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Geometric Tail Approximation for Reliability and Survivability

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ABSTRACT

A common problem in developing high-reliability systems is estimating the reliability for a population of components that cannot be 100% tested. The radiation survivability of a population of components is often estimated by testing a very small sample to some multiple of the required specification level, known as an overtest. Given a successful test with a sufficient overtest margin, the population of components is assumed to have the required survivability or radiation reliability. However, no mathematical justification for such claims has been crafted without making aggressive assumptions regarding the statistics of the unknown distribution. Here we illustrate a new approach that leverages geometric bounding arguments founded on relatively modest distribution assumptions to produce conservative estimates of component reliability.

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EXECUTIVE SUMMARY

GTARS, the Geometric Tail Approximation for Reliability and Survivability is a data analysis technique. It provides a mathematical basis for predicting reliability and confidence values for components when using very small sample sizes and an overtest to some stress level in excess of the requirement.

In the past, overtesting has found application in some areas of qualifying components for use in weapons or on spacecraft. The most well-known example is radiation survivability testing, where it is somewhat common practice to test samples as small as five components, but overtest them to some higher radiation stress level, and then assert that the small sample passing the overtest suggests that the population of components would pass the required stress with some unknown but high probability. While this approach is attractive in some scenarios, there has been no mathematical justification for such assertions.

GTARS was intended to solve the problem of predicting reliability and confidence when small samples are subjected to an overtest. While GTARS cannot provide exact predictions for reliability and confidence, it does provide a lower bounding value that can be mathematically justified. This bounding value is a function of both the sample size and the overtest factor. In many instances, overtest factors as small as two or three can result in high reliability estimates for a population of components.

While GTARS was developed to address the topic of predicting population survivability for radiation testing, the approach can be useful for any area of reliability testing where an overtest can be employed with pass/fail criteria.

ACRONYMS AND DEFINITIONS

Abbreviation	Definition
CDF	Cumulative Density Function
GTARS	Geometric Tail Approximation for Reliability and Survivability
PDF	Probability Density Function
ELDRS	Enhanced Low Dose Rate Sensitivity
HALT	Highly Accelerated Lifecycle Testing

1. INTRODUCTION AND BACKGROUND

Assessing the reliability of components when the entire population can be tested to 100% of the required operating environment is relatively simple, if time consuming. Unfortunately, when tests are destructive in nature, testing the entire population is not possible. The more difficult problem is to predict the reliability of a population of components from a small sample. In such cases, predictions come with confidence intervals as one is, in effect, assessing the probability of the reliability, or in more general terms, dealing with the probability of probabilities. In such cases, interpretation of the data is more difficult, and any prediction of reliability comes with caveats.

When expensive components are in short supply, such as early in the development of a new system, or when the tests in question are expensive, time consuming or otherwise difficult to perform, an approach sometimes used is to test a very small quantity of components to a stress much greater than required. Given that this very small sample passes this *overtest*, the population is deemed to be reliable or survivable at the lower, required stress. This approach is often pursued when testing components for survivability in a radiation environment.

Overtesting attempts to trade margin for the quantity of components tested. Overtesting is not an uncommon practice in reliability qualification as certain well accepted test approaches, such as the highly accelerated lifecycle testing (HALT) trades margin for time, but not margin for quantity. The problem with testing a reduced quantity of components is that in the general case there has been no mathematical justification for doing so. Asserting that a population will survive a required stress simply because a sample survived when subjected to a significant stress above the requirement is not technically defendable. It may look right. It may seem as if it should be true. But without making significant assumptions regarding the parameters of the statistical distribution, such assertions have lacked credible mathematical foundations.

This report introduces a new approach to overtesting, where mathematically defensible approximations are used to produce an upper bound to the failure probability (or a lower bound to the reliability) for a population. Similar to other statistical techniques, reliability predictions are bound to confidence estimates, but with this new approach, very small sample sizes, when combined with an overtest, can be used to predict the reliability or survivability of a larger population.

1.1. Binomial Estimates of Reliability

When attempting to predict the reliability for a population of components, where 100% testing is not possible, a common approach is to test a small sample in a given environment and use the results of these tests to predict the characteristics of the population. One approach would be to test each component in the small sample to failure and then use the distribution of failures from the sample to make a prediction regarding the distribution of failures for the population. A problem with this approach is that it can be quite expensive, determining the exact failure threshold of each part in every environment, and the statistics of the sample might be such that they incorrectly predict an unacceptable fraction of the population would fail. This could easily result from small samples having greater standard deviations than larger samples.

Another approach, and the one most often encountered, is to again test a small sample of components, but only test them to the required survivability or reliability. Based on the number of components in the sample, and the number of survivors and failures, binomial statistics[1] are used to calculate a family of possible reliability values, with each such prediction being paired with an associated confidence. The reliability is really a probability of survival or a probability of passing, and the confidence is in effect a probability that the reliability value is equal to or greater than the value chosen. Stated another way, reliability predictions are all about probabilities of probabilities. There are no definitive answers without 100% testing.

When calculating reliability and confidence estimate pairs, one can either start with a reliability and calculate the confidence or start with a desired confidence and calculate the associated reliability. Either can be determined from the other, given the sample size and the number of successes or failures in the test series. For a sample size, n, if the test series results in no failures, equations (1) and (2) can be used to calculate either the reliability, r, or the confidence, c. When a test series includes failures, a greater sample size, n, is required to achieve a given combination of reliability and confidence, and the equations are more complex than those seen in (1) and (2).

$$c = 1 - r^n \tag{1}$$

$$r = \sqrt[n]{1 - c} \tag{2}$$

As an example, high reliability systems often seek a reliability value of r=0.97 (97%) or greater. This can be achieved by testing 23 components with zero failures, provided a confidence value of c=0.5 (50%) is acceptable. Higher confidence values either require larger samples, or lower reliability values. A more detailed discussion of binomial statistics and reliability predictions can be found in Appendix A.

Table I presents some of the possible combinations of reliability and confidence that may be derived from testing 23 components with zero failures. With this sample size, one can be almost completely certain that the population reliability is at least 80%, or, one could be 91% certain that the population reliability is at least 90%. The choice of 50% certainty that the reliability is at least 97% might leave room for some anxiety regarding the actual reliability. This is a choice both for those who are designing and for those who will be operating the system.

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[1] When testing without replacement, technically one should use hypergeometric statistics as binomial statistics assume replacement. However, given a sufficiently large population and a relatively small sample size, the differences are minor and binomial statistics are used as their mathematical form is more easily employed.

Table I. Statistics for 23/0 Pass/Fail Testing

Reliability (%)	Confidence (%)
97	50.4
95	69.3
93	81.2
90	91.1
85	97.6
80	99.4

Rather than arbitrarily selecting a reliability and accepting the calculated confidence, one might ask what the most probable reliability is, given the results of pass/fail testing of the small sample. A mathematical approach to selecting a reliability is to use Laplace's Rule of Succession [1]. This rule is most useful in situations where a modest number of components have been tested but no failures have been observed. In the absence of failures, it leaves one to ponder what the probability is that the next component tested will result in a success or a failure. Laplace rigorously demonstrated that in the case of n successes with zero failures, the probability that the next component tested will pass (probability of next success, pns) is given by the simple formula found in equation (3).

$$pns = \frac{n+1}{n+2} \tag{3}$$

For the present example, with 23 successes and zero failures, equation (3) gives a probability (or most probable reliability) of 96% which would then have a confidence of 60.9%, calculated from equation (1).

1.2. Traditional Overtest Approach

When components are expensive or in short supply, or when the tests themselves are difficult, expensive, or excessively time consuming, destructively testing even a small sample of 23 components is a difficult choice. An approach sometimes encountered is to overtest a much smaller sample. Then if all components from this much smaller sample pass when subjected to a multiple of the required environment, the argument made is that this clearly demonstrates that the population of components would pass the much less demanding required environmental stress with a high probability.

As an example, total dose radiation testing of electronic components used in satellites or nuclear weapons is by its nature, a destructive test. Frequently, some of the components will be very expensive and in limited supply as it is difficult to make radiation hardened components. How then should test engineers certify that the population of components would survive a specified radiation dose, s?

An approach that has been used frequently in the radiation effects community is to overtest a very small sample of, for example, five parts. The magnitude of the overtest depends on the components, the radiation environment, and the system the parts will be used in, but overtest factors ranging from two to ten are not uncommon. For those who are unfamiliar with overtesting in radiation environments, the approach is often used at Sandia National Laboratories, NASA, the USAF, and some of their supporting contractors.

For the purposes of this report, when overtesting, sample sizes of five and seven are emphasized, but sample sizes as small as four are sometimes encountered in the literature. If the specified survivability stress is, s, and an overtest factor of, k, is used, then the actual environmental stress applied to the components will be, q, as given in equation (4).

$$q = k * s \tag{4}$$

For values of k in the range of two to ten (or greater), the assumption is that if the small sample of components survive an overtest with zero failures, then the remaining population of components would certainly survive the required stress, s. One could also assume that the greater the overtest, the more likely the population is to survive the required stress. This is shown graphically in Figure 1. While this seems reasonable and the approach appears to be an attractive option for reliability testing, there exists no rigorous mathematical basis for making this assessment. It is simply an unsupportable assumption. Stated another way, the assumption is nothing more than a guess.

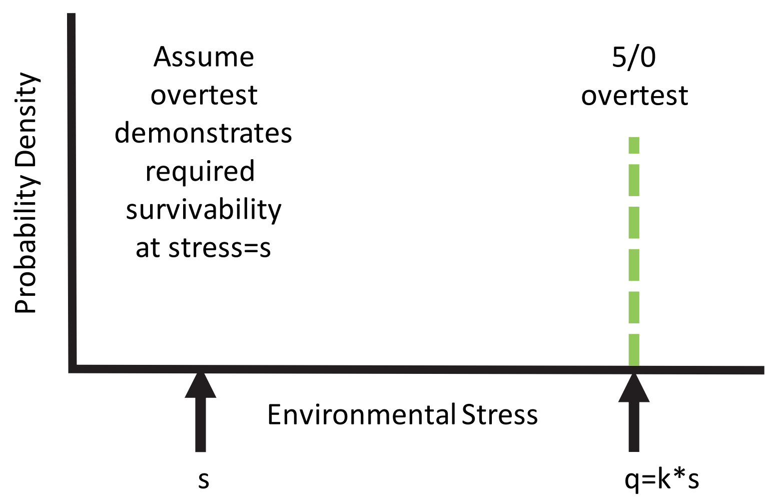

Figure 1. Overtest scheme using five parts.

1.3. The Power of Belief in Mathematical Discourse

It may be difficult to accept that a methodology developed to qualify components for high-consequence systems could have weak mathematical foundations. However, human beings have a substantial capacity for pattern-matching biases that drive them to create and build attachments to hypotheses absent any proof. The following vignettes from the history of mathematics demonstrate key principles informing the development of a sensible approach for interpreting overtest results.

In approximately the year, 1637, Pierre de Fermat, a French jurist and amateur mathematician wrote a note (one of many) in the margin of his copy of Bachet's translation of Diophantus's Arithmetica:

> *Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere: cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*

An approximate translation of this note is:

> The equation $x^n+y^n=z^n$ has no integral solutions when n>2. I have discovered a perfectly marvelous proof, but this margin is not big enough to hold it.

This note went on to become known as Fermat's Last Theorem. Essentially all the notes Fermat left in his book were eventually proven correct or false, save his last claim, which is more properly identified as a conjecture rather than a theorem, since Fermat left no proof of his claim. Fermat's note was published by his son in 1670. Over the intervening decades, even centuries, most of the best mathematicians in the world attempted to prove the conjecture. Much progress was made on the proof of special cases, but a general and complete proof was elusive.

Fermat's Last Theorem was interesting because it was so simple to state and remained interesting because it was so difficult to prove. The attempts to prove the conjecture resulted in the discovery and development of significant volumes of new math, but all without a solution to the conjecture itself. For hundreds of years, everyone knew Fermat's Last Theorem was correct. Based on the work of hundreds of mathematicians and countless attempts to find a counterexample, it just seemed to be right, but no one could prove it to be correct until 1994, when Andrew Wiles published his proof for a special case of the Taniyama-Shimura conjecture, thus completing the missing link in a long line of proofs that together demonstrated Fermat's Last Theorem to be correct[2-4].

In 1769, the legendary Swiss mathematician, Leonard Euler, proposed what has come to be known as, Euler's Sum of Powers Conjecture. Euler was generalizing Fermat's Last Theorem to higher powers and a greater number of terms. Stated simply, Euler proposed that for any prime integer, k, it would require at least k integers raised to the k^{th} power to sum to another integer raised to the k^{th} power. In mathematical terms, for all integers n and k greater than 1, if the sum of n, k^{th} powers of positive integers is itself a k^{th} power, then n is greater than or equal to k:

$$a_1^k + a_2^k + a_3^k + \cdots + a_n^k = b^k \tag{5}$$

for $n \geq k$.

As an example, for k=5, it would require at least 5 integers raised to the k^{th} power to sum to another integer raised to the k^{th} power, as

$$a_1^5 + a_2^5 + a_3^5 + a_4^5 + a_5^5 = b^5 \tag{6}$$

Similar to Fermat's Theorem, the Euler Sum of Powers conjecture attracted the attention of many mathematicians. None were able to prove the conjecture, but over time, everyone came to believe it was correct. Based on experience, it just seemed to be right. Euler's conjecture remained proven until 1966 when L. J. Lander and T. R. Parkin published a counter example [5].

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5 \tag{7}$$

Everyone knew Euler's Sum of Powers conjecture was correct, until they found out it wasn't. The practice of using an overtest on a small sample to demonstrate the reliability of a population should not be assumed to be correct, until it is proven so. Just because it seems to be correct, it feels right, and it is economically advantageous, does not mean that the assumption is correct.

When a mathematical problem proves difficult, or even intractable, one approach towards a solution is to work on a related problem that if proven correct, would help to bound or in some way limit the original problem. An example is the harmonic series. Consider a series of integers, *H*, defined as

$$H_n = \sum_1^n \frac{1}{n} \tag{8}$$

Equation ($H_n = \sum_1^n \frac{1}{n}$ (8) defines what is known as a harmonic series. The n[th] term in the series can be expressed as,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{9}$$

As n becomes infinite, does the series converge towards a finite value, or does it diverge towards infinity? As defined, the series passes the convergence test, but this does not mean that it converges (a common misunderstanding of the convergence test). The answer is not obvious and either way, this is difficult to prove. However, it is relatively easy to bound the harmonic series with another series where the terms are always less than or equal to those of the harmonic series. This alternate series can easily be shown to expand towards infinity, thereby implying that the harmonic series is also unbounded.

Rewriting (9) to include more terms, note that the series in (11) is, term by term, always less than or equal to the series in (10).

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \ldots \tag{10}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} \ldots \tag{11}$$

Next note that

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2} \tag{12}$$

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \tag{13}$$

$$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{2} \tag{14}$$

and thus, the series in (11) can be expressed as

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \ldots \tag{15}$$

which is clearly infinite. While it is difficult to prove the harmonic series itself is infinite, it is easy to prove that the bounding series seen in (11) is infinite, thus confirming the harmonic series to be infinite.

The foregoing historical vignettes clearly demonstrate two key principles:

1. An idea may be attractive and widely accepted, yet nevertheless be flatly wrong.

2. Bounding approximations may be very useful when exact determinations cannot be made.

2. GEOMETRIC TAIL APPROXIMATION FOR RELIABILITY AND SURVIVABILITY (GTARS)

The mathematical sidebar in the previous section is interesting by itself, but it helps to illustrate two important points regarding the efficacy of the overtest approach for predicting reliability for a population of components from a small sample. First, even though the approach seems to be correct, it is not possible to state that it is correct without proof. Second, while it is impossible to prove the correctness of results from the overtest approach, it might be possible to prove the correctness of an approach that provides an upper limit, bounding the value for the probability of failure, and therefore, also provides a lower limit, bounding the value for the reliability.

When one tests a very small random sample of components, n, and zero failures are observed, all that can be said about the population is that a random sample of n components were tested to some stress level and they all passed. It is not possible to make any statements regarding the population distribution itself. Based on binomial statistics, one can use the sample size to calculate a family of reliability and confidence pairs. As an example, for a sample size of five components, if tested to stress level q=k*s with zero failures (noted as a 5/0 test), survivability and confidence estimates are shown in Table II. (Note that the terms survivability and reliability are used interchangeably in this paper.)

Table II. Statistics for 5/0 Pass/Fail Overtest at Stress Level q=k*s

Survivability (%)	Confidence (%)
97	14.1
95	22.6
93	30.4
90	41.0
85	55.6
80	67.2

The available reliability (survivability) and confidence estimates shown in Table II are rather unattractive for those attempting to develop high reliability systems. It is, however, critical to note that the reliability (survivability) calculated for the sample of n components, is useful only for predicting the reliability (survivability) at the stress level used for the test. If the parts are overtested by some factor, k, then the reliability predictions are for that stress level, q=k*s, and, this point being critical to understand, no information regarding the reliability of the population at the required stress level, s, is known or can be directly determined.

To find an upper bound for the probability of failure for the population when exposed to the required stress level, s, an approach known as the Geometric Tail Approximation for Reliability and Survivability (GTARS) is developed here. For many applications, the only assumption required for GTARS is that the distribution of failure thresholds for the population can be described as being *normal like*. This simply means that the distribution has a single mode (hump), and no values less than or equal to zero, except at zero environmental stress. The height and width of the distribution are unknown. While many systems or components will have failure distributions for some applied stress that are *normal like*, this is not a hard requirement for using the GTARS approach. Variations of GTARS are available that should be useful regardless of the actual shape of the population failure distribution. Appendix B presents details for *normal-like* distributions, while Appendix C deals with distributions that are uniform, or box-like on their left extreme. For failure distributions that decrease with increasing stress, Appendix D discusses exponential decay functions while Appendix E presents an approach for distributions better described as a decreasing triangle or wedge. Without loss of generality, in this section, only *normal-like* distributions are considered.

A *normal-like* distribution, while not needing to be strictly normal, must be somewhat similar to a normal distribution. Specifically, the left tail of the distribution must not extend to stress values less than zero. Second, the left tail must start from zero and then curve upwards. The behavior of the distribution towards the right tail is of no concern, but the left tail needs to be as described here. These characteristics are shown in Figure 2. A multitude of probability distribution functions satisfy these requirements, including the normal distribution (provided it is clipped at and below zero), the log normal distribution, variants of the Weibull, Poisson, Chi Square and Gamma distributions as well as a myriad of others not mentioned here.

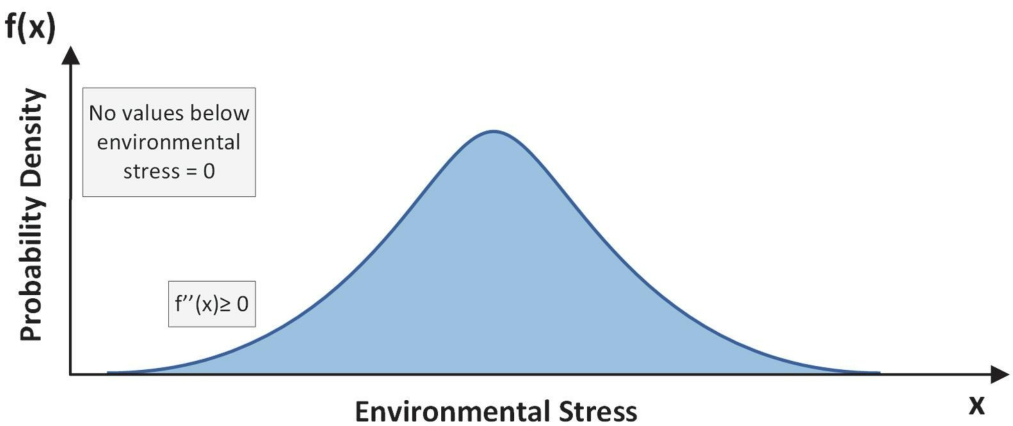

Figure 2. Key characteristics of a Normal-like distribution.

Starting from the 5/0 overtest, the area in the left tail of the distribution can be found from Table II (with failure percentage being 100 – Survivability percentage), or from equation (2). The area in the left tail is the probability of failure which is 1-r. With the area of the left tail known, a triangle is laid over the distribution such that this triangle starts from zero stress and zero probability density on the left, and extends to the right to a stress of q=k*s. The triangle height is adjusted so that the triangle's area is equal to the area in the left tail of the probability distribution. This is explored in detail in Appendix B but can also be seen in Figure 3.

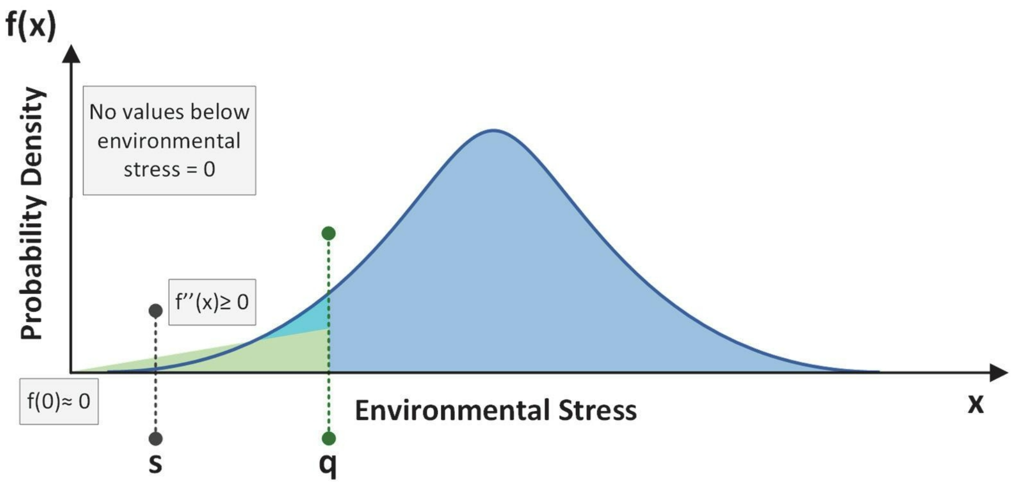

Figure 3. GTARS triangle approximation for Normal-like distribution.

Graphically, starting from a stress of zero and moving to the right, it is clear that the area in the triangle to the left of any stress less than k*s, is always larger than the area in the left tail of the distribution until the right end of the triangle where the stress is equal to k*s and the area of the triangle is equal to the area of the left tail. Because this is true, for any stress less than k*s, the area in the triangle is an upper bound on the failure probability. If the area in the left tail of the probability distribution is, f, then for some stress, s, the area, e, in the triangle is given as $e=f/k^2$. For a 5/0 overtest to some multiple, k, of the required stress, for a confidence of c=0.537, the resulting reliability, calculated as h=1-e, for several overtest factors, k, is shown in Table III.

Table III. Survival of Components Calculated from a 5/0 Overtest.

Overtest Factor	Survivability at Spec (%)	Confidence (%)
2	96.43	53.7
3	98.41	53.7
5	99.43	53.7
10	99.86	53.7

Appendix F presents tables with final reliability (survivability) values for 5/0 and 7/0 sampling using overtest factors ranging from two to ten.

In some circumstances the assumptions necessary to the triangle approximation may not be met. The assumption of a positive second derivative for the probability density may be false for a normal-like distribution if the overtest level exceeds the inflection point on the PDF. This will readily occur if the distribution is highly disperse. Many real failure probability density functions do not have a finite density at zero stress; however, the exponential distribution has a constant hazard rate and is often used in survival analysis. The exponential might be applicable or represent a contributing mechanism if there were a background failure mode that was roughly constant any time the component was in operation. Regardless of the rationale, the more conservative uniform approximation may be used when the triangle approximation is untenable, or when the analyst prefers a lower-risk estimate.

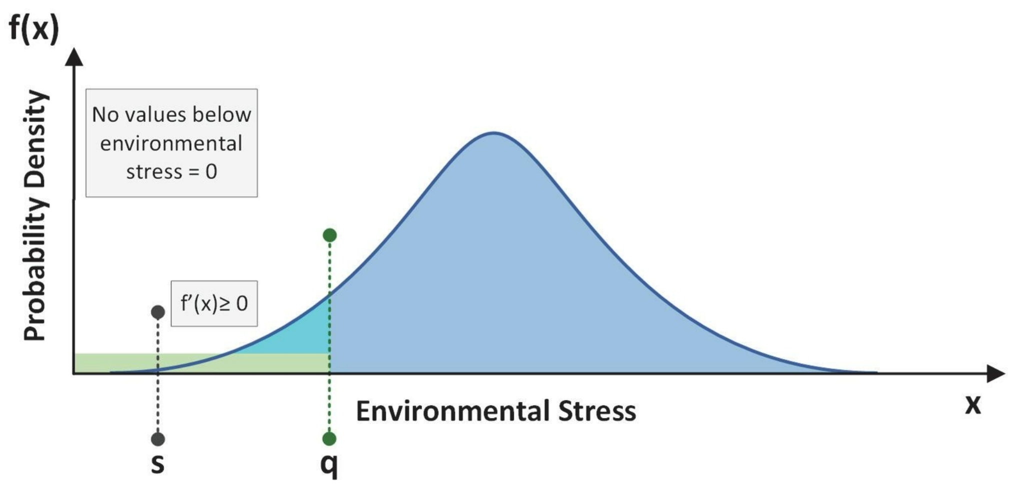

Figure 4. GTARS uniform approximation for Normal-like distribution.

Unlike the triangle approximation, the uniform approximation does not require f(0)=0, nor does it require f''(x)≥0. Instead, it requires f'(x)≥0, which is met for a Normal-like distribution up to the mode, beyond which testing will generally be inappropriate. In correspondence with the triangle approximation the area for the uniform approximation is given as e=f/k. The reliability estimate of h=1-e will be smaller for the uniform approximation, but it will also tend to be more conservative. This is discussed in greater detail in Appendix C.

Regardless of the GTARS approximation used, it is worth noting that if a small sample of components are intended for an overtest of, k=10, and parts begin to experience failures at some k factor less than k=10, it is not immediately necessary to test more parts. One might simply accept a lower overtest factor, where all parts passed, and use that to calculate the probability of failure and associated reliability at stress level, s. If for example, a small sample of parts survive to k=9 times the required environment but fail at k=10 times that which is required, then for the triangle approximation one simply has an upper bound on the failure rate of f/81 rather than f/100. Or, for the uniform approximation, an upper bound of f/9 instead of f/10. This of course results in a slightly lower reliability than would have been achieved with a k=10 overtest.

2.1. Other Considerations

Any component will experience failure in a given environment if subjected to a sufficiently large stress. A population of such components will have a probability density function (PDF) describing the specific stress at which individual components fail. This PDF exists independent of whether or not any components are tested. Other than the small effects resulting from sampling without replacement, the population PDF remains unchanged by the testing. The purpose of pass/fail testing on small samples from the population is to collect data to support estimates for the percentage of the population that will fail above, or below the test threshold.

Components fail when physical parameters change as a result of an applied stress. Stresses can vary in nature from simple aging with time to harsh external environments such as temperature extremes or the presence of radiation. The functional dependence of the component parameter shift leading to failure may be known, partially known, or possibly unknown. When the functional dependence is known, it can be exploited to design more specific tests, but for complex components, many parameters may be changing at once and the exact cause of failure may vary somewhat from part to part.

Consider a component with critical operating parameter, m. The value of this parameter as a function of stress, x, has the form

$$m_i(x) = d_i * f(x) \tag{16}$$

where m_i is the shift in parameter m, for the ith component, d_i is the sensitivity of the ith component to the specific applied stress, f(x) is the functional relationship, and x is the magnitude of the applied stress. Clearly, the collection of d values for the entire population will form a sensitivity distribution. For some value of m at which component failure is declared, equation (16) can be solved to find the failure stress for the ith component.

$$m_{fail} = m_i(x_i) = d_i * f(x_i) \tag{17}$$

Rearranging equation (17), the failure threshold for the ith component is given as

$$x_i = f^{-1}\left(\frac{m_{fail}}{d_i}\right) \tag{18}$$

The collection of failure x_i values for the population is, by definition, the distribution of failure thresholds mentioned throughout this paper.

The easiest situation occurs when the functional form is linear in the applied stress, x, such as

$$f(x_i) = x_i \tag{19}$$

but much more complex forms are possible, and highly likely. No matter what the form of equation (19), the distribution of failure x_i values for the population forms the failure PDF for the particular stress being considered. When dealing with more complex forms of equation (19), it is important to test using the environmental stress as specified and not change it into some equivalent form. As an example, some components might be susceptible to mechanical damage specified as being able to survive a drop from a given height onto a hard surface. The height corresponds to an impact velocity, but overtesting with a higher velocity is not the same as overtesting with a greater energy unless the exact functional form of equation (19) is known. While kinetic energy and velocity are functionally related, switching from one to another will necessarily change the shape of the component failure PDF. Unless this is identified and accounted for, the GTARS correction factors may overestimate or underestimate the failure probability for the population at the specified reliability stress, s. It is, in general, easiest to test to either the specification as written, or some multiple of it to achieve an overtest.

In many situations, reliability testing requires some form of temporal compression. For example, it is not possible to test components required to survive 30 years sitting on the shelf without either conducting a 30-year long test or designing some type of test that simulates 30 years of aging over a much shorter period of time. One such approach is known as Highly Accelerated Lifetime Testing (HALT) which relies on increased temperature to simulate the migration of contaminants that would occur over many years at normal temperatures. While aging is a complex process, the basic physics of many aging related failures is understood, thereby allowing tests with such temporal compression.

Because the functional relationship between stress and failure are understood, some types of HALT testing may benefit from an overtest and use of the GTARS approach to predict population reliability using very small sample sizes. For example, some aging-related failures in microelectronic components results from the time-dependent migration of contaminants, the physics of which are described by Fick's second law, sometimes referred to as the heat transport equation. Solutions of this equation differ based on initial boundary conditions, but a known solution form is

$$\frac{c}{c_{max}} = e^{\left(\frac{-y^2}{4Dt}\right)} \tag{20}$$

where c is the concentration of contaminant, y is position, D is diffusivity and t, is time. The diffusivity has a temperature dependence, given as

$$D = D_o * e^{\left(\frac{-w}{T}\right)} \tag{21}$$

where D_o is the initial concentration, w is a collection of physical constants, and T is the temperature. For a test to compress time, the terms D*t in equation (20) must remain constant. For shorter times, this requires the diffusivity to increase, which, according to equation (21), requires an increase in temperature.

$$D_{nominal} * t_{spec} = D_{increased} * t_{reduced} \tag{22}$$

The same approach can be used to provide an overtest in time, by increasing the final time by an overtest factor of, k.

$$D_{nominal} * k * t_{spec} = D_{new} * t_{reduced} \tag{23}$$

There are situations where temporal compression does not work without difficulty. For example, where more than one Arrhenius coefficient impacts reliability. In such cases, it does not matter if one is trying to use the more traditional reliability testing approach, where a sample is tested to spec, or if one is trying to use an overtest with the GTARS correction. The same problems with temporal compression that impact one test will impact the other. An example of such a situation is sometimes encountered with testing of bipolar electronic components in an ionizing radiation environment. The effect seen is that at very low dose rates, component parameter shifts are greatly enhanced as when compared to testing at the relatively high dose rates required for temporal compression. This effect is known as Enhanced Low Dose Rate Sensitivity (ELDRS).

As an example, consider a bipolar transistor that will operate on a spacecraft in low earth orbit. Perhaps for this component, the estimated accumulated dose over a ten-year mission will be 100,000 rads. This corresponds to a mission dose rate of approximately 3.17E-4 rads per second. As it is not practical to test components over a ten-year period, a test engineer might increase the dose rate to 9.26 rads per second, thereby completing the test in only 3 hours. Experiments on certain bipolar components show a substantially greater deterioration in performance for dose accumulated at the estimated mission rate when compared to the test dose rate. It is easy to see that if an overtest with a factor of k=3 was to be performed; this would extend the test to 9 hours, but the dose rate would remain unchanged. Clearly, the effect of ELDRS is present whether the test engineer was pursuing a test at 100,000 rads, or an overtest at 300,000 rads. The ELDRS problem is not related to the GTARS approach of predicting reliability from sparse overtest data.

In the case of components that exhibit ELDRS, the test engineer can still use GTARS with an overtest, but it will be necessary to characterize and understand the impact of the low dose rate sensitivity. For other situations where temporal compression exhibits nonlinearities, the test engineer will need to find ways to characterize the effect and compensate whether using GTARS or not.

GTARS can be a useful tool for providing mathematically sound estimates of reliability from pass/fail tests combining very small sample sizes with an overtest, or a test to some stress level in excess of that required. As mentioned above, overtesting allows the test engineer to trade test margin for sample size. GTARS provides the mathematical justification for the resulting reliability predictions. Appendix G presents information to aid engineers in designing reliability tests that take advantage of the GTARS approach.

3. NUMERICAL SIMULATIONS

In the straightforward case of a normal-like failure PDF with the mode several standard deviations from zero the application of the GTARS triangular (wedge) approximation yields a conservative estimate of reliability at the specification stress level. In general, however, for circumstances where the GTARS approach is of value, the shape of the distribution is unknown. Even if a Normal distribution can be safely assumed, the mode and dispersion are unknown. Consequently, the assumptions of the triangular approximation may not be satisfied at the overtest level.

Consider the visualizations of Normal probability density vs. standard deviation in Figure 5 below. The mode has been normalized to unity; therefore, the overtest level may be contemplated as a fraction of unity, or as a percentage of the mode. At 100% of the mode – in this case equivalent to the median – the overtest level would comprise half of the failure PDF and reliability would not be expected. In realistic use-cases for product development the specification and overtest levels will be well below the failure distribution median.

It is important to keep in mind that the visualization in Figure 5 is for the overtest level, not the overtest factor. Also, in general, the relative magnitudes of the specification and the failure PDF median are unknown.

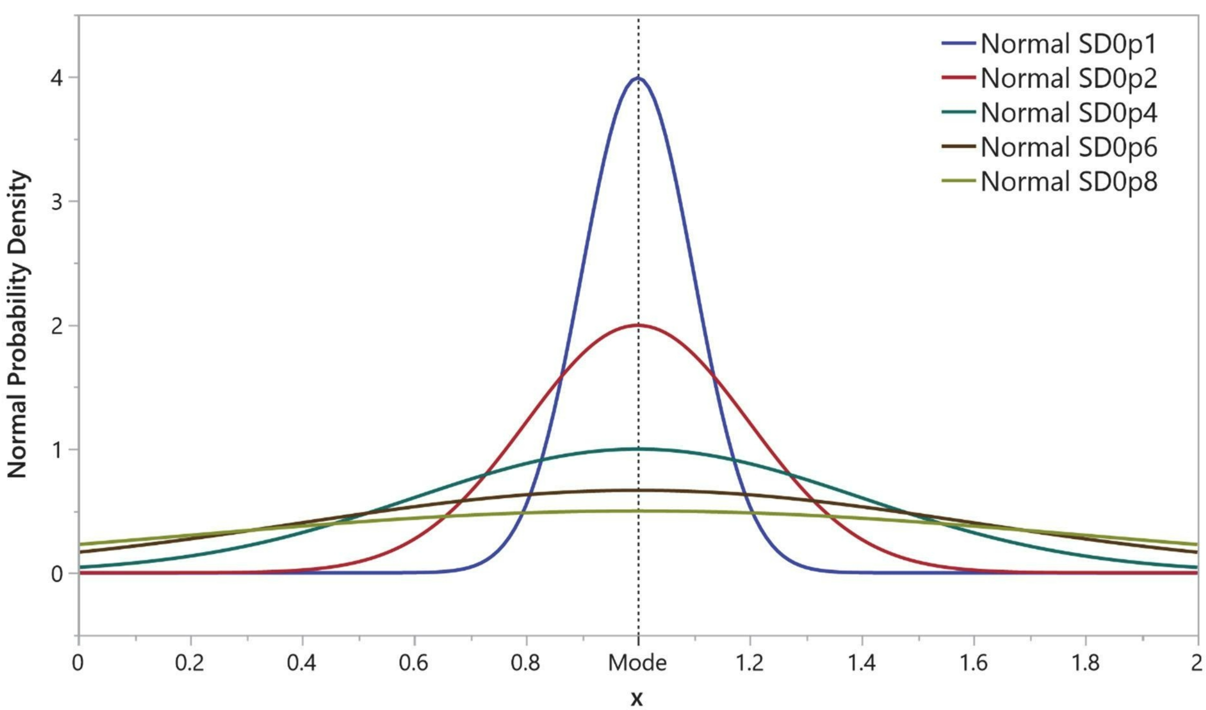

Figure 5. Normal probability density versus dispersion.

Assuming a normal-like but unknown failure threshold distribution, passing an over-test yields:

- Calculated component reliability at spec*k
- Calculated confidence in calculated reliability
- Projected component reliability at spec

But should we believe the calculated confidence applies to projected component reliability at spec? Simulations were performed for a broad range of distribution parameters to explore the robustness of the projected reliability to changes in the shape of the distribution.

The credibility of the projected component reliability over a range of distribution parameters is shown in Figure 6. This visualization was generated from numerical simulations with 50,000 repetitions for each point on the graphic. The main simulation was performed in the Julia programming language, and the outputs were visualized using SAS JMP Pro, Version 14. Details are provided in Appendix H.

For the visualization in Figure 6, the passing rate vs. overtest level is given for various sample sizes. The passing rate is the probability that a given overtest will produce zero component failures. Overtests performed at a particular sample size and overtest level will have a probability of passing based on the cumulative density function (CDF) up to that level. The simulated reliability for a sample size and overtest level is estimated as the passing rate for all individual components in the simulation vs. the specification level regardless of whether they belong to a passing or failing sample at overtest.

The visualization in Figure 6 shows that depending on the sample size and the dispersion of the failure PDF (the SD) there is a range of circumstances where the component reliability estimate obtained using the GTARS approach will be optimistic. Without further information about the distribution it is not possible to know how the overtest level compares with the median nor how the dispersion compares with the median level. The probability of getting a passing result in a region where the estimated reliability will be optimistic is substantial.

Notably, these problematic regions are consistent with operating parameters in severe violation of the assumptions necessary to the wedge approximation. For SD ≥0.4 the PDF is clearly non-zero at the origin, and the second derivative of the PDF is negative at all overtest levels. The GTARS wedge approximation is robust to minor violations of its underlying assumptions but thorough disregard for those assumptions invalidates the approach.

If the uniform approximation is used instead, as shown in Figure 7, then the requisite assumption of a non-negative slope is met at all overtest levels and the GTARS method yields optimistic estimates only at very high levels of dispersion. If the analyst using the method has good reason to believe that the distribution is not so dispersed (flat) then it might be reasonable to use the uniform approximation when conservative reliability estimates are required.

Another consideration that should be kept in mind is that the truncated Normal distribution was only used in the examples above for ease of explanation. The Normal is rarely used in survival analysis because it describes the failure behavior of few real components. Philosophically, the Normal is based on the summation of small random fluctuations approximating a random walk. Component failure is typically more related to the compounding of accumulated damages. The compounding of small fluctuations leads to the LogNormal distribution[6], which is widely observed in nature[7].

The LogNormal distribution is shown in Figure 8. For small values of sigma in comparison with the median, the distribution appears very similar to the Normal. With increasing sigma, the distribution demonstrates an initial rapid rise followed by a more gradually decreasing tail. Unlike the truncated Normal, the LogNormal always satisfies the assumption of zero value at the origin as required for the wedge approximation. As highlighted in Figure 9, the second derivative is always positive starting from the origin, but only for a narrow range as sigma is increased.

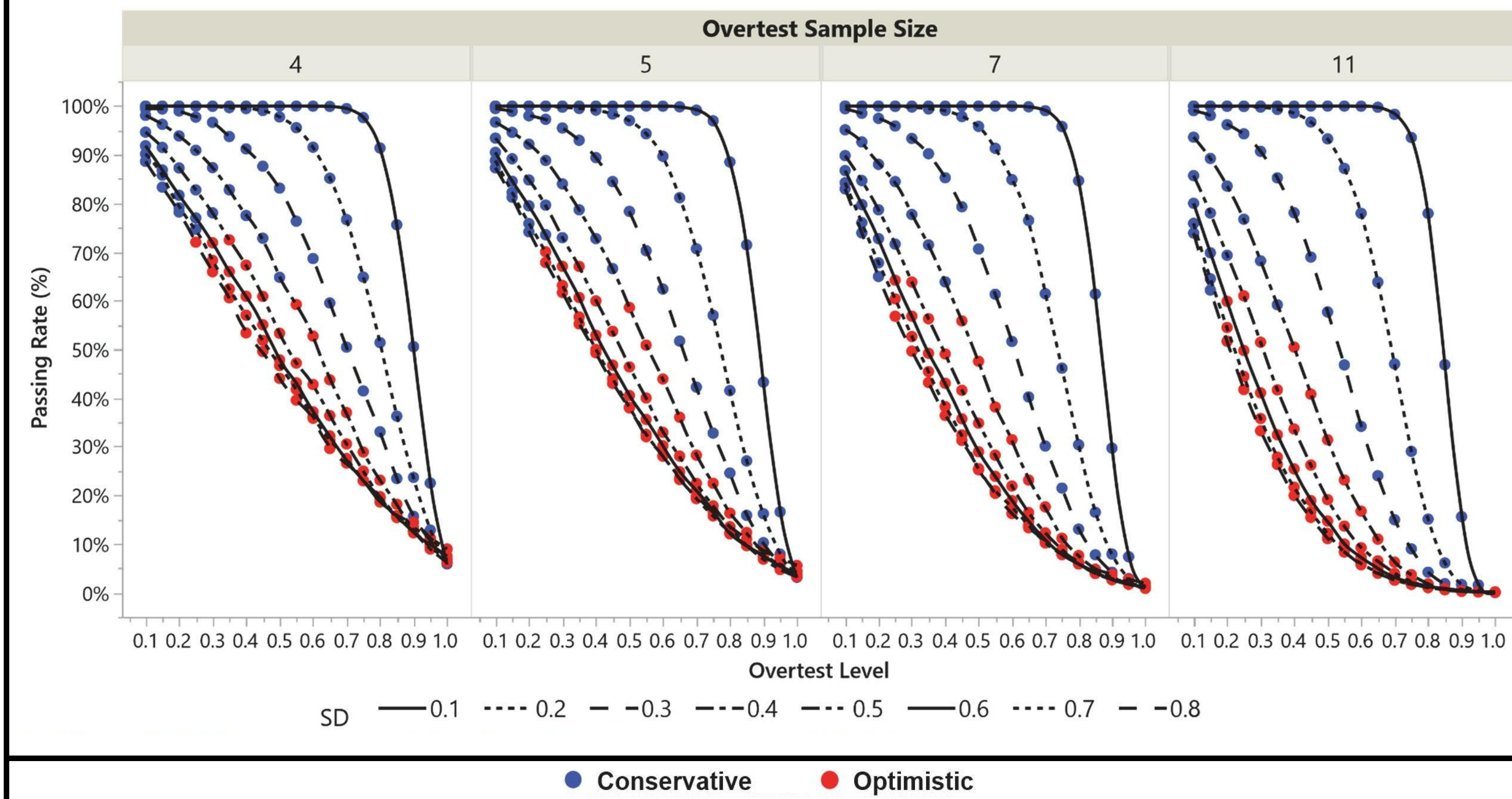

Figure 6. Credibility of reliability estimates for truncated Normal PDF with wedge approximation.

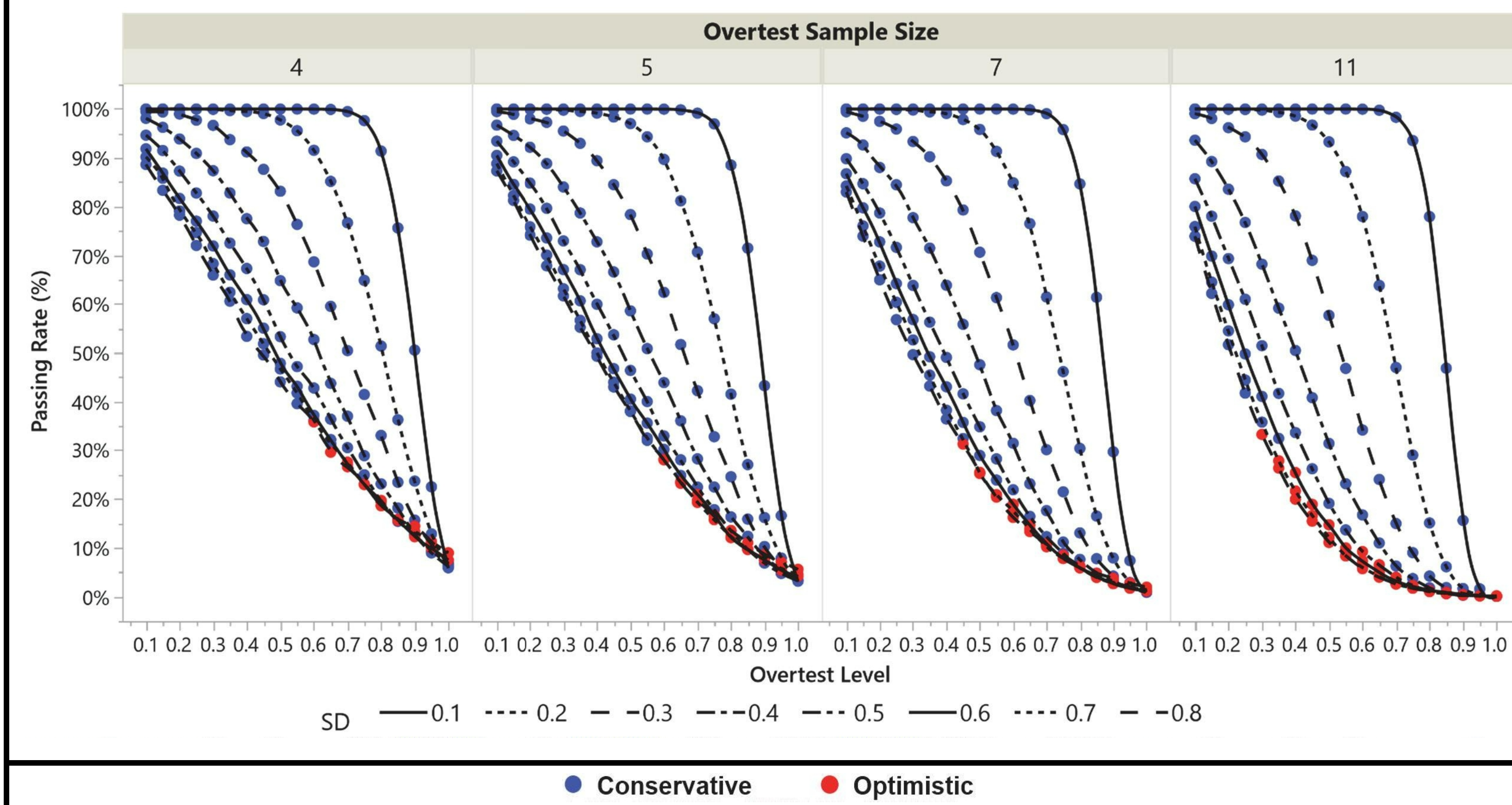

Figure 7. Credibility of reliability estimates for truncated Normal PDF with uniform approximation.

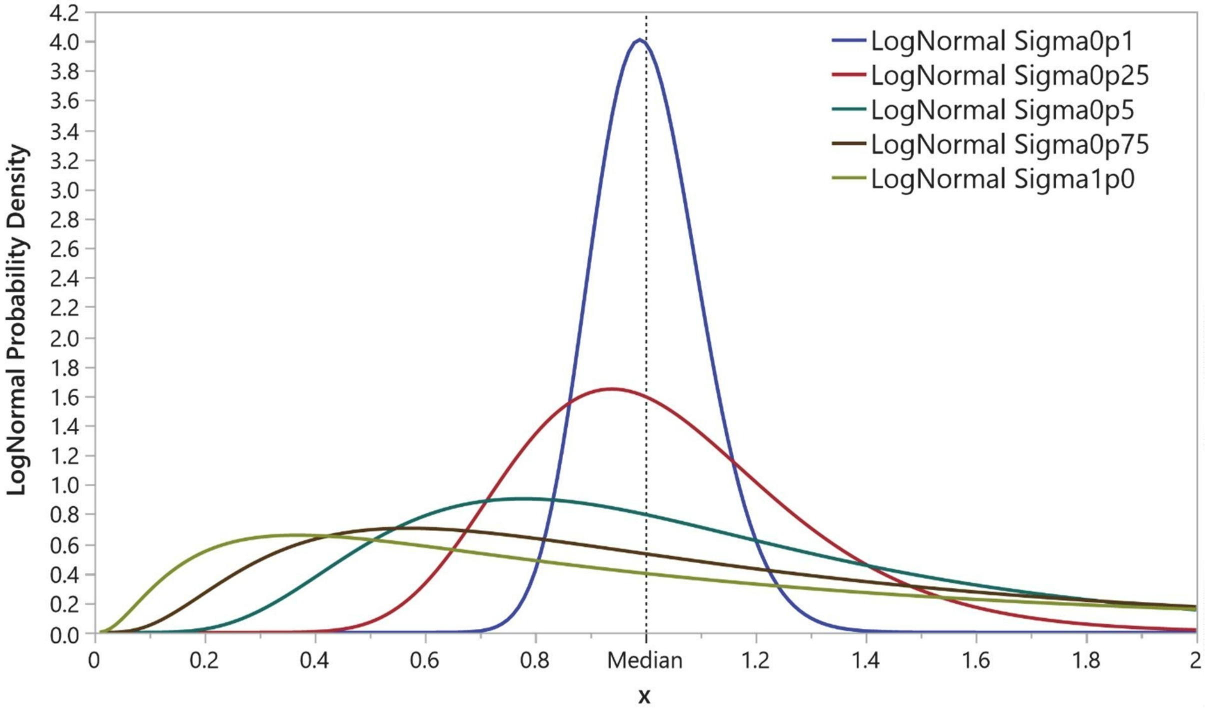

Figure 8. LogNormal probability density versus sigma.

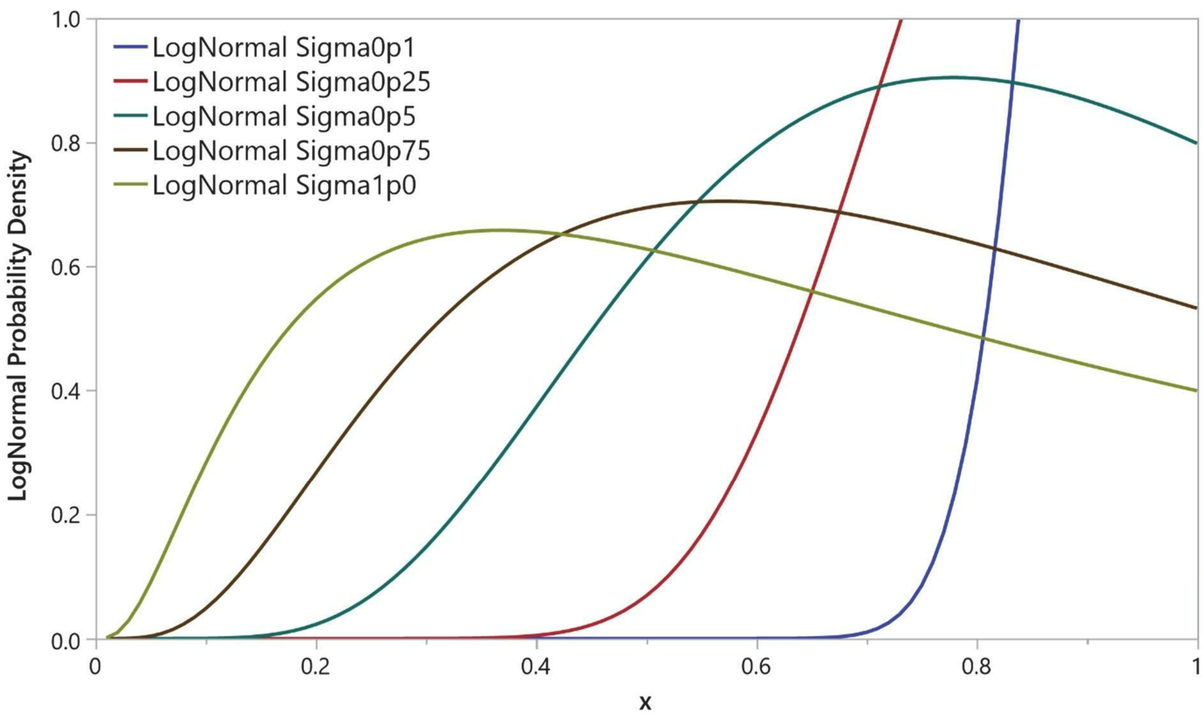

Figure 9. LogNormal probability density near zero.

As shown in Figure 10, the wedge approximation may be satisfactory if the dispersion of the distribution is known to be modest. Otherwise, as shown in Figure 11, the uniform (or box) approximation may be used with very little risk of producing an optimistic reliability estimate because the test will rarely pass in the region where an optimistic estimate would be generated.

The foregoing rationale all depends on assumptions about the failure distribution that may not be fully satisfied. The LogNormal is a typical failure distribution but without further evidence it amounts to a reasonable starting point. The analyst need not fully establish the distribution in order to use GTARS, but the method will be more valuable, and the estimates will be more credible if the characteristics of the distribution are qualitatively or semi-quantitatively understood.

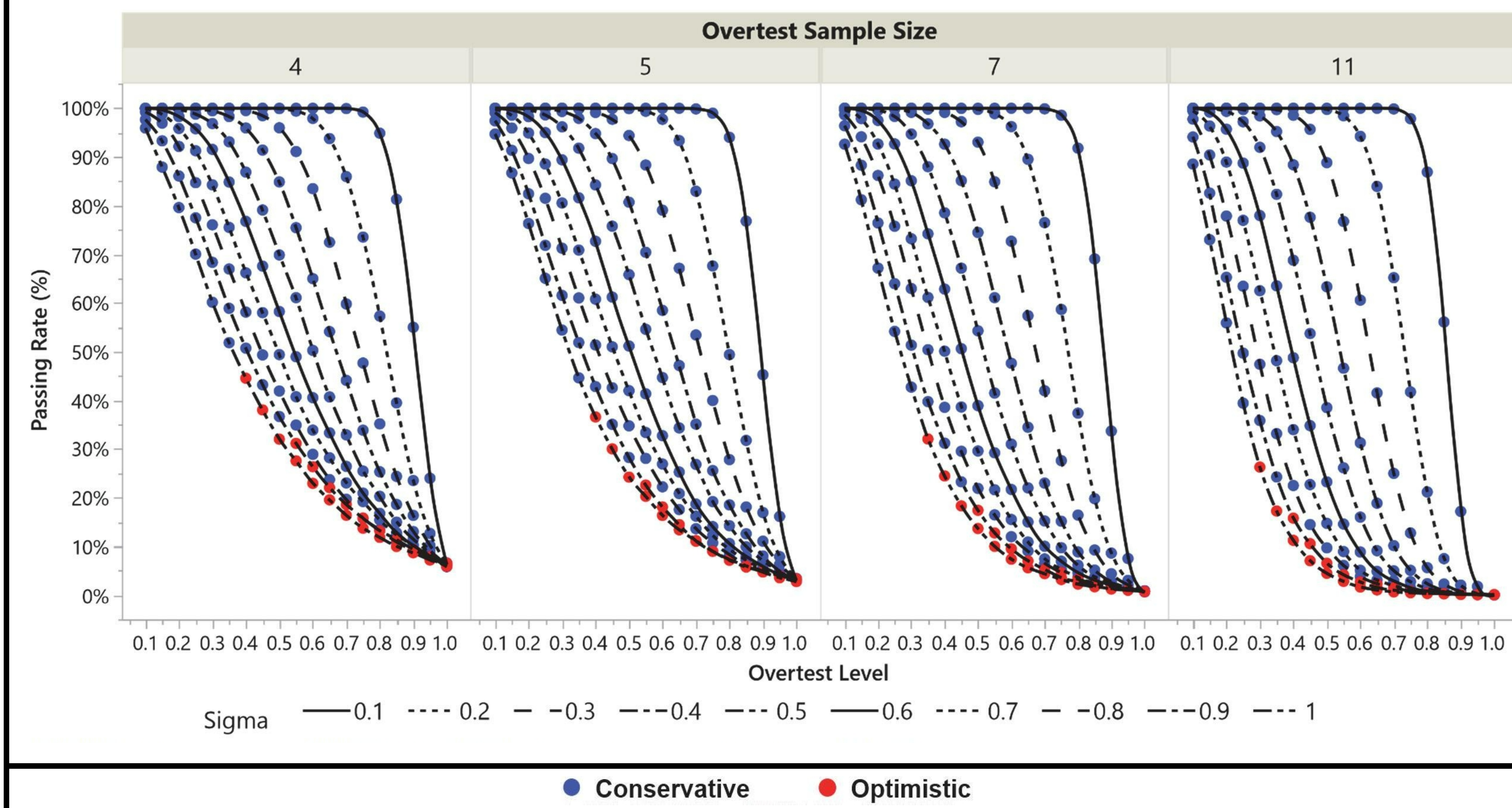

Figure 10. Credibility of reliability estimates for LogNormal PDF with wedge approximation.

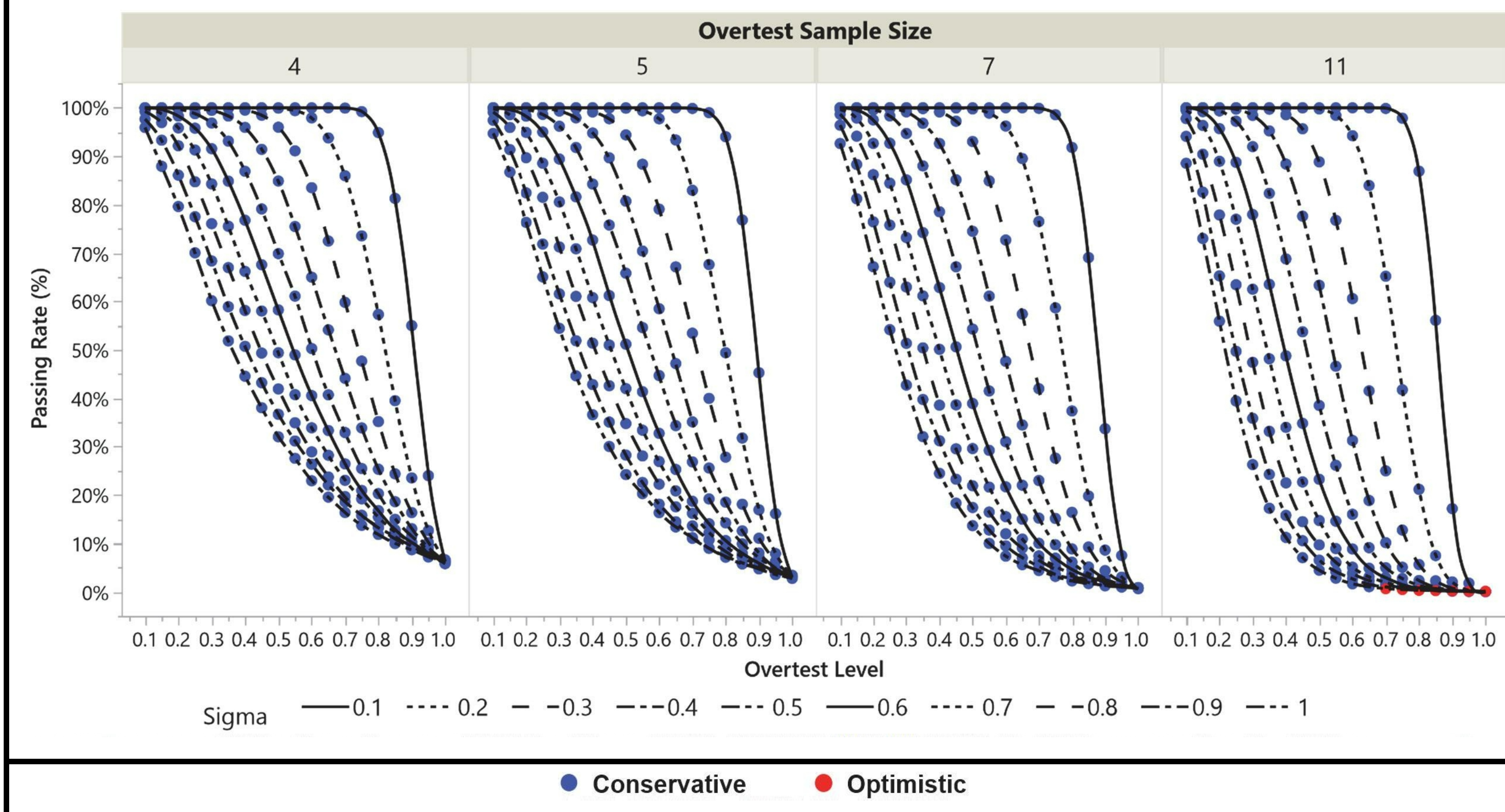

Figure 11. Credibility of reliability estimates for LogNormal PDF with uniform approximation.

4. SUMMARY

The GTARS approach provides a mathematically defensible method for bounding reliability predictions resulting from small sample testing to some stress level in excess of that required for the system. With the proper selection of an overtest factor, reliability values well in excess of 90% with confidence in excess of 50% can be obtained, rivaling the results of a 23/0 test that only stresses components to their required environmental stress.

Success with GTARS hinges on appropriately conservative assumptions about the left "tail" of the failure probability density. More aggressive assumptions enable higher reliability estimates, but the choice of assumptions depends on the sound judgment of the analyst: when little is known or may be reasonably inferred about the distribution, more conservative assumptions should be applied. However, unlike the typical overtest approach where reliability is simply asserted, with GTARS the estimates are grounded in a known construct that makes the analyst's assumptions explicit. Cleanly stated assumptions may be professionally reviewed and revisited in light of new evidence in ways that unvalidated assertions cannot. When using GTARS, reliability estimates are traceable.

The GTARS approach was developed for radiation testing and to produce defensible survivability estimates of a population, relying on small sample testing. It is possible this approach might be applied to other types of reliability estimates for components, but no specific cases have been extensively explored at this time.

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APPENDIX A. BINOMIAL STATISTICS AND RELIABILITY PREDICTIONS

When it is not possible to directly measure the reliability of a population, the alternative is to predict the reliability, relying on results of tests on small samples drawn from the population. Often, a small sample of n components will be tested to some stress threshold, s, with the results recorded as the number of parts passing and the number failing. Ideally, a test would result in zero failures. This is abbreviated as an n/0 test. Test series that experience failures can be acceptable but require a larger number of parts be tested. Without loss of generality, here only tests with zero failures are considered.

The problem with an n/0 test is that after the test, all that is known is that n components passed with zero failures. No other information is known. The basic shape of the distribution, such as normal-like, uniform, or decreasing exponential, might be reasonably assumed from past experience and other knowledge related to the components. However, important parameters such as the mean, standard deviation, and the percentage of the population that would fail at or below the stress level used for the test, are unknown. One then might ask, what good is the n/0 test?

For the moment, without loss of generality, assume that the distribution of failure thresholds for a population of components is normal-like, meaning that it has most of the characteristics of a normal distribution but with no negative or zero failure values. There are variants of the Weibull, log normal, Poisson, Chi Square and Gamma distributions that have normal-like characteristics. A normal-like distribution will have a mean and standard deviation and will more or less look like a traditional bell-shaped curve.

For an n/0 test to some stress level, s, any of the normal-like distributions shown in Figure A- 1 could represent the failure distribution of the population. The range of possible distributions is in fact, infinite. The choice of the distribution relative to the stress level, s, determines the reliability of the population. As all distributions are possible, how does one make a reliability estimate, and of what value is it? The answer is that why all distributions are possible, some are more probable than others and each reliability estimate comes with an associated probability of being correct, also known as a confidence estimate.

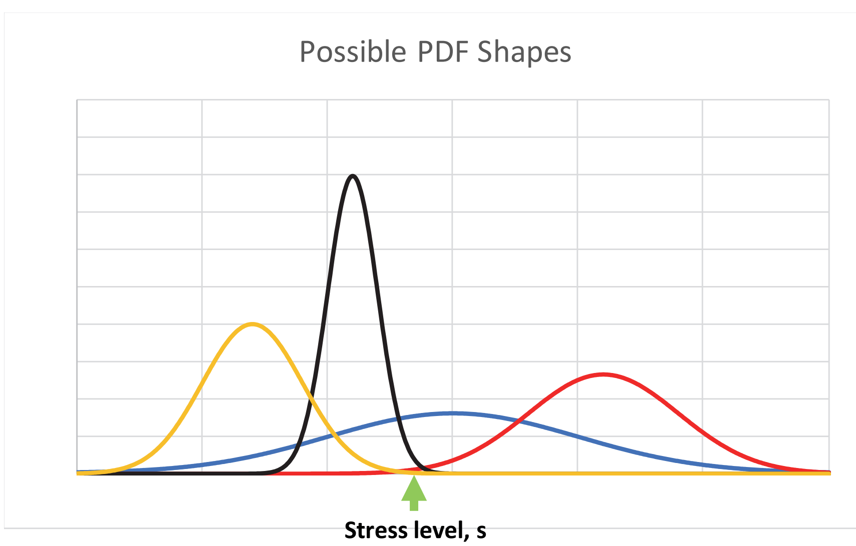

Figure A- 1. Four possible normal-like distributions for n/0 test at stress level s.

For a 23/0 test, a plot of possible reliability values with their calculated confidence levels is shown in figure A2. The sample size of 23 was selected as that gives, with zero failures, 97% reliability with 50% confidence. As seen in figure A2, a myriad of reliability and confidence combinations are possible. At this point, three questions emerge. First, how are the values plotted in figure A2 calculated. Second, what is the proper interpretation of any pair of values – what do they mean. Finally, what is the most probable value for the reliability with any n/0 test.

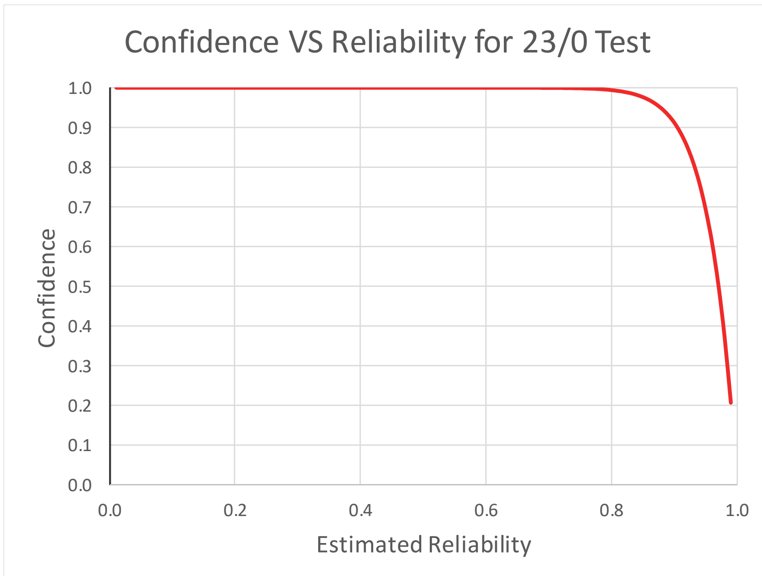

Figure A- 2. Plot of confidence VS reliability for 23/0 test.

Given an n/0 test series, to calculate the confidence for an assumed reliability, one uses equation (A1). When failures occur, a more complicated version of equation (A1) is necessary.

$$c = 1 - r^n \tag{A1}$$

Then as an example, using equation (A1), if the assumed reliability is, r=0.97, then the confidence is calculated to be c=0.504, or essentially 50%. For convenience, this pair will be referred to as coordinates in the Cartesian format, (0.97, 0.5). The confidence value is actually a probability expressing how likely it is that the actual reliability is equal to or greater than the assumed value. For the (0.97, 0.5) combination, one should interpret the values as though there is a 50% chance the population reliability is at least 97%. There is, however, also a 50% chance that the population reliability is less than 97%, but given the fact that the test series had 23 successes with zero failures, checking other reliability and confidence coordinate pairs suggests the reliability of the population is still quite high for the stress level, s, used for the 23/0 test. This can easily be seen in figure A2.

Finally, there is the question regarding the most probable value of reliability. Stated another way, given a test series of 23 events with zero failures, what is the probability that testing a 24th part would again result in a success. The answer to this question comes from Laplace's rule of succession. Laplace was theorizing on the morning sunrise. Given that the sun had risen every morning for some extremely long period of time, if sunrise was a probabilistic rather than deterministic event, what would be the probability of the sun rising the next day. The answer requires a careful mathematical analysis, but the result is surprisingly simple. For a 23/0 test, the most likely probability that the 24th event would also result in a success is pns=0.96. This is found from equation (A2), where pns is the probability of the next success.

$$pns = \frac{n+1}{n+2} \qquad \text{(A2)}$$

Equation (A2) does not provide any information on what the actual population reliability at stress level, s, will be, it only calculates the most likely reliability which, in turn, can be used to calculate a confidence value. For a 23/0 test, the most likely reliability is r=0.96 with an associated confidence value of c=0.609.

There exists a Wikipedia article suggests that Laplace's rule of succession does not apply for situations with zero failures or zero successes, but the mathematical analysis followed by Laplace and reproduced by others contains no such limitations. Also, Laplace himself developed the approach for an n/0 series of events. There are mathematical developments for n/m series where m failures are included with the n successes (n>m) and some of these developments do not generalize to an n/0 series, but Laplace's rule of succession, as represented in equation (A2) is correct.

For situations where very small samples are overtested by a factor of, k, to some stress level, q=k*s, the analysis presented above remains correct, but much lower values of confidence will be calculated for each estimate of reliability. Examples of a 5/0 and 7/0 test series are plotted in figures A3 and A4. What is critical to remember is that when using an overtest approach on very small samples, the initial calculated confidence for an estimated reliability (or survivability) is for the higher stress level of q=k*s, and the estimated reliability and confidence do not directly apply to the larger population.

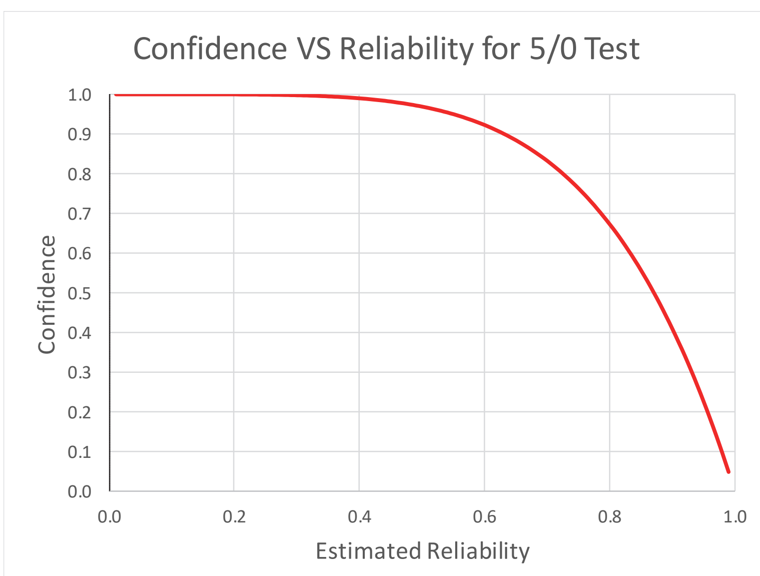

Figure A- 3. Plot of confidence VS reliability for 5/0 test.

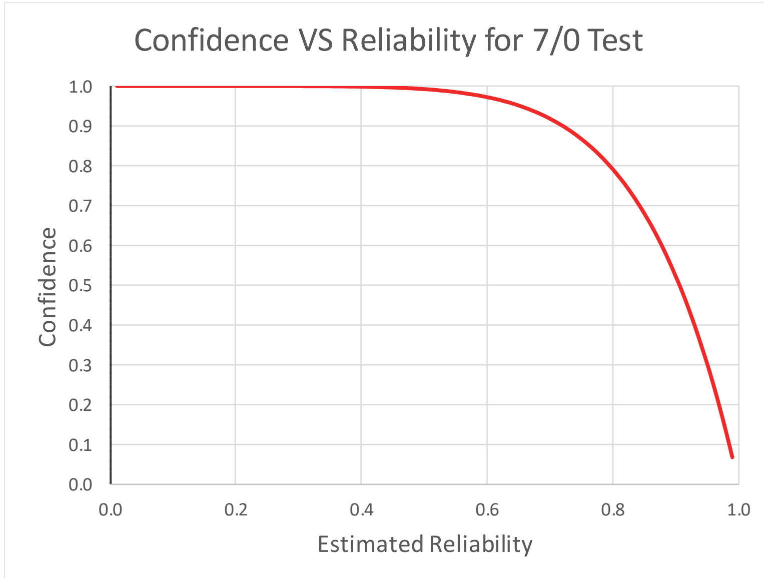

Figure A- 4. Plot of confidence VS reliability for 7/0 test.

For clarity, when discussing the required reliability, the symbol, r, should be used. When discussing a calculated reliability resulting from testing a sample to some stress level, s, the mathematical notation should be, rs, and when discussing a calculated reliability resulting from testing a sample to some stress level, q=k*s, the mathematical notation, rq, should be used.

APPENDIX B. GTARS APPLIED TO NORMAL-LIKE DISTRIBUTION

Many probability distribution functions will have characteristics similar to a normal distribution. These are referred to as normal-like distributions. They need not be ideally normal as they cannot have probability for zero or negative values, but for the most part, they appear somewhat bell-shaped. Probability distribution functions that have these characteristics include variants of the Weibull, log normal, Chi Square, Poisson and Gamma. For a normal-like PDF, the GTARS approach can be used to predict the reliability of a population from a small sample size.

For a component with a required reliability or survivability threshold of, s, an overtest by a factor of, k, to a stress level of q=k*s, for n components with zero failures results in an assessed failure fraction of, f, and a surviving fraction of g, such that f+g=1. As discussed previously, from binomial statistics, there are multiple possible values of, f, each with a confidence, c. The goal of the GTARS approach is to provide a mathematically derived value for the reliability of the population when encountering an environment with an exposure or stress no greater than, s. The GTARS approach for a normal-like PDF is shown in figure B1.

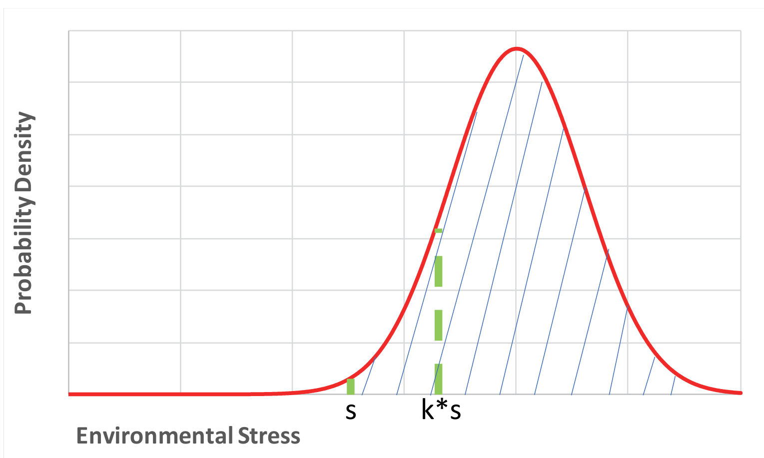

Figure B- 1. Normal-like probability distribution function.

The shaded area under the red line has an area equal to one and represents the PDF. The n parts are tested to a stress of q=k*s. The failure fraction, f, for test level q=k*s, is graphically the area under the red line and to the left of the vertical dashed line at k*s. This fraction, f, with an associated confidence, c, can be calculated from binomial statistics for n parts with zero failures.

The desired information is the survivability fraction, for the population of components when experiencing the required stress, s. The population failure fraction is referred to as area, e, and is the area under the red line and to the left of the vertical dashed line at stress, s, seen shaded in the figure B2. The population reliability or survivability is the area under the red line and to the right of the vertical dashed line at stress, s. This area is given the symbol, h.

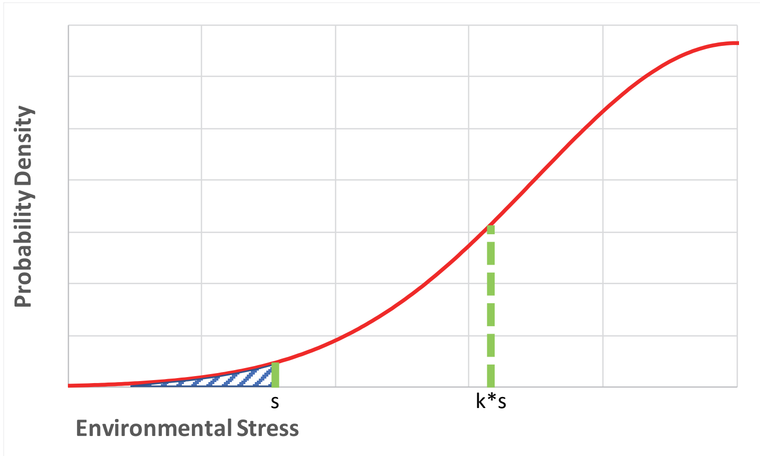

Figure B- 2

Figure B2. The shaded area is the population failure fraction, e. The unshaded area represents the population survivability fraction, h.

The area, f, can be calculated from binomial statistics for an n/0 test, with an associated confidence, c. Knowing only the overtest factor, k, and the estimated failure fraction, f, at the stress level of k*s, one can calculate the estimated population failure fraction, e, and the population estimated reliability, h, both with the same confidence, c.

To find a conservative estimate for the area, e, a triangle is fit to the left tail of the distribution starting at zero and extending to the right to an environmental stress of k*s. The height of the triangle is adjusted so that it has the same area as the failure fraction, f, as defined above. The triangle, as described here, is easiest to visualize in figure B3.

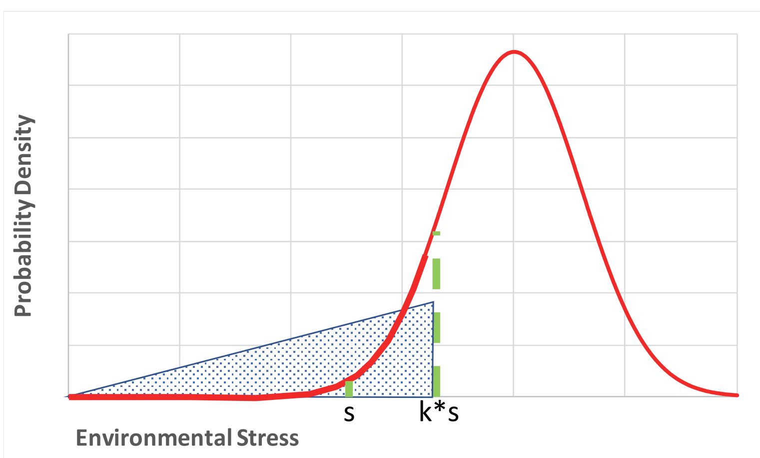

Figure B- 3.

Figure B3. Triangle adjusted to equal the area under the PDF to the left of environmental stress level k*s.

An important point to grasp, is that starting from zero, the area of the triangle always exceeds the cumulative area under the PDF up until a stress of k*s is reached, at which point the areas are equal. This characteristic results from the nature of a normal-like PDF. The left tail of any normal-like PDF is concave and curves upward. This can be confirmed by looking at the derivative of the normal-like PDF. The equation for a normal distribution with mean, m, and variance, s2, is given as

$$PDF = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{\sigma^2}} \qquad \text{(B1)}$$

and the first derivative is

$$\frac{d(PDF)}{dx} = \frac{-2(x-\mu)}{\sigma^3\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{\sigma^2}} = \frac{-2(x-\mu)}{\sigma^2} PDF \qquad \text{(B2)}$$

Since the PDF has only positive values, from equation (B2), one sees that for values of x<m, which would be the left tail of the distribution, the derivative of the PDF is positive. The second derivative of equation (B1) is

$$\frac{d^2(PDF)}{dx^2} = \frac{\left(4(x-\mu)^2-2\sigma^2\right)}{\sigma^5\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{\sigma^2}} = \frac{\left(4(x-\mu)^2-2\sigma^2\right)}{\sigma^4} PDF \qquad \text{(B3)}$$

An analysis of equation (B3) indicates that for the left tail of the distribution, specifically, for stress levels where $(x-\mu) < \frac{\sigma}{\sqrt{2}}$, the second derivative of the PDF is positive, indicating that it is concave upward, thereby satisfying the conditions for the triangle shaped GTARS approximation always having greater area, up until the point where the triangle and the tail have equal area at a stress level of x=q=k*s.

For normal-like distributions, with an overtest factor of k, the area, e, being the estimated failure probability for the population at stress level, s, is given as

$$e = \frac{f}{k^2} \qquad \text{(B4)}$$

with the estimated reliability or survivability, h, at stress level, s, given as

$$h = 1 - \frac{f}{k^2} \qquad \text{(B5)}$$

APPENDIX C. GTARS APPLIED TO DISTRIBUTION WITH BOX-LIKE TAIL

The left tail of some probability distribution functions can bear a strong resemblance to a box, or a nearly uniform distribution. Certainly, a uniform PDF itself will appear box-like on the left tail, but more complex distribution functions can still take on a box-like appearance on their left-most extreme without being truly uniform in nature. For distributions with a box-like left tail, the GTARS approach can be used to predict the reliability of a population from a small sample size.

The function, $f(x) = \sqrt{x}$, where x is a real number greater than or equal to zero, rises sharply at first, then more slowly as x gets larger. The integral of this function diverges, and it is therefore not a good example of a PDF, but the concept of rising sharply at first, then flattening out is important. A more complex function is required to fully illustrate the characteristic yet have the quality that the integral from x=0 to infinity is equal to one.

Consider the function,

$$f(x) = 0.0817628 * \sqrt[9]{x} * e^{-\left(\frac{x}{10}\right)} \quad \text{(C1)}$$

The integral of this function, from x=0 to infinity, is equal to one. A plot of f(x) over the domain of x=0 to x=20 is shown in figure C1.

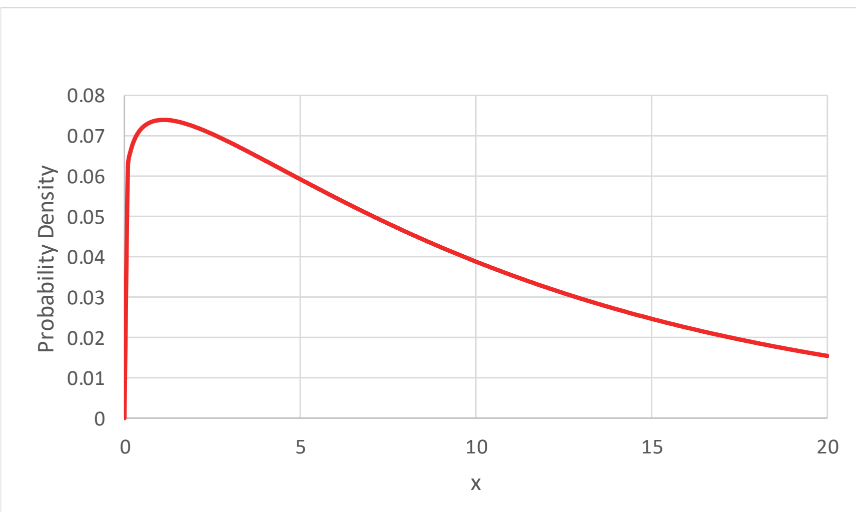

Figure C1. Plot of $f(x) = 0.0817628 * \sqrt[9]{x} * e^{-\left(\frac{x}{10}\right)}$

The plot shown in figure x, exhibits the expected characteristics. What is more interesting is to consider a plot of this same function over the more restricted domain of x=0 to x=2, as shown in figure C2.

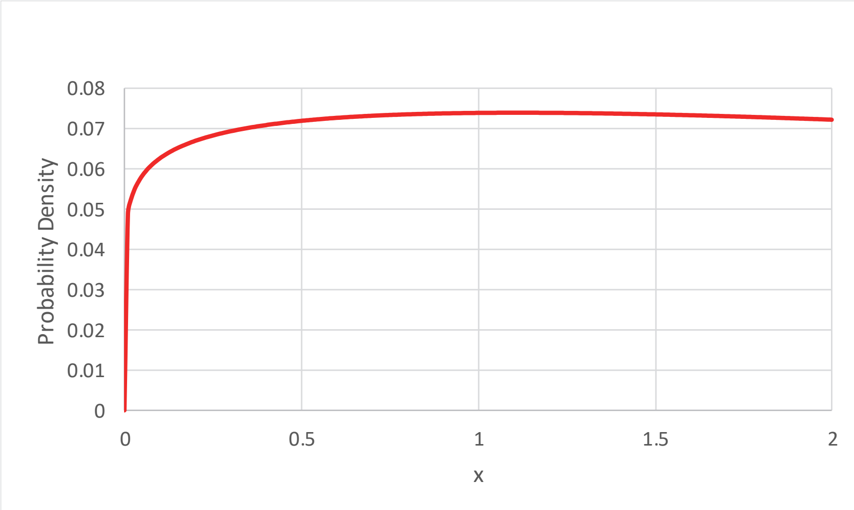

Figure C2. Plot of $f(x) = 0.0817628 * \sqrt[9]{x} * e^{-\left(\frac{x}{10}\right)}$

Figure C2 shows that the function has the desired box-like characteristic on its left tail. The function f(x) as defined in equation (C1) is not unique in having a box-like left tail. For example, variations of the Weibull, Gamma, Chi Square, and Poisson distribution functions exhibit this same characteristic.

For a component with a required reliability or survivability threshold of, s, an overtest by a factor of, k, for n components with zero failures results in an assessed failure fraction of, f, and a surviving fraction of g, such that f+g=1. As discussed previously, from binomial statistics, there are multiple possible values of, f, each with a confidence, c. The goal of the GTARS approach is to provide a mathematically derived value for the reliability of the population when encountering an environment with an exposure or stress no greater than, s. The GTARS approach for either a uniform or *box-like* tail PDF is shown in figure C3.

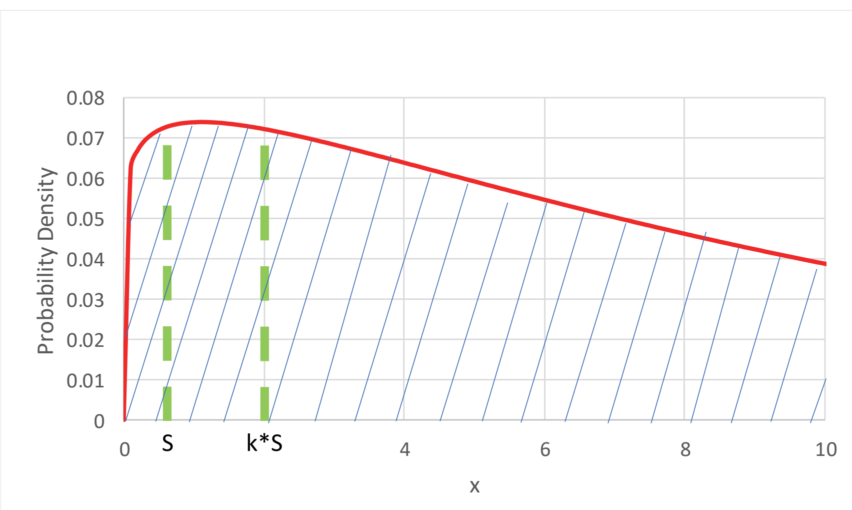

Figure C3. *Box-like* tail probability distribution function

The shaded area under the red line has an area equal to one and represents the PDF. The n parts are tested to a stress of k*s. The failure fraction, f, for test level k*s, is graphically the area under the red line and to the left of the vertical dashed line at k*s. This fraction, f, with an associated confidence, c, can be calculated from binomial statistics for n parts with zero failures (noted as an n/0 test).

The desired information is the survivability fraction, for the population of components when experiencing the required stress, s. The population failure fraction is referred to as area, e, and is the area under the red line and to the left of the vertical dashed line at stress, s, seen shaded in figure C4. This is the same function as plotted in figure C3, but with the x-axis scale changed to more clearly show the box-like nature of the left tail of the distribution. The population reliability or survivability is the area under the red line and to the right of the vertical dashed line at stress, s. This area is given the symbol, h.

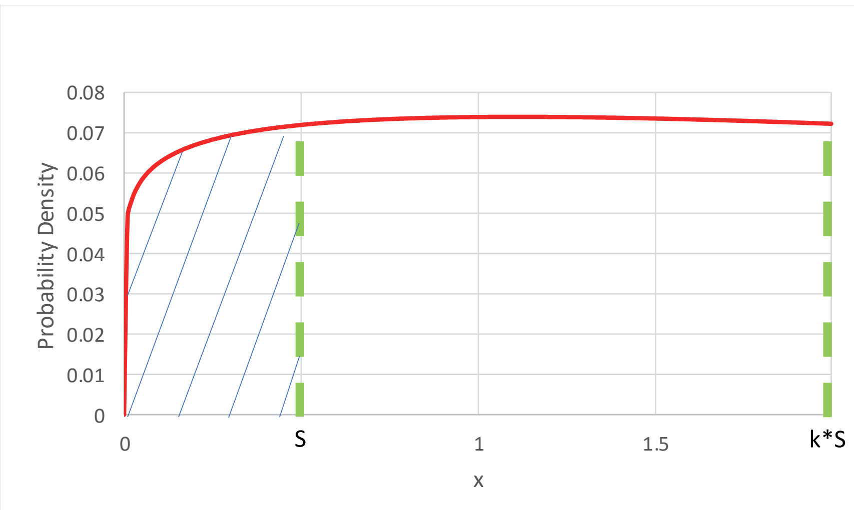

Figure C4. The shaded area is the population failure fraction, e. The unshaded area represents the population survivability fraction, h.

The area, f, can be calculated from binomial statistics for an n/0 test, with an associated confidence, c. Knowing only the overtest factor, k, and the estimated failure fraction, f, at the stress level of k*s, one can calculate the estimated population failure fraction, e, and the population estimated reliability, h, both with the same confidence, c.

The area, e, is calculated from the area, f, as

$$e = f * \frac{s}{ks} = \frac{f}{k} \tag{C1}$$

Then, the reliability estimate, h, for stress level, s, is given as

$$h = 1 - e = 1 - \frac{f}{k} \tag{C2}$$

Equations (C1) and (C2) provide a mathematically derived lower estimate for the reliability or survivability of components with a PDF that features a box-like left tail and are exact for a true uniform probability distribution function.

APPENDIX D. GTARS APPLIED TO DECAYING EXPONENTIAL DISTRIBUTION

Some possible probability distribution functions bear a strong resemblance to a decaying exponential, at least at their left most extremes. For a decaying exponential PDF, the GTARS approach can be used to predict the reliability of a population from a small sample size.

For a component with a required reliability or survivability threshold of, s, an overtest by a factor of, k, for n components with zero failures results in an assessed failure fraction of, f, and a surviving fraction of g, such that f+g=1. As discussed previously, from binomial statistics, there are multiple possible values of, f, each with a confidence, c. The goal of the GTARS approach is to provide a mathematically derived value for the reliability of the population when encountering an environment with an exposure or stress no greater than, s. The GTARS approach for a decaying exponential PDF is shown in figure D1.

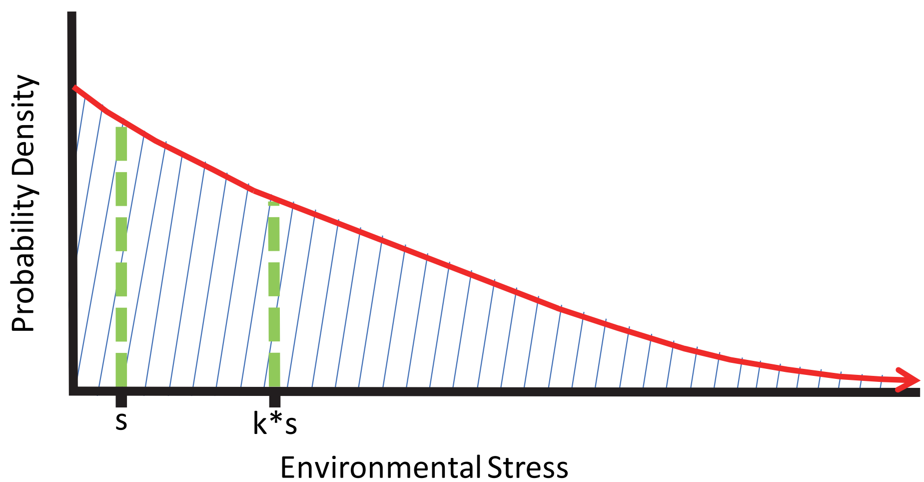

Figure D1. Decaying exponential probability distribution function

The shaded area under the red line has an area equal to one and represents the PDF. The n parts are tested to a stress of k*s. The failure fraction, f, for test level k*s, is graphically the area under the red line and to the left of the vertical dashed line at k*s. This fraction, f, with an associated confidence, c, can be calculated from binomial statistics for n parts with zero failures (noted as an n/0 test).

The desired information is the survivability fraction, for the population of components when experiencing the required stress, s. The population failure fraction is referred to as area, e, and is the area under the red line and to the left of the vertical dashed line at stress, s, seen shaded in figure D2. The population reliability or survivability is the area under the red line and to the right of the vertical dashed line at stress, s. This area is given the symbol, h.

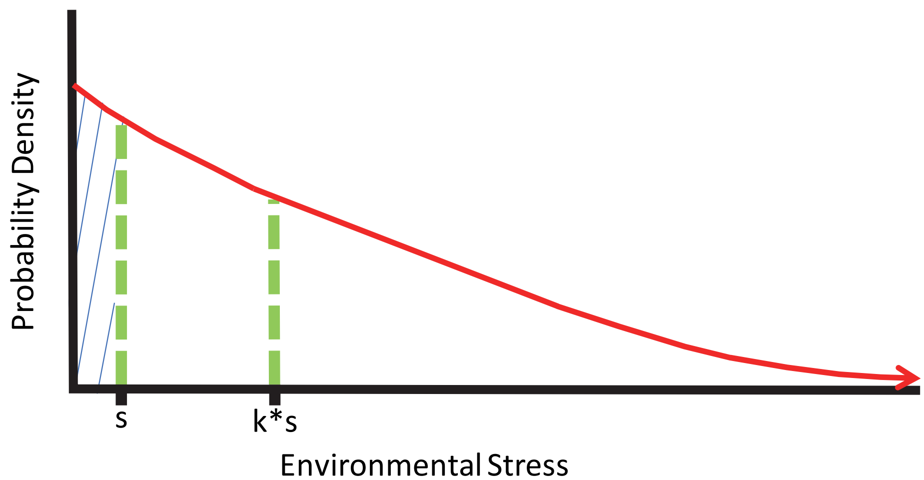

Figure D2. The shaded area is the population failure fraction, e. The unshaded area represents the population survivability fraction, h.

The area, f, can be calculated from binomial statistics for an n/0 test, with an associated confidence, c. Knowing only the overtest factor, k, and the estimated failure fraction, f, at the stress level of k*s, one can calculate the estimated population failure fraction, e, and the population estimated reliability, h, both with the same confidence, c.

The equation for a decaying exponential PDF is given as,

$$PDF(x) = w\, e^{-wx} \tag{D1}$$

which has the required property of having an integrated probability of one.

$$\int_0^{\infty} we^{-wx}\, dx = 1 \tag{D2}$$

The fraction of the population failing, f, at stress level k*s, can be used to determine the constant w. Recall that there exist an entire range of possible values of, f, each with an associated confidence, c. The constant, w, is found from

$$f = \int_0^{ks} we^{-wx}\, dx = 1 - e^{-wks} \tag{D3}$$

Because g=1-f, equation (D3) can be rewritten as

$$ws = -\frac{\ln(g)}{k} \tag{D4}$$

The reason for arranging with the product, ws, on the left side of equation (D4) will become obvious below.

The final goal is to find an expression for, e, the fraction of the population failing at stress level, s. This fraction is then used to calculate the fraction of the population, h, surviving stress level, s. This fraction, h, is the expected reliability or survivability of the population, given with confidence, c. Similar to equation (D3), an equation for, e, is

$$e = \int_0^s we^{-ws}\,dx = 1 - e^{ws} \tag{D5}$$

Then using a Taylor series approximation, the fraction, e, is given as

$$e = 1 - e^{ws} \approx ws = -\frac{\ln(g)}{k} = -\frac{\ln(1-f)}{k} \tag{D6}$$

Finally, the reliability estimate, h, for stress level, s, for values of k>1, is given as

$$h \approx 1 + \frac{\ln(1-f)}{k} \tag{D7}$$

When k=1, then h=g.

APPENDIX E. GTARS APPLIED TO DECREASING TRIANGULAR DISTRIBUTION

Some possible probability distribution functions bear a strong resemblance to a decreasing triangle, at least at their left most extremes. For a decreasing triangular PDF, the GTARS approach can be used to predict the reliability of a population from a small sample size.

For a component with a required reliability or survivability threshold of, s, an overtest by a factor of, k, for n components with zero failures results in an assessed failure fraction of, f, and a surviving fraction of g, such that f+g=1. As discussed previously, from binomial statistics, there are multiple possible values of, f, each with a confidence, c. The goal of the GTARS approach is to provide a mathematically derived value for the reliability of the population when encountering an environment with an exposure or stress no greater than, s. The GTARS approach for a decreasing triangular PDF is shown in figure E1.

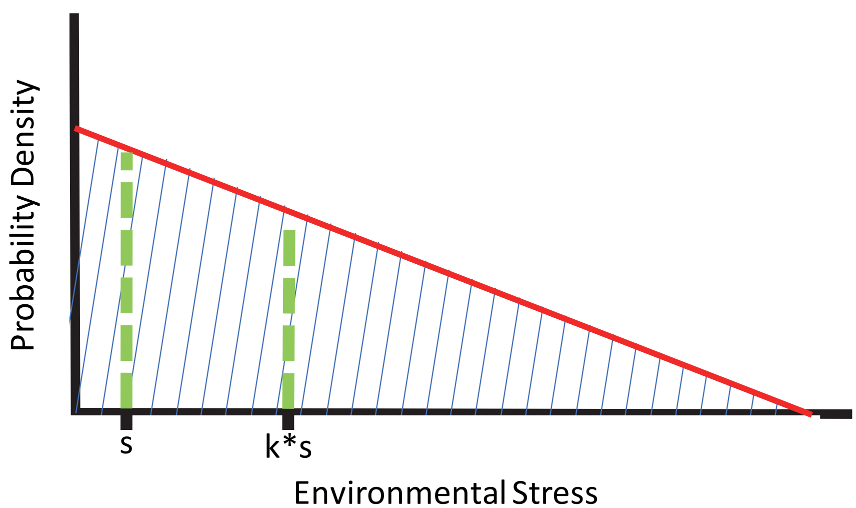

Figure E1. Decreasing triangular probability distribution function

The shaded area under the red line has an area equal to one and represents the PDF. The n parts are tested to a stress of k*s. The failure fraction, f, for test level k*s, is graphically the area under the red line and to the left of the vertical dashed line at k*s. This fraction, f, with an associated confidence, c, can be calculated from binomial statistics for n parts with zero failures.

The desired information is the survivability fraction, for the population of components when experiencing the required stress, s. The population failure fraction is referred to as area, e, and is the area under the red line and to the left of the vertical dashed line at stress, s, seen shaded in the following figure. The population reliability or survivability is the area under the red line and to the right of the vertical dashed line at stress, s. This area is given the symbol, h.

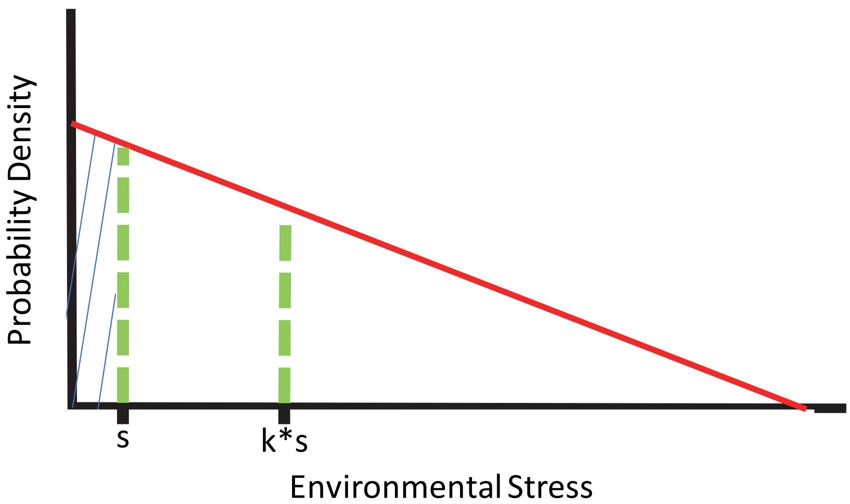

Figure E2. The shaded area is the population failure fraction, e. The unshaded area represents the population survivability fraction, h.

The area, f, can be calculated from binomial statistics for an n/0 test, with an associated confidence, c. Knowing only the overtest factor, k, and the estimated failure fraction, f, at the stress level of k*s, one can calculate the estimated population failure fraction, e, and the population estimated reliability, h, both with the same confidence, c.

An equation for e, in terms of f, and k, is derived by defining point, w, as the point at which the PDF reaches a value of zero. Then, from g=1-f, an equation for g is given as

$$g = \left[\frac{w-ks}{w}\right]^2 \tag{E1}$$

Solving for w, one finds

$$w = \frac{k^2s^2}{f^2}\left(1+\sqrt{g}\right)^2 \tag{E2}$$

Again, given that f=1-g, equation (E2) can be rearranged to give

$$f = \frac{2ksw-k^2s^2}{w^2} \tag{E3}$$

Next,

$$e = 1 - \frac{(w-s)^2}{w^2} \tag{E4}$$

which can also be expressed as,

$$e = \frac{2ws-s^2}{w^2} \tag{E5}$$

Combining equations (E3) and (E5), an alternate expression for the area, e, is found as

$$e = \frac{f}{k} + \frac{s^2}{w^2}(k-1) \tag{E6}$$

Then substituting equation (E2) in for w, one finds the desired result of

$$e = \frac{f}{k} + \frac{(k-1)f^2}{k^2(1+\sqrt{g})^2} \tag{E7}$$

Finally, the estimated population reliability is found as, h=1-e.

APPENDIX F. RELIABILITY PREDICTIONS FROM GTARS USING AN OVERTEST

To make use of the GTARS approach for different distribution shapes, it is easiest to start with a desired confidence, and then calculate the reliability resulting from an n/0 test with an overtest factor of k.

The reliability for no overtest, or an overtest factor of k=1, results from the specified confidence, c, as

$$r = \sqrt[n]{1 - c} \tag{F1}$$

Since the reliability for an n/0 test is also the same as the surviving fraction, g, then the fraction of the population failing, assuming a confidence value of c, is given as f=1-g=1-r. To estimate the fraction of the population surviving at stress level, s, resulting from the n/0 overtest by a factor of k, given a confidence value of c, one employs the appropriate GTARS equation.

As mentioned in appendix A1, to avoid confusion, when discussing the reliability requirement, the symbol, r, should be used. When testing to some stress level, s, the calculated reliability should be noted as rs, while for tests to an overtest stress level of q=k*s, the calculated reliability should be noted as rq. Note that the goal is to get rs equal to r. Note also that rq is not the same as r or rs. The estimated reliability at stress level, s, given as rs, can only be related to rq through use of the GTARS approach.

For a 5/0 test series, tables F1-F4 provide calculated reliability estimates for overtest factors ranging from k=1 to k=10. Table F1 uses the GTARS approach for a distribution with a box-like tail. Table F2 uses the GTARS approach for a normal-like distribution. Table F3 uses the GTARS approach for a distribution resembling a decaying exponential, while Table F4 uses the GTARS approach for a distribution where the left-most tail resembles a decreasing triangle. The final row in each table is for the confidence level associated with the reliability estimate resulting from Laplace's rule of succession, believed to be the most likely reliability for the 5/0 test.

Table F - 1. GTARS Reliability Estimates for Box-Like Tail Distribution.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.87	0.94	0.96	0.97	0.97	0.98	0.99	0.99
0.55	0.85	0.93	0.95	0.96	0.97	0.98	0.98	0.99
0.60	0.83	0.92	0.94	0.96	0.97	0.98	0.98	0.98
0.65	0.81	0.91	0.94	0.95	0.96	0.97	0.98	0.98
0.70	0.79	0.89	0.93	0.95	0.96	0.97	0.98	0.98
0.75	0.76	0.88	0.92	0.94	0.95	0.97	0.97	0.98
0.80	0.72	0.86	0.91	0.93	0.94	0.96	0.97	0.97
0.85	0.68	0.84	0.89	0.92	0.94	0.95	0.96	0.97
0.90	0.63	0.82	0.88	0.91	0.93	0.95	0.96	0.96
0.95	0.55	0.77	0.85	0.89	0.91	0.94	0.95	0.95
0.54	0.86	0.93	0.95	0.96	0.97	0.98	0.98	0.99

Note: Using 5/0 sample test, with overtest factors ranging from 1-10

Table F - 2. GTARS Reliability Estimates for Normal-Like Distribution.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.87	0.97	0.99	0.99	0.99	1.00	1.00	1.00
0.55	0.85	0.96	0.98	0.99	0.99	1.00	1.00	1.00
0.60	0.83	0.96	0.98	0.99	0.99	1.00	1.00	1.00
0.65	0.81	0.95	0.98	0.99	0.99	1.00	1.00	1.00
0.70	0.79	0.95	0.98	0.99	0.99	1.00	1.00	1.00
0.75	0.76	0.94	0.97	0.98	0.99	1.00	1.00	1.00
0.80	0.72	0.93	0.97	0.98	0.99	0.99	1.00	1.00
0.85	0.68	0.92	0.96	0.98	0.99	0.99	1.00	1.00
0.90	0.63	0.91	0.96	0.98	0.99	0.99	1.00	1.00
0.95	0.55	0.89	0.95	0.97	0.98	0.99	0.99	1.00
0.54	0.86	0.96	0.98	0.99	0.99	1.00	1.00	1.00

Note: Using 5/0 sample test, with overtest factors ranging from 1-10.

Table F - 3. GTARS Reliability Estimates for Decaying Exponential Distribution.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.87	0.93	0.95	0.97	0.97	0.98	0.98	0.99
0.55	0.85	0.92	0.95	0.96	0.97	0.98	0.98	0.98
0.60	0.83	0.91	0.94	0.95	0.96	0.97	0.98	0.98
0.65	0.81	0.90	0.93	0.95	0.96	0.97	0.98	0.98
0.70	0.79	0.88	0.92	0.94	0.95	0.97	0.97	0.98
0.75	0.76	0.86	0.91	0.93	0.94	0.96	0.97	0.97
0.80	0.72	0.84	0.89	0.92	0.94	0.95	0.96	0.97
0.85	0.68	0.81	0.87	0.91	0.92	0.95	0.96	0.96
0.90	0.63	0.77	0.85	0.88	0.91	0.93	0.95	0.95
0.95	0.55	0.70	0.80	0.85	0.88	0.91	0.93	0.94
0.54	0.86	0.92	0.95	0.96	0.97	0.98	0.98	0.98

Note: Using 5/0 sample test, with overtest factors ranging from 1-10.

Table F - 4. GTARS Reliability Estimates for Distribution with Decreasing Triangular Tail.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.87	0.93	0.96	0.97	0.97	0.98	0.99	0.99
0.55	0.85	0.92	0.95	0.96	0.97	0.98	0.98	0.98
0.60	0.83	0.91	0.94	0.96	0.97	0.98	0.98	0.98
0.65	0.81	0.90	0.93	0.95	0.96	0.97	0.98	0.98
0.70	0.79	0.89	0.93	0.94	0.96	0.97	0.97	0.98
0.75	0.76	0.87	0.92	0.94	0.95	0.96	0.97	0.97
0.80	0.72	0.86	0.90	0.93	0.94	0.96	0.97	0.97
0.85	0.68	0.83	0.89	0.92	0.93	0.95	0.96	0.97
0.90	0.63	0.80	0.87	0.90	0.92	0.94	0.95	0.96
0.95	0.55	0.76	0.83	0.87	0.90	0.93	0.94	0.95
0.54	0.86	0.93	0.95	0.96	0.97	0.98	0.98	0.99

Note: Using 5/0 sample test, with overtest factors ranging from 1-10.

For a 7/0 test series, tables F5-F8 provide calculated reliability estimates for overtest factors ranging from k=1 to k=10 similar to the above.

Table F - 5. GTARS Reliability Estimates for Box-Like Tail Distribution.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.91	0.95	0.97	0.98	0.98	0.99	0.99	0.99
0.55	0.89	0.95	0.96	0.97	0.98	0.98	0.99	0.99
0.60	0.88	0.94	0.96	0.97	0.98	0.98	0.99	0.99
0.65	0.86	0.93	0.95	0.97	0.97	0.98	0.98	0.99
0.70	0.84	0.92	0.95	0.96	0.97	0.98	0.98	0.98
0.75	0.82	0.91	0.94	0.96	0.96	0.97	0.98	0.98
0.80	0.79	0.90	0.93	0.95	0.96	0.97	0.98	0.98
0.85	0.76	0.88	0.92	0.94	0.95	0.97	0.97	0.98
0.90	0.72	0.86	0.91	0.93	0.94	0.96	0.97	0.97
0.95	0.65	0.83	0.88	0.91	0.93	0.95	0.96	0.97
0.56	0.89	0.94	0.96	0.97	0.98	0.98	0.99	0.99

Note: Using 7/0 sample test, with overtest factors ranging from 1-10.

Table F - 6. GTARS Reliability Estimates for Normal-Like Distribution.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.91	0.98	0.99	0.99	1.00	1.00	1.00	1.00
0.55	0.89	0.97	0.99	0.99	1.00	1.00	1.00	1.00
0.60	0.88	0.97	0.99	0.99	1.00	1.00	1.00	1.00
0.65	0.86	0.97	0.98	0.99	0.99	1.00	1.00	1.00
0.70	0.84	0.96	0.98	0.99	0.99	1.00	1.00	1.00
0.75	0.82	0.96	0.98	0.99	0.99	1.00	1.00	1.00
0.80	0.79	0.95	0.98	0.99	0.99	1.00	1.00	1.00
0.85	0.76	0.94	0.97	0.99	0.99	1.00	1.00	1.00
0.90	0.72	0.93	0.97	0.98	0.99	0.99	1.00	1.00
0.95	0.65	0.91	0.96	0.98	0.99	0.99	1.00	1.00
0.56	0.89	0.97	0.99	0.99	1.00	1.00	1.00	1.00

Note: Using 7/0 sample test, with overtest factors ranging from 1-10.

Table F - 7. GTARS Reliability Estimates for Decaying Exponential Distribution.

Confidence	k=1 Reliability	k=2 Reliability	k=3 Reliability	k=4 Reliability	k=5 Reliability	k=7 Reliability	k=9 Reliability	k=10 Reliability
0.50	0.91	0.95	0.97	0.98	0.98	0.99	0.99	0.99
0.55	0.89	0.94	0.96	0.97	0.98	0.98	0.99	0.99
0.60	0.88	0.93	0.96	0.97	0.97	0.98	0.99	0.99
0.65	0.86	0.93	0.95	0.96	0.97	0.98	0.98	0.99
0.70	0.84	0.91	0.94	0.96	0.97	0.98	0.98	0.98
0.75	0.82	0.90	0.93	0.95	0.96	0.97	0.98	0.98
0.80	0.79	0.89	0.92	0.94	0.95	0.97	0.97	0.98
0.85	0.76	0.86	0.91	0.93	0.95	0.96	0.97	0.97
0.90	0.72	0.84	0.89	0.92	0.93	0.95	0.96	0.97
0.95	0.65	0.79	0.86	0.89	0.91	0.94	0.95	0.96
0.56	0.89	0.94	0.96	0.97	0.98	0.98	0.99	0.99

Note: Using 7/0 sample test, with overtest factors ranging from 1-10.

Table F - 8. GTARS Reliability Estimates for Distribution with Decreasing Triangular Tail.

	k=1	k=2	k=3	k=4	k=5	k=7	k=9	k=10
Confidence	Reliability	Reliability	Reliability	Reliability	Reliability	Reliability	Reliability	Reliability
0.50	0.91	0.95	0.97	0.98	0.98	0.99	0.99	0.99
0.55	0.89	0.95	0.96	0.97	0.98	0.98	0.99	0.99
0.60	0.88	0.94	0.96	0.97	0.97	0.98	0.99	0.99
0.65	0.86	0.93	0.95	0.96	0.97	0.98	0.98	0.99
0.70	0.84	0.92	0.95	0.96	0.97	0.98	0.98	0.98
0.75	0.82	0.91	0.94	0.95	0.96	0.97	0.98	0.98
0.80	0.79	0.89	0.93	0.95	0.96	0.97	0.98	0.98
0.85	0.76	0.88	0.92	0.94	0.95	0.96	0.97	0.97
0.90	0.72	0.85	0.90	0.93	0.94	0.96	0.97	0.97
0.95	0.65	0.82	0.88	0.91	0.92	0.95	0.96	0.96
0.56	0.89	0.94	0.96	0.97	0.98	0.98	0.99	0.99

Note: Using 7/0 sample test, with overtest factors ranging from 1-10.

APPENDIX G. GTARS BASED COMPONENT RELIABILITY TEST DESIGN

The purpose of component reliability testing is to provide a measure of confidence that the system will operate as intended. Component reliability requirements are negotiated with the system developer such that the overall system reliability is expected to meet or exceed requirements. Translating the system-level requirements to the component level is essentially a budgeting exercise, resulting in a component reliability specification and, whether explicitly stated or not, an acceptable level of confidence in the assessed reliability.

The GTARS approach to component reliability testing employs an amplification or overtest factor to reduce the number of components that must be tested to destruction in order to provide assurances of reliability under specified stressors (e.g., for radiation testing a total dose or dose rate might be a relevant stressor). The component reliability tester must design an appropriate tradeoff between sample size and overtest factor, given the specified component reliability and assessment confidence. This paper is intended to demonstrate the necessary mathematics leading to a design equation for the overtest factor based on reliability, confidence, and sample size.

<u>Definition of Terms</u>

Several variables are necessary or helpful for describing the GTARS method and estimating the overtest factor.

s = specification stressor level

k = overtest factor

q = applied stressor level under test conditions (q=k*s)

f = failure probability under test conditions (stressor level q)

e = failure probability under specified stressor level s (estimated as e = f/k2)

h = component reliability under specified stressor level s (h = 1-e)

c = confidence attributed to test

rq = estimated component reliability under test conditions (stressor level q)

rs = estimated component reliability at required stress (stressor level s)

r = reliability in general (stressor level to be specified)

n = component sample size (count)

Note that h=rs

Mathematical Approach

Start with previously established binomial formula relating sample count, confidence, and reliability. The confidence level for the overtest and for the component reliability at spec are identical. The incremental cost of sample size may be clearly defined by economic factors. The incremental cost of overtest factor may be influenced by technical limitations of the test apparatus. Assume for this analysis that there is a reason for engaging in a tradeoff between sample size and overtest level.

$$r = (1 - c)^{1/n} \qquad \text{(G1)}$$

Note: If testing to required stressor level, s, then the reliability in equation (G1) is rs. If testing to an overtest stressor level, q=k*s, then the reliability in equation (G1) is rq. With stressor level unspecified, only the symbol, r, is used.

Survival or reliability probability is the complement of the failure probability.

$$r = 1 - f \qquad \text{(G2)}$$

When overtesting by some factor, k, with k>1, such that the stressor level is given as q=k*s, the estimated component reliability at the required stressor level, s, is higher than that given in equation (G2). The estimated reliability, h=rs, is then given as

$$h = 1 - \frac{f}{k^2} \qquad \text{(G3)}$$

Solving, equation (G3) for f,

$$f = (1 - h)k^2 \qquad \text{(G4)}$$

Then substituting for h=rs,

$$(1 - c)^{1/n} = 1 - (1 - h)k^2 \qquad \text{(G5)}$$

Solving equation (G5) for k2,

$$k^2 = \frac{1-(1-c)^{1/n}}{1-h} \qquad \text{(G6)}$$

Equation (G6) can be stated in other terms to provide some mathematical insight into the relationship between the overtest factor, k, the failure probability, f=(1-rq), resulting from the n/0 test at stressor q-k*s, and the estimated population failure probability, e=(1-rs).

$$k^2 = \frac{1-r_q}{1-r_s} \tag{G7}$$

Finally, equation (G6) can be reduced to yield

$$k = \sqrt{\frac{1-(1-c)^{\frac{1}{n}}}{1-r_s}} \tag{G8}$$

The utility of equation (G8) is that it conveniently expresses the necessary overtest factor, k, as a function of the desired confidence, population reliability estimate, rs, and sample size, n, assuming an n/0 test.

As a demonstration of the utility of equation (G8), consider an example with n=5, c=0.9, and rs=0.91. From Table F2 in appendix F, the necessary value of the overtest factor, k, is approximately k=2. Equation (G8) yields an overtest factor of, k=2.024. The difference between Table F2 and equation (G8) is negligible, given that the values in Table F2 were only reported to two significant digits.

As an additional demonstration, consider an example with n=5, c=0.9, and h=0.95. The value of k, as calculated from equation (G8) is k=2.72 which is consistent with values that can be interpolated from Table F2 in Appendix F.

APPENDIX H. NUMERICAL SIMULATION APPROACH AND CODES

H.1. Simulation Approach

The ThresholdArray holds the failure thresholds for SigmaStepN component populations. Each component population has PopN virtual component sets. Each component set is TestN units. The ThresholdArray is a PopN by SigmaStepN by TestN array.

The failure thresholds for the component populations follow a zero-truncated Normal distribution with mean = 1 and SD = SigmaStep*1:SigmaStepN. As SD increases the distortion of the Normal distribution due to truncation also increases.

There are OverSpecN items in the OverSpecFactors vector.

In simulation, each overtest design is repeated OTDesignRepeats times to enable estimation of average outcomes.

ResultsDT is the central simulation table.

ResultsDTN = OverSpecN*OTDesignRepeats*SigmaStepN is the number of rows in ResultsDT

The columns of ResultsDT are:

SampleNumber represents all results for a given SD

Repetition represents all overtest levels for a single overtest design Repetition

SD is the standard deviation for the SampleNumber

Specification is the performance requirement for the component

OvertestIndex is the index for the OverSpecFactors Vector

OvertestLevel is the actual level at over-test

OTSampleSize is the number of components in an overtest (=TestN)

FailureCount is the number of failed components in an overtest

Passing is marked as 1 if the FailureCount is 0, otherwise 0

ProjRelSpec is the projected reliability at Spec by GTARS

SimRelSpec is the simulated reliability at Spec

RelConservative is true if ProjRelSpec < SimRelSpec

RelCon is 1 if ProjRelSpec < SimRelSpec

ProjConfidence is the projected confidence based on TestN

SuccessDT is the subset of ResultsDT where the overtest was successful, meaning that no failures were observed.

Simulation was performed in the Julia programming language.

H.2. GTARS Simulation Script for LogNormal Distribution

```
#Revision of truncated Normal PDF based GTARS simulation script to instead use the LogNormal distribution.
cd("C:\\Users\\tdosbor\\julia_GTARS")
println(pwd())
#
#Activate necessary packages
println("Initiating packages for data processing and visualization")
using OhMyREPL
using Revise
using Statistics
using Gadfly
using Cairo
using Fontconfig
using DataFrames
using CSV
using Feather
using Distributions
println("Packages initiated")
#
#=
Package status as of 09Apr2021
(v1.0) pkg> status
    Status `C:\Users\tdosbor\.juliapro\JuliaPro_v1.0.5-2\environments\v1.0\Project.toml`
  [c52e3926] Atom v0.11.2
```

[336ed68f] CSV v0.7.7

[159f3aea] Cairo v0.8.0

[a93c6f00] DataFrames v0.21.5

[31c24e10] Distributions v0.23.8

[becb17da] Feather v0.5.6

[186bb1d3] Fontconfig v0.3.0

[c91e804a] Gadfly v1.3.0

[7073ff75] IJulia v1.21.2

[682c06a0] JSON v0.21.0

[e5e0dc1b] Juno v0.7.2

[5fb14364] OhMyREPL v0.5.5

[69de0a69] Parsers v1.0.7

[91a5bcdd] Plots v0.28.4

[295af30f] Revise v2.7.3

[2cb19f9e] StatsKit v0.3.0

=#

#Since the LogNormal is supported by the Distributions package, and since the LogNormal distribution doesn't extend below zero, a few changes are necessary.

#Note that the LogNormal mu parameter, when set to zero, makes the median unity.

#The LogNormal sigma parameter is not equivalent to the SD, but plays a similar role. It must be greater than zero but less than or equal to unity. Otherwise the second derivative will initially be negative and the GTARS wedge approximation assumes it is >=0.

#

#

PopN = 10000#Count of points (components) generated for each distribution

MaxOverSpecLevel = 1.0#Purpose is to avoid cutting too close to the median - was 0.32

MaxSigma = 1.0#Limit sigma to no larger than unity

DistMedian = 1.0#Standardize so the first half of the distribution is in the space between zero and unity.

DistMu = log(DistMedian)#In Julia the natural logarithm is log() and the base 10 logarithm is log10()

OverSpecFactors = [1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5 8 8.5 9 9.5 10]#Vector of overspec multiplication factors.

OverSpecN = size(OverSpecFactors)[2]#Number of overspec levels to test

```
TestN = 11#Number of components in reliability test.  A requirement of zero failures is assumed (a TestN/0 testing approach).
Spec = 0.1#Specified stress level for component reliability
OTDesignRepeats = 50000#Number of times to repeat each overtest design at each overtest level
MaxOverSpec = maximum( OverSpecFactors )
MinOverSpec = minimum( OverSpecFactors )
MaxSpec = MaxOverSpecLevel / MaxOverSpec
if Spec > MaxSpec
	Spec = MaxSpec
end
#Reduce Spec to MaxSpec if it would cause the overtest stress level to exceed MaxOverSpecLevel
SigmaStepN = 10#Number of sigma values used to map out response space.  Sigma will vary from MaxSigma/SigmaSteps up to MaxSigma.
SigmaStep = MaxSigma / SigmaStepN
#
#
function Rand_LogNormal(work_mu, work_sigma)
	tempval=rand(LogNormal(work_mu, work_sigma))
	return tempval
end
#
show(Rand_LogNormal(.5, Spec))
#
ThresholdArray = [Rand_LogNormal(DistMu, SigmaStep*j) for i=1:PopN, j=1:SigmaStepN, k=1:TestN]
#

#First column to add is SampleNumber
#SampleNumber repeats by OverSpecN *OTDesignRepeats
SampleNumberReps = OverSpecN*OTDesignRepeats
ResultsDTN = SampleNumberReps*SigmaStepN
OvertestBlocks=floor(Int, ResultsDTN/OverSpecN)
OTBnew = OTDesignRepeats*SigmaStepN
#
```

```
DistributionDT=DataFrame(SigmaStep=repeat(1:SigmaStepN, inner=PopN), Outcome=0.0)
for i=1:SigmaStepN
	Baseline=(i-1)*PopN
	for j=1:PopN
		Marker=Baseline+j
		#DistributionDT.SigmaStep[Marker]=i
		DistributionDT.Outcome[Marker]=ThresholdArray[j, i, 1]
	end
end

#
ResultsDT = DataFrame(SampleNumber=repeat(1:SigmaStepN, inner=SampleNumberReps))
#
RepetitionX=repeat(1:OTDesignRepeats, inner=OverSpecN, outer=SigmaStepN)
ResultsDT.Repetition = RepetitionX
#
SDX=ResultsDT.SampleNumber*SigmaStep
ResultsDT.SD=SDX
#
SpecificationX = fill(Spec, ResultsDTN)
ResultsDT.Specification = SpecificationX
#
OvertestIndexX=repeat(1:OverSpecN, outer=OvertestBlocks)
ResultsDT.OvertestIndex = OvertestIndexX
#
OvertestFactorX = OverSpecFactors[ResultsDT.OvertestIndex]
ResultsDT.OvertestFactor = OvertestFactorX
#
OvertestLevelX = ResultsDT.OvertestFactor*Spec
ResultsDT.OvertestLevel = OvertestLevelX
#
OTSampleSizeX=fill(TestN, ResultsDTN)
ResultsDT.OTSampleSize = OTSampleSizeX
```

```
#

#The Failure Count and Passing vectors are set to -1 so that if any of these values survive further processing it will be obvious that an error occurred.

#

FailureCountX = fill(-1, ResultsDTN)

ResultsDT.FailureCount = FailureCountX

#

PassingX = fill(-1, ResultsDTN)

ResultsDT.Passing = PassingX

#

#The following loop randomly picks one of the PopN sets of TestN components for each row of the ResultsDT dataframe.  This sampling with replacement since the set of TestN components is not retired when drawn.  The number of components with threshold less than the overtest level is counted.  If the count is nonzero then that test is considered to have failed (Passing = 0).

#

for i=1:ResultsDTN

	ResultsDT.FailureCount[i]=sum(ThresholdArray[ceil(Int, PopN*rand()),ResultsDT.SampleNumber[i],:].<ResultsDT.OvertestLevel[i])

	if ResultsDT.FailureCount[i] == 0

		ResultsDT.Passing[i]=1

	else

		ResultsDT.Passing[i]=0

	end

end

#

#The next chunk calculates the projected reliability at spec using Laplace's rule.  Squaring the individual values of the overtest factor means that the projection is for a wedge, or "Normal-like" component failure threshold distribution.  Fpr the more conservative flat-tailed distribution the overtest factor is not squared.

#

ProjRelSpecX = 1 .- (1 - (TestN + 1) / (TestN + 2)) ./ (OvertestFactorX.^2)

ResultsDT.ProjRelSpec = ProjRelSpecX

#

#The next chunk repeats the projected reliability calculation as above, but for the flat-tailed or uniform distribution assumption.

#
```

```
ProjRelSpecXU = 1 .- (1 - (TestN + 1) / (TestN + 2)) ./ (OvertestFactorX)
ResultsDT.ProjRelSpecU = ProjRelSpecXU
#
#The next chunk takes all of the values in the Threshold Array for each level of SD and compares them with the spec level.  The count of values exceeding the spec is compared with the count of values in the array for that SD, yielding a simulated reliability at spec.  That value is placed in the Reliability by Standard Deviation (RelBySD) dictionary and then in the next loop the values are placed in the SimRelSpec vector in the ResultsDT dataframe.
RelBySD = Dict()
SimCount = PopN*TestN
for i=1:SigmaStepN
    TempSum = sum(ThresholdArray[:, i, :] .> Spec)
    TempRel = TempSum/SimCount
    setindex!(RelBySD, TempRel, i)
end
#
SampleNumberX=ResultsDT.SampleNumber
SimRelSpecX=zeros(Float64, ResultsDTN)
for i=1:ResultsDTN
    SimRelSpecX[i]=RelBySD[SampleNumberX[i]]
end
ResultsDT.SimRelSpec=SimRelSpecX
#
RelConservativeX=trues(ResultsDTN)
RelConservativeX=SimRelSpecX.>ProjRelSpecX
ResultsDT.RelConservative=RelConservativeX
#
#
RelConservativeXU=trues(ResultsDTN)
RelConservativeXU=SimRelSpecX.>ProjRelSpecXU
ResultsDT.RelConservativeU=RelConservativeXU
#
RelConX=zeros(Float64, ResultsDTN)
for i=1:ResultsDTN
```

```
        if SimRelSpecX[i] > ProjRelSpecX[i]
                RelConX[i]=1.0
        end
end
ResultsDT.RelCon=RelConX
#
#
RelConXU=zeros(Float64, ResultsDTN)
for i=1:ResultsDTN
        if SimRelSpecX[i] > ProjRelSpecXU[i]
                RelConXU[i]=1.0
        end
end
ResultsDT.RelConU=RelConXU
#
ProjConfidenceX=zeros(Float64, ResultsDTN)
TestNConf=1 - ((TestN + 1) / (TestN + 2)) ^ TestN
for i=1:ResultsDTN
        ProjConfidenceX[i]= TestNConf
end
ResultsDT.ProjConfidence=ProjConfidenceX
show(ResultsDT, allcols=true)
#
ResultsDTComboA=combine(groupby(ResultsDT, [:SampleNumber, :OvertestIndex]), df -> mean(df.RelCon))
show(ResultsDTComboA, allcols=true)
#
ResultsDTComboB=combine(groupby(ResultsDT, [:SampleNumber, :OvertestIndex]), :Repetition => maximum, :OvertestLevel => mean, :SimRelSpec => mean, :FailureCount => mean, :ProjRelSpec => mean, :RelCon => mean)
show(ResultsDTComboB, allcols=true, allrows=true)
#
#Generate SuccessDT, a subset of ResultsDT where the overtest was a success (no failures observed)
```

SuccessDT = ResultsDT[ResultsDT.Passing .== 1, :]

#

#Generate SummaryDT, an aggregation of SuccessDT that enables comparison of the simulated confidence with the projected confidence

SummaryDT=combine(groupby(SuccessDT, [:OvertestFactor, :OvertestLevel, :SD]), :Specification => mean, :RelCon => mean, :ProjConfidence => mean, nrow)

show(SummaryDT, allcols=true, allrows=true)

#

#Need to plot SummaryDT showing simulated and projected confidence vs. overtest level and/or overtest factor

#Concerned that RelCon is not giving an accurate mean.

#Next visualize the data using Gadfly

set_default_plot_size(15cm, 10cm)

#Make a scatterplot

ConfidenceVSOvertestFactor = plot(SummaryDT, x=:OvertestFactor, y=:RelCon_mean, color = :SD, Theme(point_size = 1.5mm))

ConfidenceVSOvertestFactor |> SVG("ConfidenceVSOvertestFactor.svg")

ConfidenceVSOvertestFactor |> PNG("ConfidenceVSOvertestFactor.png")

#

#Save dataframes for examination with other tools (e.g., JMP, Excel)

ResultsDT |> CSV.write("ResultsDT.csv")

SuccessDT |> CSV.write("SuccessDT.csv")

SummaryDT |> CSV.write("SummaryDT.csv")

DistributionDT |> CSV.write("DistributionDT.csv")

#

#=

How this script works:

The ThresholdArray holds the failure thresholds for SigmaStepN component populations. Each component population has PopN virtual component sets. Each component set is TestN units. The ThresholdArray is a PopN by SigmaStepN by TestN array.

There are OverSpecN items in the OverSpecFactors vector.

In simulation, each overtest design is repeated OTDesignRepeats times to enable estimation of average outcomes.

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SimRelSpec is the simulated reliability at Spec

RelConservative is true if ProjRelSpec < SimRelSpec

RelCon is 1 if ProjRelSpec < SimRelSpec

ProjConfidence is the projected confidence based on TestN

SuccessDT is the subset of ResultsDT where the overtest was successful, meaning that no failures were observed.

=#

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