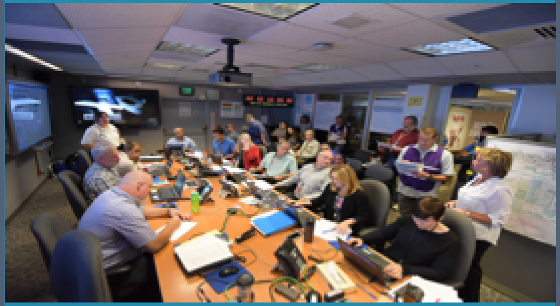
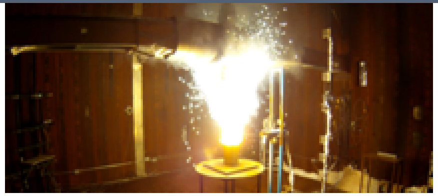
This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

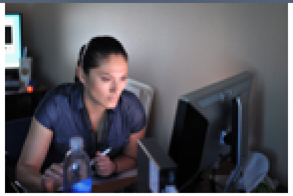
Sandia National Laboratories

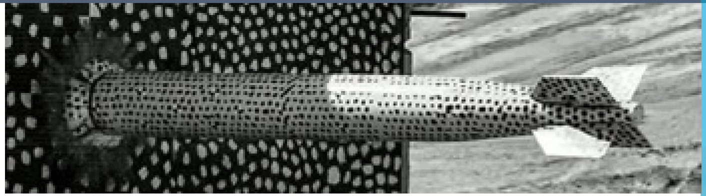
SAND2021-2217C

Coupling 1D Telegrapher Equations to 3D Maxwell's Equations with Applications to Pulsed Power









- *SIAM CSE, March 2020*

D. McGregor, E. Phillips, D. Sirajuddin, T. Pointon

U.S. DEPARTMENT OF ENERGY NNSA National Nuclear Security Administration

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology & Engineering Solutions of Sandia, LLC, a wholly owned subsidiary of Honeywell International Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

Acknowledgement

Methods discussed today were implemented in the SNL ASC code EMPIRE.
We would like to acknowledge our team for their support.

- This work has been supported Sandia National Laboratories Laboratory Directed Research & Development Program.

  Plasma Science and Engineering Grand Challenge LDRD Project #209240 (2018-2020)

  Coupling Powerflow to Target Simulations LDRD Project #210398 (2021-2022)

Pulsed Power

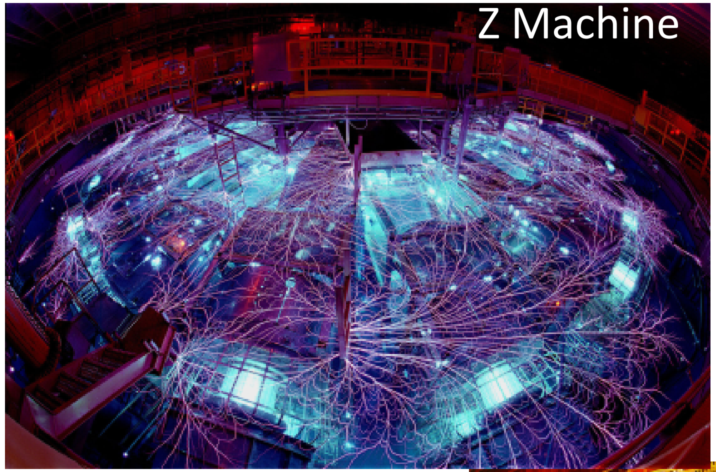
Large scale experimental facilities at SNL

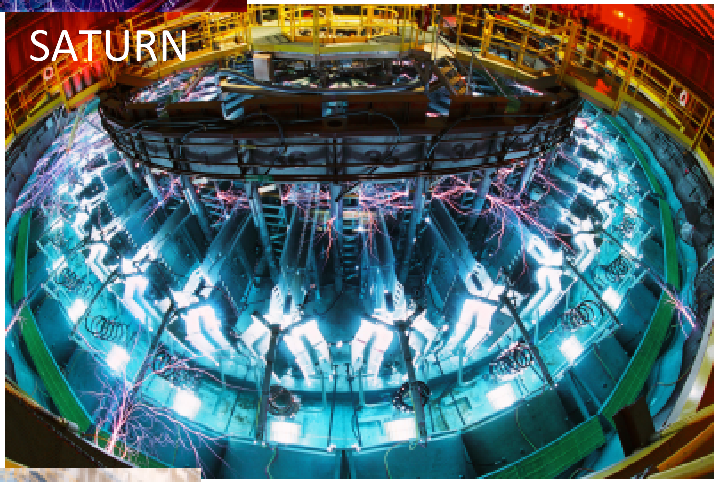
- Radiation sources
- High energy density material science
- Create astrophysical conditions in a laboratory setting

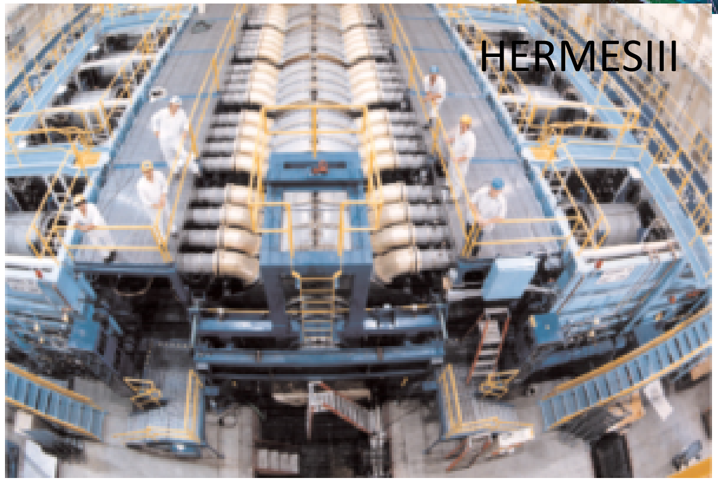
Magnetically Insulated Transmission Lines

Legacy development of these platforms has been experimental/empirical. **Programmatic desire for science based design (i.e. computation) for future systems.**

Issues:
Multiscale problem
Challenging (non-linear) models physics

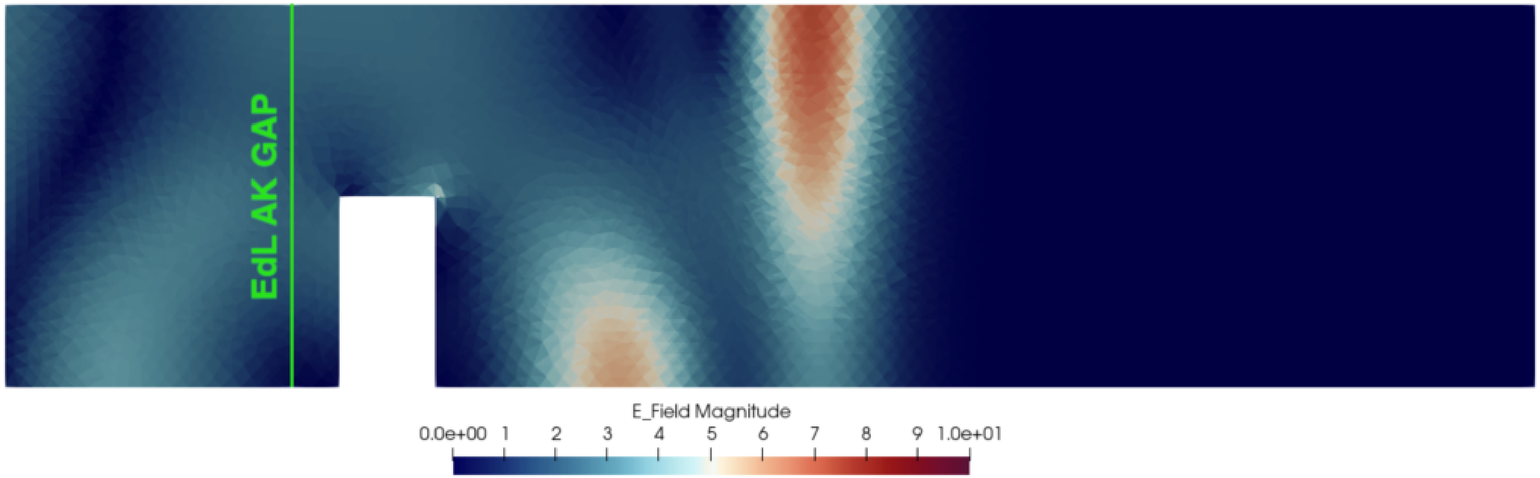


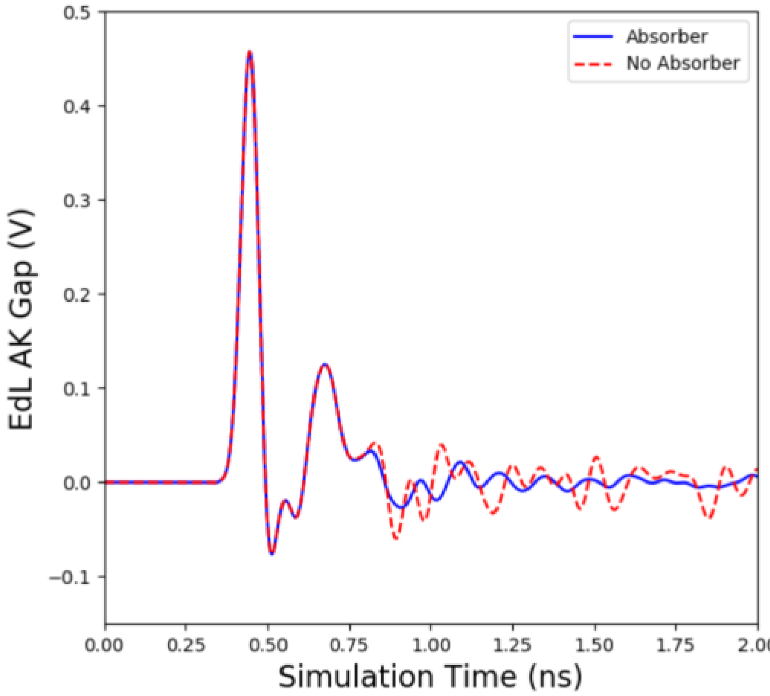




New Features of Our Coupling Algorithm

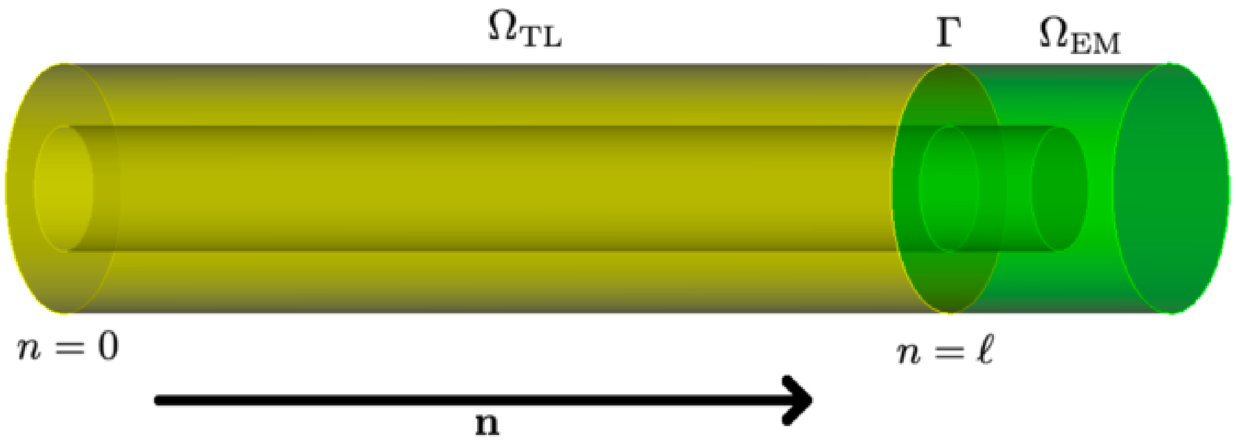
- Variational – couple through surface integrals – this allows it to apply to arbitrary meshes
- Implicit coupling – introduces no new stability constraints – we developed an efficient linear solver to handle implicit solves
- Self-consistent – coupling is based directly on assumptions made to by the TL model – we enforce continuity of voltage and current at the interface

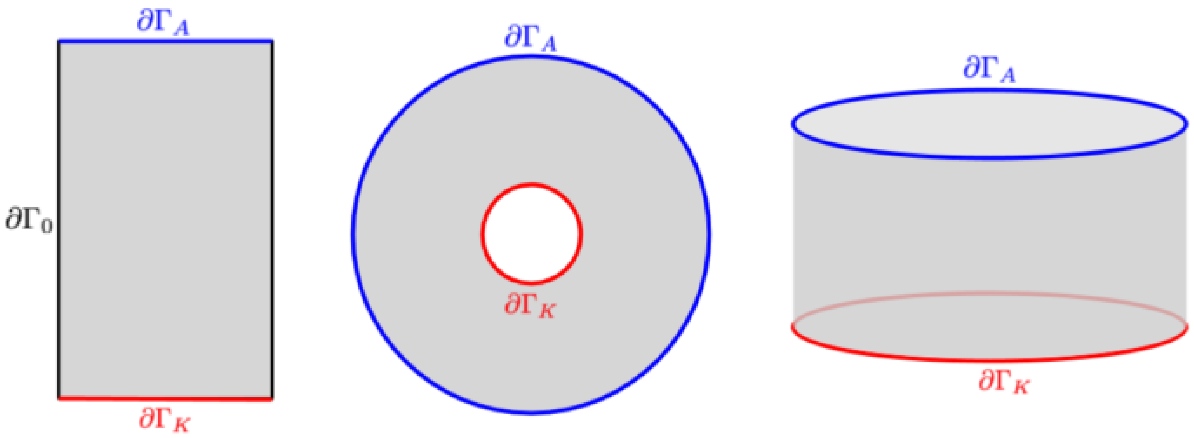




- We enforce voltage continuity via a constraint and a Lagrange multiplier
- Allows us to apply an additional boundary conditional at the EM/TL interface
- We apply an absorbing BC at the interface that absorbs non-TEM modes
- Reduces unphysical ringing due to reflection of non-TEM modes

Abstract Modeling Problem





Maxwell's Equations

$$\begin{cases} \dfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J} - \mathbf{curl}\ \mathbf{H} = \mathbf{0} \\ \dfrac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\ \mathbf{E} = \mathbf{0} \\ \text{div}\ \mathbf{D} = \rho \\ \text{div}\ \mathbf{B} = 0 \end{cases}$$

Simple Dielectric

$$\begin{cases} \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{B} = \mu \mathbf{H} \end{cases}$$

Homogeneous BCs

$$\begin{cases} \mathbf{E} \times \mathbf{n} = \mathbf{0} & \text{on conductors} \\ \mathbf{H} \times \mathbf{n} = \mathbf{0} & \text{on symmetry} \end{cases}$$

$\mathbf{J}$ is data

Transverse Electromagnetic Mode

Fundamental mode for these geometries.
Higher modes (TE, TM) have a high pass like dispersion relationship
TEM is the ideal mode for powerflow

IF ϵ, μ constant $\mathbf{J} \equiv \mathbf{0}$

on Γ

$$\begin{cases} -\mathrm{div}_\tau \epsilon_r \mathbf{grad}_\tau \varphi = 0 \\ \mathbf{E}_0 = -\mathbf{grad}_\tau \varphi \\ \varphi|_{\partial\Gamma_A} = 0 \\ \varphi|_{\partial\Gamma_K} = 1 \end{cases}$$

$$\mathbf{H}_0 = \frac{\mathbf{n} \times \mathbf{E}_0}{\|\mathbf{E}_0\|^2_{L^2(\Gamma)}}$$

on $[0, \ell]$

$$\begin{cases} C\frac{\partial V}{\partial t} + \frac{\partial I}{\partial n} = 0 \\ L\frac{\partial I}{\partial t} + \frac{\partial V}{\partial n} = 0 \end{cases}$$

$$C := \int_\Gamma \epsilon |\mathbf{E}_0|^2 \, dA$$

$$L := \int_\Gamma \mu |\mathbf{H}_0|^2 \, dA$$

THEN

$$\mathbf{E}_\tau = V(n,t)\mathbf{E}_0(\tau_1, \tau_2)$$
$$\mathbf{H}_\tau = I(n,t)\mathbf{H}_0(\tau_1, \tau_2)$$

ARE A SOLUTION TO MAXWELL'S EQAUTIONS ON

$$\Omega_{\mathrm{TL}} = \Gamma \times [0, \ell]$$

WITH DISPERSION RELATIONSHIP

$$\epsilon\mu\omega^2 = k^2$$

Sketching the Variational Formulation

Our notion of spaces for our fields

$$(\mathbf{E}, \mathbf{B}, V, I) \in \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{EM}}) \times \mathbf{H}(\mathrm{div}, \Omega_{\mathrm{EM}}) \times H^1(0,\ell) \times L^2(0,\ell)$$

$$\begin{cases} \int_{\Omega_{\mathrm{EM}}} \epsilon \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{\Psi} + \mathbf{J} \cdot \mathbf{\Psi} - \mu^{-1} \mathbf{B} \cdot \mathbf{curl}\ \mathbf{\Psi}\ dV & \forall \mathbf{\Psi} \in \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{EM}}) \\ \int_{\Omega_{\mathrm{EM}}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{\Phi} + \mathbf{curl}\ \mathbf{E} \cdot \mathbf{\Phi} dV & \forall \mathbf{\Phi} \in \mathbf{H}(\mathrm{div}, \Omega_{\mathrm{EM}}) \\ \int_0^\ell C \frac{\partial V}{\partial t} \psi - I \frac{\partial \psi}{\partial n} dS & \forall \psi \in H^1(0,\ell) \\ \int_0^\ell L \frac{\partial I}{\partial t} \phi + \frac{\partial V}{\partial n} \phi\ dS & \forall \phi \in L^2(0,\ell \end{cases}$$

We are going to add coupling conditions

Spectral Coupling

Our notion of spaces for our fields

$$(\mathbf{E}, \mathbf{B}, V, I) \in \mathbf{H}(\mathbf{curl}, \Omega_{\text{EM}}) \times \mathbf{H}(\text{div}, \Omega_{\text{EM}}) \times H^1(0, \ell) \times L^2(0, \ell)$$

Use L2 projections to "pick off the TEM part"

$$\int_\Gamma \epsilon \underbrace{\mathbf{E}(t, \ell, \tau_1, \tau_2)}_{\text{3D Electric Field}} \cdot \underbrace{\mathbf{E}_0(\tau_1, \tau_2)}_{\text{TEM } \mathbf{E} \text{ Profile}} \, dA := C \underbrace{V(t, \ell)}_{\text{TEM Voltage}}$$

Dirichlet condition

$$\int_\Gamma \mu \underbrace{\mathbf{H}(t, \ell, \tau_1, \tau_2)}_{\text{3D Magnetic Field}} \cdot \underbrace{\mathbf{H}_0(\tau_1, \tau_2)}_{\text{TEM } \mathbf{H} \text{ Profile}} \, dA := L \underbrace{I(t, \ell)}_{\text{TEM Current}}$$

Neumann condition

If you enforce these two conditions then the TEM component of the Maxwell's equations solution will match at the boundary

Naïve couplings

Dirichlet Maxwell/ Neumann Telegrapher

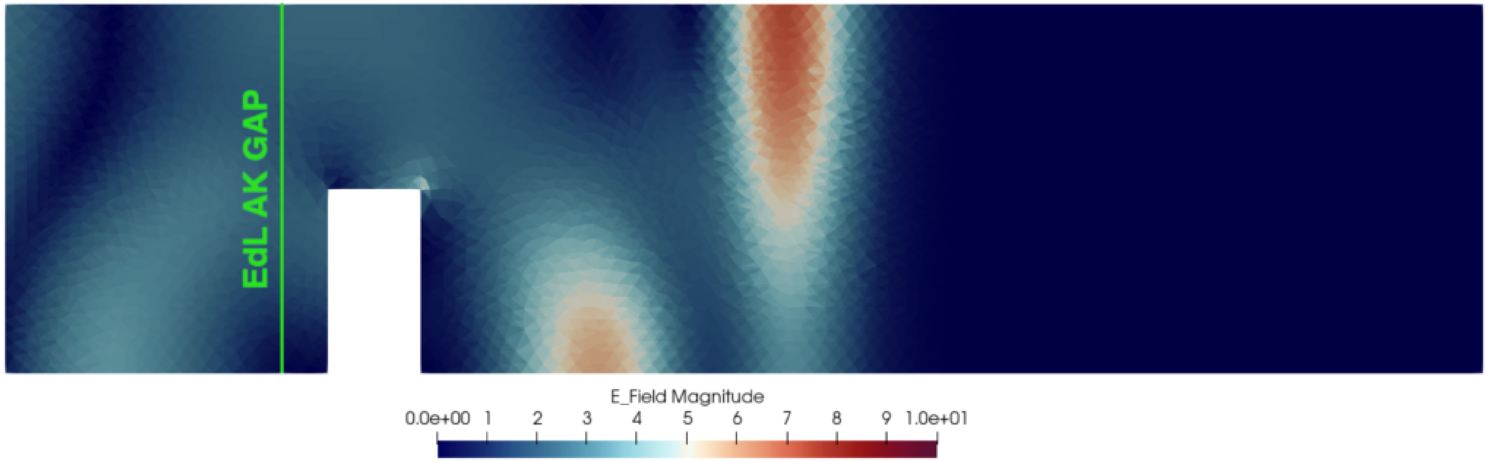
$$\begin{cases} \mathbf{E} \times \mathbf{n}|_\Gamma = V(t,\ell)\mathbf{E}_0 \times \mathbf{n} \\ LI(\ell) = \int_\Gamma \mu \mathbf{H} \cdot \mathbf{H}_0 \, dA \end{cases}$$

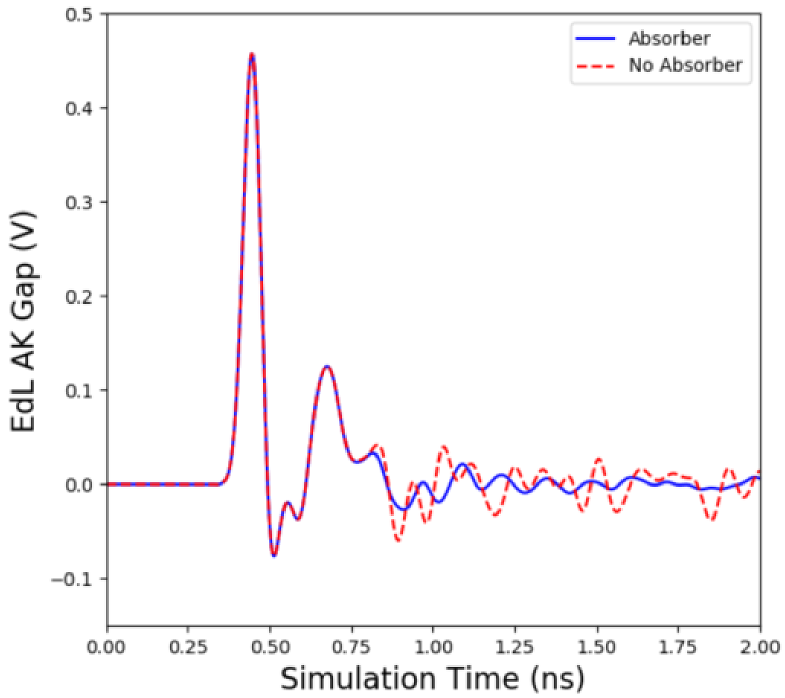
Neumann Maxwell/ Dirichlet Telegrapher

$$\begin{cases} \mathbf{H} \times \mathbf{n}|_\Gamma = I(t,\ell)\mathbf{H}_0 \times \mathbf{n} \\ CV(\ell) = \int_\Gamma \epsilon \mathbf{E} \cdot \mathbf{E}_0 \, dA \end{cases}$$

Both methods have the problem of reflecting non-TEM components off coupling interface in the EM domain. Problematic for plasmas!

Red lineout is equiv to Neumann Maxwell/ Dirichlet Telegrapher solution





Non-TEM absorber and voltage coupling

$$\Pi_{\text{TEM}} : \mathbf{L}^2(\Gamma) \to \mathbf{L}^2(\Gamma) :$$

$$\Pi_{\text{TEM}}(\mathbf{E}) := \mathbf{E}_0 \frac{\int_\Gamma \epsilon \mathbf{E} \cdot \mathbf{E}_0 \; dA}{\int_\Gamma \epsilon \mathbf{E}_0 \cdot \mathbf{E}_0 \; dA}$$

We can use this projection to define a first order outgoing wave condition for non-TEM Waves! Add this to variational Ampere's law

$$\int_\Gamma Z^{-1}(\mathbf{E} - \Pi_{\text{TEM}}\mathbf{E}) \times \mathbf{n} \cdot \mathbf{\Psi} \times \mathbf{n} \; dA$$

How do we impose 2 boundary conditions at the same time?

Impose voltage coupling as a constraint and relax with a Lagrange multiplier $\lambda \in \mathbb{R}$

$$\begin{cases} \int_{\Omega_{\text{EM}}} \epsilon \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{\Psi} + \mathbf{J} \cdot \mathbf{\Psi} - \mu^{-1} \mathbf{B} \cdot \mathbf{curl} \; \mathbf{\Psi} \; dV \\ + \int_\Gamma \left(Z^{-1}(\mathrm{I} - \Pi_{\text{TEM}})(\mathbf{E}) + \lambda \epsilon \mathbf{E}_0 \right) \times \mathbf{n} \cdot \mathbf{\Psi} \times \mathbf{n} & \forall \mathbf{\Psi} \in \mathbf{H}(\mathbf{curl}, \Omega_{\text{EM}}) \\ \theta \int_\Gamma \epsilon \mathbf{E} \times \mathbf{n} \cdot \mathbf{E}_0 \times \mathbf{n} \; dA = \theta C V & \forall \theta \in \mathbb{R} \end{cases}$$

Current coupling

$$I_{\mathrm{EM}} = \frac{\int_\Gamma \mu \mathbf{H} \cdot \mathbf{H}_0 \; dA}{\int_\Gamma \mu \mathbf{H}_0 \cdot \mathbf{H}_0 \; dA}$$

$$= -\int_\Gamma \mathbf{H} \times \mathbf{n} \cdot \mathbf{E}_0 \; dA$$

$$= \int_{\Omega_{\mathrm{EM}}} \epsilon \frac{\partial \mathbf{E}}{\partial t} \cdot \overline{\mathbf{E}}_0 + \mathbf{J} \cdot \overline{\mathbf{E}}_0 - \mu^{-1} \mathbf{B} \cdot \mathbf{curl}\; \overline{\mathbf{E}}_0 \; dV \qquad \overline{\mathbf{E}}_0|_\Gamma = \mathbf{E}_0$$

Introduce a Neumann condition to couple

$$\int_0^\ell C \frac{\partial V}{\partial t} \psi - I \frac{\partial \psi}{\partial n} \; dS + I_{\mathrm{EM}} \psi(\ell)$$

Test Ampere's law against $\overline{\mathbf{E}}_0$ we can solve for λ

$$C\lambda = -I_{\mathrm{EM}}$$

This gives us exact energy conservation !

Variational Formulation Summary

$$(\mathbf{E}, \mathbf{B}, \lambda, V, I) \in \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{EM}}) \times \mathbf{H}(\mathrm{div}, \Omega_{\mathrm{EM}}) \times \mathbb{R} \times H^1(0,\ell) \times L^2(0,\ell)$$

$$\begin{cases} \displaystyle\int_{\Omega_{\mathrm{EM}}} \epsilon \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{\Psi} + \mathbf{J} \cdot \mathbf{\Psi} - \mu^{-1} \mathbf{B} \cdot \mathbf{curl}\ \mathbf{\Psi}\ dV & \\ \displaystyle + \int_{\Gamma} \left(Z^{-1} (\mathrm{I} - \Pi_{\mathrm{TEM}})(\mathbf{E}) + \lambda \epsilon \mathbf{E}_0 \right) \times \mathbf{n} \cdot \mathbf{\Psi} \times \mathbf{n} & \forall \mathbf{\Psi} \in \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{EM}}) \\ \displaystyle \theta \int_{\Gamma} \epsilon \mathbf{E} \times \mathbf{n} \cdot \mathbf{E}_0 \times \mathbf{n}\ dA = \theta C V & \forall \theta \in \mathbb{R} \\ \displaystyle \int_{\Omega_{\mathrm{EM}}} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{\Phi} + \mathbf{curl}\ \mathbf{E} \cdot \mathbf{\Phi} dV & \forall \mathbf{\Phi} \in \mathbf{H}(\mathrm{div}, \Omega_{\mathrm{EM}}) \\ \displaystyle \int_0^{\ell} C \frac{\partial V}{\partial t} \psi - I \frac{\partial \psi}{\partial n} dS + I_{\mathrm{EM}} \psi(\ell) & \forall \psi \in H^1(0,\ell) \\ \displaystyle \int_0^{\ell} L \frac{\partial I}{\partial t} \phi + \frac{\partial V}{\partial n} \phi\ dS & \forall \phi \in L^2(0, \ell \end{cases}$$

Discretization

Lowest order Nedelec elements for **E**

Lowest order Raviart Thomas elements for **B**

P1 elements for V

P0 elements for I

λ is discrete

Implicit RK method for time integration – i.e. implicit midpoint or Crank Nicolson.

Linear Solver for the Coupled System

- Fully coupled linear system

$$\mathcal{A}\mathbf{x} = \begin{pmatrix} \mathcal{A}_M & f & g\Pi_{V_s} \\ f^t & 0 & -C\Pi_{V_s} \\ \Pi_{V_s}^t k^t & 0 & \mathcal{A}_{TL} \end{pmatrix} \begin{pmatrix} \mathbf{x}_M \\ \lambda \\ \mathbf{x}_{TL} \end{pmatrix}$$

- Exact block LU decomposition

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_M & 0 & 0 \\ f^t & s_\lambda & 0 \\ \Pi_{V_s}^t k^t & -\Pi_{V_s}^t k^t \mathcal{A}_M^{-1} f & \mathcal{S}_{TL} \end{pmatrix} \begin{pmatrix} \mathcal{I} & \mathcal{A}_M^{-1} f & \mathcal{A}_M^{-1} g \Pi_{V_s} \\ 0 & 1 & (-C - f^t \mathcal{A}_M^{-1} g) s_\lambda^{-1} \Pi_{V_s} \\ 0 & 0 & \mathcal{I} \end{pmatrix}$$

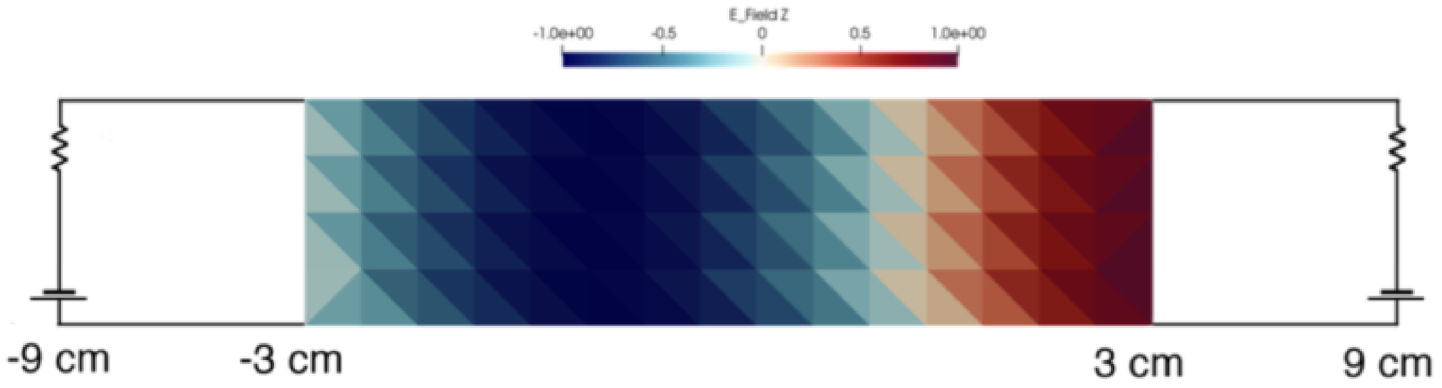
$$s_\lambda = -f^t \mathcal{A}_M^{-1} f,$$

$$\mathcal{S}_{TL} = \mathcal{A}_{TL} - [k^t \mathcal{A}_M^{-1} g + k^t \mathcal{A}_M^{-1} f (C + f^t \mathcal{A}_M^{-1} g) s_\lambda^{-1}] \Pi_{V_s}^t \Pi_{V_s}$$

- Exact inverse requires 1 EM solve plus 2 for each TL whenever the time-step changes
- An augmented TL system solve (S_{TL}) for each TL (assumed small compared to EM solve – we do a small direct solve on each MPI rank)
- For fixed time-step simulations, the cost per time-step is essentially the same as solving EM and TL decoupled

Verification: O-Wave between Parallel Plates

- Simulating a transient O-wave where the middle of the domain is 3D EM and the sides are 1D TLs



(a) CFL varies from (4.35, 7.53)

p.p.w.	p.p.p.	$\mathscr{E}_E$	rate	$\mathscr{E}_B$	rate
21.8	2.5	1.43	–	1.64	–
43.6	5	8.24e-1	0.79	1.10	0.69
87.2	10	1.99e-1	2.04	2.56e-1	1.98
174.5	20	4.99e-2	1.99	6.11e-2	2.06

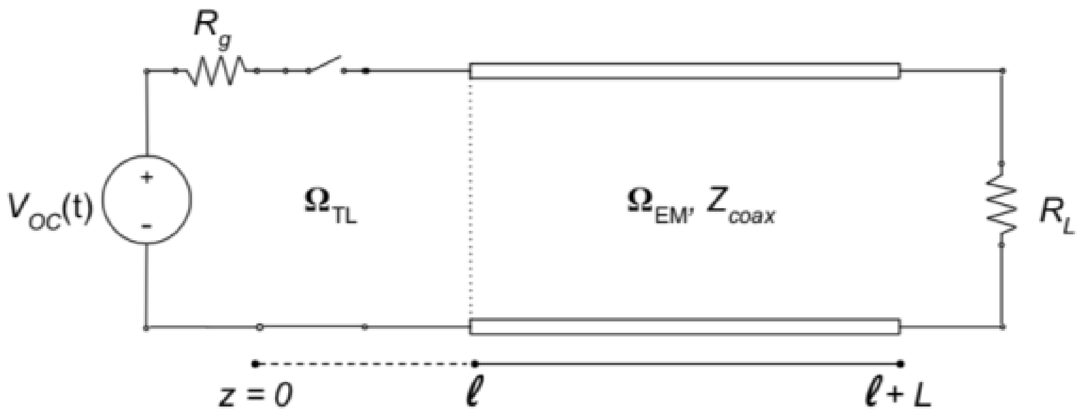
(b) CFL varies from (0.54, 0.94)

p.p.w.	p.p.p.	$\mathscr{E}_E$	rate	$\mathscr{E}_B$	rate
21.8	10	7.84e-2	–	7.06e-2	–
43.6	20	3.37e-2	1.21	2.41e-2	1.55
87.2	40	1.61e-2	1.07	1.03e-2	1.22
174.5	80	7.97e-3	1.01	5.16e-3	0.99

- Using implicit midpoint time-stepping (second order) and first order conforming finite elements for the EM
- Large CFL – error is dominated by time discretization – obtain expected second order convergence
- Small CFL – error is dominated by spatial discretization – obtain expected first order convergence

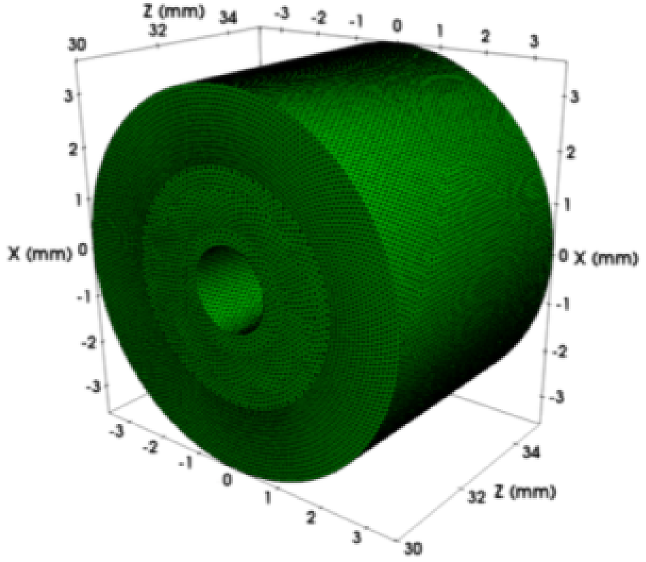
Verification: Coaxial Waveguide

- Coaxial waveguide driven by an equivalent circuit
- Obtained first order convergence to analytic steady-state solution



(a) 1D circuit representation of $\Omega_{TL} \cup \Omega_{EM}$

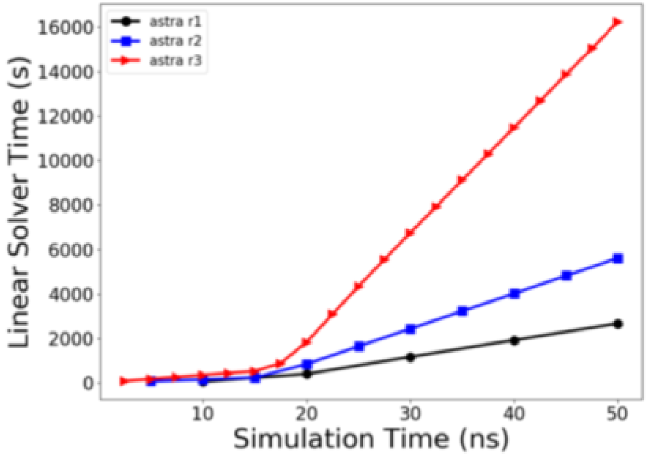
h (m)	$\frac{\text{Gap}}{h}$	$\mathscr{E}_E$	rate	$\mathscr{E}_B$	rate
$1.00e-03$	5.70	$7.89e-02$	–	$5.01e-02$	–
$5.99e-04$	9.70	$7.30e-02$	0.15	$4.70e-02$	0.12
$4.64e-04$	12.3	$6.77e-02$	0.29	$4.49e-02$	0.17
$3.59e-04$	15.9	$5.57e-02$	0.76	$3.64e-02$	0.82
$2.78e-04$	20.5	$4.21e-02$	1.09	$2.67e-02$	1.20
$2.15e-04$	26.5	$3.25e-02$	1.00	$2.03e-02$	1.07
$1.67e-04$	34.2	$2.52e-02$	1.00	$1.55e-02$	1.05
$1.29e-04$	44.3	$1.91e-02$	1.08	$1.17e-02$	1.07
$1.00e-04$	57.0	$1.45e-02$	1.06	$8.96e-03$	1.06

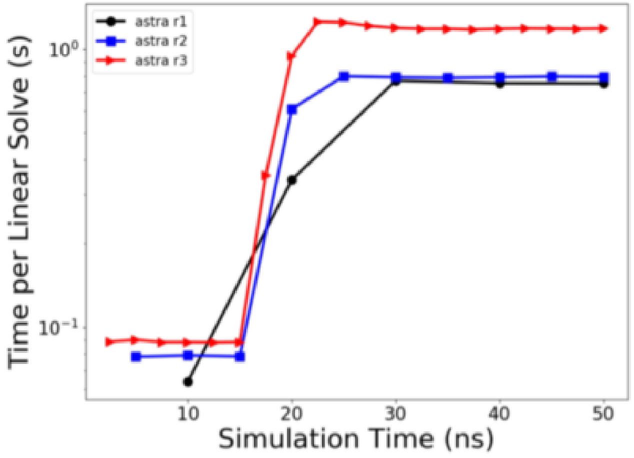


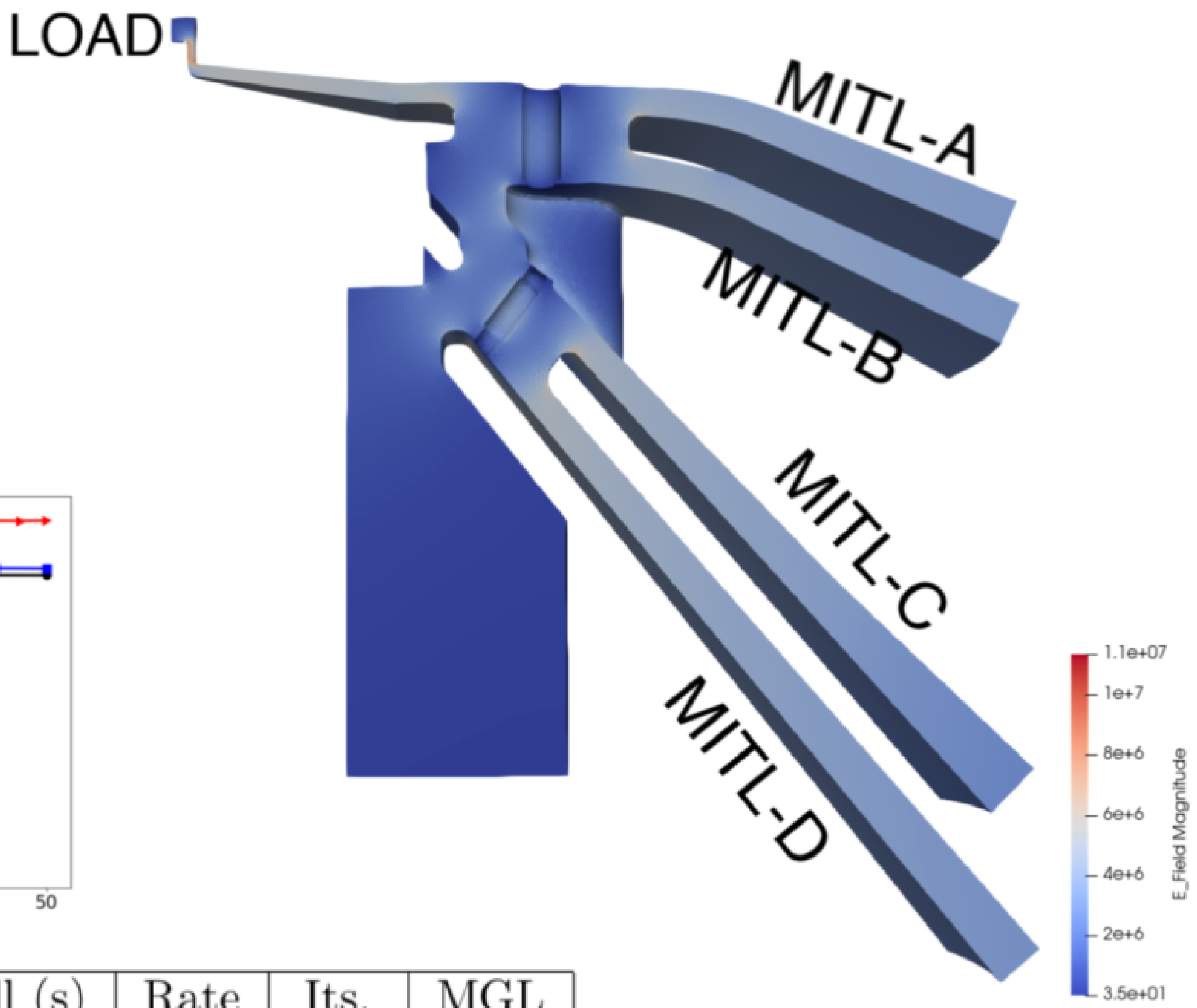
(b) 3D domain Ω_{EM}, 0.1 mm mesh

Demonstration: 18a Convolute

- EM simulation on 18a geometry driven by 4 TLs equivalent to BERTHA model
- Have demonstrated good agreement with CHICAGO for same problem elsewhere
- Coupled linear solver scales similarly to EM alone
- Early time-steps are cheap until pulses reach the EM domain







Ref.	Els.	Rate	Steps	CFL DR	Nodes	Wall (s)	Rate	Its.	MGL
1	2.14 M	–	5000	19.7	2	2667	–	23.4	2
2	19.3 M	9.01	10000	19.6	16	5614	2.10	18.4	3
3	154.7 M	8.0	20000	23.3	128	16241	2.89	27.9	4

Questions

Maxwell's Equations on $\Omega_{\mathrm{TL}} = \Gamma \times [0, \ell]$

$$\mathbf{E} = (\mathbf{E}_\tau, E_n)$$
$$\mathbf{H} = (\mathbf{H}_\tau, H_n)$$

$$\begin{pmatrix} 0 & -\partial_z & \partial_{\tau_2} \\ \partial_z & 0 & -\partial_{\tau_1} \\ -\partial_{\tau_2} & \partial_{\tau_1} & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} \partial_n \mathbf{n}\times & \mathbf{curl}_\tau \\ \mathrm{rot}_\tau & 0 \end{pmatrix}$$

$$\begin{cases} \epsilon \dfrac{\partial \mathbf{E}_\tau}{\partial t} - \mathbf{curl}_\tau H_n - \dfrac{\partial}{\partial n}\mathbf{n} \times \mathbf{H}_\tau = \mathbf{0} \\ \epsilon \dfrac{\partial E_n}{\partial t} - \ \mathrm{rot}_\tau \mathbf{H}_\tau = 0 \\ \mu \dfrac{\partial \mathbf{H}_\tau}{\partial t} + \mathbf{curl}_\tau E_n + \dfrac{\partial}{\partial n}\mathbf{n} \times \mathbf{E}_\tau = \mathbf{0} \\ \mu \dfrac{\partial H_n}{\partial t} + \ \mathrm{rot}_\tau \mathbf{E}_\tau = 0 \\ \dfrac{\partial}{\partial n}\epsilon E_n + \mathrm{div}_\tau \epsilon \mathbf{E}_\tau = 0 \\ \dfrac{\partial}{\partial n}\mu H_n + \mathrm{div}_\tau \mu \mathbf{H}_\tau = 0 \end{cases}$$

Transverse Electromagnetic Mode (TEM Mode)

$$E_n = B_n = 0$$

$$\begin{cases} \epsilon \frac{\partial \mathbf{E}_\tau}{\partial t} - \frac{\partial}{\partial n} \mathbf{n} \times \mathbf{H}_\tau = \mathbf{0} \\ \mu \frac{\partial \mathbf{H}_\tau}{\partial t} + \frac{\partial}{\partial n} \mathbf{n} \times \mathbf{E}_\tau = \mathbf{0} \\ \mathrm{rot}_\tau \mathbf{E}_\tau = 0 \\ \mathrm{rot}_\tau \mathbf{H}_\tau = 0 \\ \mathrm{div}_\tau \epsilon \mathbf{E}_\tau = 0 \\ \mathrm{div}_\tau \mu \mathbf{H}_\tau = 0 \end{cases}$$

Separate variables

$$\mathbf{E}_\tau = V(n,t)\mathbf{E}_0(\tau_1, \tau_2)$$

$$\mathbf{H}_\tau = I(n,t)\mathbf{H}_0(\tau_1, \tau_2)$$

Evolution equations:

$$\begin{cases} \frac{\partial V}{\partial t} \epsilon \mathbf{E}_0 - \frac{\partial I}{\partial n} \mathbf{n} \times \mathbf{H}_0 = \mathbf{0} \\ \frac{\partial I}{\partial t} \mu \mathbf{H}_0 + \frac{\partial V}{\partial n} \mathbf{n} \times \mathbf{E}_0 = \mathbf{0} \end{cases}$$

Mode condition: $\mathbf{E}_0 \propto \mathbf{n} \times \mathbf{H}_0$

Cross sectional profiles: Laplace problems

$$\epsilon_0\epsilon_r = \epsilon \qquad\qquad \mu_0\mu_r = \mu$$

$$\begin{cases} -\mathrm{div}_\tau \epsilon_r \mathbf{grad}_\tau \varphi = 0 \\ \mathbf{E}_0 = -\mathbf{grad}_\tau \varphi \\ \varphi|_{\partial\Gamma_A} = 0 \\ \varphi|_{\partial\Gamma_K} = 1 \end{cases} \qquad \begin{cases} \mathrm{rot}_\tau \mu_r^{-1} \mathbf{curl}_\tau \psi = 0 \\ \widetilde{\mathbf{H}}_0 = \mu_r^{-1} \mathbf{curl}_\tau \psi \\ \psi|_{\partial\Gamma_A} = 0 \\ \widetilde{\mathbf{H}}_0 \cdot \mathbf{s}|_{\partial\Gamma_K} = \frac{1}{|\Gamma_K|} \end{cases}$$

ϵ, μ constant $\implies \varphi \propto \psi \implies \mathbf{E}_0 \propto \mathbf{n} \times \widetilde{\mathbf{H}}_0$

For MITLs this is reasonable : the medium is vacuum

These transverse profiles satisfy the mode condition

We're going to chose a convenient normalization

$$\mathbf{H}_0 = \frac{\mathbf{n} \times \mathbf{E}_0}{\|\mathbf{E}_0\|^2_{L^2(\Gamma)}}$$

Projection out transverse profile

$$\begin{cases} \frac{\partial V}{\partial t}\int_\Gamma \epsilon|\mathbf{E}_0|^2\ dA - \frac{\partial I}{\partial n}\int_\Gamma \mathbf{n}\times\mathbf{H}_0\cdot\mathbf{E}_0\ dA = \mathbf{0} \\ \frac{\partial I}{\partial t}\int_\Gamma \mu|\mathbf{H}_0|^2\ dA + \frac{\partial V}{\partial n}\int_\Gamma \mathbf{n}\times\mathbf{E}_0\cdot\mathbf{H}_0\ dA = \mathbf{0} \end{cases}$$

$$\int_\Gamma \mathbf{H}_0\cdot\mathbf{n}\times\mathbf{E}_0\ dA = 1$$

$$C := \int_\Gamma \epsilon|\mathbf{E}_0|^2\ dA \qquad L := \int_\Gamma \mu|\mathbf{H}_0|^2\ dA \qquad \Longrightarrow \quad CL = \epsilon\mu$$

$$\Downarrow$$

$$\begin{cases} C\frac{\partial V}{\partial t} + \frac{\partial I}{\partial n} = 0 \\ L\frac{\partial I}{\partial t} + \frac{\partial V}{\partial n} = 0 \end{cases}$$