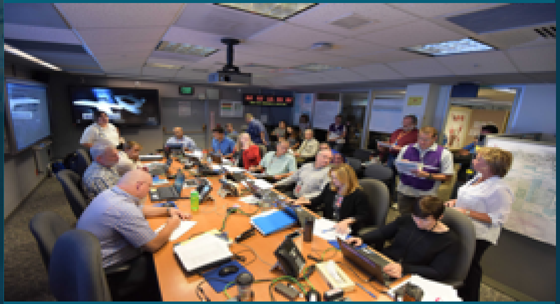
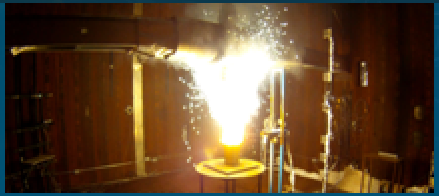
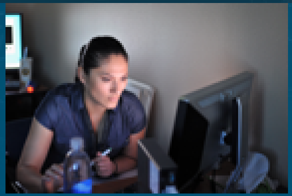
Sandia National Laboratories

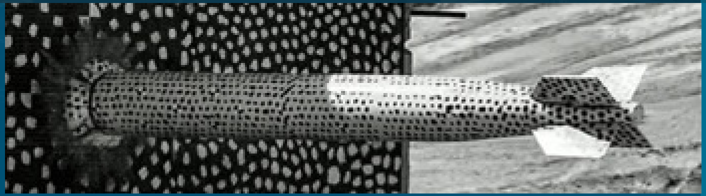
SAND2021-2253C

Implementing Calving Laws in Ice-Sheet Models using Level Set Methods









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K. Chad Sockwell*, Mauro Perego*

* Center for Computing Research, Sandia National Laboratories

Acknowledgement to Doug Brinkerhoff, University of Montana

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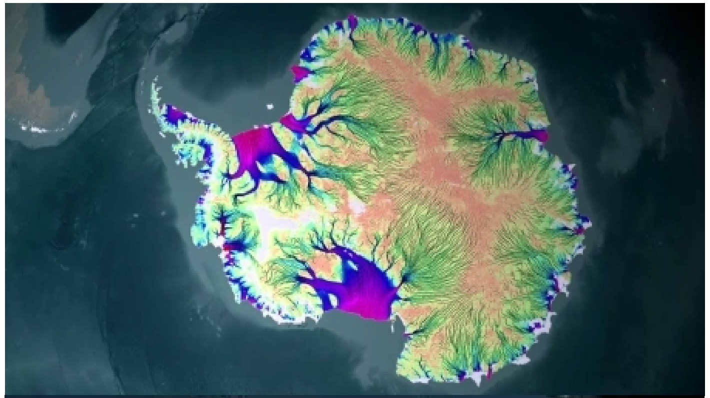
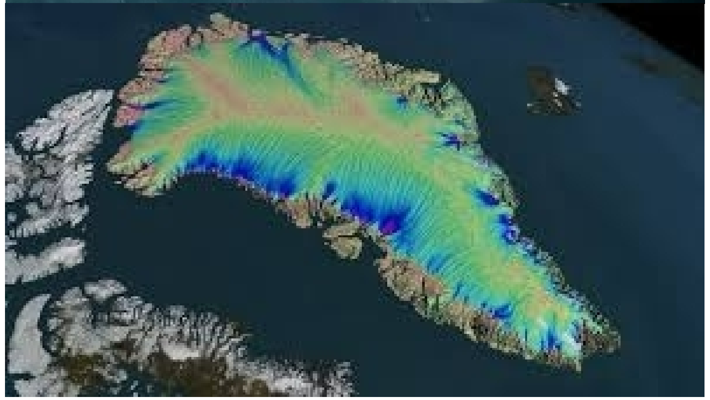
Ice Sheet Modeling and Calving

Ice Sheet Modeling

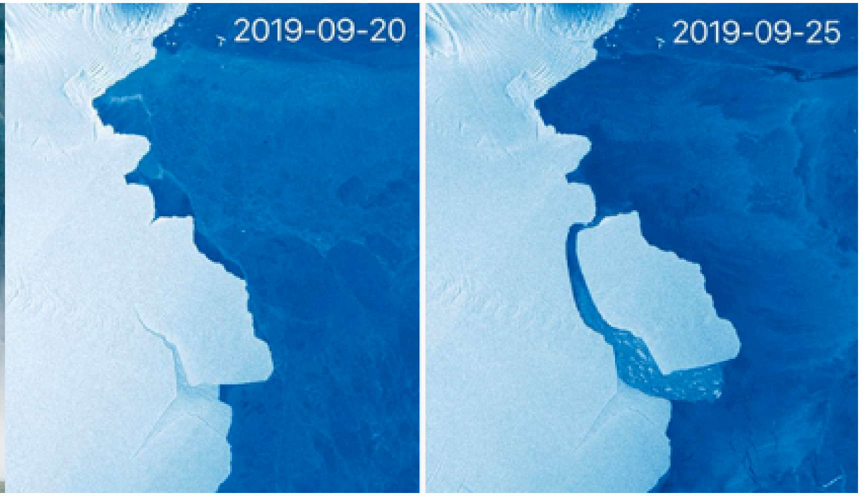
- The ice in Greenland and Antarctica store most of the Fresh Water on Earth
- Modeling ice sheet dynamics is critical to predict sea level rise





Calving

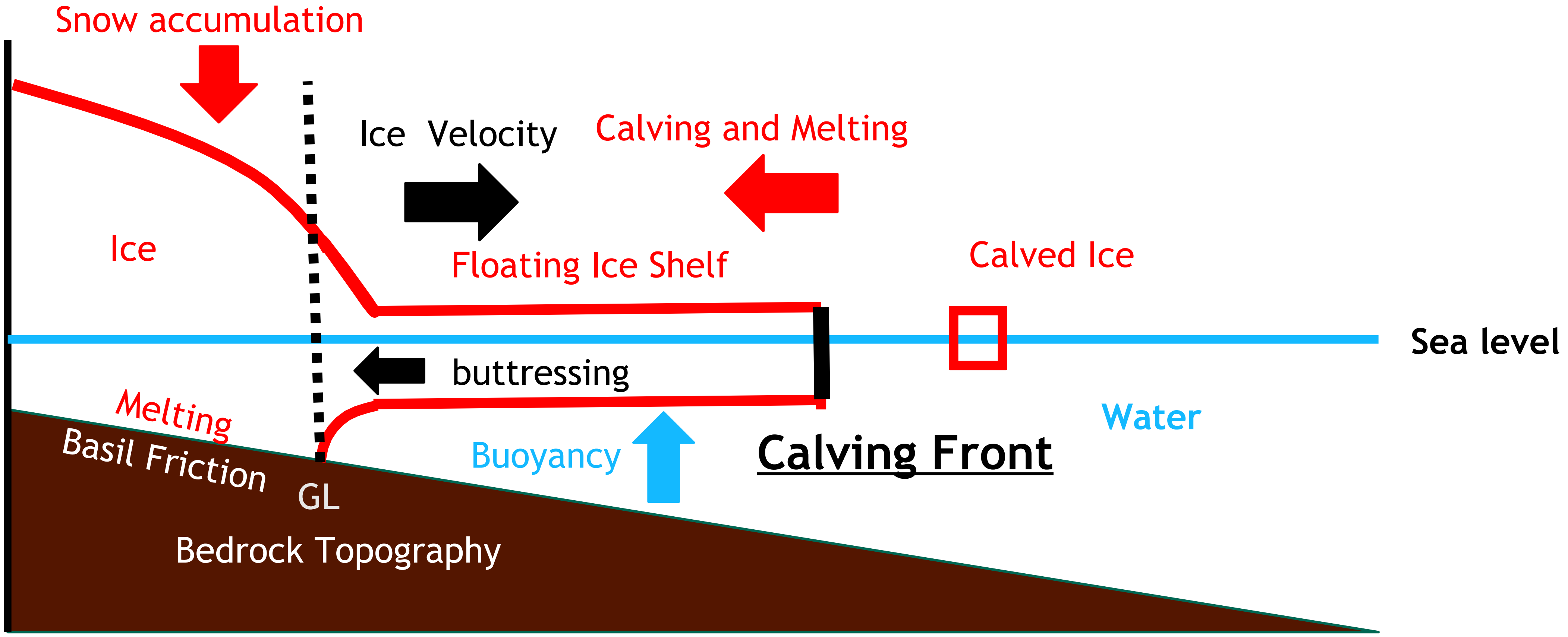
- Calving is the process where ice blocks break off glaciers or ice sheets
- Physical considerations:
  - Individual events on a much faster time scales and smaller spatial scales than ice sheet dynamics
  - “[Calving] accounts for about half of ice discharge from the Greenland and Antarctic ice sheets” (Bondzio 2016)
- Modeling considerations
  - Critical to mass consistency – Mass conserving ESM
  - Accurate boundary condition location for ice sheet
  - Accurate modeling of Buttressing effect



General Problem

- Ice flows like a very viscous thin fluid driven by gravity:
- Sources: snow accumulation
- Sinks: melting, **calving**

Typical Ice sheet geometry spans 1000 of km, while being at most ~ 4 km in height.
Lets zoom into ocean boundary
Modeling the calving front location and mass flux through the front are the central interest of this talk



Modeling Ice Flow

Momentum Balance: Stokes

$$\begin{cases} -\nabla\cdot\boldsymbol{\sigma} = \rho g \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

Where

Stress $\boldsymbol{\sigma} = 2\mu\mathbf{D} - pI$

Viscosity $\mu = \frac{1}{2}\alpha(T)|\mathbf{D}(\mathbf{u})|^{\frac{1}{n}-1}, \quad n = 3 \qquad \mathbf{D_{ij}}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_i}\right)$

Mass Balance

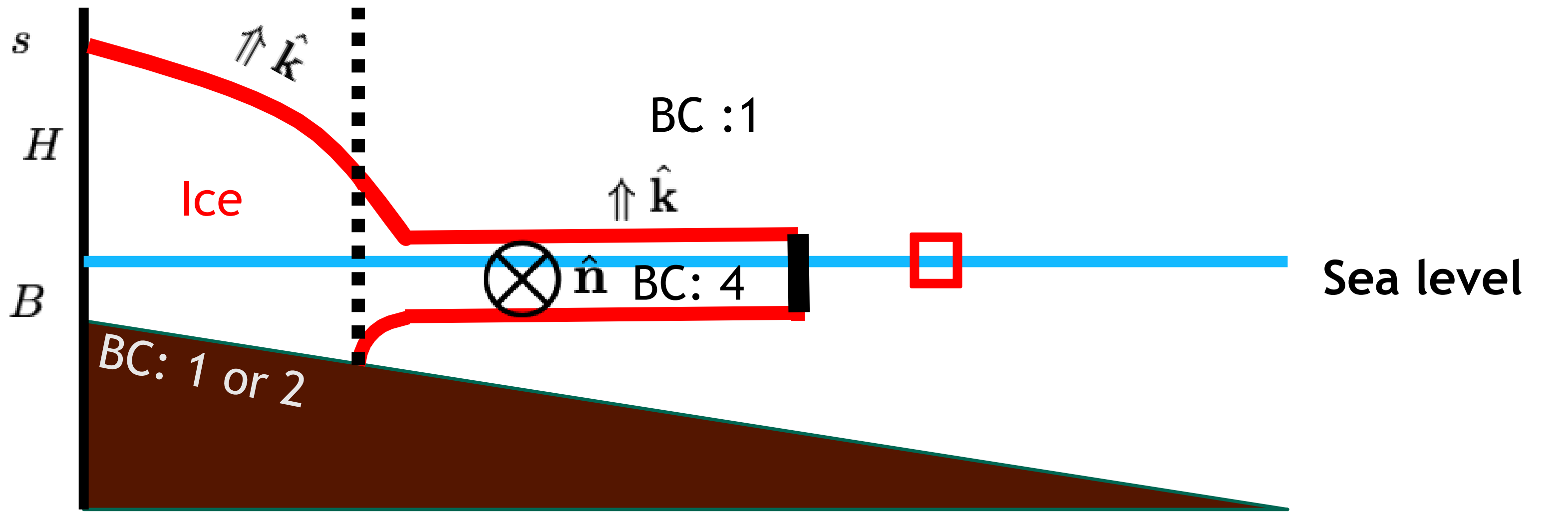
$$\frac{\partial H}{\partial t} = -\nabla\cdot(\bar{\mathbf{u}}H) + \dot{a}$$

$$\bar{\mathbf{u}} = \frac{1}{H}\int_z \mathbf{u}\,\mathrm{d}z$$

Vertical Boundary Conditions:

1. $\boldsymbol{\sigma}\cdot\hat{\mathbf{k}} = 0$ — Top
2. $u_1 = u_2 = 0$ no slip — Bedrock
3. $\boldsymbol{\sigma}\cdot\hat{\mathbf{k}} + \beta u_i^m = 0$ sliding — Bedrock
4. $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}} - \rho g(s-z)\hat{\mathbf{n}} = \rho_w g\max(z,0)\hat{\mathbf{n}}$ floating

Lateral ice-water boundary of floating ice



Lateral Boundary conditions on velocity

No Penetration $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$

No Slip $\mathbf{u}\perp\hat{\mathbf{n}} = 0$

Free Flow

Free Slip

Shallow Shelf Approximation

- Equations can be reduced to first-order-accurate Stokes approximation (also referred to as the Blatter–Pattyn approximation) due to scale L>>Z. Asymptotic analysis with Z/L.
- The equations can be further reduced to the Shallow Ice Approximation (SIA) [Deformation] or the **Shallow Shelf Approximation (SSA) [Sliding]**
- Vertically integrating Blatter–Pattyn approximation gives SSA: $\frac{\partial \bar{\boldsymbol{\sigma}}_{i1}}{\partial x_1} + \frac{\partial \bar{\boldsymbol{\sigma}}_{i2}}{\partial x_2} + \tau_{b,i} = \rho g \frac{\partial s}{\partial x_i}$

$$\{x_1, x_2\} \in \Omega \subset \mathbb{R}^2 \qquad \mathcal{T} = [t_o, t]$$

$$\bar{\mathbf{u}} : \Omega \to \mathbb{R}^2 \qquad H : \Omega \times \mathcal{T} \to \mathbb{R}$$

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\bar{\mathbf{u}} H) + \dot{a}$$

$$s = H + b$$

$$\frac{\partial}{\partial x_1}\left[2\bar{\mu}H\left(2\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2}\right)\right] + \frac{\partial}{\partial x_2}\left[\bar{\mu}H\left(\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1}\right)\right] + \tau_{b,1} = \rho g H \frac{\partial s}{\partial x_1}$$

$$\frac{\partial}{\partial x_1}\left[\bar{\mu}H\left(\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1}\right)\right] + \frac{\partial}{\partial x_2}\left[2\bar{\mu}H\left(2\frac{\partial \bar{u}_1}{\partial x_1} + \frac{\partial \bar{u}_2}{\partial x_2}\right)\right] + \tau_{b,2} = \rho g H \frac{\partial s}{\partial x_2}$$

$$\begin{cases} s = H + B \;\text{ and }\; \tau_{b,i} = C|u_i|^{m-1}u_i & \text{if grounded} \\ s = \rho\left(1 - \frac{\rho}{\rho_w}\right)H \;\text{ and }\; \tau_{b,i} = 0 & \text{if floating} \end{cases}$$

$$\bar{\boldsymbol{\sigma}} \cdot \hat{\mathbf{n}} = \frac{1}{2}\rho\left(1 - \frac{\rho}{\rho_w}\right) g H^2 \quad \text{on } \partial\Omega \quad \text{if floating}$$

Lateral BC on stress

Lateral Boundary conditions on velocity

No Penetration $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$
No Slip $\mathbf{u} \perp \hat{\mathbf{n}} = 0$
Free Flow
Free Slip

Tracking the Calving Front

Why Track the Calving Front?
- Precise boundary condition
- Mass consistency
- Coupling to ESM
- Increased accuracy

What is currently done in MALI
- Masking cells with minimum thickness -> adhoc, mass conservation
- Parameterizations
- Methods do not play well with optimization

What is desired
- Implicit time-stepping and nonlinear solves
- Optimization – Monolithic system
  - Initialization and Calibration
- No remeshing
- Mass consistency
- Provable accuracy
- Changes in topology
- Load Balancing
- Long time accuracy

Want robust method that can be used within optimization algorithms that track calving front only
- Multiphase flow problems: Explicit, fast velocities, fast changing topology
- Topology optimization: nonlinear, within an optimization algorithm, well studied
- FSI: outlook to coupling with ocean and sea-ice

Other Works
- Bondzio et al. 2016- decoupled ice-sheet 2d level-set method to track calving Front: Extensions, No Reinit, unstructured grid
- Hossain et al. 2020- decoupled ice-sheet 3d level-set method to track ice surface: FMM, extensions, structured grid

The Level-Set Method

Interface Capturing Core Idea

- Function or field of values is advected in some manner
- Only values that identify interface are physical
  - Field over domain can identify changes in topology
- First Choice: Heaviside Function-> All the difficulties of shocks
  - Dispersion and dissipation

The Level-Set function is described by

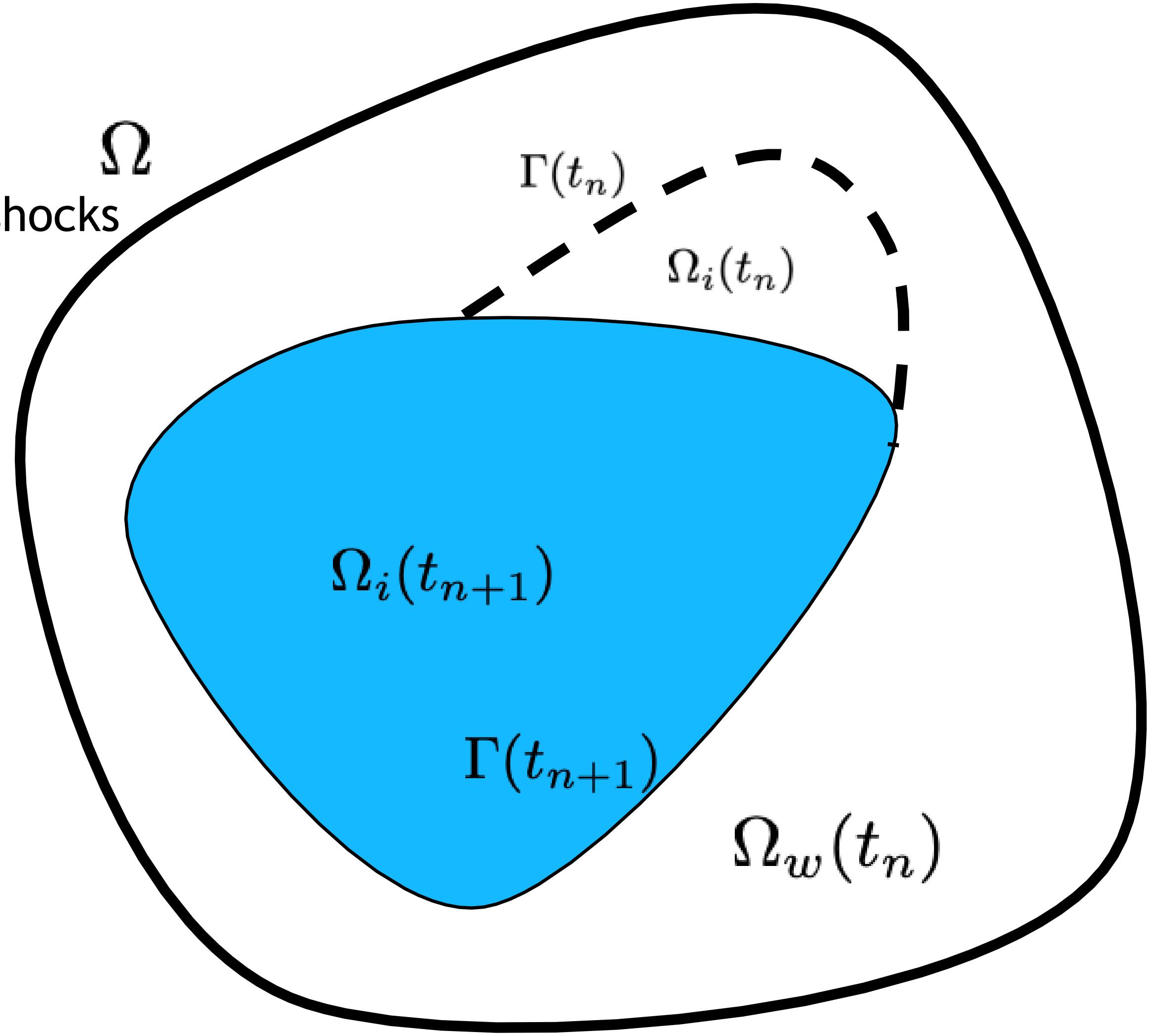
$$\begin{cases} \phi(x,t) = 0 & \forall x \in \Gamma(t) \\ \phi(x,t) < 0 & \forall x \in \Omega_i(t) \\ \phi(x,t) > 0 & \forall x \in \Omega_w(t) \end{cases}$$

The Interface is defined implicit by

$$\{x : \phi(x,t) = 0\}$$

The Level-set Equation

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0$$



Level-Set Function

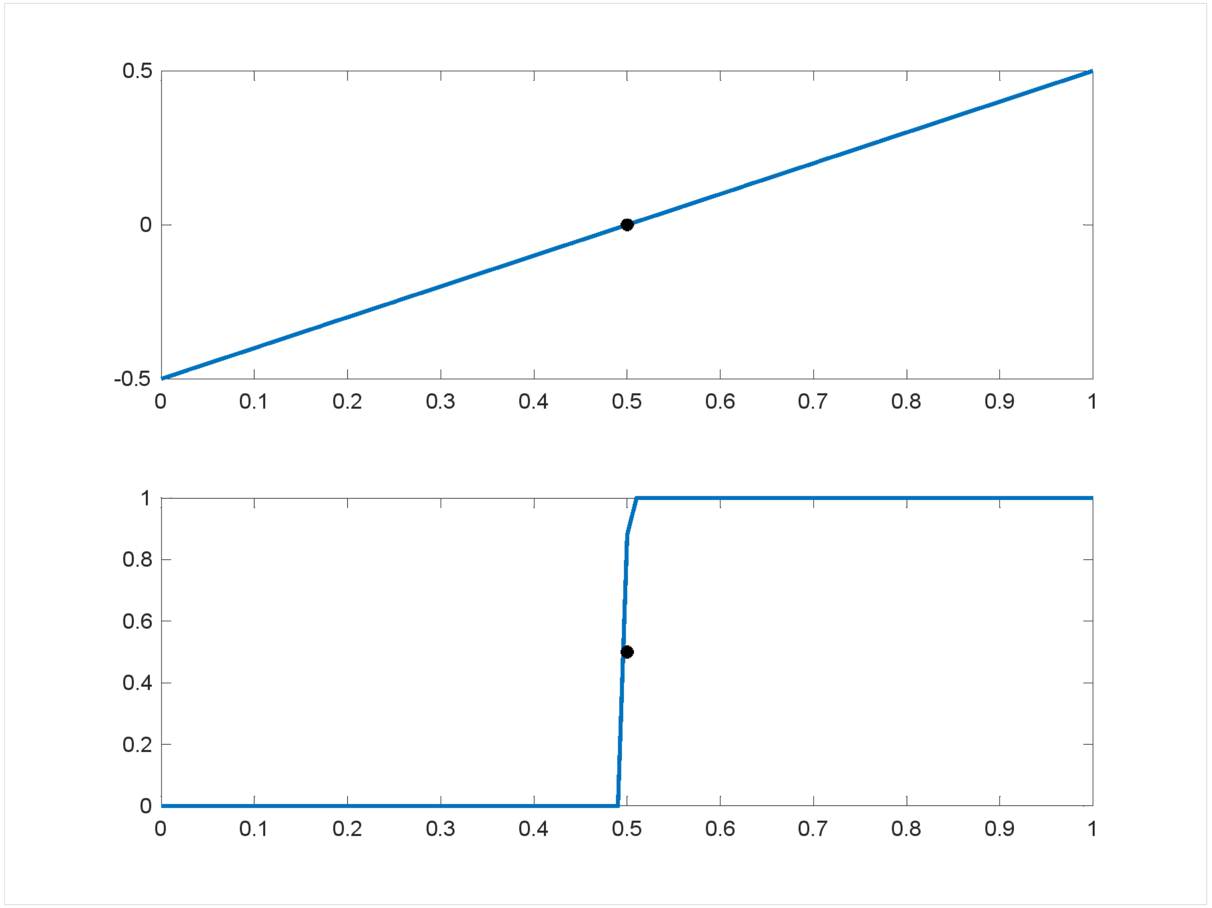
Function Choices: Initial Condition for Advection scheme

Step
- **Pros**
  - Precise interface location
- **Cons**
  - Advecting a shock

Signed Distance Function
- **Pros**
  - |Gradients| = 1 a.e.
  - Simplified Model
- **Cons**
  - Mass sensitive to smoothing

Tanh function
- **Pros**
  - Volume of function is mass of level set
  - Less sensitive to smoothing
- **Cons**
  - Steep Gradients

Choices for the Level-Set function

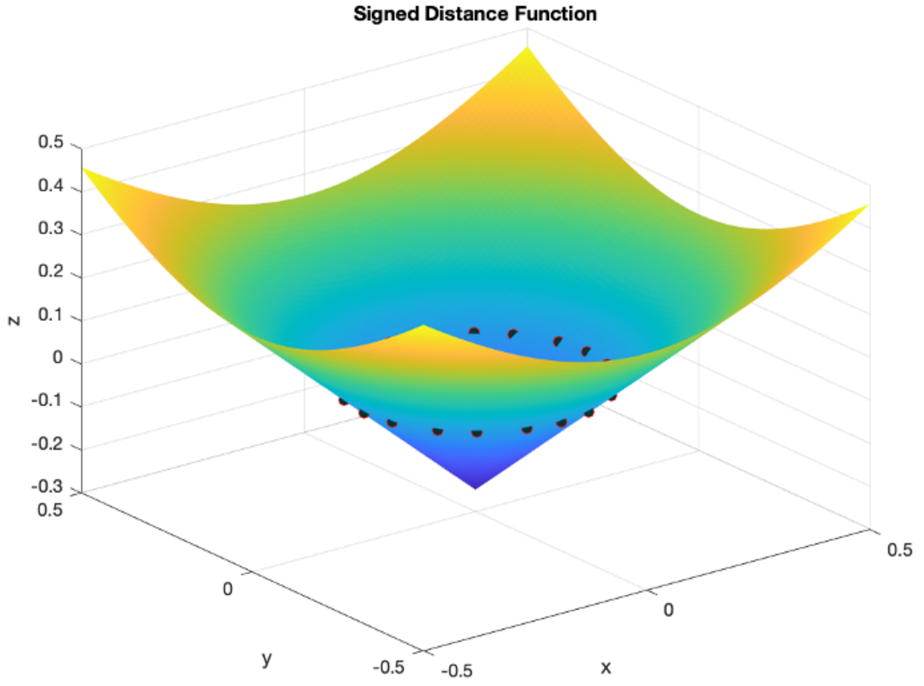


Signed Distance Function

- Geometrically, interface is given intersection of signed distance (SD) function by plane at z=0 (conic section)
- What happens to SD under diffusion of dispersion? The volume grows
- In General: Bunching or Dispersion lead to integration issues or spurious changes in topology

$$\phi(x,t) = d(x, \Gamma(t))$$

$$A_{\text{in}} = \int_{\Omega} H_{\text{step}}(\phi)\, \mathrm{d}A$$

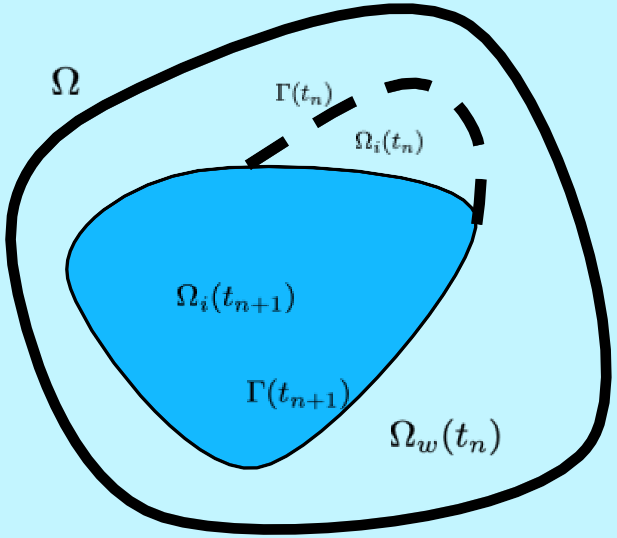


SD can lose defining property: $|\nabla\phi| = 1$
requiring expensive re-initilation
We want to AVIOD

Hamilton-Jacobi Equation and Extension Velocities

Defining property of SD function: $|\nabla\phi| = 1 \text{ a.e.}$

Normal function: On interface gives normal interface $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|}$



$$\frac{\partial\phi}{\partial t} + \mathbf{u}\cdot\nabla\phi = 0 \qquad \frac{\partial\phi}{\partial t} = -\mathbf{u}\cdot\mathbf{n}|\nabla\phi| = -F|\nabla\phi|$$

Hamilton-Jacobi

SD representation exposes scalar interface speed F that is only physical at $\{x : \phi(x,t) = 0\}$

Physical ice located only in $\Omega_i(t)$

Solve auxiliary equation for **Extension Velocity** that is equal to interface velocity but maintain SD function

$$\begin{cases} (S(\phi)\nabla\phi)\cdot\nabla F = 0 & \text{in } \Omega \\ F = \mathbf{u}\cdot\mathbf{n}(\phi) & \text{on } \Gamma(t) \end{cases}$$

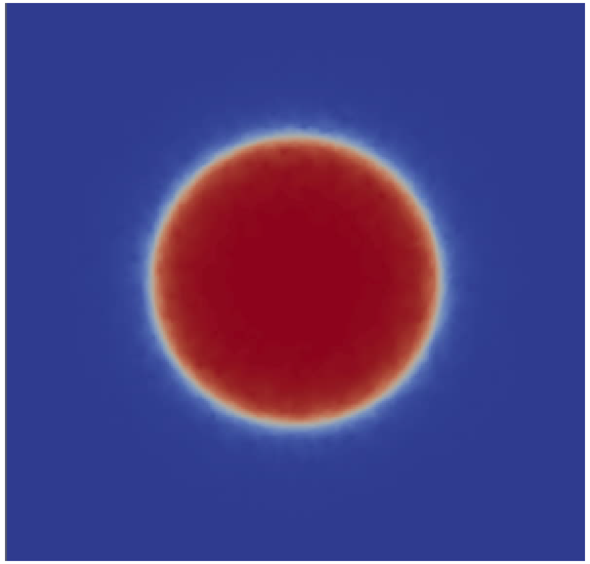
$$S(\phi) = \frac{\phi}{\sqrt{\phi^2+\epsilon}}$$

Is required to ensure inflow condition on each side of interface

If decoupled, boundary condition is Dirichlet

If coupled must be enforced weakly- We want robust monolithic method, **avoid reinitialization if we can**

Velocity Extensions and Signed Distance Functions



12000 a oscillating circle (10 periods)

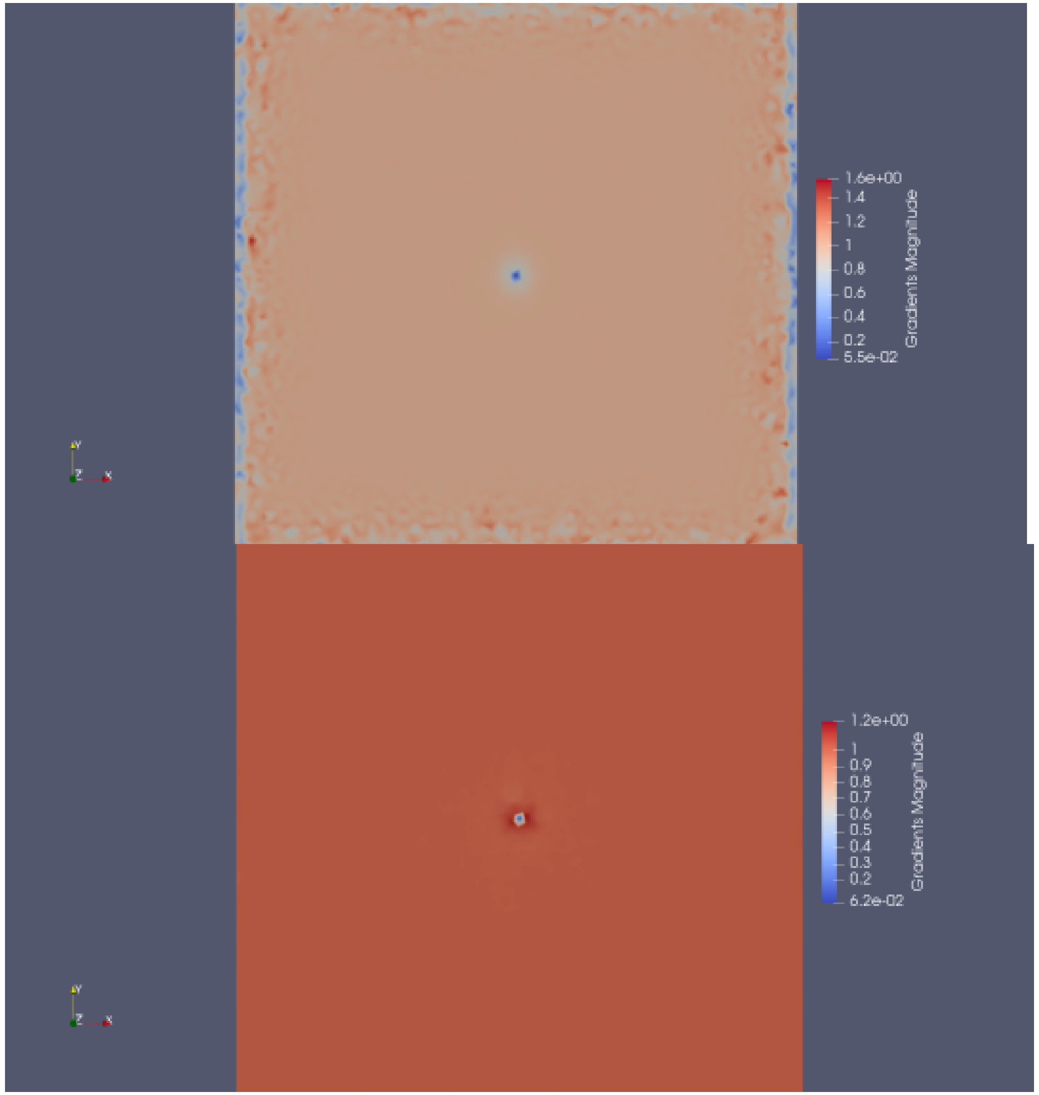
$$\Omega = [0, 1e6] \times [0, 1e6]\ [\mathrm{m}]$$

$$\vec{u} = (\omega B cos(\omega t), \omega B cos(\omega t))$$

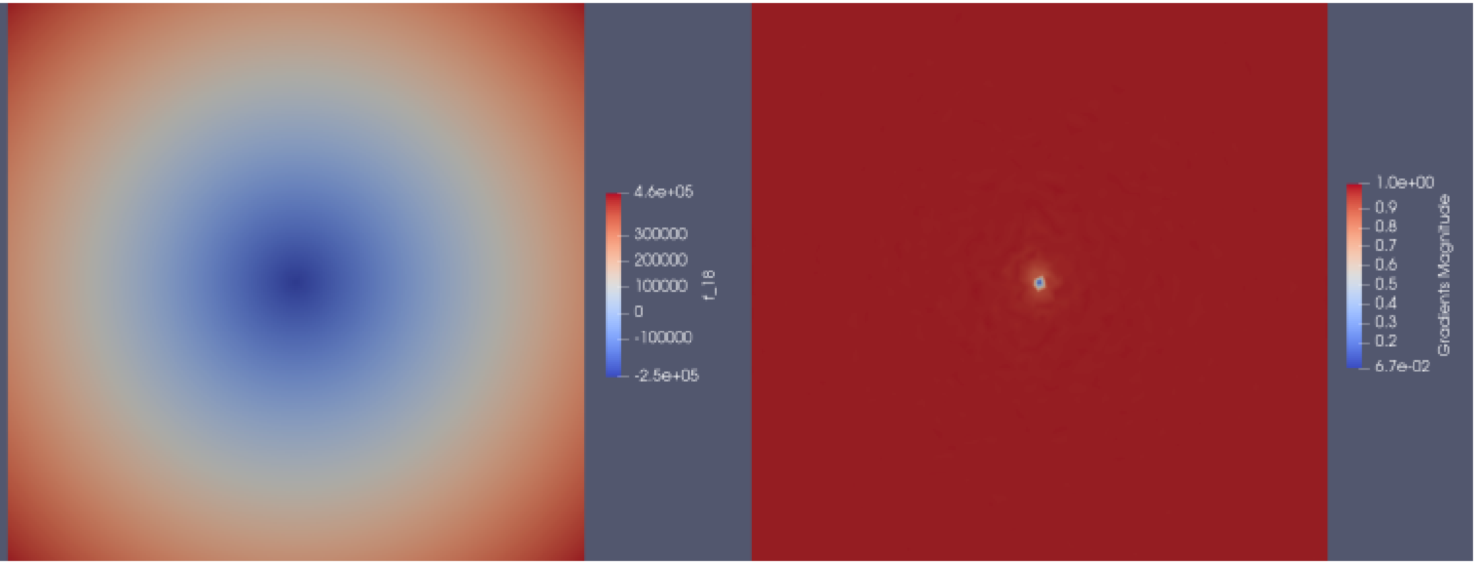
$$B = 1e6\ [\mathrm{m}]$$

$$\omega = (2\pi)1200\ [\mathrm{a}^{-1}]$$

Grad of level set without ext. Vel (top) and with (bottom) at 12000 a



Initial level set function and gradient



Level-Set Method Discretization

The finite element method is chosen, linear Lagrange elements on Tri's:

Linear advection (LA)

seek $\phi_h \in V_h$ such that

$$(v_h, \dot{\phi}_h)_\Omega + (v_h, \mathbf{u} \cdot \boldsymbol{\nabla}\phi_h)_\Omega + a(v_h, \phi_h) = 0 \quad \forall v_h \in V_h \ .$$

Stabilization

$$a_\mu(v, \phi) = (\boldsymbol{\nabla} v, \mu \boldsymbol{\nabla} \varphi)_\Omega \ ,$$
$$a_\nu(v, \phi) = (\boldsymbol{\nabla} v, \nu \boldsymbol{\nabla} \varphi)_\Omega \ ,$$
$$\nu(\vec{x}, t) = \alpha h/2 |\vec{u}(\vec{x}, t)| \ ,$$
$$\mu(\vec{x}, t) = \beta h/2 |\vec{u}(\vec{x}, t)| \ .$$

Linear Velocity Extension (LVE)

Bondzio 2016

seek $\phi_h \in V_h$

$$(v_h, \dot{\phi})_\Omega + (v_h, F_h |\boldsymbol{\nabla}\phi_h|)_\Omega + a_\nu(v_h, \phi_h) = 0 \quad \forall v_h \in V_h \ ,$$

seek $F_h \in V_{F,h}$

$$(\xi_h, S(\phi_h)(\mathbf{n}(\phi_h) \cdot \boldsymbol{\nabla} F_h))_\Omega + a_\mu(\xi_h, F_h) = 0 \quad \forall \xi_h \in V_{h,F} \ .$$

Theta Method – Implicit Midpoint

$$\dot{v} = N(v) \Rightarrow$$

$$v^{n+1} = v^n + \Delta t N\left(\frac{1}{2}[\theta v^{n+1} + (1-\theta) v^n]\right)$$

Nonlinear Velocity Extension (NVE)

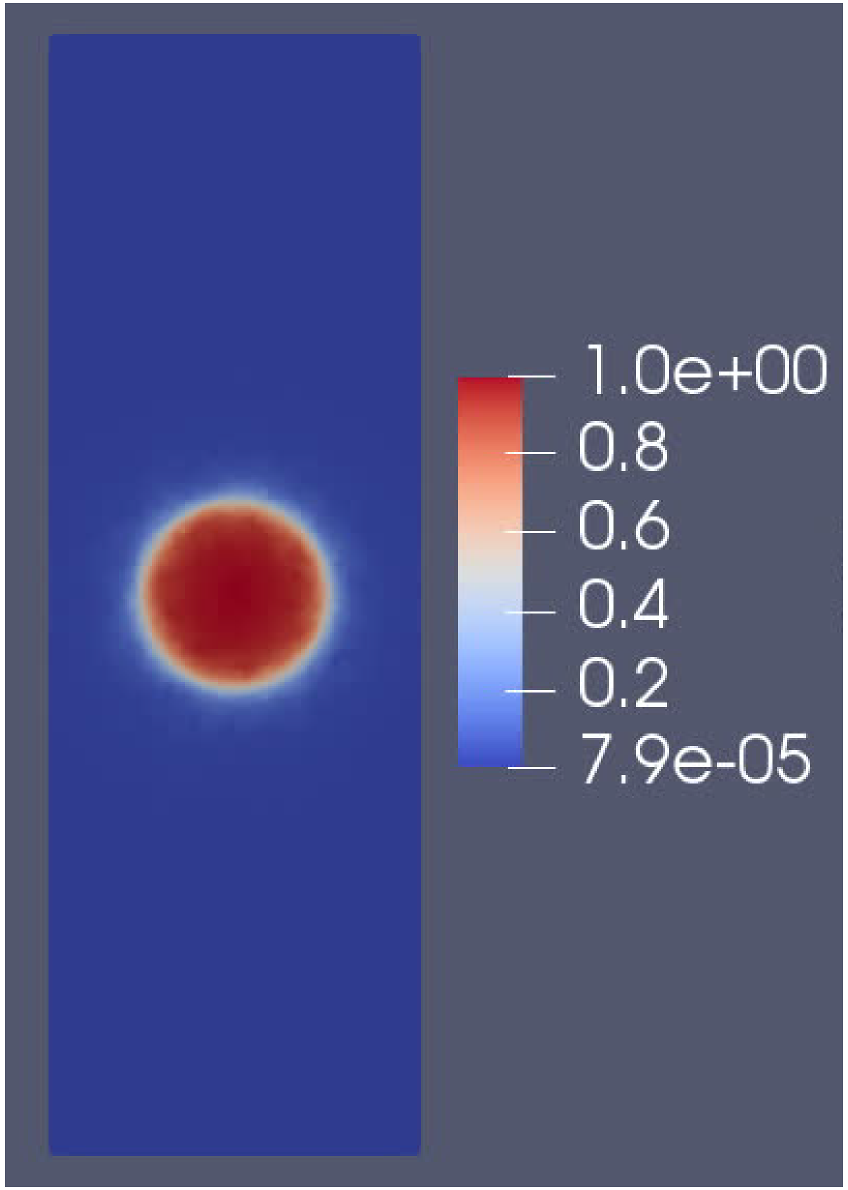
seek $\{\phi_h, w_{x,h}, w_{x,h}\} \in \{V_h, V_{h,F,x}, V_{h,F,y}\}$

$$(v_h, \dot{\phi})_\Omega + (v_h, \vec{w}_h \cdot \boldsymbol{\nabla}\phi_h)_\Omega + a_\nu(v_h, \phi_h) = 0 \quad \forall v^h \in V_h$$

$$(\xi_{x,h}, S(\phi_h)(\mathbf{n}(\phi_h) \cdot \boldsymbol{\nabla} w_{x,h})) + a_\mu(\xi_{x,h}, w_{x,h})) + \frac{1}{h}(\xi_{x,h}, (w_{x,h}) - u_1)_{\Gamma_\varphi} = 0 \quad \forall \xi_{x,h} \in V_{h,F,x} \ ,$$

$$(\xi_{y,h}, S(\phi_h)(\mathbf{n}(\phi_h) \cdot \boldsymbol{\nabla} w_{y,h})) + a_\mu(\xi_{y,h}, w_{y,h})) + \frac{1}{h}(\xi_{y,h}, (w_{y,h}) - u_2)_{\Gamma_\varphi} = 0 \quad \forall \xi_{y,h} \in V_{h,F,y} \ ,$$

Verification Tests



40 by 120 DOF grid, $\theta = 0.6,\ \alpha = 0.1,\ \beta = 0.1$

dt[a]	Error(φ)	Conv(φ)	Error(θ_{Ω_i})	Conv(θ_{Ω_i})	Vol(θ_{Ω_i}) [m^2]
80	3.1610e-02	—	1.3876e-01	—	2.0838e+11
40	1.6439e-02	0.9431	6.8821e-02	1.01171	2.0838e+11
20	8.4349e-03	0.9627	3.5301e-02	0.9631	2.0838e+11

Temporal Conv LA for 1200 [a] h=4.7428e+04 m

dt[a]	Error(φ)	Conv(φ)	Error(θ_{Ω_i})	Conv(θ_{Ω_i})	Vol(θ_{Ω_i}) [m^2]	h [m]
0.1	1.3799e-02	—	6.0166e-02	—	3.6136e+11	2.8251e+05
0.1	4.1003e-03	1.7677	1.9134e-02	1.6687	2.6258e+11	1.4219e+05
0.1	1.5253e-03	2.4419	8.9116e-03	1.8870	2.3485e+11	9.4850e+04
0.1	8.5715e-04	2.0021	5.4086e-03	1.73453	2.2101e+11	7.1123e+04

Spatial Conv LA

dt[a]	Error(φ)	Conv(φ)	Error(θ_{Ω_i})	Conv(θ_{Ω_i})	Vol(θ_{Ω_i}) [m^2]
80	3.260250e-02	—	1.484734e-01	—	2.084935e+11
40	1.664381e-02	0.969997	7.333399e-02	1.017651	2.084935e+11
20	8.461814e-03	0.975947	3.744001e-02	0.969901	2.084935e+11

Temporal Conv NVE for 1200 [a] $h = 4.7242e{+}04$ m

dt[a]	Error(φ)	Conv(φ)	Error(θ_{Ω_i})	Conv(θ_{Ω_i})	Vol(θ_{Ω_i}) [m^2]	h [m]
0.1	4.1003e-03	—	1.9134e-02	—	2.6258e+11	1.4219e+05
0.1	8.5715e-04	2.2592	5.4086e-03	1.8236	2.2101e+11	7.1123e+04
0.1	4.2419e-04	1.7359	2.9711e-03	1.4784	2.0945e+11	4.7428e+04
0.1	2.9473e-04	1.2660	1.8945e-03	1.5644	2.0441e+11	3.5573e+04

Spatial Conv NVE

Ice – Interface Coupling

Fully Decoupled: Solve Ice Then Solve Level set, Then Update domains after solve

- LA (NVE)+ SSA
  1. Construct $(H, \bar{\mathbf{u}}, \phi, \Omega_i(t_0), \Omega_w(t_0), \Gamma(t_0))$
  2. Solve SSA for $(H, \bar{\mathbf{u}})$
  3. Solve LA (NVE) for $(\phi, (F))$
  4. Update BCs and $(H, \bar{\mathbf{u}}, \Omega_i(t_{n+1}), \Omega_w(t_{n+1}), \Gamma(t_{n+1}))$

Semi-Coupled : Solve Ice and Level set together, update domains after solve

- LA (NVE)+ SSA
  1. Construct initial $(H, \bar{\mathbf{u}}, \phi, \Omega_i(t_0), \Omega_w(t_0), \Gamma(t_0))$
  2. Solve coupled SSA and LA (NVE) for $(H, \bar{\mathbf{u}}, \phi, (F))$
  3. Update BCs and $(H, \bar{\mathbf{u}}, \phi, \Omega_i(t_0), \Omega_w(t_0), \Gamma(t_0))$

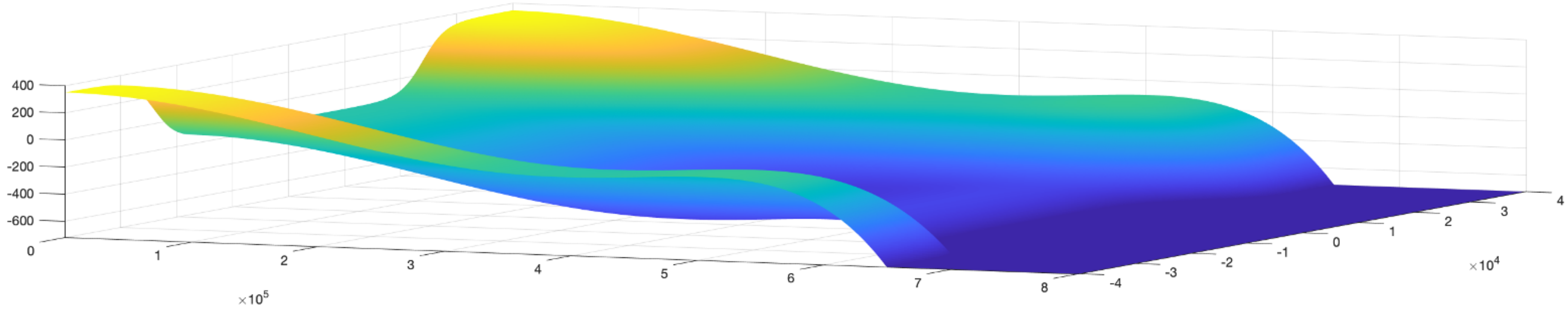
Fully Coupled: Solve Ice and Level set together, multiply ice equations by step function, update domains during solve

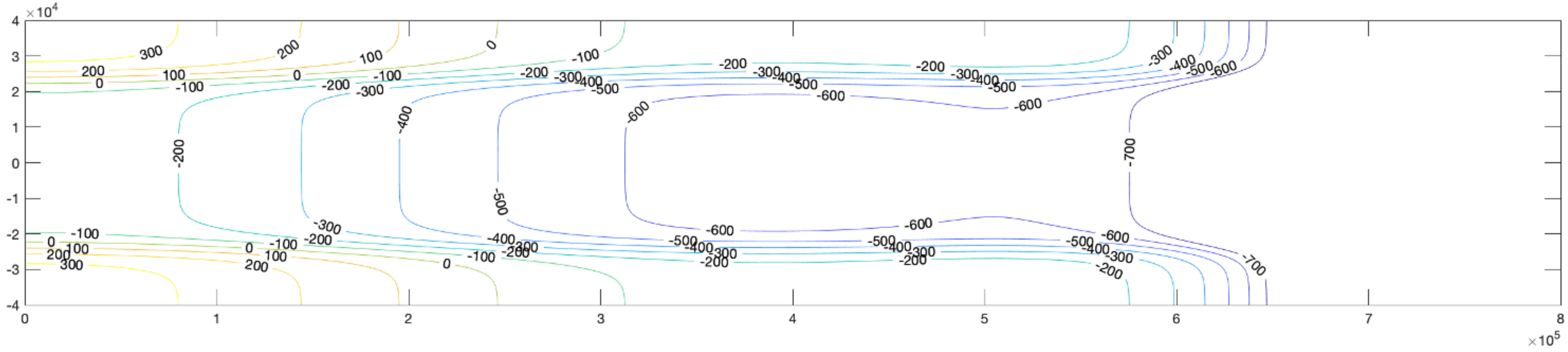
- LA (NVE)+ SSA
  1. Construct $(H, \bar{\mathbf{u}}, \phi, \Omega_i(t_0), \Omega_w(t_0), \Gamma(t_0))$
  2. Multiply SSA equations by $H_{\text{step}}(\phi) = (1 - S(\phi))/2$
  3. Solve coupled SSA and LA
  4. Domains updated by H_{step} within solve
  5. Update BCs and $(H, \bar{\mathbf{u}}, \phi, \Omega_i(t_0), \Omega_w(t_0), \Gamma(t_0))$

- LA: Liner advection level-set Model
- NVE: Nonlinear Velocity extension model
- LVE: Linear velocity extension model
- HJ: Hamilton-Jacobi Equation

Validation Tests

MISMIP+ (Cornford Et al.) bedrock topology and forcing with LA





MISMIP + bedrock topography: A very “cool” schoolyard slide! (Probably scary too)

Very Preliminary MISMIP+ Test

- 20 x 100 DOF grid of 80 km by 800 km for 3e4 [a]
- Initial thickness is 100 m from x=0 to x=500 km
- Turn on Fully Coupled LA level set at start.
- Level set "clips" thickness prescribed by Calving – melting rate
- FENICS CODE BASES – Doug Brinkerhoff

$$h_{\Delta x} = 8000 \text{ m}$$
$$h_{\Delta y} = 400 \text{ m}$$

Prescribing the Calving Law:
Calving is "Faster" where Thickness is smaller and bed topography is deeper in the water.

Level set Equation $$\frac{\partial \phi}{\partial t} + \vec{u}_c \cdot \nabla \phi = 0$$

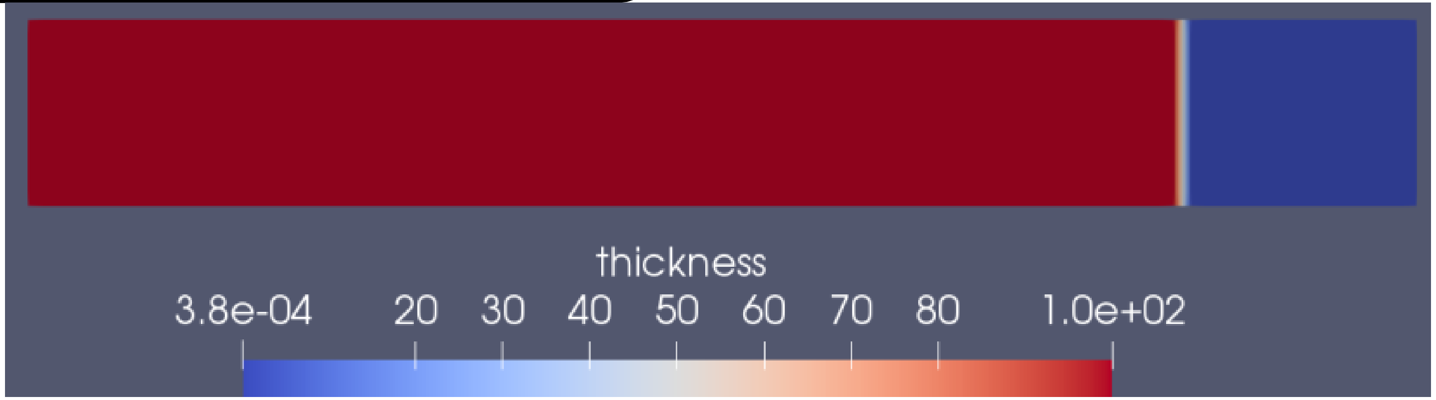
Calving "Velocity" $$u_c = \vec{u} - k * C(H) * \vec{u}$$

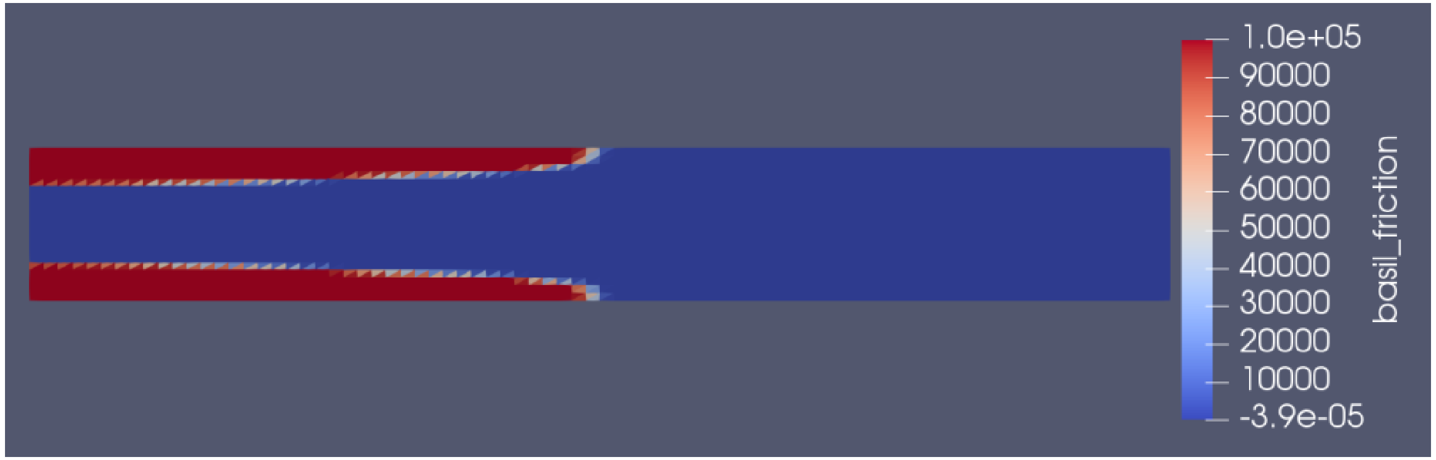
Calving rate $$C(H) = \frac{\left(\frac{\rho_w}{\rho} \max(-B, 0)\right)}{H}$$

$$k = 0.3$$

One Again $\theta = 0.6,\ \alpha = 0.1,\ \beta = 0.1$

Adpative Time-stepping : $\Delta t = [0.1, 200]$ a



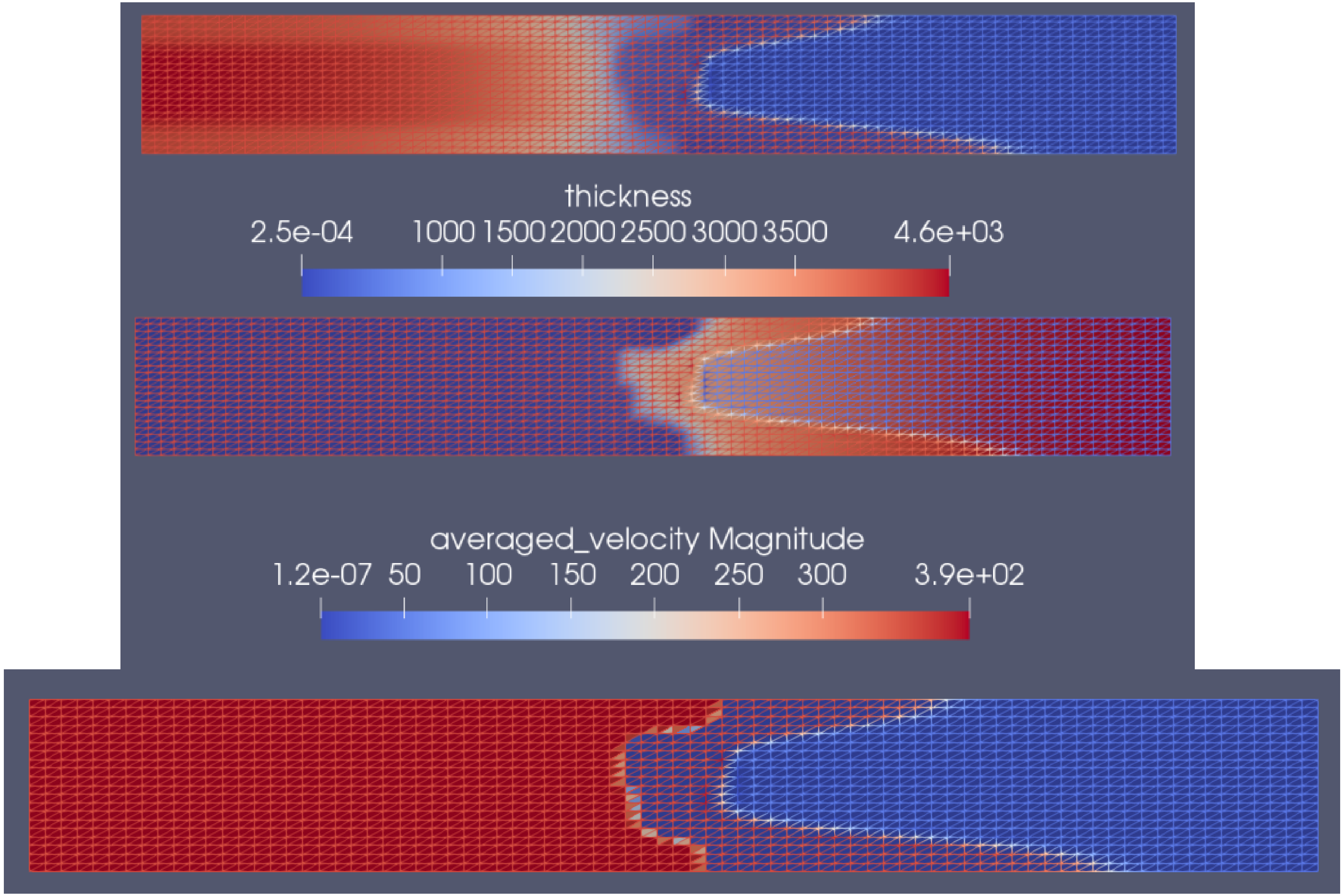


Very Preliminary MISMIP+ Test

Thickness with Heaviside Superimposed

Velocity Mag. with Heaviside Superimposed

Basil Fric. Coeff. with Heaviside Superimposed

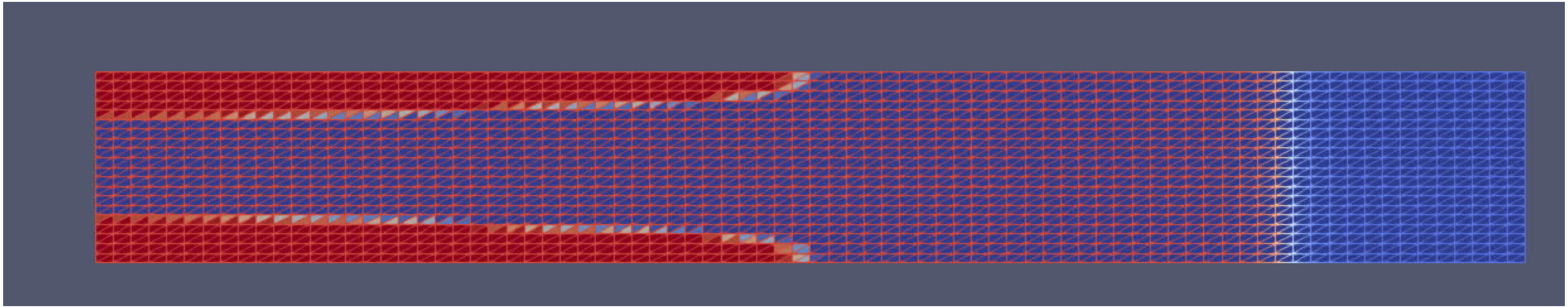


Grounding Line

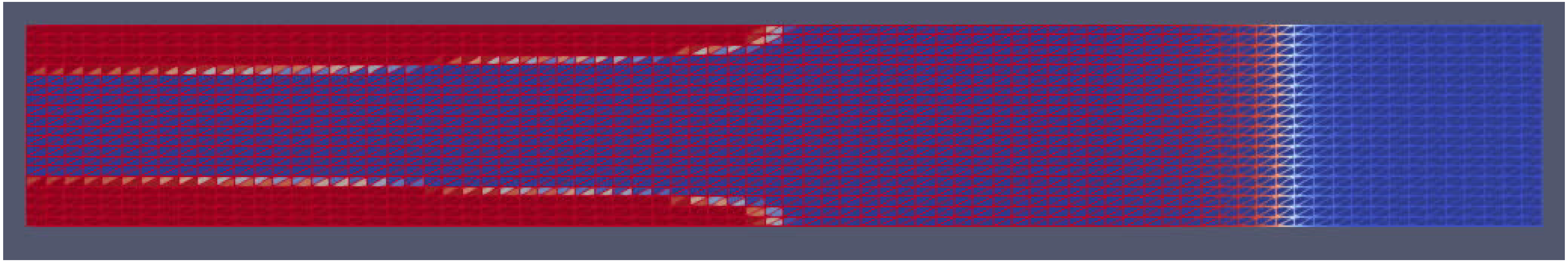
Calving Front

Very Preliminary MISMIP+ Test

Movie of Groundling and Calving Front: Basil Friction Coefficient with Heaviside superimposed (k=0.3)



Movie of Groundling and Calving Front: Basil Friction Coefficient with Heaviside superimposed (k=0.4)



In Progress and Outlook

In Progress

- MISMIP+ metrics and effects in FENICS code -> Effect of calving laws on grounding line position
- Apply to higher fidelity ice sheet models – FO stokes
  - Implementation of level set in MALI
- Implement NVE in partially coupled or fully coupled MISMIP+ simulation

Outlook

- Explore Fully Coupled Method -> Multiply Ice Equations by Heaviside - > Shape optimization
- Explore Level set methods within optimization / initialization / calibration -> Shape optimization
- At some point we will be forced to use Reinitialization but want to minimize as much as possible
- Explore shifted boundary methods to apply precise boundary condition to Calving Front

Interface Capturing and Level-Set Methods (backup was slide 6)

(Front) Interface Capturing Vs. Tracking

- Explicitly tracking interface by remeshing -> BAD
- Implicitly tracking mesh using a function

Interface Capturing Core Idea

- Function or field of values is advected in some manner
- Only values that identify interface are physical
  - Field over domain can identify changes in topology
- First Choice: Heaviside Function-> All the difficulties of shocks
  - Dispersion and dissipation

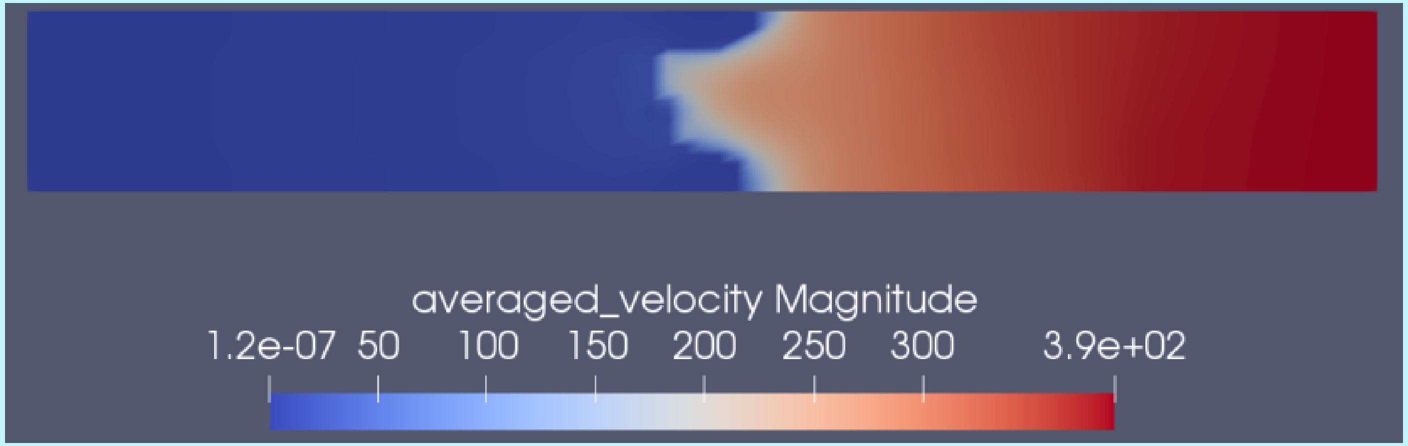
Method	Mass Cons	Remeshing	Topo change	Load Bal
FV	Yes	No	Yes	Good
Level Set	Harder	No	Yes	Good
Volume of Fluid	Trivial	No	No	Bad
MAC	Yes	No	No	Bad
Phase Field	Yes	No	Yes	Good
Front Tracking	Yes	Yes	Yes	Good

Very Preliminary MISMIP+ Test (backup was slide 17)

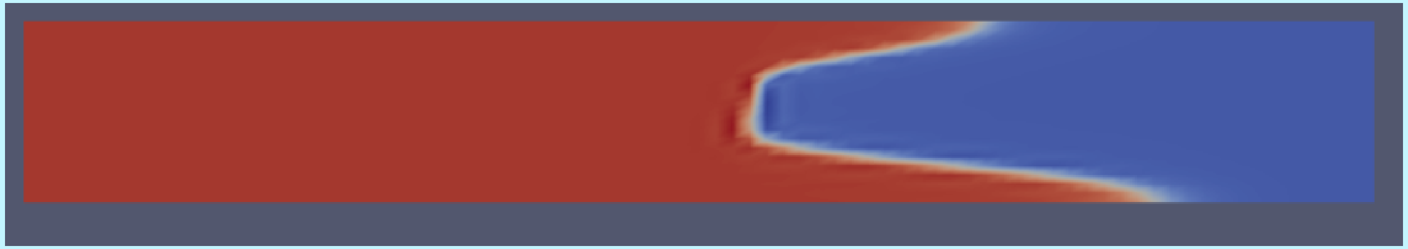
Results at 3e4 [a]

Thickness

Magnitude of Average Velocity



Heaviside of Level Set
Red = ICE
Blue = WATER



Backup Mismip+ data

Basil Drag: $\tau_{i,b} = 10^5 u_i$ if grounded

$\tau_{i,b} = 0$ if floating

$$A = 10^{-16}\ \text{Pa}^3\ \text{a}^{-1}$$

$$\rho = 900\ \text{kg m}^{-3}$$

$$\rho_i = 1000\ \text{kg m}^{-3}$$

$$\dot{a} = 0.3\text{m a}^{-1}$$