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A 6th Order Mehrstellen Finite Volume Discretization of Poisson's Equation in Three Dimensions

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A 6th Order Mehrstellen Finite Volume Discretization of Poisson's Equation in Three Dimensions

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Abstract

We discuss the derivation of a new, sixth-order finite volume scheme for Poisson's equation on 3D Cartesian equispaced grids. The scheme is based on a discretization of the Laplace operator with a compact (*Mehrstellen*) *27-point* stencil. To achieve sixth order convergence the right hand side of the equation is replaced with a discrete operator that involves the discrete Laplace and Biharmonic operators and the sum of discrete fourth-order cross derivatives applied to the charge function. Numerical tests demonstrate the superiority of the proposed method compared to the well known schemes associated with the 7-point and 19-point discretizations of the Laplacian.

Introduction

We present a sixth-order compact finite volume scheme for Poisson's equation in a three-dimensional rectangular domain Ω

$$\Delta\phi = f \tag{1}$$

We consider a cell-centered discretization Ω^h of Ω as a union of size h cubic cells $V_{\mathbf{i}}$ centered at points $\mathbf{i}$ in index space. The general form of a cell-centered stencil for discretizing the Laplace operator as applied to a function ϕ is given by

$$\Delta^h \phi^h_{\boldsymbol{i}} = \sum_{\boldsymbol{s}\in[-l,\,l]^3} a_{\boldsymbol{s}} \phi^h_{\boldsymbol{i}+\boldsymbol{s}}$$

Here $\mathbf{i}$, the point of evaluation in index space, is the centroid of a mesh cell. The mesh size is denoted by h, l is the radius of the stencil and $a_{\boldsymbol{s}}$ are the stencil weights. With $l = 1$, $a_{(0,0,0)} = -\frac{6}{h^2}$ and $a_{(\pm1,0,0)} = a_{(0,\pm1,0)} = a_{(0,0,\pm1)} = \frac{1}{h^2}$ we obtain the well known seven-point discrete Laplace operator L^h_7 that is second-order accurate. To attain higher accuracy, a larger stencil radius is typically required. However, high order stencils can also be obtained using small values of l, the so called compact or Mehrstellen stencils [4]. A fourth-order compact finite volume scheme has been discussed in [3]. The associated stencil L^h_{19} has radius 1 and involves function evaluations at 19 cell centers. The stencil coefficients are given by

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$$a_{\boldsymbol{s}} = \begin{cases} -\frac{4}{h^2}, & \text{if } |\boldsymbol{s}| = 0 \\ \frac{1}{3h^2}, & \text{if } |\boldsymbol{s}| = 1 \\ \frac{1}{6h^2}, & \text{if } |\boldsymbol{s}| = 2 \\ 0, & \text{if } |\boldsymbol{s}| = 3 \end{cases}$$

where by $|\boldsymbol{s}|$ we denote the number of non-zero components of multi-index $\boldsymbol{s}$. To actually derive a fourth-order scheme for Poisson's equation, the forcing term f must be replaced by $f^* := f + \frac{h^2}{12}\Delta f$, otherwise the scheme is merely second-order accurate. This substitution is the so called Mehrstellen correction. In this work, we extend the approach discussed in [3] and show that a stencil of radius 1 is still sufficient for a sixth-order finite volume scheme.

Finite Volume scheme

The starting point in deriving a sixth-order stencil is the finite volume discrete formulation of Poisson's equation [3]:

$$\langle f \rangle_{\mathbf{i}} = \frac{1}{h} \sum_{d=0}^{2} \left(\left\langle \frac{\partial \phi}{\partial x_d} \right\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} - \left\langle \frac{\partial \phi}{\partial x_d} \right\rangle_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d} \right) = \frac{1}{h} \sum_{d=0}^{2} \left[\left\langle \frac{\partial \phi}{\partial x_d} \right\rangle \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} \tag{2}$$

which follows directly from integrating Poisson's equation (1) over a cell $V_{\mathbf{i}}$ and applying Gauss's theorem. Here we have defined $[\psi]_{\mathbf{j}}^{\mathbf{i}} := \psi_{\mathbf{i}} - \psi_{\mathbf{j}}$ as the value difference of a grid function ψ at points i and j in index space and by $\langle \cdot \rangle$ we denote the average value over a control volume or a face of a control volume.

We begin our discussion by considering the following estimates about finite difference stencils for partial derivatives of order up to 4

$$\delta_d^2 \psi_{\mathbf{j}} := \frac{1}{h^2} \left(\psi_{\mathbf{j}+\mathbf{e}^d} + \psi_{\mathbf{j}-\mathbf{e}^d} - 2\psi_{\mathbf{j}} \right) = \frac{\partial^2 \psi_{\mathbf{j}}}{\partial x_d^2} + O(h^2) \tag{3}$$

$$\delta_d^3 \psi_{\mathbf{j}+\frac{1}{2}\mathbf{e}^d} := \frac{1}{h^3} \left(-\psi_{\mathbf{j}-\mathbf{e}^d} + 3\psi_{\mathbf{j}} - 3\psi_{\mathbf{j}+\mathbf{e}^d} + \psi_{\mathbf{j}+2\mathbf{e}^d} \right) = \frac{\partial^3 \psi_{\mathbf{j}+\frac{1}{2}\mathbf{e}^d}}{\partial x_d^3} + O(h^2) \tag{4}$$

$$\delta_d^4 \psi_{\mathbf{j}} := \frac{1}{h^4} \left(\psi_{\mathbf{j}-2\mathbf{e}^d} - 4\psi_{\mathbf{j}-\mathbf{e}^d} + 6\psi_{\mathbf{j}} - 4\psi_{\mathbf{j}+\mathbf{e}^d} + \psi_{\mathbf{j}+2\mathbf{e}^d} \right) = \frac{\partial^4 \psi_{\mathbf{j}}}{\partial x_d^4} + O(h^2) \tag{5}$$

It is also noted that

$$\left[\delta_d^3 \psi \right]_{\mathbf{j}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{j}+\frac{1}{2}\mathbf{e}^d} = h \delta_d^4 \psi_{\mathbf{j}} \tag{6}$$

The following 6th order estimates for the average values of f and $\frac{\partial \phi}{\partial x_d}$ over $V_{\mathbf{i}}$ and face $A_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}$, expressed in terms of point values, can be easily shown using Taylor expansions and integration over the respective domains

Lemma 1.

$$\langle \psi \rangle_{\mathbf{i}} = \psi_{\mathbf{i}} + \frac{h^2}{24} \Delta \psi_{\mathbf{i}} + \frac{h^4}{1920} B \psi_{\mathbf{i}} + \frac{h^4}{1440} D \psi_{\mathbf{i}} + O(h^6)$$

$$\langle \psi \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \psi_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^2}{24} \Delta^{\perp,d} \psi_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^4}{1920} B^{\perp,d} \psi_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^4}{1440} D^{\perp,d} \psi_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + O(h^6)$$

where B is the 3D Biharmonic operator, $D = \sum_{d' \neq d''} \frac{\partial^4}{\partial^2 x_{d'} \partial^2 x_{d''}}$ is the sum of the fourth order cross derivatives and $\Delta^{\perp,d}$, $B^{\perp,d}$ and $D^{\perp,d}$ are the corresponding transverse operators (i.e. the 2D operators with respect to the directions d', d'' perpendicular to d). We also define $T := B - 2D = \sum_{d=0}^{2} \frac{\partial^4}{\partial^4 x_d}$ as the sum of fourth-order derivatives in each coordinate direction. Because of Lemma 1 we infer

$$\langle f \rangle_{\mathbf{i}} = f_{\mathbf{i}} + \frac{h^2}{24} \Delta f_{\mathbf{i}} + \frac{h^4}{1920} B f_{\mathbf{i}} + \frac{h^4}{1440} D f_{\mathbf{i}} + O(h^6) \tag{7}$$

$$\left\langle \frac{\partial \phi}{\partial x_d} \right\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \frac{\partial \phi}{\partial x_d} + \frac{h^2}{24} \Delta^{\perp,d} \frac{\partial \phi}{\partial x_d} + \frac{h^4}{1920} B^{\perp,d} \frac{\partial \phi}{\partial x_d} + \frac{h^4}{1440} D^{\perp,d} \frac{\partial \phi}{\partial x_d} + O(h^6) \tag{8}$$

where the indices of the right hand side in (8) have been omitted to simplify notation. Using Taylor expansions we easily derive the following estimates

Lemma 2.

$$\left. \frac{\partial \psi}{\partial x_d} \right|_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \frac{1}{h} \left(\psi_{\mathbf{i}+\mathbf{e}^d} - \psi_{\mathbf{i}} \right) - \frac{h^2}{24} \frac{\partial^3 \psi}{\partial x_d^3} - \frac{h^4}{1920} \frac{\partial^5 \psi}{\partial x_d^5} + O(h^6) \tag{9}$$

$$\left[\frac{\partial \psi}{\partial x_d} \right]_{\mathbf{j}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{j}+\frac{1}{2}\mathbf{e}^d} = h \delta_d^2 \psi_{\mathbf{j}} - \frac{h^2}{24} \left[\frac{\partial^3 \psi}{\partial x_d^3} \right]_{\mathbf{j}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{j}+\frac{1}{2}\mathbf{e}^d} - \frac{h^4}{1920} \left[\frac{\partial^5 \psi}{\partial x_d^5} \right]_{\mathbf{j}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{j}+\frac{1}{2}\mathbf{e}^d} + O(h^7) \tag{10}$$

From (8) and (9) we infer

$$\begin{aligned} \left\langle \frac{\partial \phi}{\partial x_d} \right\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= \frac{1}{h} \left(\phi_{\mathbf{i}+\mathbf{e}^d} - \phi_{\mathbf{i}} \right) - \frac{h^2}{24} \frac{\partial^3 \phi}{\partial x_d^3} - \frac{h^4}{1920} \frac{\partial^5 \phi}{\partial x_d^5} \\ &+ \frac{h^2}{24} \Delta^{\perp,d} \frac{\partial \phi}{\partial x_d} + \frac{h^4}{1920} B^{\perp,d} \frac{\partial \phi}{\partial x_d} + \frac{h^4}{1440} D^{\perp,d} \frac{\partial \phi}{\partial x_d} + O(h^6) \end{aligned} \tag{11}$$

and using Poisson's equation (1) to express the third and fifth order derivatives of ϕ in each direction d we obtain

$$\begin{aligned} \frac{\partial^3 \phi}{\partial x_d^3} &= \frac{\partial}{\partial x_d} \left(f - \Delta^{\perp,d} \phi \right) = \frac{\partial f}{\partial x_d} - \Delta^{\perp,d} \frac{\partial \phi}{\partial x_d} \\ \frac{\partial^5 \phi}{\partial x_d^5} &= \frac{\partial^3 f}{\partial x_d^3} - \Delta^{\perp,d} \frac{\partial f}{\partial x_d} + B^{\perp,d} \frac{\partial \phi}{\partial x_d} \end{aligned}$$

Substituting the latter algebraic relations in (11) we have

$$\begin{aligned} \left\langle \frac{\partial \phi}{\partial x_d} \right\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= \frac{1}{h} \left(\phi_{\mathbf{i}+\mathbf{e}^d} - \phi_{\mathbf{i}} \right) + \frac{h^2}{12} \Delta^{\perp,d} \frac{\partial \phi}{\partial x_d} + \frac{h^4}{1440} D^{\perp,d} \frac{\partial \phi}{\partial x_d} \\ &- \frac{h^2}{24} \frac{\partial f}{\partial x_d} - \frac{h^4}{1920} \frac{\partial^3 f}{\partial x_d^3} + \frac{h^4}{1920} \Delta^{\perp,d} \frac{\partial f}{\partial x_d} + O(h^6) \end{aligned} \tag{12}$$

and

$$\begin{aligned} \left[\left\langle \frac{\partial \phi}{\partial x_d} \right\rangle \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= h \delta_d^2 \phi_{\mathbf{i}} + \frac{h^2}{12} \left[\Delta^{\perp,d} \frac{\partial \phi}{\partial x_d} \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^4}{1440} \left[D^{\perp,d} \frac{\partial \phi}{\partial x_d} \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} \\ &- \frac{h^2}{24} \left[\frac{\partial f}{\partial x_d} \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} - \frac{h^4}{1920} \left[\frac{\partial^3 f}{\partial x_d^3} \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^4}{1920} \left[\Delta^{\perp,d} \frac{\partial f}{\partial x_d} \right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + O(h^7) \end{aligned} \tag{13}$$

We denote by S_h^d the sum of the first three terms in (13), that is

$$S_h^d\phi_{\mathbf{i}} = h\delta_d^2\phi_{\mathbf{i}} + \frac{h^2}{12}\left[\Delta^{\perp,d}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^4}{1440}\left[D^{\perp,d}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} \tag{14}$$

from which part of the 6th order operator stencil will be derived. The last three terms of (13) contribute to the Mehrstellen correction of the right hand side as we will discuss later. We proceed with approximating the second and third term of (14) in a compact fashion. At this point we recall the well known estimate for the 2D Laplacian as approximated with a 5-point stencil

$$\Delta^{\perp,d}\psi_{\mathbf{j}} = \sum_{s\neq d}\delta_s^2\psi - \frac{h^2}{12}\sum_{s\neq d}\frac{\partial}{\partial x_s^4}\frac{\partial\phi}{\partial x_d} + O(h^4)$$

This estimate together with (10) (applied with $\mathbf{j} = \mathbf{i}\pm\mathbf{e}^d$) and $\delta_s^2\psi_{\mathbf{j}} = \frac{\partial^2\psi}{\partial x_s^2} + O(h^2)$ imply

$$\begin{aligned}
\left[\Delta^{\perp,d}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= \frac{1}{h}\sum_{s\neq d}\left(\delta_d^2\phi_{\mathbf{i}+\mathbf{e}^s} + \delta_d^2\phi_{\mathbf{i}-\mathbf{e}^s} - 2\delta_d^2\phi_{\mathbf{i}}\right) \\
&- \frac{h^2}{24}\sum_{s\neq d}\left[\frac{\partial^2}{\partial x_s^2}\frac{\partial^3\phi}{\partial x_d^3}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} \\
&- \frac{h^4}{1920}\sum_{s\neq d}\left[\frac{\partial^2}{\partial x_s^2}\frac{\partial^5\phi}{\partial x_d^5}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} \\
&- \frac{h^2}{12}\sum_{s\neq d}\left[\frac{\partial^4}{\partial x_s^4}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + O(h^5)
\end{aligned} \tag{15}$$

In the following, using (3), (4) and (6) we find

$$\left[\frac{\partial^2}{\partial x_s^2}\frac{\partial^3\phi}{\partial x_d^3}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \left[\delta_d^3\frac{\partial^2\phi}{\partial x_s^2}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + O(h^3) = h\delta_d^4\delta_s^2\phi_{\mathbf{i}} + O(h^3)$$

Furthermore, it is obvious that $\left[\frac{\partial^2}{\partial x_s^2}\frac{\partial^5\phi}{\partial x_d^5}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = O(h)$ and because of (10) we obtain

$$\left[\frac{\partial^4}{\partial x_s^4}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = h\delta_d^2\delta_s^4\phi + O(h^3)$$

so that finally (15) becomes

$$\begin{aligned}
\left[\Delta^{\perp,d}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= h\sum_{s\neq d}\delta_d^2\delta_s^2\phi_{\mathbf{i}} - \frac{h^3}{24}\sum_{s\neq d}\delta_d^4\delta_s^2\phi_{\mathbf{i}} \\
&- \frac{h^3}{12}\sum_{s\neq d}\delta_d^2\delta_s^4\phi_{\mathbf{i}} + O(h^5)
\end{aligned} \tag{16}$$

Moreover, using (10) again, the third term of (14) is estimated as follows

$$\begin{aligned}
\left[D^{\perp,d}\frac{\partial\phi}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= h\delta_d^2 D^{\perp,d}\phi_{\mathbf{i}} + O(h^3) \\
&= h\delta_d^2\delta_{d'}^2\delta_{d''}^2\phi_{\mathbf{i}} + O(h^3)
\end{aligned} \tag{17}$$

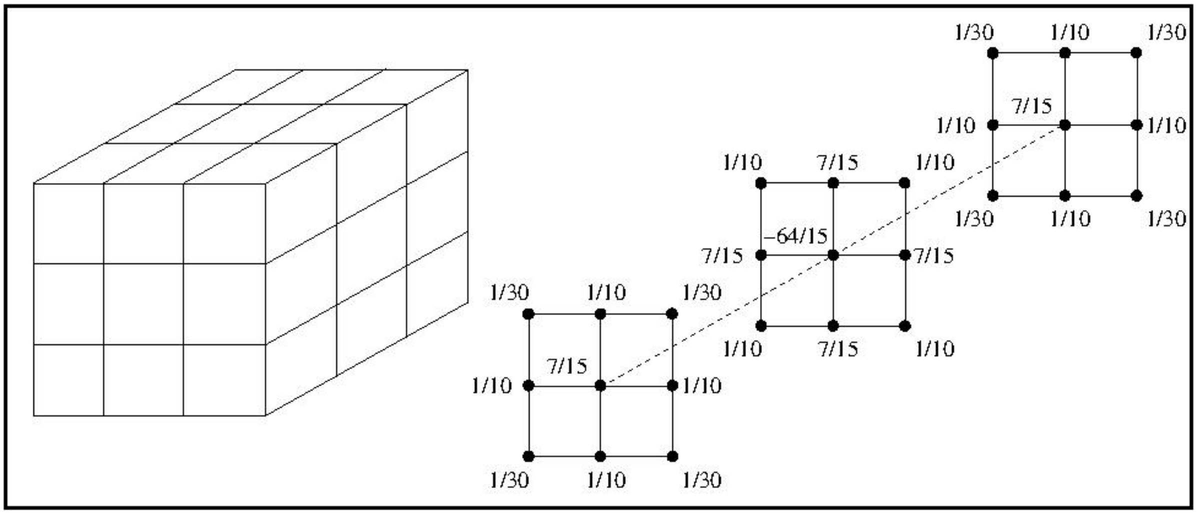

Figure 1: Stencil of the L_{27}^h Mehrstellen discrete Laplace operator. Function values are assigned at the cell centers and the actual stencil weights are the numbers depicted divided by h^2.

In the following, using (16) and (17) we obtain

$$\begin{aligned} \frac{1}{h}\sum_{d=0}^{2} S_h^d \phi_{\mathbf{i}} &= \overbrace{\sum_{d=0}^{2}\delta_d^2\phi_{\mathbf{i}} + \frac{h^2}{6}\sum_{s<d}\delta_d^2\delta_s^2\phi_{\mathbf{i}}}^{L_{19}^h} \\ &- \frac{h^4}{96}\sum_{d=0}^{2}\sum_{s\neq d}\delta_d^4\delta_s^2\phi_{\mathbf{i}} + \frac{h^4}{480}\delta_x^2\delta_y^2\delta_z^2\phi_{\mathbf{i}} + O(h^6) \end{aligned} \quad (18)$$

Poisson's equation (1) implies that [8]

$$\frac{\partial^4\phi}{\partial x_d^4} = \frac{\partial^2 f}{\partial x_d^2} - \frac{\partial^4\phi}{\partial x_d^2\partial x_{d'}^2} - \frac{\partial^4\phi}{\partial x_d^2\partial x_{d''}^2}$$

and hence [8]

$$\frac{\partial^6\phi}{\partial x^2\partial y^2\partial z^2} = \frac{\partial^4 f}{\partial x_d^2\partial x_{d''}^2} - \frac{\partial^6\phi}{\partial x_d^4\partial x_{d''}^2} - \frac{\partial^6\phi}{\partial x_{d''}^4\partial x_d^2}$$

As a result

$$3\frac{\partial^6\phi}{\partial x^2\partial y^2\partial z^2} = Df - \sum_{d=0}^{2}\sum_{s\neq d}\frac{\partial^6\phi}{\partial x_d^4\partial x_s^2}$$

which is rewritten in discrete form as

$$-\sum_{d=0}^{2}\sum_{s\neq d}\delta_d^4\delta_s^2\phi_{\mathbf{i}} = -D^h f_{\mathbf{i}} + 3\delta_x^2\delta_y^2\delta_z^2\phi_{\mathbf{i}} + O(h^2) \quad (19)$$

where $D^h = \delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2$ is the discrete sum of fourth-order cross derivatives which is second-order accurate. At this point we define the following stencil of radius 1

$$L_{27}^h\phi_{\mathbf{i}} := L_{19}^h\phi_{\mathbf{i}} + \frac{h^4}{30}\delta_x^2\delta_y^2\delta_z^2\phi_{\mathbf{i}} \quad (20)$$

that involves function evaluations at 27 points and is depicted in Figure 1. We will prove that it results in a sixth-order scheme for Poisson's equation. Indeed, (18) and the latter estimate (19) yield

$$\frac{1}{h}\sum_{d=0}^{2} S_h^d \phi_{\mathbf{i}} = L_{27}^h \phi_{\mathbf{i}} - \frac{h^4}{96} D^h f_{\mathbf{i}} + O(h^6) \tag{21}$$

In the sequel, we treat the last three terms of Equation (13) to derive the Mehrstellen correction of the scheme. Indeed, (4) and (6) imply

$$\sum_{d=0}^{2} \left[\frac{\partial^3 f}{\partial x_d^3}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = hT^h f_{\mathbf{i}} + O(h^3) \tag{22}$$

Using the latter and (10) we derive

$$\begin{aligned} \sum_{d=0}^{2} \left[\frac{\partial f}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= \sum_{d=0}^{2} h\delta_d^2 f_{\mathbf{i}} - \frac{h^2}{24}\sum_{d=0}^{2}\left[\frac{\partial^3 f}{\partial x_d^3}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + O(h^5) \\ &= hL_7^h f_{\mathbf{i}} - \frac{h^3}{24} T^h f_{\mathbf{i}} + O(h^5) \end{aligned} \tag{23}$$

In addition, (10) implies that

$$\begin{aligned} \sum_{d=0}^{2} \left[\Delta^{\perp,d}\frac{\partial f}{\partial x_d}\right]_{\mathbf{i}-\frac{1}{2}\mathbf{e}^d}^{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} &= h\sum_{d=0}^{2}\delta_d^2 \Delta^{\perp,d} f_{\mathbf{i}} + O(h^3) \\ &= h\sum_{d=0}^{2}\delta_d^2\left(\delta_{d'}^2 + \delta_{d''}^2\right) f_{\mathbf{i}} + O(h^3) \\ &= 2hD^h f_{\mathbf{i}} + O(h^3) \end{aligned} \tag{24}$$

Equation (2) because of (7), (13) and (21)-(24) becomes

$$\begin{aligned} &L_{27}^h \phi_{\mathbf{i}} - \frac{h^4}{96} D^h f_{\mathbf{i}} - \frac{h^2}{24} L_7^h f_{\mathbf{i}} + \frac{h^4}{576} T^h f_{\mathbf{i}} - \frac{h^4}{1920} T^h f_{\mathbf{i}} + \frac{2h^4}{1920} D^h f_{\mathbf{i}} = \\ &f_{\mathbf{i}} + \frac{h^2}{24}\Delta f_{\mathbf{i}} + \frac{h^4}{1920} B f_{\mathbf{i}} + \frac{h^4}{1440} D f_{\mathbf{i}} + O(h^6) \end{aligned}$$

hence, considering second order approximations B^h, T^h, D^h of B, T, D, respectively, we obtain

$$L_{27}^h \phi_{\mathbf{i}} = f_{\mathbf{i}} + \frac{h^2}{24}\Delta f_{\mathbf{i}} + \frac{h^2}{24} L_7^h f_{\mathbf{i}} - \frac{h^4}{1440} T^h f_{\mathbf{i}} + \frac{h^4}{90} D^h f_{\mathbf{i}} + O(h^6) \tag{25}$$

Next, using the truncation error of the L_7^h Laplacian we have

$$\Delta f_{\mathbf{i}} + L_7^h f_{\mathbf{i}} = -\frac{h^2}{12} T^h f_{\mathbf{i}} + 2L_7^h f_{\mathbf{i}} + O(h^4)$$

so that (25) becomes

$$L_{27}^h \phi_{\mathbf{i}} = f_{\mathbf{i}} + \frac{h^2}{12} L_7^h f_{\mathbf{i}} - \frac{h^4}{240} B^h f_{\mathbf{i}} + \frac{7h^4}{360} D^h f_{\mathbf{i}} + O(h^6) \tag{26}$$

or since $L_{19}^h = L_7^h + \frac{h^2}{6} D^h$

$$L_{27}^h \phi_{\mathbf{i}} = f_{\mathbf{i}} + \frac{h^2}{12} L_{19}^h f_{\mathbf{i}} - \frac{h^4}{240} B^h f_{\mathbf{i}} + \frac{h^4}{180} D^h f_{\mathbf{i}} + O(h^6) \tag{27}$$

As is evident from equation (27), a sixth order discretization of Poisson's equation can be derived by employing the L_{27}^h discrete Laplacian and a Mehrstellen correction of the right hand side given by

$$f_{\mathbf{i}}^* = f_{\mathbf{i}} + \frac{h^2}{12} L_{19}^h f_{\mathbf{i}} - \frac{h^4}{240} B^h f_{\mathbf{i}} + \frac{h^4}{180} D^h f_{\mathbf{i}}$$

It is noted that using the original charge function results in a second order scheme.

Numerical Results

As a representative test case we consider Poisson's equation in the unit cube $\Omega = [0,\, 1]^3$ with the following charge

$$f(\mathbf{x}) = -12\pi^2 cos(2\pi x) cos(2\pi y) cos(2\pi z)$$

and periodic boundary conditions. The compatibility condition $\int_\Omega f d\mathbf{x} = 0$ is satisfied for this problem and hence the solution which is given by

$$\phi(\mathbf{x}) = cos(2\pi x) cos(2\pi y) cos(2\pi z)$$

is unique up to a constant. We solve this boundary problem numerically with Martin's relaxation method [6] and employ the L_7^h, L_{19}^h and L_{27}^h discretizations of the Laplacian for a comparison. The associated max-norm errors are presented in Table 1 and as is evident the 6th order compact scheme outperforms the other two. A log-log plot of the max-norm error against mesh size for the new scheme demonstrates its 6th order convergence rate (Figure 2).

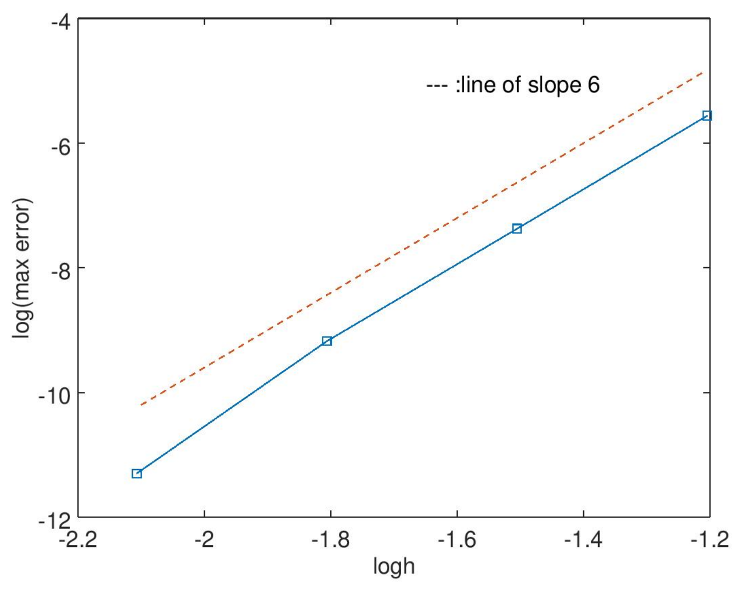

Figure 2: Convergence plot using the L_{27}^h discrete compact Laplace operator.

N	L_7^h	L_{19}^h	L_{27}^h
16	1.29507e-02	2.34866e-04	2.75100e-06
32	3.21896e-03	1.45076e-05	4.23717e-08
64	8.03577e-04	9.04328e-07	6.65203e-10
128	2.00820e-04	5.75698e-08	5.01377e-12

Table 1: Max-norm errors $max_{\mathbf{i}} \|\phi_{\mathbf{i}}^h - \phi\|_\infty$ with periodic boundary conditions using the L_7^h, L_{19}^h and L_{27}^h discrete Laplacians.

Conclusions

We have introduced a provably sixth-order finite volume discretization of Poisson's equation based on a compact Laplace stencil and a suitable modification of the charge function. Compact stencils for the Laplace operator are essential to a new class of fast adaptive Poisson solvers [2, 7, 5] that are based on Anderson's Method of Local Corrections (MLC) [1]. Future work will include the application of the present scheme to the MLC method for Poisson's equation. Of interest is the extension of the MLC method to other equations of Mathematical Physics via the derivation of compact stencils for the associated differential operators.

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