot X)^{2h}}. \tag{5.7}$$

These obey, by construction, the wave equation as well as

$$H_1^{\hat{x}} \varphi_h(X; \hat{x}) = \bar{H}_1^{\hat{x}} \varphi_h(X; \hat{x}) = 0, \tag{5.8}$$

$$H_0^{\hat{x}} \varphi_h(X; \hat{x}) = \bar{H}_0^{\hat{x}} \varphi_h(X; \hat{x}) = h \varphi_h(X; \hat{x}), \tag{5.9}$$

$$H_{-1}^{\hat{x}} \varphi_h(X; \hat{x}) = -2 \hat{\partial}_+ \varphi_h(X; \hat{x}), \tag{5.10}$$

$$\bar{H}_{-1}^{\hat{x}} \varphi_h(X; \hat{x}) = -2 \hat{\partial}_- \varphi_h(X; \hat{x}). \tag{5.11}$$

Scattering amplitudes of particles with these wave functions are Mellin transforms of plane wave amplitudes, and are identified with conformal primary correlation functions on the celestial torus. These wave functions have branch cuts for generic h and are periodic in both the time and space directions ψ and ϕ. One may also consider the shadows of these solutions

$$\widetilde{\varphi}_h(X; \hat{x}) = \varphi_h(X; \hat{x})(X^2)^{2h-1}, \tag{5.12}$$

which obey (5.8)–(5.11).

We can put our $(1,1)$ CFT on the Lorentzian cylinder just as well as the Lorentzian torus, and it is instructive to see how they are related. In a conventional $(1,1)$ CFT on the cylinder there is a canonical map from primary operators at a point to operator modes on the circle, given by an integral over a causal diamond

$$O_{m,n} = \int_0^{2\pi} d\hat{x}^+ \int_0^{2\pi} d\hat{x}^- e^{-im\hat{x}^+ - in\hat{x}^-} \mathcal{O}_{h,\bar{h}}(\hat{x}^+, \hat{x}^-), \tag{5.13}$$

where $h - \bar{h} \in \mathbb{Z}$. Spatial periodicity requires $m - n \in \mathbb{Z}$. In order that our modes create the primary and descendant states associated with $\mathcal{O}_{h,\bar{h}}$, we need $m, n \in \mathbb{Z} - h$. When we instead consider this mode expansion on the torus, the timelike periodicity further requires $m + n \in \mathbb{Z}$, which means that the only consistent operators of the type (5.13) arise from H-primaries with $h + \bar{h} \in \mathbb{Z}$. Accordingly we henceforth restrict to $h \pm \bar{h} \in \mathbb{Z}$.

The analog of this map (on the torus now) at the level of the wave functions (5.7) is

$$\Phi_{m,n}(X) = \int_0^{2\pi} d\hat{x}^+ \int_0^{2\pi} d\hat{x}^- e^{-im\hat{x}^+ - in\hat{x}^-} \varphi_h(X; \hat{x}^+, \hat{x}^-). \tag{5.14}$$

Using

$$L_0 = -\frac{i}{2}(H_1^{\hat{x}} + H_{-1}^{\hat{x}}), \quad L_{\pm 1} = \frac{e^{\mp i \hat{x}^+}}{2}(i H_1^{\hat{x}} - i H_{-1}^{\hat{x}} \pm 2 H_0^{\hat{x}}), \tag{5.15}$$

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[5]In Minkowski space the $\pm i\varepsilon$ prescription at $\hat{p} \cdot X = 0$ distinguishes ingoing and outgoing solutions. In Klein space changing the sign in front of ε is equivalent to changing the sign of $\hat{p}$ and so does not give a new solution, in accord with the fact that $\mathcal{I}$ has only one connected component. In the case when $2h - 1$ is a negative integer — which is related to soft currents in the spin-one case — the wave functions should be normalized so as to cancel the Γ-function singularities [26–29].

together with (5.8), and integrating by parts with respect to $\hat{x}^+$ one finds the standard mode relations

$$L_1 \Phi_{m,n} = (h - 1 - m)\Phi_{m+1,n}, \tag{5.16}$$

$$L_0 \Phi_{m,n} = -m\Phi_{m,n}, \tag{5.17}$$

$$L_{-1} \Phi_{m,n} = (1 - h - m)\Phi_{m-1,n}. \tag{5.18}$$

Any solutions Φ constructed from linear combinations of $\varphi_h(X; \hat{x})$ obey

$$(L_0(L_0 - 1) - L_{-1}L_1)\Phi = (L_0(L_0 + 1) - L_1 L_{-1})\Phi = h(h - 1)\Phi. \tag{5.19}$$

Hence highest-weight solutions obeying $L_1\Phi = 0$ have $m = -h$ or $m = h - 1$, while lowest-weight solutions obeying $L_{-1}\Phi = 0$ have $m = h$ or $m = 1 - h$. Similar relations hold for the $\bar{L}_n$. The highest-weight solution with $m = n = -h$ is

$$\Phi_{-h,-h} = \int_0^{2\pi} d\hat{x}^+ \int_0^{2\pi} d\hat{x}^- e^{ih(\hat{x}^+ + \hat{x}^-)}\varphi_h(X; \hat{x}^+, \hat{x}^-) \tag{5.20}$$

$$= 2^{2+2h}\pi^2 e^{i\pi h}\Gamma(2h)\bar{w}^{-2h} \propto \Phi_h^{++}, \tag{5.21}$$

in agreement with (4.12). The integral here is performed in appendix A. The poles at negative half-integer h are inherited from the normalization of the wave functions (5.7) and can be absorbed by a redefinition of the wave functions resulting in finite amplitudes [27–29].

For $h > 0$, taking descendants of the primary generates the standard infinite-dimensional unitary $\mathrm{SL}(2, \mathbb{R})_L \times \mathrm{SL}(2, \mathbb{R})_R$ representation. For $h < 0$, after taking $2h$ descendants (on either left or right) we reach $m = h$ and the representation terminates. This is a non-unitary finite-dimensional representation.

Similarly for $m = n = h$ we have

$$\Phi_{h,h} = \int_0^{2\pi} d\hat{x}^+ \int_0^{2\pi} d\hat{x}^- e^{-ih(\hat{x}^+ + \hat{x}^-)}\varphi_h(X; \hat{x}^+, \hat{x}^-) \tag{5.22}$$

$$= 2^{2+2h}\pi^2 e^{i\pi h}\Gamma(2h)w^{-2h} \propto \Phi_h^{--}. \tag{5.23}$$

This is a lowest-weight solution. The representations are filled out by acting with powers of L_1, $\bar{L}_1$. Mixed primary solutions can also be obtained by taking (m, n) as $(h, -h)$ or $(-h, h)$.

6 Comments on scattering

Celestial $\mathcal{S}$-vector elements of particles with Klein space H-primary wave functions are given by Mellin transforms of momentum space $\mathcal{S}$-vector elements. The kth external particle is labeled by 3 continuous parameters: h_k, x_k^+, x_k^-. They take the form of CFT correlation functions on a Lorentzian torus.

Celestial $\mathcal{S}$-vector elements of particles with L-primary wave functions, and their descendants, can also be computed from momentum space $\mathcal{S}$-vector elements, with the additional weighted integral over the celestial torus given in (5.13). The kth external particle

is labeled by 3 *discrete* parameters: h_k and the left and right levels of the descendant. It is interesting that the *L*-primary scattering problem has a discrete character. This resonates with the results of [17] where it was shown, for the Minkowskian four-particle amplitude, that the Wilsonian coefficients are encoded in poles at discrete integral conformal weights. These are likely probed by *L*-primary scattering amplitudes.

We defer a more detailed analysis of properties of the *L*-primary solutions to future work.

Acknowledgments

A Mapping *H*-primaries to *L*-primaries

In this appendix we evaluate the integral (5.20) mapping conformal *H*-primary solutions to *L*-primary ones. We start with

$$I = \int_0^{2\pi} \int_0^{2\pi} d\hat{x}^+ d\hat{x}^- e^{-im\hat{x}^+} e^{-in\hat{x}^-} \frac{e^{-i\pi h}\Gamma(2h)}{(\hat{p}\cdot X)^{2h}}. \tag{A.1}$$

This integral consists of points inside the causal diamond, covering half of the celestial torus. It can be related to an integral over the full torus by noticing that the integral over the other half is obtained from the integral over the diamond by shifting $\hat{x}^+ \to \hat{x}^+ + 2\pi$ for fixed $\hat{x}^-$. This transformation amounts to taking $\hat{p}$ to its antipodal point $-\hat{p}$. Under this transformation,

$$e^{-im\hat{x}^+ - in\hat{x}^-}\Phi_h(\hat{x}^+, \hat{x}^-) \to e^{-2\pi i h}e^{-im\hat{x}^+ - in\hat{x}^-}e^{\pm 2\pi i h}\Phi_h(\hat{x}^+, \hat{x}^-). \tag{A.2}$$

For $h \in \frac{1}{2}\mathbb{Z}$ the phases on the r.h.s. cancel in which case the integrals over the diamond and its complement are equal. The integral over the causal diamond can then be replaced by half the integral over the torus. The change of variables to $\hat{\psi}, \hat{\phi}$ then leads to

$$I = \int_{T^2} d\hat{\psi}d\hat{\phi}\, e^{-i(m+n)\hat{\psi}}e^{-i(m-n)\hat{\phi}} \frac{e^{-i\pi h}\Gamma(2h)}{\left[r\cos(\hat{\phi}-\phi) - q\cos(\hat{\psi}-\psi)\right]^{2h}}. \tag{A.3}$$

Setting $m = n$ and

$$w = e^{i(\psi-\hat{\psi})}, \quad z = e^{i(\hat{\phi}-\phi)}, \tag{A.4}$$

we find

$$I = -2^{2h} e^{-2im\psi} \oint \frac{dw}{w} \oint \frac{dz}{z} w^{2m} \frac{e^{-i\pi h}\Gamma(2h)}{[r(z+z^{-1}) - q(w+w^{-1})]^{2h}}, \tag{A.5}$$

where the contours are the unit circles $|z| = |w| = 1$. For fixed w and $|r(z + z^{-1})| < |q(w + w^{-1})|$ we can expand the denominator[6]

$$I = -2^{2h} e^{-2im\psi} e^{-i\pi h}\Gamma(2h) \oint \frac{dw}{w} \oint \frac{dz}{z} w^{2m} \sum_{k=0}^{\infty} (-q)^{-2h}(w+w^{-1})^{-2h}\left(-\frac{r}{q}\frac{z+z^{-1}}{w+w^{-1}}\right)^k \binom{-2h}{k} \tag{A.6}$$

and first evaluate the z integral,

$$\oint \frac{dz}{z}(z + z^{-1})^k = 2\pi i \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k/2+1)^2}, & k \in 2\mathbb{Z}_+, \\ 0, & \text{else.} \end{cases} \tag{A.7}$$

Then, upon a redefinition of the summation variable,

$$I = -2^{2h} e^{-2im\psi} e^{-i\pi h}\Gamma(2h) 2\pi i \sum_{k=0}^{\infty} \oint \frac{dw}{w} w^{2m}(w + w^{-1})^{-2h-2k}(-q)^{-2h}\left(-\frac{r}{q}\right)^{2k}$$
$$\times \frac{\Gamma(-2h+1)}{\Gamma(1+k)^2\Gamma(-2h-2k+1)}. \tag{A.8}$$

We now easily see that the remaining integral above simplifies and gives us the solutions in (4.12). First set $m = -h$. The integral over w reduces to

$$\oint \frac{dw}{w}(w^2 + 1)^{-2h-2k} w^{2k}. \tag{A.9}$$

For $k > 0$, there are no poles inside the contour $|z| = 1$, so this integral vanishes.[7] The only term that survives in (A.8) is the $k = 0$ one which gives

$$I_{m=-h} = 4\pi^2 2^{2h} e^{-i\pi h}\Gamma(2h) e^{2ih\psi}(-q)^{-2h} = 4\pi^2 2^{2h} e^{i\pi h}\Gamma(2h)\bar{w}^{-2h}. \tag{A.10}$$

The answer is well-defined for $h \in \frac{1}{2}\mathbb{Z}_+$ and diverges for $h \in \frac{1}{2}\mathbb{Z}_-$. The latter divergences follow from the normalization of the conformal primary wave functions (5.7) and were related to conformally soft poles in [27–29].

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[6]It can be shown that the analogous expansion for $|r(z + z^{-1})| > |q(w + w^{-1})|$ gives the same result for $m = -h$.

[7]The branch cuts at $\pm i$ can be pushed away from the contour by an $i\varepsilon$ prescription.

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