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Larry K. Warne, Luis San Martin, and Alden R. Pack

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico
87185 and Livermore,
California 94550

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Estimated Maximum Electric Field In A Cavity Supporting Eccentric Coaxial Modes

Larry K. Warne, Luis San Martin, and Alden R. Pack
Electromagnetic Theory & Simulation Dept.
Sandia National Laboratories
P. O. Box 5800
Albuquerque, NM 87185-1152

Abstract

Using the power balance method we estimate the maximum electric field on a conducting wall of a cavity containing an interior structure supporting eccentric coaxial modes in the frequency regime where the resonant modes are isolated from each other.

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1 INTRODUCTION

The power balance approach to estimating interior cavity fields uses estimates of net power entering an aperture, along with estimates of interior wall losses to estimate mean interior squared fields. Then extreme field levels are extrapolated from the mean levels. When interior structures are present which locally enhance the fields, these extrapolations must take into account such structures. This report shows an example where an interior structure has the form of an eccentric coax and the maximum field exists on the side of the coax in proximity to the cavity wall. This example is a nice illustration of how such structures can locally enhance the cavity fields.

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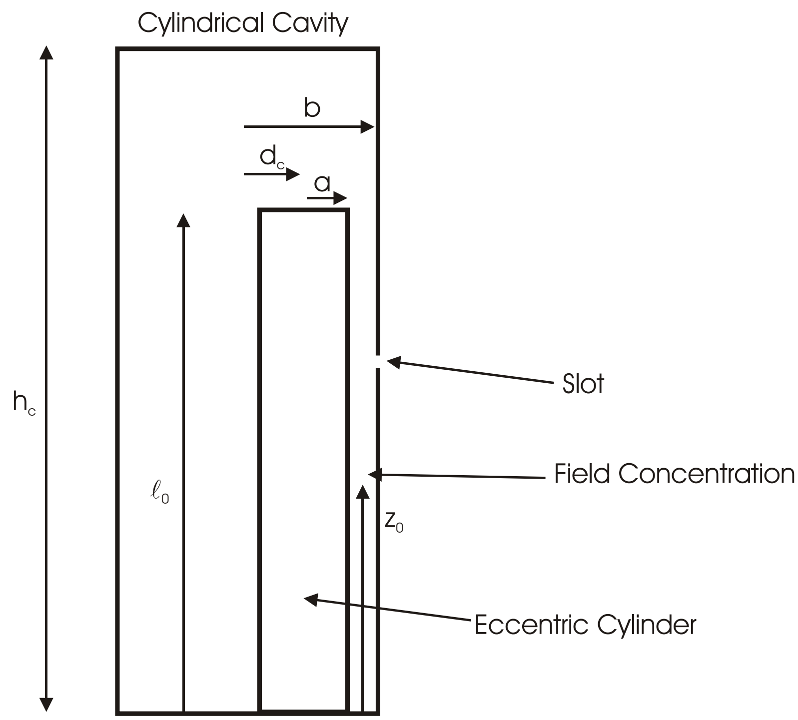

Figure 1: Geometry of cylindrical cavity containing eccentrered coaxial conductor.

2 MAXIMUM ELECTRIC FIELD IN ECCENTRIC COAXIAL MODE

We examine the maximum electric field associated with the eccentric coaxial mode within a cylindrical cavity driven by an azimuthal slot as illustrated in Figure 1.

2.1 TEM-Like Mode Potential & Electric Field

The potential in an eccentric coax can be written as [1], [2]

$$\phi_0 = -\frac{q_0}{2\pi\varepsilon_0} \ln \sqrt{\frac{x^2 + (y + y_1 - h - y_c)^2}{x^2 + (y + y_1 - h + y_c)^2}} - U_1$$

where q_0 is the charge per unit length, $\varepsilon_0 = 8.854188$ pF/m is the permittivity of free space, and

$$U_1 = \frac{q_0}{2\pi\varepsilon_0} \text{Arccosh}\,(y_1/b)$$

The potential $\phi_0 = 0$ on the outer cylinder of radius b and $\phi_0 = V_0$ on the inner cylinder of radius a, where the charge per unit length can be found from the voltage and the capacitance per unit length

$$q_0 = V_0 2\pi\varepsilon_0 / \text{Arccosh}\left(\frac{a^2 + b^2 - d_c^2}{2ab}\right) = V_0 C_0$$

The origin in this expression is shifted to $y = h$ from the center of the outer cylinder of radius b, and the center of the inner cylinder is at $y = h - d_c$. The quantity y_c can be found from

$$2y_c d_c = \sqrt{\left[(b-a)^2 - d_c^2\right]\left[(b+a)^2 - d_c^2\right]}$$

and y_1 can be found from

$$y_1 = \sqrt{y_c^2 + b^2} = \left(-a^2 + b^2 + d_c^2\right)/\left(2d_c\right)$$

where we also note that

$$\left(y_1 - y_c\right)\left(y_1 + y_c\right) = \left(y_1^2 - y_c^2\right) = b^2$$

Taking $h = 0$ with $y = \rho \sin\varphi$ and $x = \rho \sin\varphi$ the radial field near the outer conductor $\rho = b$ is

$$E_{0\rho}\left(\rho, \varphi\right) = \frac{q_0}{2\pi\varepsilon_0}\left[\frac{\rho + \left(y_1 - y_c\right)\sin\varphi}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - y_c\right)^2} - \frac{\rho + \left(y_1 + y_c\right)\sin\varphi}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 + y_c\right)^2}\right]$$

which at $\rho = b$ can be written as

$$E_{0\rho}\left(b, \varphi\right) = \frac{q_0}{2\pi b\varepsilon_0}\frac{y_c}{y_1 + b\sin\varphi}$$

Alternatively, taking $h = d_c$ with $y = \rho\sin\varphi$ and $x = \rho\cos\varphi$ the radial field near the inner conductor $\rho = a$ is

$$E_\rho\left(\rho, \varphi\right) = -\frac{\partial}{\partial\rho}\phi_0 = \frac{q_0}{2\pi\varepsilon_0}\frac{\partial}{\partial\rho}\ln\sqrt{\frac{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - d_c - y_c\right)^2}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - d_c + y_c\right)^2}}$$

$$= \frac{q_0}{2\pi\varepsilon_0}\left[\frac{\rho + \left(y_1 - d_c - y_c\right)\sin\varphi}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - d_c - y_c\right)^2} - \frac{\rho + \left(y_1 - d_c + y_c\right)\sin\varphi}{\rho^2\cos^2\varphi + \left(\rho\sin\varphi + y_1 - d_c + y_c\right)^2}\right]$$

$$= \frac{q_0}{2\pi\varepsilon_0}\left[\frac{\rho + \left(y_1 - d_c - y_c\right)\sin\varphi}{\rho^2 + 2\left(y_1 - d_c - y_c\right)\rho\sin\varphi + \left(y_1 - d_c - y_c\right)^2} - \frac{\rho + \left(y_1 - d_c + y_c\right)\sin\varphi}{\rho^2 + 2\left(y_1 - d_c + y_c\right)\rho\sin\varphi + \left(y_1 - d_c + y_c\right)^2}\right]$$

$$E_\rho\left(a, \varphi\right) = \frac{q_0}{2\pi\varepsilon_0}\left[\frac{a + \left(y_1 - d_c - y_c\right)\sin\varphi}{a^2 + 2a\left(y_1 - d_c - y_c\right)\sin\varphi + \left(y_1 - d_c - y_c\right)^2} - \frac{a + \left(y_1 - d_c + y_c\right)\sin\varphi}{a^2 + 2a\left(y_1 - d_c + y_c\right)\sin\varphi + \left(y_1 - d_c + y_c\right)^2}\right]$$

$$= \frac{q_0}{2\pi\varepsilon_0}\left[\left\{a + \left(y_1 - d_c - y_c\right)\sin\varphi\right\}\left\{a^2 + 2a\left(y_1 - d_c + y_c\right)\sin\varphi + \left(y_1 - d_c + y_c\right)^2\right\}\right.$$

$$\left. - \left\{a + \left(y_1 - d_c + y_c\right)\sin\varphi\right\}\left\{a^2 + 2a\left(y_1 - d_c - y_c\right)\sin\varphi + \left(y_1 - d_c - y_c\right)^2\right\}\right] /$$

$$\left[\left\{a^2 + 2a\left(y_1 - d_c - y_c\right)\sin\varphi + \left(y_1 - d_c - y_c\right)^2\right\}\left\{a^2 + 2a\left(y_1 - d_c + y_c\right)\sin\varphi + \left(y_1 - d_c + y_c\right)^2\right\}\right]$$

Simplifying the numerator

$$\left\{a + \left(y_1 - d_c - y_c\right)\sin\varphi\right\}\left\{a^2 + 2a\left(y_1 - d_c + y_c\right)\sin\varphi + \left(y_1 - d_c + y_c\right)^2\right\}$$

$$- \left\{a + \left(y_1 - d_c + y_c\right)\sin\varphi\right\}\left\{a^2 + 2a\left(y_1 - d_c - y_c\right)\sin\varphi + \left(y_1 - d_c - y_c\right)^2\right\}$$

$$= a\left\{(y_1 - d_c + y_c)^2 - (y_1 - d_c - y_c)^2\right\}$$

$$+a^2\left\{a + (y_1 - d_c - y_c)\sin\varphi - a - (y_1 - d_c + y_c)\sin\varphi\right\}$$

$$+\left\{a + (y_1 - d_c - y_c)\sin\varphi\right\} 2a\,(y_1 - d_c + y_c)\sin\varphi - \left\{a + (y_1 - d_c + y_c)\sin\varphi\right\} 2a\,(y_1 - d_c - y_c)\sin\varphi$$

$$+ (y_1 - d_c - y_c)\sin\varphi\,(y_1 - d_c + y_c)^2 - (y_1 - d_c + y_c)\sin\varphi\,(y_1 - d_c - y_c)^2$$

$$= 4ay_c\,(y_1 - d_c) - 2y_c a^2\sin\varphi$$

$$+2a^2\,(y_1 - d_c + y_c)\sin\varphi - 2a^2\,(y_1 - d_c - y_c)\sin\varphi$$

$$+ (y_1 - d_c - y_c)\sin\varphi 2a\,(y_1 - d_c + y_c)\sin\varphi - (y_1 - d_c + y_c)\sin\varphi 2a\,(y_1 - d_c - y_c)\sin\varphi$$

$$+\left\{(y_1 - d_c)^2 - y_c^2\right\}\sin\varphi\,(y_1 - d_c + y_c) - \left\{(y_1 - d_c)^2 - y_c^2\right\}\sin\varphi\,(y_1 - d_c - y_c)$$

$$= 4ay_c\,(y_1 - d_c) - 2y_c a^2\sin\varphi + 4a^2 y_c\sin\varphi$$

$$+2y_c\left\{(y_1 - d_c)^2 - y_c^2\right\}\sin\varphi$$

$$= 4ay_c\,(y_1 - d_c) + 2a^2 y_c\sin\varphi$$

$$+2y_c\left\{(y_1 - d_c)^2 - y_c^2\right\}\sin\varphi$$

$$= 2y_c\left[2a\,(y_1 - d_c) + \left\{a^2 + (y_1 - d_c)^2 - y_c^2\right\}\sin\varphi\right]$$

$$= 2y_c\left[2a\,(y_1 - d_c) + \left\{b^2 + a^2 + d_c^2 - 2d_c y_1\right\}\sin\varphi\right]$$

$$= 2y_c\left[2a\,(y_1 - d_c) + \left\{b^2 + a^2 + d_c^2 - \left(-a^2 + b^2 + d_c^2\right)\right\}\sin\varphi\right]$$

$$= 4ay_c\left[(y_1 - d_c) + a\sin\varphi\right]$$

Then

$$E_\rho\,(a,\varphi) = \frac{2y_c a q_0}{\pi\varepsilon_0}\left[\frac{y_1 - d_c + a\sin\varphi}{\left\{a^2 + 2a\,(y_1 - d_c - y_c)\sin\varphi + (y_1 - d_c - y_c)^2\right\}\left\{a^2 + 2a\,(y_1 - d_c + y_c)\sin\varphi + (y_1 - d_c + y_c)^2\right\}}\right]$$

Simplifying the denominator

$$\left\{a^2 + 2a\,(y_1 - d_c - y_c)\sin\varphi + (y_1 - d_c - y_c)^2\right\}\left\{a^2 + 2a\,(y_1 - d_c + y_c)\sin\varphi + (y_1 - d_c + y_c)^2\right\}$$

$$= a^4 + a^2 \left\{ (y_1 - d_c - y_c)^2 + (y_1 - d_c + y_c)^2 \right\} + \left\{ (y_1 - d_c)^2 - y_c^2 \right\}^2$$

$$+2a^3 (y_1 - d_c - y_c + y_1 - d_c + y_c) \sin\varphi + 2a \left\{ (y_1 - d_c - y_c)(y_1 - d_c + y_c)^2 + (y_1 - d_c + y_c)(y_1 - d_c - y_c)^2 \right\} \sin\varphi$$

$$+4a^2 (y_1 - d_c - y_c)(y_1 - d_c + y_c) \sin^2\varphi$$

$$= a^4 + 2a^2 \left\{ (y_1 - d_c)^2 + y_c^2 \right\} + \left\{ (y_1 - d_c)^2 - y_c^2 \right\}^2$$

$$+4a^3 (y_1 - d_c) \sin\varphi + 4a (y_1 - d_c) \left\{ (y_1 - d_c)^2 - y_c^2 \right\} \sin\varphi$$

$$+4a^2 \left\{ (y_1 - d_c)^2 - y_c^2 \right\} \sin^2\varphi$$

$$= 2a^2 \left\{ a^2/2 + (y_1 - d_c)^2 + y_c^2 \right\} + \left\{ (y_1 - d_c)^2 - y_c^2 \right\}^2$$

$$4a (y_1 - d_c) \left\{ a^2 + (y_1 - d_c)^2 - y_c^2 \right\} \sin\varphi$$

$$+4a^2 \left\{ (y_1 - d_c)^2 - y_c^2 \right\} \sin^2\varphi$$

$$= 2a^2 \left\{ a^2/2 - b^2 + (y_1 - d_c)^2 + y_1^2 \right\} + \left\{ b^2 + (y_1 - d_c)^2 - y_1^2 \right\}^2$$

$$4a (y_1 - d_c) \left\{ a^2 + b^2 + (y_1 - d_c)^2 - y_1^2 \right\} \sin\varphi$$

$$+4a^2 \left\{ b^2 + (y_1 - d_c)^2 - y_1^2 \right\} \sin^2\varphi$$

$$= 2a^2 \left\{ a^2/2 - b^2 + 2y_1^2 - 2d_c y_1 + d_c^2 \right\} + \left\{ b^2 - 2dy_1 + d_c^2 \right\}^2$$

$$4a (y_1 - d_c) \left\{ a^2 + b^2 - 2d_c y_1 + d_c^2 \right\} \sin\varphi$$

$$+4a^2 \left\{ b^2 - 2d_c y_1 + d_c^2 \right\} \sin^2\varphi$$

$$= 4a^2 \left\{ a^2/4 + a^2/2 - b^2 + y_1^2 \right\} + a^4$$

$$+8a^3 (y_1 - d_c) \sin\varphi + 4a^4 \sin^2\varphi$$

$$= 4a^2 \left[\left\{ -a^2/4 + a^2 - b^2 + y_1^2 \right\} + \frac{1}{4} a^2 + 2a (y_1 - d_c) \sin\varphi + a^2 \sin^2\varphi \right]$$

$$= 4a^2 \left[(y_1 - d_c)^2 + 2a (y_1 - d_c) \sin\varphi + a^2 \sin^2\varphi \right]$$

$$= 4a^2 (y_1 - d_c + a \sin\varphi)^2$$

where

$$(y_1 - d_c + a\sin\varphi)^2 = (y_1 - d_c)^2 + 2a(y_1 - d_c)\sin\varphi + a^2\sin^2\varphi$$

$$(y_1 - d_c)^2 = d_c^2 - 2d_c y_1 + y_1^2 = a^2 - b^2 + y_1^2$$

$$E_\rho(a,\varphi) = \frac{q_0}{2\pi a\varepsilon_0}\left[\frac{y_c}{y_1 - d_c + a\sin\varphi}\right]$$

2.2 Magnetic Field & Quality Factor

Noting that (the plus sign is for propagation in the z direction and the minus sign is for propagation in the $-z$ direction)

$$\omega\mu_0 \underline{H}_t = \pm k\underline{e}_z \times \underline{E}_t$$

or

$$\eta_0 H_\varphi(a,\varphi) = \pm E_\rho(a,\varphi)$$

$$\eta_0 H_\varphi(b,\varphi) = \pm E_\rho(b,\varphi)$$

where $\mu_0 = 4\pi\times10^{-7}$ H/m is the free space magnetic permeability and $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \approx 120\pi$ ohms is the impedance of free space. We can find the losses per unit length along the eccentric coax

$$P = R_s\int_{-\pi}^{\pi}|H_\varphi(a,\varphi)|^2 a d\varphi + R_s\int_{-\pi}^{\pi}|H_\varphi(b,\varphi)|^2 b d\varphi$$

$$= 2R_s\left(\frac{cC_0}{2\pi a}\right)^2|V_0|^2\int_{-\pi/2}^{\pi/2}\left(\frac{y_c}{y_1 - d_c + a\sin\varphi}\right)^2 a d\varphi + 2R_s\left(\frac{cC_0}{2\pi b}\right)^2|V_0|^2\int_{-\pi/2}^{\pi/2}\left(\frac{y_c}{y_1 + b\sin\varphi}\right)^2 b d\varphi$$

where $\varepsilon_0\eta_0 = 1/c$, $c = 1/\sqrt{\mu_0\varepsilon_0}$ is the vacuum velocity of light. Letting $u = \sin\varphi$ and $\varphi = \arcsin(u)$ and $d\varphi = du/\sqrt{1-u^2}$

$$P = \left(\frac{C_0^2/\varepsilon_0^2}{2\pi^2 a}\right)R_s|V_0/\eta_0|^2\int_{-1}^{1}\left(\frac{y_c}{y_1 - d_c + au}\right)^2\frac{du}{\sqrt{1-u^2}} + \left(\frac{C_0^2/\varepsilon_0^2}{2\pi^2 b}\right)R_s|V_0/\eta_0|^2\int_{-1}^{1}\left(\frac{y_c}{y_1 + bu}\right)^2\frac{du}{\sqrt{1-u^2}}$$

Taking

$$(y_1 - d_c)/a = \left(b^2 - a^2 - d_c^2\right)/(2d_c a) = p_0$$

$$b - a > d_c \rightarrow p_0 = \left(\sqrt{b^2 - a^2} - d_c\right)\left(\sqrt{b^2 - a^2} + d_c\right)/(2d_c a)$$

$$\sqrt{b^2 - a^2} = \sqrt{(b-a)(b+a)} > b - a \rightarrow p_0 > 0$$

$$1 - p_0 = \frac{2d_c a - \left(b^2 - a^2 - d_c^2\right)}{2d_c a} = -\frac{b^2 - (a + d_c)^2}{2d_c a} = -\frac{(b - a - d_c)(b + a + d_c)}{2d_c a} < 0$$

$$y_1/b = \left(b^2 - a^2 + d_c^2\right)/(2d_c b) = p_1 > 0$$

$$1-p_1=\frac{2d_cb-\left(b^2-a^2+d_c^2\right)}{2d_cb}=-\frac{\left(b-d_c\right)^2-a^2}{2d_cb}=-\frac{\left(b-d_c-a\right)\left(b-d_c+a\right)}{2d_cb}<0$$

$$P=\left(\frac{C_0^2/\varepsilon_0^2}{2\pi^2 a}\right)\left(\frac{y_c}{a}\right)^2 R_s\left|V_0/\eta_0\right|^2\int_{-1}^{1}\frac{du}{\left(p_0+u\right)^2\sqrt{1-u^2}}+\left(\frac{C_0^2/\varepsilon_0^2}{2\pi^2 b}\right)\left(\frac{y_c}{b}\right)^2 R_s\left|V_0/\eta_0\right|^2\int_{-1}^{1}\frac{du}{\left(p_1+u\right)^2\sqrt{1-u^2}}$$

Using [3]

$$\int_{-1}^{1}\frac{du}{\left(p+u\right)^2\sqrt{1-u^2}}=-\int_{1/(p-1)}^{1/(p+1)}\frac{tdt}{\sqrt{-1+2pt+\left(1-p^2\right)t^2}}$$
$$=-\frac{1}{2\left(1-p^2\right)}\int_{1/(p-1)}^{1/(p+1)}\frac{\left[2\left(1-p^2\right)t+2p-2p\right]dt}{\sqrt{-1+2pt+\left(1-p^2\right)t^2}}$$

$$=-\frac{1}{2\left(1-p^2\right)}\int_{1/(p-1)}^{1/(p+1)}d\sqrt{-1+2pt+\left(1-p^2\right)t^2}+\frac{p}{\left(1-p^2\right)}\int_{1/(p-1)}^{1/(p+1)}\frac{dt}{\sqrt{-1+2pt+\left(1-p^2\right)t^2}}$$

$$=-\frac{1}{2\left(1-p^2\right)}\left[\sqrt{-1+2pt+\left(1-p^2\right)t^2}\right]_{1/(p-1)}^{1/(p+1)}+\frac{p}{\left(p^2-1\right)\sqrt{p^2-1}}\arcsin\left[\frac{2\left(1-p^2\right)t+2p}{\sqrt{4p^2+4\left(1-p^2\right)}}\right]_{1/(p-1)}^{1/(p+1)}$$

$$=-\frac{1}{2\left(1-p^2\right)}\left[\sqrt{-1+2pt+\left(1-p^2\right)t^2}\right]_{-1/(1-p)}^{1/(1+p)}+\frac{p}{\left(p^2-1\right)\sqrt{p^2-1}}\arcsin\left[\frac{2\left(1-p^2\right)t+2p}{\sqrt{4p^2+4\left(1-p^2\right)}}\right]_{-1/(1-p)}^{1/(1+p)}$$

$$=-\frac{1}{2\left(1-p^2\right)}\left[\sqrt{-1+\frac{2p}{1+p}+\frac{1-p^2}{\left(1+p\right)^2}}-\sqrt{-1-\frac{2p}{1-p}+\frac{1-p^2}{\left(1-p\right)^2}}\right]$$
$$+\frac{p}{\left(p^2-1\right)\sqrt{p^2-1}}\left\{\arcsin\left[\frac{2\left(\frac{1-p^2}{1+p}\right)+2p}{\sqrt{4p^2+4\left(1-p^2\right)}}\right]-\arcsin\left[\frac{-2\left(\frac{1-p^2}{1-p}\right)+2p}{\sqrt{4p^2+4\left(1-p^2\right)}}\right]\right\}$$
$$=-\frac{1}{2\left(1-p^2\right)}\left[\sqrt{-1+\frac{2p}{1+p}+\frac{1-p}{1+p}}-\sqrt{-1-\frac{2p}{1-p}+\frac{1+p}{1-p}}\right]$$
$$+\frac{p}{\left(p^2-1\right)\sqrt{p^2-1}}\left\{\arcsin\left[\left(\frac{1-p^2}{1+p}\right)+p\right]-\arcsin\left[-\left(\frac{1-p^2}{1-p}\right)+p\right]\right\}$$
$$=\frac{p}{\left(p^2-1\right)\sqrt{p^2-1}}\left\{\arcsin\left(1\right)-\arcsin\left(-1\right)\right\}$$
$$=\frac{\pi p}{\left(p^2-1\right)\sqrt{p^2-1}}$$

or

$$\int_{-1}^{1}\frac{du}{\left(p+u\right)^2\sqrt{1-u^2}}=\frac{\pi p}{\left(p^2-1\right)\sqrt{p^2-1}}\ ,\ p>1$$

$$P = \left(\frac{C_0^2/\varepsilon_0^2}{2\pi a}\right)\left(\frac{y_c}{a}\right)^2 R_s \left|V_0/\eta_0\right|^2 \frac{p_0}{\left(p_0^2-1\right)^{3/2}} + \left(\frac{C_0^2/\varepsilon_0^2}{2\pi b}\right)\left(\frac{y_c}{b}\right)^2 R_s \left|V_0/\eta_0\right|^2 \frac{p_1}{\left(p_1^2-1\right)^{3/2}}$$

$$= \left(\frac{C_0^2/\varepsilon_0^2}{2\pi a}\right) R_s \left|V_0/\eta_0\right|^2 \frac{y_c^2\left(y_1-d_c\right)}{\left\{\left(y_1-d_c\right)^2-a^2\right\}^{3/2}} + \left(\frac{C_0^2/\varepsilon_0^2}{2\pi b}\right) R_s \left|V_0/\eta_0\right|^2 \frac{y_c^2 y_1}{\left(y_1^2-b^2\right)^{3/2}}$$

$$= \left(\frac{C_0^2/\varepsilon_0^2}{2\pi a}\right) R_s \left|V_0/\eta_0\right|^2 \frac{y_c^2\left(y_1-d_c\right)}{\left(y_1^2-b^2\right)^{3/2}} + \left(\frac{C_0^2/\varepsilon_0^2}{2\pi b}\right) R_s \left|V_0/\eta_0\right|^2 \frac{y_c^2 y_1}{\left(y_1^2-b^2\right)^{3/2}}$$

$$= \left(\frac{C_0^2/\varepsilon_0^2}{2\pi}\right) R_s \left|V_0/\eta_0\right|^2 \frac{y_c^2}{\left(y_1^2-b^2\right)^{3/2}} \left[\left(y_1-d_c\right)/a + y_1/b\right]$$

$$= \left(\frac{C_0^2/\varepsilon_0^2}{2\pi}\right) R_s \left|V_0/\eta_0\right|^2 \frac{1}{\sqrt{y_1^2-b^2}} \left[\left(y_1-d_c\right)/a + y_1/b\right]$$

$$P = \left(\frac{C_0^2/\varepsilon_0^2}{2\pi}\right) R_s \left|V_0/\eta_0\right|^2 \frac{\left(b^2-d_c^2-a^2\right)/a + \left(b^2+d_c^2-a^2\right)/b}{\sqrt{\left\{\left(b-d_c\right)^2-a^2\right\}\left\{\left(b+d_c\right)^2-a^2\right\}}}$$

where

$$\left(y_1-d_c\right)^2 = d_c^2 - 2d_c y_1 + y_1^2 = a^2 - b^2 + y_1^2$$

$$y_1 = \sqrt{y_c^2+b^2} = \left(-a^2+b^2+d_c^2\right)/\left(2d_c\right)$$

$$y_1 - d_c = \left(-a^2+b^2-d_c^2\right)/\left(2d_c\right)$$

To obtain the total power lost (without the end cap)

$$P_{tot} = P\int_0^{\ell_0} \cos^2\left(n\frac{\pi}{2}z/\ell_0\right)dz = P\frac{1}{2}\int_0^{\ell_0}\left[1+\cos\left(n\pi z/\ell_0\right)\right]dz = P\ell_0/2$$

where n is the number of half wave oscillations along the eccentric coax (we are neglecting capacitive loading at the open circuited end). The stored energy per unit length is

$$W = C_0\left|V_0\right|^2 = \left(C_0/\varepsilon_0\right)\mu_0\left|V_0/\eta_0\right|^2$$

To obtain the total modal stored energy

$$W_{tot} = W\int_0^{\ell_0} \sin^2\left(n\frac{\pi}{2}z/\ell_0\right)dz = W\frac{1}{2}\int_0^{\ell_0}\left[1-\cos\left(n\pi z/\ell_0\right)\right]dz = W\ell_0/2$$

The modal quality factor is then

$$Q = \frac{\omega W_{tot}}{P_{tot}} = \frac{\omega W}{P} = \frac{\omega\mu_0}{R_s}\frac{2\pi}{C_0/\varepsilon_0}\frac{\sqrt{\left\{\left(b-d_c\right)^2-a^2\right\}\left\{\left(b+d_c\right)^2-a^2\right\}}}{\left(b^2-d_c^2-a^2\right)/a+\left(b^2+d_c^2-a^2\right)/b}$$

where

$$\frac{2\pi}{C_0/\varepsilon_0} = \text{Arccosh}\left(\frac{a^2+b^2-d_c^2}{2ab}\right)$$

and

$$\mathrm{Arccosh}\,(u) = \ln\left(u + \sqrt{u^2 - 1}\right)$$

Both the numerator and denominator are weighted by the squares of the distributions of current and voltage; hence these cancel in the total quantities (again neglecting the losses on the end cap and the capacitive loading at the open circuit).

The extra losses at the shorted end cap are expected to be a small fraction and ignoring these gives an upper bound for the quality factor. For the centered coax the results are very simple

$$P = R_s \int_{-\pi}^{\pi} |H_\varphi(a,\varphi)|^2 \, a d\varphi + R_s \int_{-\pi}^{\pi} |H_\varphi(b,\varphi)|^2 \, b d\varphi$$

$$= R_s \left|\frac{I}{2\pi a}\right|^2 2\pi a + R_s \left|\frac{I}{2\pi b}\right|^2 2\pi b = R_s \frac{|V_0/\eta_0|^2}{2\pi \ln^2(b/a)} (1/a + 1/b)$$

$$P_{tot} = (\ell_0/2)\, R_s \frac{|V_0/\eta_0|^2}{2\pi \ln^2(b/a)} (1/a + 1/b)$$

$$P_{cap} \approx R_s \int_a^b |H_\varphi(\rho)|^2 \, 2\pi\rho d\rho = R_s \int_a^b \left|\frac{I}{2\pi\rho}\right|^2 2\pi\rho d\rho = R_s \int_a^b \left|\frac{V_0/\left\{\eta_0 \ln(b/a)\right\}}{2\pi\rho}\right|^2 2\pi\rho d\rho$$

$$= R_s \frac{|V_0/\eta_0|^2}{2\pi \ln(b/a)}$$

Comparing these with the coax part on the left and the centered end cap on the right

$$\frac{(\ell_0/2)}{\ln(b/a)} (1/a + 1/b) \Leftrightarrow 1$$

and if we take $a = 1$ inch, $b = 4$ inches, and $\ell_0 = 16$ inches

$$\frac{16 \text{ in}/2}{\ln(4)} (1/1 \text{ in} + 1/4 \text{ in}) \approx 7.2135 \Leftrightarrow 1$$

The eccentric coax has

$$P_{tot} = (\ell_0/2) \left(\frac{C_0^2/\varepsilon_0^2}{2\pi}\right) R_s |V_0/\eta_0|^2 \frac{\left(b^2 - d_c^2 - a^2\right)/a + \left(b^2 + d_c^2 - a^2\right)/b}{\sqrt{\left\{(b - d_c)^2 - a^2\right\}\left\{(b + d_c)^2 - a^2\right\}}}$$

Comparing this to the centered end cap gives an even larger disparity

$$(\ell_0/2)\left(C_0^2/\varepsilon_0^2\right) \frac{\left(b^2 - d_c^2 - a^2\right)/a + \left(b^2 + d_c^2 - a^2\right)/b}{\sqrt{\left\{(b - d_c)^2 - a^2\right\}\left\{(b + d_c)^2 - a^2\right\}}} \Leftrightarrow \frac{1}{\ln(b/a)}$$

and

$$\frac{4\pi^2}{1.1796875^2} \frac{16 \text{ in}/2}{0.382825 \text{ in}} \approx 592.81 \Leftrightarrow \frac{1}{\ln(4)} \approx 0.72135$$

The total magnetic field in the eccentered coax can be used to more rigorously estimate the end cap losses but it is more complicated to carry out the integrations involved.

Neglecting the end cap losses (and the loading capacitance) in this example we take $a = 1$ in, $d_c = 2.75$ in, $b = 4$ in, to find

$$\frac{2\pi}{C_0/\varepsilon_0} \approx 1.1796875$$

$$\frac{\sqrt{\left\{(b-d_c)^2 - a^2\right\}\left\{(b+d_c)^2 - a^2\right\}}}{\left(b^2 - d_c^2 - a^2\right)/a + \left(b^2 + d_c^2 - a^2\right)/b} \approx \frac{5.00663622 \text{ in}^2}{7.4375 \text{ in} + 5.640625 \text{ in}} \approx 0.382825 \text{ in} \approx 9.72376 \text{ mm}$$

$$Q = \frac{\omega W}{P} = \frac{\omega \mu_0}{R_s} 0.011471 \text{ m}$$

At $f = 1$ GHz using wall conductivity $\sigma = 2.6 \times 10^7$ S/m for 6061 aluminum, the quality factor is

$$Q \approx 7,350$$

2.3 Maximum Electric Field

The maximum radial field at axial position z_0 is

$$\begin{aligned} E_\rho(a, -\pi/2) &= V_0 \sin\left(n\frac{\pi}{2} z_0/\ell_0\right) \frac{C_0/\varepsilon_0}{2\pi} \left(\frac{y_c/a}{y_1 - d_c - a}\right) \\ &= V_0 \sin\left(n\frac{\pi}{2} z_0/\ell_0\right) \frac{C_0/\varepsilon_0}{2\pi} \frac{2y_c d_c/a}{b^2 - (a+d_c)^2} \end{aligned}$$

where

$$y_1 - d_c = \left(-a^2 + b^2 - d_c^2\right)/(2d_c)$$

$$2y_c d_c = \sqrt{\left[(b-a)^2 - d_c^2\right]\left[(b+a)^2 - d_c^2\right]}$$

This can be written as

$$E_\rho(a, -\pi/2) = \sin\left(n\frac{\pi}{2} z_0/\ell_0\right) \frac{V_0}{a} \frac{C_0/\varepsilon_0}{2\pi} \frac{\sqrt{\left[(b-a)^2 - d_c^2\right]\left[(b+a)^2 - d_c^2\right]}}{b^2 - (a+d_c)^2}$$

where in this example

$$\frac{\sqrt{\left[(b-a)^2 - d_c^2\right]\left[(b+a)^2 - d_c^2\right]}}{b^2 - (a+d_c)^2} \approx 2.5840703$$

or using the preceding capacitance per unit length

$$E_\rho(a, -\pi/2) = \sin\left(n\frac{\pi}{2} z_0/\ell_0\right) \frac{V_0}{a} 2.19047$$

Noting

$$\int_0^{\ell_0} \cos^2\left(n\frac{\pi}{2} z/\ell_0\right) dz = \frac{1}{2}\int_0^{\ell_0} \left[1 + \cos\left(n\pi z/\ell_0\right)\right] dz = \ell_0/2$$

we have

$$QP_{tot}/\omega = W_{tot} = (\ell_0/2)\, W = (\ell_0/2)\, \frac{(C_0/\varepsilon_0)}{2\pi} 2\pi\varepsilon_0 \left|V_0\right|^2$$

or

$$\frac{2\pi}{C_0/\varepsilon_0} QP_{tot}/\left(\ell_0 \pi a \omega \varepsilon_0 a\right) = \frac{\left|V_0\right|^2}{a^2}$$

If we take the slot net input cross section to be σ_{in} and the incident Poynting vector amplitude to be

$$S_0 = \left|E_0\right|^2/\eta_0$$

where E_0 is the incident wave amplitude, then we can write the total input power (equal to the power lost as) [4]

$$P_{tot} = \sigma_{in} S_0$$

Neglecting any radiation leaving the slot in order to bound the interior modal amplitude, and assuming the resonant modes are isolated from each other, we can then write

$$\left|E_\rho\left(a, -\pi/2\right)/E_0\right|^2 = \sin^2\left(n\frac{\pi}{2} z_0/\ell_0\right) \frac{Q\sigma_{in}}{\ell_0 \pi a k a} \left(\frac{C_0/\varepsilon_0}{2\pi}\right) \left\{\frac{\sqrt{\left[(b-a)^2 - d_c^2\right]\left[(b+a)^2 - d_c^2\right]}}{b^2 - (a+d_c)^2}\right\}^2$$

Because we have not included the capacitive load at the open circuited end of the eccentric coax, there may be a shift in the peak locations of $\sin\left(n\pi z/\left(2\ell_0\right)\right)$ as a function of z. Hence we will estimate the average and peak values

$$\left\langle \left|E_\rho\left(a, -\pi/2\right)/E_0\right|^2 \right\rangle_z = \frac{Q\sigma_{in}}{2\ell_0 \pi a k a} \left(\frac{C_0/\varepsilon_0}{2\pi}\right) \left\{\frac{\sqrt{\left[(b-a)^2 - d_c^2\right]\left[(b+a)^2 - d_c^2\right]}}{b^2 - (a+d_c)^2}\right\}^2$$

$$\left|E_\rho\left(a, -\pi/2\right)/E_0\right|_{pk}^2 = 2\left\langle \left|E_\rho\left(a, -\pi/2\right)/E_0\right|^2 \right\rangle_z$$

2.4 Slot Received Power

The slot input power is bounded by the matched received power [4]

$$\sigma_{in} = \sigma_{rec} \sim q_{rec}^{tl} \frac{\lambda\ell}{1.38 \mathrm{Cin}\left(4\pi\ell/\lambda\right)} \sim q_{rec}^{tl} \frac{\lambda\ell}{1.38\left[\gamma' + \ln\left(4\pi\ell/\lambda\right)\right]}$$

where $\gamma' \approx 0.5772$ is Euler's constant, $\ell = 2h = 5.5$ in is the slot length, and the slot mismatch factor is

$$q_{rec}^{tl} = \frac{G_{rad}/2}{G_{rad}/2 + R_{int}/Z_0^2}$$

The slot characteristic impedance is

$$Z_0 \sim \eta_0 \pi/\Omega_e$$

with fatness parameter

$$\Omega_e = 2\ln\left(2h/a_e\right) + C_e \sim 2\left\{\ln\left(\pi\ell/w\right) - 4/3\right\} + \pi d/w$$

where

$$C_e = 2\left(\ln 2 - 7/3\right)$$

and equivalent radius

$$a_e \sim \frac{2w}{\pi e} e^{-\pi d/(2w)} \ , \ d > w/3$$

where the slot depth is $w = 0.01$ in and the slot depth is $d = 0.25$ in. The radiation conductance is taken as the value at the resonances

$$\pi \eta_0 h G_{rad} = \mathrm{Cin}\left(4kh\right) \ , \ kh = n\pi/2 \tag{1}$$

$$\sim \gamma' + \ln\left(4kh\right) = \gamma' + \ln\left(4\pi\ell/\lambda\right) \ , \ kh >> 1$$

The slot wall losses give

$$R_{int} \approx \left(\frac{L}{L^{intr}}\right)^2 2R_s/d$$

where the slot inductance per unit length is

$$L \sim \mu_0 \pi/\Omega_e$$

and the interior slot inductance per unit length is

$$L^{intr} = \mu_0 w/d \tag{2}$$

the surface impedance of the metal is

$$R_s = 1/\left(\sigma\delta\right)$$

with skin depth

$$\delta = \sqrt{2/\left(\omega\mu\sigma\right)}$$

where for 6061 aluminum the magnetic permeability can be taken as free space $\mu = \mu_0$ and the electric conductivity is $\sigma \approx 2.6 \times 10^7$ S/m. The first slot resonance is near $kh = \pi/2$ or with $\ell = 5.5$ in at $f \approx 1.073$ GHz. Noting $\Omega_e \approx 90.7824$ and $Z_0 \approx 13.046$ ohms, therefore near $f = 1$ GHz (assuming we are at the slot resonance in G_{rad}) we find $R_s \approx 0.01232$ ohms, $R_{int} \approx 2.905$ ohms/m, $\mathrm{Cin}(2\pi) \approx 2.43765$, $G_{rad} \approx 0.02947$ S/m

$$q_{rec}^{tl} = \frac{G_{rad}/2}{G_{rad}/2 + R_{int}/Z_0^2} \approx 0.4633$$

$$\sigma_{in} \approx 5.3757 \times 10^{-3} \ \mathrm{m}^2$$

2.5 Maximum Eccentered Coaxial Mode Field Example

Inserting dimensions $\ell_0 = 16$ in, $a = 1$ in, $d_c = 2.75$ in, $w = 0.01$ in, $d = 0.25$ in, $\ell = 5.5$ in, $\sigma = 2.6 \times 10^7$ S/m, and using the preceding value of quality factor gives

$$\left\langle \left| E_\rho \left(a, -\pi/2\right)/E_0 \right|^2 \right\rangle_z \approx \frac{(7,350)\left(5.3757 \times 10^{-3}\ \mathrm{m}^2\right)}{\left(3.4527388 \times 10^{-2}\ \mathrm{m}^2\right)} \frac{(2.5840703)^2}{1.1796875} \approx 6477\,(+38.1\ \mathrm{dB})$$

and

$$\left| E_\rho \left(a, -\pi/2\right)/E_0 \right|^2_{pk} = 2 \left\langle \left| E_\rho \left(a, -\pi/2\right)/E_0 \right|^2 \right\rangle_z \approx +41.1\ \mathrm{dB}$$

3 TYPICAL CAVITY MODES

Now we examine the usual power balance procedure for an empty cavity as a comparison to the preceding eccentric coaxial mode results [4].

3.1 Cavity Without Inner Cylinder

If there were no inner cylinder with cavity dimensions having radius $b = 4$ in and height $h_c = 24$ in

$$S = 2\pi b^2 + 2\pi b h_c \approx 0.45401 \text{ m}^2$$

$$V = \pi b^2 h_c \approx 0.01976888 \text{ m}^3$$

The stored energy can be written as

$$W_{tot} = V\varepsilon_0 \left\langle |\underline{E}|^2 \right\rangle_V = V\mu_0 3 \left\langle |\underline{H}|^2 \right\rangle_V = V\varepsilon_0 3 \left\langle |E_j|^2 \right\rangle_V = V\mu_0 3 \left\langle |H_j|^2 \right\rangle_V$$

where E_j and H_j are components. The power lost on the cavity walls is

$$P_{tot} = SR_s \left\langle |\underline{H}|^2 \right\rangle_S = SR_s 2 \left\langle |H_j|^2 \right\rangle_S = SR_s 4 \left\langle |H_j|^2 \right\rangle_V$$

The quality factor is then

$$Q = \frac{\omega W_{tot}}{P_{tot}} = \frac{\omega V \varepsilon_0 \left\langle |\underline{E}|^2 \right\rangle_V}{SR_s \left\langle |\underline{H}|^2 \right\rangle_S} = \frac{3}{4}\frac{\omega \mu_0 V}{R_s S} = \frac{3}{2}\frac{V}{\delta S}\,(\mu = \mu_0) \approx 20,925.4$$

Taking the power lost to equal the power in

$$P_{tot} = \sigma_{in} S_0$$

where S_0 is the incident Poynting vector magnitude. Then from the formula for Q

$$\omega\varepsilon_0 V \left\langle |\underline{E}|^2 \right\rangle_V = \omega\varepsilon_0 V 3 \left\langle |E_j|^2 \right\rangle_V = QP_{tot} = Q\sigma_{in} S_0 = Q\sigma_{in} |E_0|^2 / \eta_0$$

or the mean square electric field divided by the incident squared electric field is

$$\left\langle |E_j|^2 / |E_0|^2 \right\rangle_V = \frac{1}{2} \left\langle |E_j|^2 / |E_0|^2 \right\rangle_S = Q\sigma_{in} / (3kV)$$

For a three-dimensional standing wave we extrapolate from mean square to the square of the peak

$$|E_j|^2_{\max} / |E_0|^2 \approx 8Q\sigma_{in} / (3kV)$$

Using the preceding input cross section

$$\sigma_{in} \approx 5.3757 \times 10^{-3} \text{ m}^2$$

and at 1 GHz

$$|E_j|^2_{\max} / |E_0|^2 \approx 8\,(20,925.4)\left(5.3757 \times 10^{-3} \text{ m}^2\right) / \left[3\left(20.9586 \text{ m}^{-1}\right) 0.01976888 \text{ m}^3\right]$$

$$\approx 724\,(+28.6 \text{ dB})$$

Even if we add an additional 3 dB to address extreme levels of modal indices (or possible wall increases) this is still only +31.6 dB, which is considerable less than the eccentric coaxial estimate.

3.2 Cavity With Inner Cylinder

If we do the same calculation but with the inner cylinder present we obtain

$$S = 2\pi b^2 + 2\pi b h_c + 2\pi a \ell_0 \approx 0.45401 \text{ m}^2 + 0.064858557 \text{ m}^2 \approx 0.518868557 \text{ m}^2$$

$$V = \pi b^2 h_c - \pi a^2 \ell_0 \approx 0.01976888 \text{ m}^3 - 0.00082370368 \text{ m}^3 \approx 0.01894518 \text{ m}^3$$

$$Q = \frac{\omega W_{tot}}{P_{tot}} = \frac{\omega V \varepsilon_0 \left\langle |\underline{E}|^2 \right\rangle_V}{S R_s \left\langle |\underline{H}|^2 \right\rangle_S} = \frac{3}{4} \frac{\omega \mu_0 V}{R_s S} = \frac{3}{2} \frac{V}{\delta S} \approx 17,546.85$$

$$|E_j|^2_{\max} / |E_0|^2 \approx 8\,(17,546.85)\left(5.3757 \times 10^{-3} \text{ m}^2\right) / \left[3\left(20.9586 \text{ m}^{-1}\right) 0.01894518 \text{ m}^3\right]$$

$$\approx 607.1\,(+27.8 \text{ dB})$$

Even if we add an additional 3 dB to address extreme levels of modal indices (or possible wall increases) this is still only +30.8 dB, which is considerable less than the eccentric coaxial estimate.

4 CONCLUSIONS

We see from the power balance results in this report that the estimated maximum electric field in the eccentric coaxial modes (+41.1 dB) are significantly above the maximum fields estimated by the usual power balance procedure applied to empty cavities (yielding $+30.8 \rightarrow +31.6$ dB). This is not surprising since such interior metallic structures can locally enhance the field.

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