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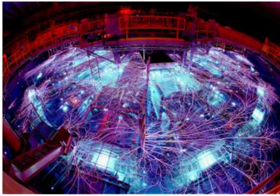

An a posteriori error analysis of the exact penalty formulation for stationary incompressible MHD

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Overview

Introduction and context

Abstract adjoint based analysis
- Linear PDE
- Nonlinear PDE

Adjoint based analysis for MHD
- Theoretical results
- Numerical results
- Well posedness for the MHD adjoint problem

Summary/further research

Why *a posteriori*?

Numerical methods often provide some "built-in" *a priori* (u_h arbitrary, not computed) error estimate

$$\|u - u_h\| \leq Ch^p$$

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- $C = C(u)$ in general depends on solution $\implies$ not computable
- Bound in predetermined norm $\implies$ not very flexible
- $\therefore$ useful e.g. for proving convergence, not so useful for problem specific analysis/adaptivity

Why *a posteriori*?

- *A posteriori* error analysis[1,2,3] leverages a computed solution to gain more information about error
- Leads to accurate and robust error estimation
- Useful for informing solver and discretization strategies as well as to understand the behavior of errors

[1] M. Ainsworth and T. Oden. A posteriori *error estimation in finite element analysis*. John Wiley-Teubner, 2000.

[2] D. J. Estep, M. G. Larson, R. D. Williams, and American Mathematical Society. *Estimating the error of numerical solutions of systems of reaction-diffusion equations*. American Mathematical Society, 2000.

[3] Michael B. Giles and Endre Süli. "Adjoint methods for PDEs: a posteriori error analysis and postprocessing by duality". In: *Acta Numerica 2002* (2002), pp. 145–236.

Why adjoint based?

Adjoint based *a posteriori* error analysis (ABAPEA) is

1. Accurate: error representations are theoretically exact
2. Goal oriented: error is computed in a user specified quantity of interest (QoI)
3. Error decomposition: error estimates naturally lead to an additive decomposition

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Adjoint based analysis for MHD

Summary/further research

Abstract setting

- V Hibert space with inner product $\langle \cdot, \cdot \rangle$
- $L : V \to V$ bounded linear operator
- Want to compute the QoI $\mathcal{Q}(u) = \langle \psi, u \rangle$ for u solving

$$
\begin{aligned}
Lu &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
$$

Approximate solution $u_h \in V_h \subset V$, denote $e = u - u_h$

Strong adjoint operator

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Definition

The adjoint to L is the unique linear operator $L^* : V \to V$ defined by

$$\langle Lu, v\rangle = \langle u, L^* v\rangle, \quad \forall u, v \in V.$$

e.g. If $V = \mathbb{R}^n$, $Lu = Au$ for some $A \in \mathbb{R}^{n\times n}$ and

$$(v^T Au)^T = u^T A^T v$$

We now consider the associated adjoint problem: find $\phi \in V$ such that

$$\begin{aligned} L^*\phi &= \psi, \quad \text{in } \Omega, \\ \phi &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

Recall $\mathcal{Q}(u) = \langle \psi, u\rangle$

Error representation

Theorem

The error in the QoI, $\mathcal{Q}(u) - \mathcal{Q}(u_h) = \mathcal{Q}(e)$, is computable as

$$\mathcal{Q}(e) = \langle \psi, e \rangle = \langle \phi, f \rangle - \langle \phi, Lu_h \rangle.$$

Proof

$$\langle \psi, e \rangle = \langle L^* \phi, e \rangle = \langle \phi, Lu - Lu_h \rangle = \langle \phi, f \rangle - \langle \phi, Lu_h \rangle.$$

Adjoints for bilinear forms

Definition

Given a bilinear form $a : V \times V \to \mathbb{R}$ the adjoint bilinear form $a^* : V \times V \to \mathbb{R}$ is defined by the relation[4,5]

$$a^*(w, v) = a(v, w), \quad \forall w, v \in V.$$

Remark

Agrees with strong definition for $a_L(v, w) = \langle Lv, w \rangle$.

[4] Michael B. Giles and Endre Süli. "Adjoint methods for PDEs: a posteriori error analysis and postprocessing by duality". In: *Acta Numerica 2002* (2002), pp. 145–236.

[5] Roland Becker and Rolf Rannacher. *An optimal control approach to a posteriori error estimation in finite element methods: Acta Numerica*. Jan. 2003.

Outline

Defining an adjoint operator for nonlinear problems

Compute $\mathcal{Q}(u) = \langle \psi, u \rangle$ where u solves nonlinear problem

$$\begin{aligned} \mathcal{F}(u(x)) &= f(x), \quad && x \text{ in } \Omega, \\ u(x) &= 0, && x \text{ on } \partial\Omega. \end{aligned}$$

Seek a linear operator $\bar{\mathcal{F}}$ for adjoint problem $\bar{\mathcal{F}}^* \phi = \psi$ so that

$$\langle \bar{\mathcal{F}}^* \phi, e \rangle = \langle \phi, \mathcal{F}(u) - \mathcal{F}(u_h) \rangle.$$

Leads to error represenation $\langle \psi, e \rangle = \langle f, \phi \rangle - \langle \mathcal{F}(u_h), \phi \rangle$

Choice of nonlinear operator

$$\text{Define} \quad \bar{\mathcal{F}} v := \int_0^1 \frac{\partial \mathcal{F}}{\partial u}(su + (1-s)u_h)\, ds\, v. \qquad (*)$$

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$$\implies \mathcal{F}(u) - \mathcal{F}(u_h) \overset{\text{(FTC)}}{=} \int_0^1 \frac{d}{ds}\mathcal{F}(su + (1-s)u_h)\, ds$$

$$\overset{\text{(chain rule)}}{=} \int_0^1 \frac{\partial \mathcal{F}}{\partial u}(su + (1-s)u_h)\, ds (u - u_h) = \bar{\mathcal{F}} e.$$

Choice of nonlinear operator

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$$\overset{\text{(chain rule)}}{=} \int_0^1 \frac{\partial \mathcal{F}}{\partial u}(su + (1-s)u_h)\, ds (u - u_h) = \bar{\mathcal{F}} e.$$

Remarks

- Precise interpretation of $\frac{\partial \mathcal{F}}{\partial u}$ where $\mathcal{F}$ is an operator is nontrivial.
- In practice don't have u (true solution) approximate u by u_h in $(*)$ induces so-called linearization error

Nonlinear variational framework

- $\mathscr{W} = \prod_{i=1}^{n} \mathscr{W}_i$ product Hilbert space with inner product $\langle \cdot, \cdot \rangle$
- $\mathscr{V} = \prod_{i=1}^{n} \mathscr{V}_i$ dense subspace such that $\mathscr{V}_i$ is a dense subspace of $\mathscr{W}_i$ for all i
- E.g. $\mathscr{W} = \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ and $\mathscr{V} = \boldsymbol{H}^1(\Omega) \times \boldsymbol{H}^1(\Omega)$

Nonlinear variational framework II

Want to evaluate $\mathcal{Q}(u) = \langle \psi, u \rangle$ where $u \in \mathscr{V}$ solves

$$\mathcal{N}(u, v) = \langle f, v \rangle, \quad \forall v \in \mathscr{V}$$

$\mathcal{N} : \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ has the specific structure

$$\mathcal{N}(v, w) = \sum_{i=1}^{m} \langle N_i(v), w_{\ell_i} \rangle + a(v, w)$$

$a(\cdot, \cdot)$ is a bilinear form, $\ell_i \in \{1, \ldots, n\}$ and $N_i : \mathscr{V} \to \mathscr{W}_{\ell_i}$

Remark

$\mathcal{N}(\cdot, \cdot)$ abstract version of exact penalty weak form- closely related to MHD

For a solution/approximation pair (u/u_h), define

- the matrix $\overline{\mathcal{J}} \in \mathbb{R}^{m\times n}$

$$\overline{\mathcal{J}}_{ij} = \int_0^1 \frac{\partial N_i}{\partial u_j}(su + (1-s)u_h)ds,$$

- the linearized operator $\bar{N}_i : \mathscr{V} \to \mathscr{W}_{\ell_i}$,

$$\bar{N}_i v = \sum_{j=1}^{n} \overline{\mathcal{J}}_{ij} v_j.$$

- the bilinear forms

$$\overline{\nu}_i(v,w) = \langle \bar{N}_i v, w_{\ell_i} \rangle = \left\langle \sum_{j=1}^{n} \overline{\mathcal{J}}_{ij} v_j, w_{\ell_i} \right\rangle = \sum_{j=1}^{n} \left\langle \overline{\mathcal{J}}_{ij} v_j, w_{\ell_i} \right\rangle .$$

Adjoint objects and error representation

As established before,

- $\overline{\nu}_i^*(v, w) := \overline{\nu}_i(w, v)$ (weak definition of adjoint)
- $\overline{\mathcal{J}}_{ij}^*$ such that $\langle \overline{\mathcal{J}}_{ij} v, w \rangle = \langle v, \overline{\mathcal{J}}_{ij}^* w \rangle$ (strong definition of the adjoint)

Now set $\overline{\mathcal{N}}^*(\phi, v) := \sum_{i=1}^m \overline{\nu}_i^*(\phi, v) + a^*(\phi, v)$

Theorem

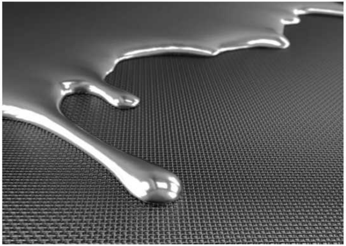
if ϕ solves the dual problem,

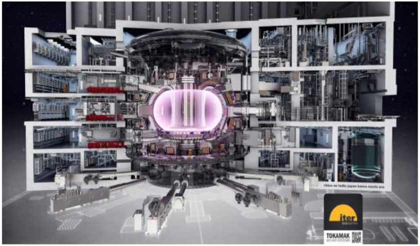
$$\overline{\mathcal{N}}^*(\phi, v) = \langle \psi, v \rangle, \ \forall v \in \mathscr{V},$$

the error in a QoI represented by $\mathcal{Q}(u) = \langle \psi, u \rangle$ is computable as $\langle \psi, e \rangle = \langle f, \phi \rangle - \mathcal{N}(u_h, \phi)$.

Outline

What is MHD?





- Model for electrically conducting fluids, e.g. liquid metals and plasmas
- MHD for plasma valid as continuum representation of collisional plamsa systems
- Structure of resistive MHD is Navier-Stokes + Lorentz force coupled with low frequency Maxwell equations

Mathematical model

Stationary incompressible MHD in $\Omega \subset \mathbb{R}^d$ convex, polygonal

$$
\begin{aligned}
-\frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p - \kappa(\nabla\times\boldsymbol{b})\times\boldsymbol{b} &= \boldsymbol{f}, && \text{(momentum)}\\
\nabla\cdot\boldsymbol{u} &= 0, && \text{(continuity)}\\
\frac{\kappa}{\mathrm{Re}_m}\nabla\times(\nabla\times\boldsymbol{b}) - \kappa\nabla\times(\boldsymbol{u}\times\boldsymbol{b}) &= \boldsymbol{0}, && \text{(induction)}\\
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- Unknowns: velocity $\boldsymbol{u}$, magnetic field $\boldsymbol{b}$, pressure p

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- Unknowns: velocity $\boldsymbol{u}$, magnetic field $\boldsymbol{b}$, pressure p
- Nondimensionalized parameters
- Incompressible Navier-Stokes substructure
- Three coupled nonlinear terms

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- Unknowns: velocity $\boldsymbol{u}$, magnetic field $\boldsymbol{b}$, pressure p
- Nondimensionalized parameters
- Incompressible Navier-Stokes substructure
- Three coupled nonlinear terms
- $2d+2$ equations only $2d+1$ unknowns $\Longrightarrow$ overdetermined

Augment MHD system with boundary conditions,

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{g}, && \text{on } \partial\Omega, \\ \boldsymbol{b} \times \boldsymbol{n} &= \boldsymbol{q} \times \boldsymbol{n}, && \text{on } \partial\Omega. \end{aligned}$$

Define in the sense of the trace operator,

$$\begin{aligned} \boldsymbol{H}_0^1(\Omega) &:= \{\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) : \boldsymbol{w}|_{\partial\Omega} \equiv \boldsymbol{0}\} \\ \boldsymbol{H}_\tau^1(\Omega) &:= \{\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) : (\boldsymbol{w} \times \boldsymbol{n})|_{\partial\Omega} \equiv \boldsymbol{0}\}. \end{aligned}$$

We also define the product space,

$$\mathscr{P} := \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_\tau^1(\Omega) \times L^2(\Omega)$$

Exact penalty problem

Find $U = (\boldsymbol{u}, \boldsymbol{b}, p) \in \mathscr{P}$ such that

$$\mathcal{N}_{EP}(U, V) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall\, V \in \mathscr{P},$$

The nonlinear form $\mathcal{N}_{EP}$ is defined by

$$\mathcal{N}_{EP}(U, V) := \frac{1}{\mathrm{Re}}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + (\boldsymbol{\mathcal{C}}(\boldsymbol{u}), \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u})$$
$$+ \kappa(\boldsymbol{\mathcal{Y}}(\boldsymbol{b}), \boldsymbol{v}) - \kappa(\boldsymbol{\mathcal{Z}}(\boldsymbol{u}, \boldsymbol{b}), \boldsymbol{c}) + \frac{\kappa}{\mathrm{Re}_m}(\nabla \times \boldsymbol{b}, \nabla \times \boldsymbol{c}) + \frac{\kappa}{\mathrm{Re}_m}(\nabla \cdot \boldsymbol{b}, \nabla \cdot \boldsymbol{c}),$$

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- Nonlinear operators

$$\mathcal{C}(\boldsymbol{u}) := (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \quad \mathcal{Y}(\boldsymbol{b}) := (\nabla \times \boldsymbol{b}) \times \boldsymbol{b}, \quad \mathcal{Z}(\boldsymbol{u}, \boldsymbol{b}) := \nabla \times (\boldsymbol{u} \times \boldsymbol{b})$$

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- Penalty term

Exact penalty and MHD

- First 7 terms of EP weak form arise from multiplying MHD equations by test functions and performing integration by parts
- $\frac{\kappa}{\mathrm{Re}_m}(\nabla \cdot \boldsymbol{b}, \nabla \cdot \boldsymbol{c})$, enforces the solenoidal involution
- Existence and uniqueness of discrete problem proven under certain assumptions on data[6]

[6] Max D. Gunzburger, Amnon J. Meir, and Janet S. Peterson. "On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics". In: *Mathematics of Computation* 56.194 (1991), pp. 523–523.

Matching up with theoretical derivations

For the exact penalty weak form, we have that

$$\mathcal{N}_{EP}(U, V) = \sum_{i=1}^{3} (N_{EP,i}(U), V_{\ell_i}) + a_{EP}(U, V),$$

where

$$(N_{EP,1}(U), V_2) = (\mathcal{Z}(\boldsymbol{u}, \boldsymbol{b}), \boldsymbol{c}),$$
$$(N_{EP,2}(U), V_1) = (\mathcal{Y}(\boldsymbol{b}), \boldsymbol{v}),$$
$$(N_{EP,3}(U), V_1) = (\mathcal{C}(\boldsymbol{u}), \boldsymbol{v}),$$

and

$$a_{EP}(U, V) = \frac{1}{\mathrm{Re}}(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) + (q, \nabla \cdot \boldsymbol{u})$$
$$+ \frac{\kappa}{\mathrm{Re}_m}(\nabla \times \boldsymbol{b}, \nabla \times \boldsymbol{c}) + \frac{\kappa}{\mathrm{Re}_m}(\nabla \cdot \boldsymbol{b}, \nabla \cdot \boldsymbol{c}).$$

Averaged entries

As outlined in the theoretical error analysis, the entries $\overline{\mathcal{J}}^*_{11} V_2 = \overline{\boldsymbol{\mathcal{Z}}}^*_{\boldsymbol{u}} \boldsymbol{c}$, $\overline{\mathcal{J}}^*_{12} V_2 = \overline{\boldsymbol{\mathcal{Z}}}^*_{\boldsymbol{b}} \boldsymbol{c}$, $\overline{\mathcal{J}}^*_{21} V_1 = \overline{\boldsymbol{\mathcal{Y}}}^* \boldsymbol{v}$ and $\overline{\mathcal{J}}^*_{31} V_1 = \overline{\boldsymbol{\mathcal{C}}}^* \boldsymbol{v}$ are,

$$
\begin{aligned}
\overline{\boldsymbol{\mathcal{Z}}}^*_{\boldsymbol{u}} \boldsymbol{c} &= \tfrac{1}{2}(\boldsymbol{u} + \boldsymbol{u}_h) \times (\nabla \times \boldsymbol{c}), \\
\overline{\boldsymbol{\mathcal{Z}}}^*_{\boldsymbol{b}} \boldsymbol{c} &= -\tfrac{1}{2}(\boldsymbol{b} + \boldsymbol{b}_h) \times (\nabla \times \boldsymbol{c}), \\
\overline{\boldsymbol{\mathcal{Y}}}^* \boldsymbol{v} &= \tfrac{1}{2}\left(-(\nabla \times (\boldsymbol{b} + \boldsymbol{b}_h) \times \boldsymbol{v}) + \nabla \times ((\boldsymbol{b} + \boldsymbol{b}_h) \times \boldsymbol{v})\right), \\
\overline{\boldsymbol{\mathcal{C}}}^* \boldsymbol{v} &= \tfrac{1}{2}\Big((\nabla \boldsymbol{u} + \nabla \boldsymbol{u}_h)^T \boldsymbol{v} - ((\boldsymbol{u} + \boldsymbol{u}_h) \cdot \nabla)\, \boldsymbol{v} - \big(\nabla \cdot (\boldsymbol{u} + \boldsymbol{u}_h)\big)\boldsymbol{v}\Big),
\end{aligned}
$$

while the remaining $\overline{\mathcal{J}}^*_{ij}$ entries are zero

Exact penalty adjoint problem

The weak dual problem is therefore be stated as: find $\Phi = (\phi, \boldsymbol{\beta}, \pi) \in \mathscr{P}$ such that

$$\overline{\mathcal{N}}^*_{EP}(\Phi, V) = (\Psi, V), \quad \forall\, V \in \mathscr{P}$$

with

$$\begin{aligned}
\overline{\mathcal{N}}^*_{EP}(\Phi, V) = {} & \frac{1}{\mathrm{Re}}(\nabla \boldsymbol{\phi}, \nabla \boldsymbol{v}) + \left(\overline{\boldsymbol{\mathcal{C}}}^* \boldsymbol{\phi}, \boldsymbol{v}\right) + (\nabla \cdot \boldsymbol{v}, \pi) - (\nabla \cdot \boldsymbol{\phi}, q) \\
& + \frac{\kappa}{\mathrm{Re}_m}(\nabla \times \boldsymbol{\beta}, \nabla \times \boldsymbol{c}) + \frac{\kappa}{\mathrm{Re}_m}(\nabla \cdot \boldsymbol{\beta}, \nabla \cdot \boldsymbol{c}) \\
& - \kappa \left(\overline{\boldsymbol{\mathcal{Y}}}^* \boldsymbol{\phi}, \boldsymbol{c}\right) - \kappa \left(\overline{\boldsymbol{\mathcal{Z}}}^*_{\boldsymbol{u}} \boldsymbol{\beta}, \boldsymbol{v}\right) - \kappa \left(\overline{\boldsymbol{\mathcal{Z}}}^*_{\boldsymbol{b}} \boldsymbol{\beta}, \boldsymbol{c}\right).
\end{aligned}$$

Error representation for exact penalty

Theorem (Chaudhry, Rappaport, Shadid, 2020)

For a given QoI represented by $\Psi = \left[\psi_{\boldsymbol{u}}, \psi_{\boldsymbol{b}}, \psi_p\right]^T$, the error $E = \left[\boldsymbol{e_u}, \boldsymbol{e_b}, e_p\right]^T$ in the numerical approximation of the QoI satisfies

$$
\begin{aligned}
(\Psi, E) = (\boldsymbol{f}, \boldsymbol{\phi}) - \Bigg[& \frac{1}{\mathrm{Re}}(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{\phi}) + ((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h, \boldsymbol{\phi}) \\
& - (p_h, \nabla \cdot \boldsymbol{\phi}) + \kappa((\nabla \times \boldsymbol{b}_h) \times \boldsymbol{b}_h, \boldsymbol{\phi}) + (\nabla \cdot \boldsymbol{u}_h, \pi) \\
& + \frac{\kappa}{\mathrm{Re}_m}(\nabla \times \boldsymbol{b}_h, \nabla \times \boldsymbol{\beta}) + \kappa(\nabla \times (\boldsymbol{u}_h \times \boldsymbol{b}_h), \boldsymbol{\beta}) \\
& + \frac{\kappa}{\mathrm{Re}_m}(\nabla \cdot \boldsymbol{b}_h, \nabla \cdot \boldsymbol{\beta}) \Bigg].
\end{aligned}
$$

Error estimator

Estimator $\eta \approx (\Psi, E)$ where,

$$\eta = E_{mom} + E_{con} + E_M,$$

and

$$E_{mom} = (\boldsymbol{f}, \boldsymbol{\phi}_h) - \left(\frac{1}{\mathrm{Re}} (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{\phi}_h) + ((\boldsymbol{u}_h \cdot \nabla)\boldsymbol{u}_h, \boldsymbol{\phi}_h) - (p_h, \nabla \cdot \boldsymbol{\phi}_h) \right.$$

$$\left. + \kappa((\nabla \times \boldsymbol{b}_h) \times \boldsymbol{b}_h, \boldsymbol{\phi}_h) \right),$$

$$E_{con} = -(\nabla \cdot \boldsymbol{u}_h, \pi_h),$$

$$E_M = -\frac{\kappa}{\mathrm{Re}_m} (\nabla \times \boldsymbol{b}_h, \nabla \times \boldsymbol{\beta}_h) + \kappa(\nabla \times (\boldsymbol{u}_h \times \boldsymbol{b}_h), \boldsymbol{\beta}_h) - \frac{\kappa}{\mathrm{Re}_m} (\nabla \cdot \boldsymbol{b}_h, \nabla \cdot \boldsymbol{\beta}_h).$$

Outline

Hartmann problem description

- Models the one-dimensional flow of a conducting fluid in a channel[7]
- analytic solution $\boldsymbol{u} = \left[u_x, 0\right]^T$, $\boldsymbol{b} = \left[b_x, 1\right]^T$, p

$$u_x(y) = \frac{G\,\mathrm{Re}(\cosh(\mathrm{Ha}/2) - \cosh(\mathrm{Ha}\,y))}{2\mathrm{Ha}\sinh(\mathrm{Ha}/2)},$$

$$B_x(y) = \frac{G(\sinh(\mathrm{Ha}\,y) - 2\sinh(\mathrm{Ha}/2)y)}{2\kappa\sinh(\mathrm{Ha}/2)},$$

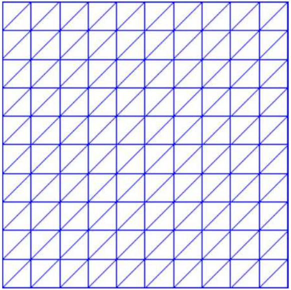
$$p(x) = -Gx - \kappa B_x^2/2,$$

- $G = -\frac{dp}{dx}$ arbitrary pressure drop

[7] Müller Ulrich and Bühler Leo. *Magnetofluiddynamics in Channels and containers*. Springer, 2010.

Discretization and QoI

- Lagrange space finite element space of order q: $\mathbb{P}^q := \left\{ v \in C(\Omega) : \forall K \in \mathcal{T}_h,\ v|_K \in \mathbb{P}^q(K) \right\}$
- Product space satisfies the LBB stability condition
- $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^2 \subset \mathbb{R}^2$
- $\Omega_c := \left[-\frac{1}{4}, \frac{1}{2}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right]$
- $\mathbb{1}_{\Omega_c}$ the characteristic function on Ω_c
- $\Psi = \left[\mathbb{1}_{\Omega_c}, 0, 0, 0, 0\right]^T$
- QoI $\mathcal{Q}(U) = (\Psi, U) = (\mathbb{1}_{\Omega_c}, u_x)$



$\mathcal{T}_h$

[7] Daniele Boffi, Franco Brezzi, Michel Fortin, et al. *Mixed finite element methods and applications*. Vol. 44. Springer, 2013.

Hartmann results

$$\text{Eff.} = \frac{\text{Error estimate}}{\text{True error}} = \frac{\eta}{(\Psi, E)}$$

# Elements	True Error	Eff.	E_{mom}	E_{con}	E_M
1600	2.76e-04	1.00	4.53e-06	-2.28e-04	5.00e-04
6400	6.98e-05	1.00	1.29e-06	-6.23e-05	1.31e-04
14400	3.11e-05	1.00	6.05e-07	-2.86e-05	5.91e-05
25600	1.75e-05	1.00	3.49e-07	-1.63e-05	3.35e-05

$(\mathbb{P}^2, \mathbb{P}^1, \mathbb{P}^1)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$.

# Elements	True Error	Eff.	E_{mom}	E_{con}	E_M
1600	-2.25e-04	1.02	1.08e-06	-2.27e-04	4.79e-06
6400	-6.13e-05	1.04	1.04e-06	-6.23e-05	2.18e-06
14400	-2.81e-05	1.04	5.98e-07	-2.86e-05	1.13e-06
25600	-1.60e-05	1.04	3.76e-07	-1.64e-05	6.81e-07

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$(\mathbb{P}^2, \mathbb{P}^2, \mathbb{P}^1)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$.

Hartmann results

2d Elem.	True Error	Eff.	E_{mom}	E_{con}	E_M
1600	1.23e-06	1.21	3.97e-07	-4.15e-06	5.24e-06
6400	1.46e-07	1.47	9.23e-08	-5.07e-07	6.29e-07
14400	4.97e-08	1.63	3.84e-08	-1.40e-07	1.83e-07
25600	2.47e-08	1.73	2.07e-08	-5.44e-08	7.64e-08

$(\mathbb{P}^3, \mathbb{P}^2, \mathbb{P}^2)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$

Hartmann results

2d Elem.	True Error	Eff.	E_{mom}	E_{con}	E_M
1600	1.23e-06	1.21	3.97e-07	-4.15e-06	5.24e-06
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2d Elem.	True Error	Eff.	E_{mom}	E_{con}	E_M
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$(\mathbb{P}^3, \mathbb{P}^2, \mathbb{P}^2)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$ using true soln in adjoint

Hartmann results

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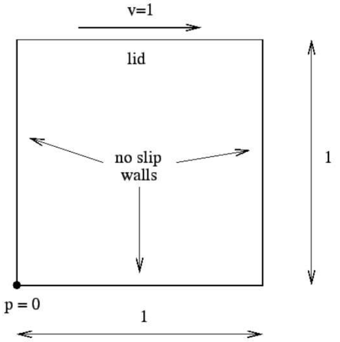
$(\mathbb{P}^3, \mathbb{P}^2, \mathbb{P}^2)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$ using true soln in adjoint

Magnetic lid driven cavity

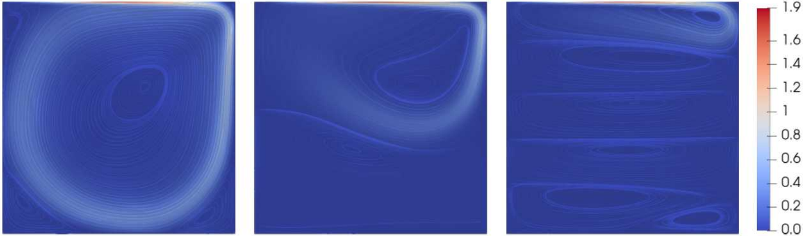
- Fluid lid driven cavity BCs for u_x (right)
- Set $\boldsymbol{b} \times \boldsymbol{n} = [-1, 0] \times \boldsymbol{n}$ on $\partial\Omega$
- Discontinous lid velocity $\implies u_x \in H^{1/2-\varepsilon}(\partial\Omega)$
- Polynomial regularization of lid velocity[8]

$$u_{top}(x) = C\left(x - \tfrac{1}{2}\right)^2 \left(x + \tfrac{1}{2}\right)^2$$



[8]Michael W. Lee, Earl H. Dowell, and Maciej J. Balajewicz. *A study of the regularized lid-driven cavity's progression to chaos.* Nov. 2018.

Magnetic lid driven cavity velocity profiles



$\mathrm{Re}_m = 0.1$ $\mathrm{Re}_m = 0.5$ $\mathrm{Re}_m = 5.0$

- Increasing $\mathrm{Re}_m \implies$ flow is weakend
- Multiple interlocked eddies appear
- Results agree qualitatively with non-regularized problem[9]

[9] Edward G. Phillips, Howard C. Elman, Eric C. Cyr, John N. Shadid, and Roger P. Pawlowski. "A Block Preconditioner for an Exact Penalty Formulation for Stationary MHD". In: *SIAM Journal on Scientific Computing* 36.6 (2014).

Remarks

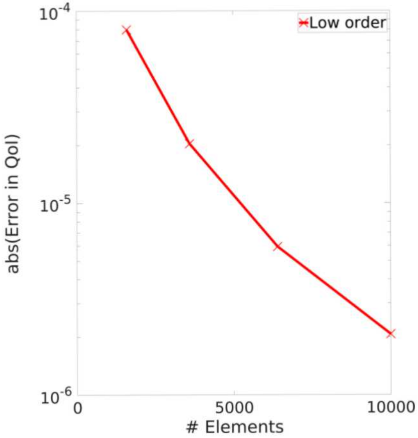
- *No analytic solution for* (Ψ, E), *use high resolution reference solution on* 400×400 *mesh using* $(\mathbb{P}^3, \mathbb{P}^2, \mathbb{P}^2)$ *for* $(\boldsymbol{u}, \boldsymbol{b}, p)$
- $\Omega_c := \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[0, \frac{1}{2}\right]$
- *QoI* $\mathcal{Q}(U) = (\mathbb{1}_{\Omega_c}, b_y)$ *average induced* b_y *in upper middle*

Adjoint results for Re=2000

# Elements	True Error	Eff.	E_{mom}	E_{con}	E_M
1600	-8.01e-05	1.10	-3.65e-05	-5.70e-05	5.63e-06
3600	-2.04e-05	0.98	-5.69e-06	-1.66e-05	2.25e-06
6400	-5.92e-06	0.96	-1.84e-06	-5.06e-06	1.19e-06
10000	-2.07e-06	0.96	-8.17e-07	-1.91e-06	7.41e-07

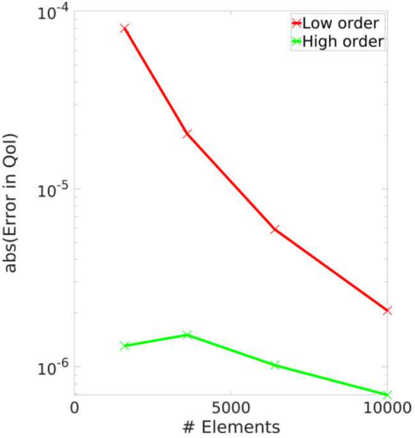
$(\mathbb{P}^2, \mathbb{P}^1, \mathbb{P}^1)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$

Adjoint results for Re=2000 II



$(\mathbb{P}^2, \mathbb{P}^1, \mathbb{P}^1)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$

Adjoint results for Re=2000 III



$(\mathbb{P}^3, \mathbb{P}^1, \mathbb{P}^2)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$

Adjoint results for Re=2000 IV

# Elements	True Error	Eff.	E_{mom}	E_{con}	E_M
1600	1.31e-06	0.78	-1.58e-06	-3.47e-06	6.08e-06
3600	1.51e-06	0.96	-1.91e-07	-5.29e-07	2.17e-06
6400	1.02e-06	0.98	-3.87e-08	-1.28e-07	1.17e-06
10000	6.94e-07	0.99	-1.07e-08	-4.04e-08	7.38e-07

$(\mathbb{P}^3, \mathbb{P}^1, \mathbb{P}^2)$ for $(\boldsymbol{u}, \boldsymbol{b}, p)$

Outline

Introduction and context

Abstract adjoint based analysis

Adjoint based analysis for MHD

- Theoretical results
- Numerical results
- Well posedness for the MHD adjoint problem

Summary/further research

Reformultaion

- Appeal to the standard theory of saddle point problem[10,11] $X := \boldsymbol{H}_0^1(\Omega) \times \boldsymbol{H}_\tau^1(\Omega)$ and $M := L^2(\Omega)$ so that $\mathscr{P} = X \times M$.
- define the bilinear form $a : X \times X \to \mathbb{R}$ by

$$
\begin{aligned}
a((\phi,\beta),(\boldsymbol{v},\boldsymbol{c})) &= \frac{1}{\mathrm{Re}}(\nabla\phi,\nabla\boldsymbol{v}) + \left(\overline{\mathcal{C}}^*\phi,\boldsymbol{v}\right)\\
&+ \frac{\kappa}{\mathrm{Re}_m}(\nabla\times\beta,\nabla\times\boldsymbol{c}) + \frac{\kappa}{\mathrm{Re}_m}(\nabla\cdot\beta,\nabla\cdot\boldsymbol{c})\\
&-\kappa\left(\overline{\mathcal{Y}}^*\phi,\boldsymbol{c}\right) - \kappa\left(\overline{\mathcal{Z}}_{\boldsymbol{u}}^*\beta,\boldsymbol{v}\right) - \kappa\left(\overline{\mathcal{Z}}_{\boldsymbol{b}}^*\beta,\boldsymbol{c}\right),
\end{aligned}
$$

and the mixed form $b : X \times M \to \mathbb{R}$ by

$$
b((\boldsymbol{v},\boldsymbol{c}),\pi) = -(\pi,\nabla\cdot\boldsymbol{v}).
$$

[10] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*. Springer, 2011.

[11] Susanne C. Brenner and L. Ridgway. Scott. *The mathematical theory of finite element methods*. Springer, 2011.

Reformultaion II

Weak exact penalty adjoint problem equivalent to: find $((\phi, \boldsymbol{\beta}), \pi) \in X \times M$ such that

$$\begin{cases} a((\phi, \boldsymbol{\beta}), (\boldsymbol{v}, \boldsymbol{c})) + b((\boldsymbol{v}, \boldsymbol{c}), p) = (\psi, \boldsymbol{v}), & \forall (\boldsymbol{v}, \boldsymbol{c}) \in X, \\ b((\phi, \boldsymbol{\beta}), q) = (\psi_p, q), & \forall q \in M. \end{cases}$$

Existence and uniqueness equivalent to

1. The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bounded on their respective domains
2. The form $a(\cdot,\cdot)$ is coercive
3. The form $b(\cdot,\cdot)$ satisfies the inf-sup condition: $\exists \beta > 0$ such that

$$\inf_{q \in M} \sup_{(\boldsymbol{v},\boldsymbol{c}) \in X} \frac{b((\boldsymbol{v}, \boldsymbol{c}), q)}{\|(\boldsymbol{v}, \boldsymbol{c})\|_X \|q\|_M} \geq \beta.$$

We consider

$$a((\phi,\boldsymbol{\beta}),(\boldsymbol{v},\boldsymbol{c})) = a_0((\phi,\boldsymbol{\beta}),(\boldsymbol{v},\boldsymbol{c})) + a_1((\phi,\boldsymbol{\beta}),(\boldsymbol{v},\boldsymbol{c}))$$

where

$$a_0((\phi,\boldsymbol{\beta}),(\boldsymbol{v},\boldsymbol{c})) = \frac{1}{\mathrm{Re}}(\nabla\phi,\nabla\boldsymbol{v}) + \frac{\kappa}{\mathrm{Re}_m}(\nabla\times\boldsymbol{\beta},\nabla\times\boldsymbol{c})$$
$$+ \frac{\kappa}{\mathrm{Re}_m}(\nabla\cdot\boldsymbol{\beta},\nabla\cdot\boldsymbol{c}),$$
$$a_1((\phi,\boldsymbol{\beta}),(\boldsymbol{v},\boldsymbol{c})) = \left(\overline{\mathcal{C}}^*\phi,\boldsymbol{v}\right) - \kappa\left(\overline{\mathcal{Y}}^*\phi,\boldsymbol{c}\right)$$
$$-\kappa\left(\overline{\mathcal{Z}}^*_{\boldsymbol{u}}\beta,\boldsymbol{v}\right) - \kappa\left(\overline{\mathcal{Z}}^*_{\boldsymbol{b}}\beta,\boldsymbol{c}\right).$$

Lemma

The form $a_1(\cdot,\cdot)$ is bounded on X.

Sketches of the proofs of well-posedness II

Definition

$\boldsymbol{s} := \boldsymbol{u} + \boldsymbol{u}_h$, $\boldsymbol{t} := \boldsymbol{b} + \boldsymbol{b}_h$

Sketch for a single term:

$$
\begin{aligned}
\int_\Omega \left|\overline{\mathcal{C}}^* \phi \cdot \boldsymbol{v}\right| dx &= \frac{1}{2}\int_\Omega \left|\left[(\nabla \boldsymbol{s})^T \phi - ((\boldsymbol{s}\cdot\nabla)\phi) - (\nabla\cdot\boldsymbol{s})\phi\right]\cdot\boldsymbol{v}\right| dx \\
&= \frac{1}{2}\int_\Omega \left|\phi^T(\nabla\boldsymbol{s})\boldsymbol{v} - \boldsymbol{v}^T(\nabla\phi)\boldsymbol{s} - (\nabla\cdot\boldsymbol{s})(\phi\cdot\boldsymbol{v})\right| dx \\
&\le \frac{1}{2}\left[\|\phi\|_{L^4}\|\boldsymbol{s}\|_1\|\boldsymbol{v}\|_{L^4} + \|\phi\|_1\|\boldsymbol{s}\|_{L^4}\|\boldsymbol{v}\|_{L^4} + \|\nabla\cdot\boldsymbol{s}\|\|\phi\cdot\boldsymbol{v}\|\right] \\
&\le \frac{1}{2}\left[\|\phi\|_{L^4}\|\boldsymbol{s}\|_1\|\boldsymbol{v}\|_{L^4} + \|\phi\|_1\|\boldsymbol{s}\|_{L^4}\|\boldsymbol{v}\|_{L^4} + \sqrt{3}\|\boldsymbol{s}\|_1\|\phi\|_{L^4}\|\boldsymbol{v}\|_{L^4}\right] \\
&\le \frac{\gamma}{2}\left(\|\phi\|_1\|\boldsymbol{s}\|_1\|\boldsymbol{v}\|_1 + \|\boldsymbol{s}\|_1\|\phi\|_1\|\boldsymbol{v}\|_1 + \sqrt{3}\|\boldsymbol{s}\|_1\|\phi\|_1\|\boldsymbol{v}\|_1\right) \\
&\le \frac{3\sqrt{3}\gamma}{2}\|\boldsymbol{s}\|_1\|\phi\|_1\|\boldsymbol{v}\|_1,
\end{aligned}
$$

Putting terms together

$$
\begin{aligned}
a_1((\phi,\beta),(\boldsymbol{v},\boldsymbol{c})) &\leq \gamma\Bigg(\frac{3\sqrt{3}}{2}\|\boldsymbol{s}\|_1\|\phi\|_1\|\boldsymbol{v}\|_1 + \kappa\sqrt{2}\|\boldsymbol{c}\|_1\|\boldsymbol{t}\|_1\|\phi\|_1 \\
&\qquad + \frac{\kappa\sqrt{2}}{2}\|\boldsymbol{v}\|_1\|\boldsymbol{t}\|_1\|\beta\|_1 + \frac{\kappa\sqrt{2}}{2}\|\boldsymbol{c}\|_1\|\boldsymbol{s}\|_1\|\beta\|_1\Bigg) \\
&\leq \gamma\Bigg(\frac{3\sqrt{3}}{2}\|\boldsymbol{s}\|_1\|\phi\|_1\|\boldsymbol{v}\|_1 + \frac{\kappa\sqrt{2}}{2}\|\boldsymbol{c}\|_1\|\boldsymbol{s}\|_1\|\beta\|_1 + \|\boldsymbol{t}\|_1\kappa\sqrt{2}\|(\boldsymbol{v},\boldsymbol{c})\|_X\|(\phi,\beta)\|_X\Bigg) \\
&\leq \gamma\Bigg(\|\boldsymbol{s}\|_1\max\left\{\frac{3\sqrt{3}}{2},\frac{\kappa\sqrt{2}}{2}\right\}\|(\boldsymbol{v},\boldsymbol{c})\|_X\|(\phi,\beta)\|_X + \|\boldsymbol{t}\|_1\|(\boldsymbol{v},\boldsymbol{c})\|_X\|(\phi,\beta)\|_X\Bigg) \\
&\leq \alpha_b\|(\boldsymbol{v},\boldsymbol{c})\|_X\|(\phi,\beta)\|_X,
\end{aligned}
$$

Sketches of the proofs of well-posedness IV

Lemma

There exists a constant $\alpha_c > 0$ such that whenever

$$\frac{k_1}{\mathrm{Re}} - \gamma \left[\frac{3\sqrt{3}}{2} \|\boldsymbol{s}\|_1 + \frac{3\kappa\sqrt{2}}{4} \|\boldsymbol{t}\|_1 \right] > 0,$$

and

$$\frac{k_2\kappa}{\mathrm{Re}_m^2} - \gamma \left[\frac{\kappa\sqrt{2}}{2} \|\boldsymbol{s}\|_1 + \frac{3\kappa\sqrt{2}}{4} \|\boldsymbol{t}\|_1 \right] > 0$$

then

$$a((\phi, \beta), (\phi, \beta)) \geq \alpha_c \|(\phi, \beta)\|_X^2, \quad \forall (\phi, \beta) \in X.$$

Sketches of the proofs of well-posedness V

$$
\begin{aligned}
a((\phi,\beta),(\phi,\beta)) &\geq \frac{k_1}{\mathrm{Re}}\|\phi\|_1^2 + \frac{k_2\kappa}{\mathrm{Re}_m^2}\|\beta\|_1^2 - |a_1((\phi,\beta),(\phi,\beta))| \\
&\geq \left(\frac{k_1}{\mathrm{Re}} - \frac{\gamma 3\sqrt{3}}{2}\|s\|_1\right)\|\phi\|_1^2 + \left(\frac{k_2\kappa}{\mathrm{Re}_m^2} - \frac{\gamma\kappa\sqrt{2}}{2}\|s\|_1\right)\|\beta\|_1^2 - \frac{\gamma 3\kappa\sqrt{2}}{2}\|\phi\|_1\|t\|_1\|\beta\|_1 \\
&\geq \left(\frac{k_1}{\mathrm{Re}} - \frac{\gamma 3\sqrt{3}}{2}\|s\|_1\right)\|\phi\|_1^2 + \left(\frac{k_2\kappa}{\mathrm{Re}_m^2} - \frac{\gamma\kappa\sqrt{2}}{2}\|s\|_1\right)\|\beta\|_1^2 \\
&\quad - \frac{\gamma 3\kappa\sqrt{2}}{4}\|t\|_1\left(\|\beta\|_1^2 + \|\phi\|_1^2\right) \\
&= \left(\frac{k_1}{\mathrm{Re}} - \gamma\left[\frac{3\sqrt{3}}{2}\|s\|_1 + \frac{3\kappa\sqrt{2}}{4}\|t\|_1\right]\right)\|\phi\|_1^2 \\
&\quad \left(\frac{k_2\kappa}{\mathrm{Re}_m^2} - \gamma\left[\frac{\kappa\sqrt{2}}{2}\|s\|_1 + \frac{3\kappa\sqrt{2}}{4}\|t\|_1\right]\right)\|\beta\|_1^2.
\end{aligned}
$$

Summary

- Developed adjoint framework for nonlinear weak forms
  1. Mean value theorem for integrals in Hilbert spaces
  2. Standard definitions and properties of strong/weak adjoint
- Applied framework to exact penalty form of (resistive) MHD
  1. Coupling provides nontrivial linearization
  2. Define adjoint directly to weak form due to solenoidal constraint ($\nabla \cdot \boldsymbol{B} = 0$)
- Proved well-posedness under assumptions on system parameters: $\mathrm{Re}, \mathrm{Re}_m, \kappa$. Not valid in the case of ideal MHD

Possible future research

1. Extend analysis to transient MHD and more complex physics
2. Explore other stabilization methods e.g. divergence cleaning or compatible discretizations
3. Use adjoint analysis on a coarse grid to inform a decoupled iterative algorithm for solving MHD[12]

Thank you! Any questions?

[12] Max D. Gunzburger, Amnon J. Meir, and Janet S. Peterson. "On the existence, uniqueness, and finite element approximation of solutions of the equations of stationary, incompressible magnetohydrodynamics". In: *Mathematics of Computation* 56.194 (1991), pp. 523–523.