SAND2020-10541C

A Convex Approach to Optimal Control Synthesis for Nonlinear Systems

Joseph Moyalan, Hyungjin Choi, Yongxin Chen, and Umesh Vaidya

***Abstract*—We consider control synthesis problem for nonlinear dynamics. We propose a convex optimization-based approach for the optimal control synthesis of a class of control-affine nonlinear systems. The proposed approach relies on exploiting the duality results in the stability theory of dynamical system. In particular, the optimal control problem is formulated as infinite dimensional convex optimization problem in the dual space of density. The sum of square computational framework is further employed for the finite dimensional approximation of the infinite dimensional convex optimization problem. Simulation results are presented to demonstrate the efficacy of the developed framework.**

I. Introduction

The control synthesis problem for nonlinear systems has been a longstanding challenge in the control community. The optimal control synthesis for general nonlinear dynamics aiming at designing a control law that minimizes some certain cost function is normally addressed using either Pontryagin's maximum principle [1] or dynamic programming [2]. With dynamic programming, the optimal control is characterized by the celebrated Hamilton-Jacobi-Bellman equation [2]. The dimension this partial differential equation is equal to the dimension of the state space. Thus, the complexity of dynamic programming grows rapids with the state dimension; this phenomenon is known as the curse of dimensionality. The Pontryagin's maximum principle, on the other hand, enjoys much better scalability. However, it only leads to local optimal control. Another limitation of the maximum principle is that the solution doesn't provide a feedback law; the solution has to be recalculated each time for different initial state.

In this work, we consider optimal control problems for control-affine systems [3], an important class of nonlinear systems widely used in applications such as robotics. We propose a novel framework to compute the optimal controllers for control-affine dynamics. The key idea of our framework is to convert the nonlinear dynamics to linear dynamics in the lifted density space. By leveraging the linear operator theory for nonlinear systems and with proper reparametrization, the optimal control problem becomes a convex optimization over the lifted density space. With a proper parametrization, this convex optimization is then solved efficiently using sum of square method. The performance of our method is illustrated with several numerical examples.

The idea of carrying out control synthesis over the density space was originally explored in [4] for stabilization problems. The celebrated Lyapunov theory [5] is a powerful framework to certify the stability of a given nonlinear dynamics. Searching for a suitable Lyapunov function for a nonlinear system is a convex problem. However, when it comes to control synthesis, this framework becomes less effective. The joint search of the Lyapunov function and the control law is in general a non-convex problem. To overcome this difficulty, [4] proposed to use the dual Lyapunov function known as Lyapunov density [6]. For control-affine dynamics, the control design can be formulated as a convex problem. The method we proposed in this paper can be viewed as a generalization of that in [4] to deal with optimal control problems. Indeed, as discussed in Section III, the method in [4] is a special case of our framework. The formulation of optimal control problem in the density space is made possible by viewing the duality in stability and stabilization results through the lense of linear operator theory involving Koopman and Perron-Frobenius operator [7]–[10].

The proposed framework requires solving Sums-of-Squares (SOS) optimization problem. DUe to the complexity of the resulting SOS optimization problem, We envision that this method can be applied to low to medium dimensional dynamical systems (e.g. robotics, distributed power-electronics control applications with energy storage systems).

The rest of the paper is structured as follows. In Section II, we provide a brief introduction to the necessary ingredient of our framework. The problem formulation and the main theoretical results are given in Section III. In Section IV, we develop the algorithm details based on the sum of square framework. This is followed by several numerical examples in Section V and a brief conclusion in Section VI.

II. Background

A. Koopman and Perron-Frobenius Operators

For a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n \tag{1}$$

H. Choi gratefully acknowledges funding from the Department of Energy, Office of Electricity's Energy Storage Program, under the direction of Dr. Imre Gyuk. Sandia National Laboratories is a multi-mission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525. This paper describes objective technical results and analysis. Any subjective views or opinions that might be expressed in the paper do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

J. Moyalan and U. Vaidya are with the Department of Mechanical Enginerring, Clemson University, Clemson SC, USA, josephraphelm@gmail.com, uvaidya@clemson.edu

H. Choi is with Sandia National Laboratories, Albuquerque NM, USA, hchoi@sandia.gov

Y. Chen is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA, USA. yongchen@gatech.edu

there are two different ways of linearly lifting the finite dimensional nonlinear dynamics from state space to infinite dimension space of functions, $\mathcal{F}$, namely Koopman and Perron-Frobenius operators. Denote the solution of system (1) at time t starting from initial condition $\mathbf{x}$ by $\mathbf{s}_t(\mathbf{x})$. The definitions of these operators along with the infinitesimal generators of these operators are defined as follows.

Definition 1 (Koopman Operator): $\mathbb{K}_t : \mathcal{L}_\infty(\mathbf{X}) \to \mathcal{L}_\infty(\mathbf{X})$ for dynamical system (1) is defined as

$$[\mathbb{K}_t\varphi](\mathbf{x}) = \varphi(\mathbf{s}_t(\mathbf{x})), \quad \varphi \in \mathcal{L}_\infty, \quad t \geq 0.$$

The infinitesimal generator for the Koopman operator is

$$\lim_{t\to 0} \frac{(\mathbb{K}_t - I)\varphi}{t} = \mathbf{f}(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x}) =: \mathcal{K}_{\mathbf{f}}\varphi \tag{2}$$

Definition 2 (Perron-Frobenius Operator): $\mathbb{P}_t : \mathcal{L}_1(\mathbf{X}) \to \mathcal{L}_1(\mathbf{X})$ for dynamical system (1) is defined as

$$[\mathbb{P}_t\psi](\mathbf{x}) = \psi(\mathbf{s}_{-t}(\mathbf{x})) \left| \frac{\partial \mathbf{s}_{-t}(\mathbf{x})}{\partial \mathbf{x}} \right|, \quad \psi \in \mathcal{L}_1, \quad t \geq 0$$

where $|\cdot|$ stands for the determinant. The infinitesimal generator for the P-F operator is given by

$$\lim_{t\to 0} \frac{(\mathbb{P}_t - I)\psi}{t} = -\nabla \cdot (\mathbf{f}(\mathbf{x})\psi(\mathbf{x})) =: \mathcal{P}_{\mathbf{f}}\psi \tag{3}$$

These two operators are dual to each other where the duality is expressed as follows.

$$\int_{\mathbb{R}^n} [\mathbb{K}_t\varphi](\mathbf{x})\psi(\mathbf{x})d\mathbf{x} = \int_{\mathbb{R}^n} [\mathbb{P}_t\psi](\mathbf{x})\varphi(\mathbf{x})d\mathbf{x} \tag{4}$$

B. Almost everywhere stability and Stabilization

The formulation for the optimal control problem we present in the dual space is intimately connected to density function and Lyapunov measure introduced for verifying the almost everywhere notion of stability defined below.

Definition 3: [Almost everywhere stable] The equilibrium point at $\mathbf{x} = 0$ is said to be almost everywhere stable w.r.t. measure, μ, if

$$\mu\{\mathbf{x} \in \mathbf{X} : \lim_{t\to\infty} \mathbf{s}_t(\mathbf{x}) \neq 0\} = 0$$

Following theorem from [6] provide condition for almost eveywhere stability with respect to (w.r.t.) Lebesgue measure.

Theorem 1: Given the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f}$ is continuous differentiable and $\mathbf{f}(0) = 0$, suppose there exists a nonnegative ρ is continuous differentiable for $\mathbf{x} \neq 0$ such that $\rho(\mathbf{x})\mathbf{f}(\mathbf{x})/|\mathbf{x}|$ is integrable on $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \geq 1\}$ and

$$[\nabla \cdot (\rho\mathbf{f})](\mathbf{x}) > 0 \text{ for almost all } \mathbf{x}. \tag{5}$$

Then, for almost all initial states $\mathbf{x}(0)$, the trajectory $\mathbf{x}(t)$ tends to zero as $t \to \infty$.

The density ρ serves as a stability certificate and can be viewed as a dual to the Lyapunov function [6]. Applying Theorem 1 to control system, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}$, we arrive at

$$\nabla \cdot (\rho(\mathbf{f} + \mathbf{g}\mathbf{u})) > 0 \text{ for almost all } \mathbf{x}. \tag{6}$$

The control synthesis problem becomes searching for a pair $(\rho, \mathbf{u})$ such that (6) holds. Even though (6) is again bilinear, it becomes linear in terms of $(\rho, \rho\mathbf{u})$. Thus, the density function based method for control synthesis is a convex problem.

C. Sum of squares

Sum of squares (SOS) optimization [11]–[14] is a relaxation of positive polynomial constraints appearing in polynomial optimization problems which are generally difficult to solve. SOS polynomials are in a set of polynomials which can be described as a finite linear combinations of monomials, i.e., $p = \sum_{i=1}^{\ell} d_i p_i^2$ where p is a SOS polynomial; p_i are polynomials; and d_i are coefficients. Hence, SOS is a sufficient condition for nonnegativity of a polynomial and thus SOS relaxation provides a lower bound on the minimization problems of polynomial optimizations. Using the SOS relaxation, any polynomial optmization problems with positive constraints can be formulated as SOS optimization as follows:

$$\min_{\mathbf{d}} \mathbf{w}^\top\mathbf{d} \text{ s.t. } p_s(\mathbf{x}, \mathbf{d}) \in \Sigma[\mathbf{x}],\ p_e(\mathbf{x}; \mathbf{d}) = 0, \tag{7}$$

where $\Sigma[\mathbf{x}]$ denotes SOS set; $\mathbf{w}$ is weighting coefficients; p_s and p_e are polynomials with coefficients $\mathbf{d}$. The problem in (7) is translated into Semidefinite Programming (SDP) [12], [15]. There are readily available SOS optimization packages such as SOSTOOLS [16] and SOSOPT [17] to solve (7).

III. Convex Formulation of Optimal Control Problem

We consider optimal control problem for input in affine control system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u} \tag{8}$$

where, $\mathbf{x} \in \mathbf{X} \subseteq \mathbb{R}^n$ is the state, $\mathbf{u} \in \mathbb{R}^m$ is the control input and $\mathbf{g}(\mathbf{x}) = (\mathbf{g}_1(\mathbf{x}), \ldots, \mathbf{g}_m(\mathbf{x}))$ with $\mathbf{g}_i \in \mathbb{R}^n$ is the input vector field. Let $\mathcal{N}$ denote the small neighborhood around the origin and we denote, $\mathbf{X}_1 := \mathbf{X} \setminus \mathcal{N}$. With some abuse of notation we denote by $\mathbf{x}(t)$ solution of system (8) starting from initial condition $\mathbf{x}$.

$$J(\mu_0) = \int_{\mathbf{X}_1} \int_0^\infty [q(\mathbf{x}(t)) + \mathbf{u}^\top(t)\mathbf{R}\mathbf{u}(t)]\, dt d\mu_0(\mathbf{x}) \tag{9}$$

where μ_0 is the initial measure assumed to be equivalent to Lebesgue measure i.e., there exits a function $0 < h \in \mathcal{L}_1(\mathbf{X})$ such that $\frac{d\mu_0}{d\mathbf{x}} = h(\mathbf{x})$. The $q : \mathbf{X} \to \mathbb{R}^+$ is a positive function such that $q(0) = 0$ and $\mathbf{R} > 0$ is positive definite. The objective is to minimize the cost starting from all initial condition supported by measure μ_0. Note that the cost function is optimized over set $\mathbf{X}_1$ and hence a small neighborhood, $\mathcal{N}$, around the origin in the computation of cost function. The reason for this is explained in the form of Remark 1.

We now make the following assumption on the optimal control input.

Assumption 2: We assume that the optimal control input is feedback form i.e., $\mathbf{u} = \mathbf{k}(\mathbf{x})$ and the system (8) with feedback control input is almost everywhere stable w.r.t. measure μ_0, Definition 3.

With the above form of optimal feedback control input, the optimal control problem can be written as

$$\begin{aligned} \min_{\mathbf{k}} \quad & \int_{\mathbf{X}_1} \left[\int_0^\infty q(\mathbf{x}) + \mathbf{k}(\mathbf{x})^\top\mathbf{R}\mathbf{k}(\mathbf{x})\, dt\right] d\mu_0(\mathbf{x}) \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}) \end{aligned} \tag{10}$$

Lemma 3: The feedback control system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}) =: \mathbf{f}_c(\mathbf{x})$ is almost everywhere stable (Definition 3) with respect to measure μ then

$$\lim_{t\to\infty}[\mathbb{P}_t^c h](\mathbf{x}) = 0 \tag{11}$$

where, $h = \frac{d\mu_0}{d\mathbf{x}}$ and $\mathbb{P}_t^c$ is P-F operator for system $\dot{\mathbf{x}} = \mathbf{f}_c(\mathbf{x})$.

Proof 4: For any set $B \subset \mathbf{X}_1$, let $B_t := \{\mathbf{x} \in \mathbf{X} : \mathbf{s}_t(\mathbf{x}) \in B\}$, then

$$\chi_{B_t}(\mathbf{x}) = \chi_B(\mathbf{s}_t(\mathbf{x})) = [\mathbb{U}_t \chi_B](\mathbf{x}).$$

where $\mathbf{s}_t(\mathbf{x})$ is the solution of system $\dot{x} = \mathbf{f}_c(\mathbf{x})$ starting from initial condition $\mathbf{x}$. Furthermore,

$$0 = \lim_{t\to\infty} \chi_{B_t}(\mathbf{x}) = \lim_{t\to\infty} \chi_B(\mathbf{s}_t(\mathbf{x})) = \lim_{t\to\infty}[\mathbb{U}_t\chi_B](\mathbf{x})$$

for all point $\mathbf{x}$ such that $\mathbf{s}_t(\mathbf{x}) \to 0$. Since the system is a.e. stable w.r.t. measure $d\mu_0(\mathbf{x}) = h(\mathbf{x})d\mathbf{x}$, we have

$$\begin{aligned} 0 &= \int_{\mathbf{X}} \lim_{t\to\infty}[\mathbb{U}_t\chi_B](\mathbf{x})h(\mathbf{x})d\mathbf{x} \\ &= \int_{\mathbf{X}} \chi_B(\mathbf{x}) \lim_{t\to\infty}[\mathbb{P}_t h](\mathbf{x})d\mathbf{x}. \end{aligned} \tag{12}$$

The above is true for arbitrary set $B \subset \mathbf{X}_1$, hence we have $\lim_{t\to\infty}[\mathbb{P}_t h](\mathbf{x}) = 0$.

■

We now state the main theorem on the convex formulation of the optimal control problem.

Theorem 5: Under the Assumption 2, the optimal control problem (10) can be written as following infinite dimensional convex optimization problem

$$\begin{aligned} \min_{\rho\geq 0,\bar{\boldsymbol{\rho}}} \quad & \int_{\mathbf{X}_1} q(\mathbf{x})\rho(\mathbf{x}) + \frac{\bar{\boldsymbol{\rho}}(\mathbf{x})^\top \mathbf{R}\bar{\boldsymbol{\rho}}(\mathbf{x})}{\rho} d\mathbf{x} \\ & \text{s.t.} \quad \nabla\cdot(\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) = h \end{aligned} \tag{13}$$

and the optimal feedback control input recovered from the solution of the above linear program as

$$\mathbf{k}(\mathbf{x}) = \frac{\bar{\boldsymbol{\rho}}(\mathbf{x})}{\rho(\mathbf{x})} \tag{14}$$

Proof 6: With the feedback control input the cost can be written as

$$J(\mu) = \int_{\mathbf{X}_1}\int_0^\infty q(\mathbf{x}(t)) + \mathbf{k}(\mathbf{x}(t))^\top \mathbf{R}\mathbf{k}(\mathbf{x}(t))dtd\mu_0 \tag{15}$$

where $x(t)$ is the solution of feedback control system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}) = \mathbf{f}_c(\mathbf{x}). \tag{16}$$

Let $\mathbb{U}_t^c$ and $\mathbb{P}_t^c$ be the Koopman and P-F semigroup for the feedback control system (16). The cost function can be written in terms of the Koopman operator as

$$\begin{aligned} J(\mu) &= \textstyle\int_{\mathbf{X}_1}\int_0^\infty [\mathbb{U}_t^c(q + \mathbf{k}^\top\mathbf{R}\mathbf{k})](\mathbf{x})\; dtd\mu_0 \\ &= \textstyle\int_0^\infty \left\langle \mathbb{U}_t^c(q + \mathbf{k}^\top\mathbf{R}\mathbf{k}), h\right\rangle_{\mathbf{X}_1} dt \end{aligned} \tag{17}$$

where $\langle\cdot,\cdot\rangle_{\mathbf{X}_1}$ stands for inner product between functions and we have used the fact that $d\mu_0 = hd\mathbf{x}$. Using the duality property between the P-F and Koopman operator, we obtain

$$J = \int_0^\infty \left\langle q + \mathbf{k}^\top\mathbf{R}\mathbf{k}, \mathbb{P}_t^c h\right\rangle_{\mathbf{X}_1} dt = \left\langle q + \mathbf{k}^\top\mathbf{R}\mathbf{k}, \rho\right\rangle_{\mathbf{X}_1}$$

where we have exchanged the integral over time with integral over space and defined

$$\rho(\mathbf{x}) := \int_0^\infty [\mathbb{P}_t^c h](\mathbf{x})dt, \quad \mathbf{x} \in \mathbf{X}_1 \tag{18}$$

It follows that $\rho(\mathbf{x})$ is a solution to the following equation

$$\nabla\cdot(\mathbf{f}_c(\mathbf{x})\rho(\mathbf{x})) = h(\mathbf{x}), \quad x \in \mathbf{X}_1 \tag{19}$$

Substituting (18) in (19), we obtain

$$\begin{aligned} \nabla\cdot(\mathbf{f}_c(\mathbf{x})\rho(\mathbf{x})) &= \int_0^\infty \nabla\cdot(\mathbf{f}_c(\mathbf{x})[\mathbb{P}_t^c h](\mathbf{x}))dt \\ = \int_0^\infty -\frac{d}{dt}[\mathbb{P}_t^c h](\mathbf{x})dt &= -[\mathbb{P}_t^c h](\mathbf{x})\Big|_{t=0}^\infty = h(\mathbf{x}) \end{aligned} \tag{20}$$

where we have used the infinitesimal generator property of P-F operator Eq. (3) and the fact that $\lim_{t\to\infty}[\mathbb{P}_t^c h](\mathbf{x}) = 0$ from Lemma 3. Furthermore, since $h > 0$, it follows that $\rho > 0$ from the positivity property of P-F semigroup $\mathbb{P}_t^c$. The OCP can then be written as

$$\begin{aligned} \min_{\mathbf{k},\rho\geq 0} \quad & \int_{\mathbf{X}_1} (q(\mathbf{x}) + \mathbf{k}(\mathbf{x})^\top\mathbf{R}\mathbf{k}(\mathbf{x}))\rho(\mathbf{x})d\mathbf{x} \\ & s.t. \quad \nabla\cdot((\mathbf{f} + \mathbf{g}\mathbf{k})\rho) = h. \end{aligned} \tag{21}$$

Using the fact that $\rho > 0$, we can write above problem as

$$\begin{aligned} \min_{\bar{\boldsymbol{\rho}},\rho\geq 0} \int_{\mathbf{X}_1} & q(\mathbf{x})\rho(\mathbf{x}) + \frac{\bar{\boldsymbol{\rho}}^\top\mathbf{R}\bar{\boldsymbol{\rho}}}{\rho}d\mathbf{x} \\ & s.t. \quad \nabla\cdot(\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) = h \end{aligned} \tag{22}$$

where $\bar{\boldsymbol{\rho}}(\mathbf{x}) = \mathbf{k}(\mathbf{x})\rho(\mathbf{x})$. Once we solve for $\bar{\boldsymbol{\rho}}$ and ρ, $\mathbf{k}$ can be recovered as $\mathbf{k}(\mathbf{x}) = \frac{\bar{\boldsymbol{\rho}}(\mathbf{x})}{\rho(\mathbf{x})}$ ■

We next consider optimization problem with $\mathcal{L}_1$ norm on control input.

$$\begin{aligned} \min_{\mathbf{k}} \quad & \textstyle\int_{\mathbf{X}_1}\left[\int_0^\infty q(\mathbf{x}) + \beta\|\mathbf{k}(\mathbf{x})\|_1\; dt\right] d\mu(\mathbf{x}) \\ \text{s.t.} \quad & \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{k}(\mathbf{x}) \end{aligned} \tag{23}$$

The solution to the above optimization problem can be obtained by solving the following infinite-dimensional linear program.

Theorem 7: Under the Assumption 2, the optimal control problem (23) can be written as following infinite dimensional linear optimization problem

$$\begin{aligned} \min_{\rho\geq 0,\bar{\boldsymbol{\rho}}} \quad & \int_{\mathbf{X}_1} q(\mathbf{x})\rho(\mathbf{x}) + \beta\|\bar{\boldsymbol{\rho}}(\mathbf{x})\|_1 d\mathbf{x} \\ & \text{s.t.} \quad \nabla\cdot(\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) = h \end{aligned} \tag{24}$$

and the optimal feedback control input recovered from the solution of the above linear program as $\mathbf{k}(\mathbf{x}) = \frac{\bar{\boldsymbol{\rho}}(\mathbf{x})}{\rho(\mathbf{x})}$.

Remark 1: The cost function is optimized in the region excluding the small neighborhood around the origin. The reason for this can be explained as follows. The optimal density function serves a density function for occupation measure. In particular, $\bar{\mu}(A) := \int_A \rho(\mathbf{x})d\mathbf{x}$ signifies the amount of time system trajectories for the close loop system spend in set A. Hence the optimal density function has singularity at the origin as all the trajectories are funnel to the origin. This singularity at the origin create computation challenge for the approximation of the cost function near the origin. There are two potential approaches to address this challenge due to singularity. The first approach is to ignore the optimality at the origin as the neighborhood around the origin is small the performance of closed loop system is not compromised. The second approach is to design a locally optimal stabilizing controller based on the linearization of the nonlinear system at the origin. The local optimal controller can be designed as a linear quadratic regulator. In this paper, we adopt the later approach.

IV. SOS-based Computation Framework for Optimal Control

In this section, we provide SOS-based computational framework for the finite dimensional approximation of optimal control formulation involving $\mathcal{L}_1$ norm (Eq. (23)) and $\mathcal{L}_2$ norm (Eq. (10)) for the control cost.

A. Optimal Control with $\mathcal{L}_1$ norm of feedback control

Consider the parameterization of $\rho(\mathbf{x})$ and $\bar{\boldsymbol{\rho}}(\mathbf{x})$ as follows:

$$\rho(\mathbf{x}) = \frac{a(\mathbf{x})}{b(\mathbf{x})^\alpha}, \quad \bar{\boldsymbol{\rho}}(\mathbf{x}) = \frac{\mathbf{c}(\mathbf{x})}{b(\mathbf{x})^\alpha}, \tag{25}$$

where the polynomials $a(\mathbf{x}) \geq 0$ and $\mathbf{c}(\mathbf{x}) = \left[c_1(\mathbf{x}), \ldots, c_m(\mathbf{x})\right]^\top$. Here, $b(\mathbf{x})$ is a positive polynomial (positive at $\mathbf{x} \neq 0$), and α is a positive constant which is sufficiently large so that the integrability condition in Theorem 1 holds. Using (25), we can restate the left-hand side of (24) as follows [4]:

$$\begin{aligned}&\nabla \cdot (\mathbf{f}\rho + \mathbf{g}\bar{\boldsymbol{\rho}}) = \nabla \cdot [\frac{1}{b}(\mathbf{f}a + \mathbf{g}\mathbf{c})]\\ =&\frac{1}{b^{\alpha+1}}[(1+\alpha)b\nabla \cdot (\mathbf{f}a + \mathbf{g}\mathbf{c}) - \alpha\nabla \cdot (b\mathbf{f}a + b\mathbf{g}\mathbf{c})].\end{aligned}$$

Thus, it becomes finding a, and $\mathbf{c}$, such that

$$[(1+\alpha)b\nabla \cdot (\mathbf{f}a + \mathbf{g}\mathbf{c}) - \alpha\nabla \cdot (b\mathbf{f}a + b\mathbf{g}\mathbf{c})] \tag{26}$$

is a non-negative polynomial function. Furthermore, $b(\mathbf{x})$ can be chosen as a quadratic control Lyapunov function for the linearized dynamics at $\mathbf{x} = 0$ to guarantee a local stabilization near the origin. Combining (25)–(26), (24) can be rewritten as follows:

$$\begin{aligned}\min_{\rho\geq 0,\bar{\boldsymbol{\rho}}} \quad & \int_{\mathbf{X}_1} \frac{q(\mathbf{x})a(\mathbf{x})}{b(\mathbf{x})^\alpha} + \frac{\beta||\mathbf{c}(\mathbf{x})||_1}{b(\mathbf{x})^\alpha} d\mathbf{x}\\ \text{s.t.} \quad & (26) \geq 0,\ a(\mathbf{x}) \geq 0,\end{aligned} \tag{27}$$

where a small neighborhood of the origin

$$\mathcal{N} = \{\mathbf{x} \in \mathbf{X} : |\mathbf{x}| \leq \epsilon,\ \epsilon > 0\}$$

is chosen as a polytope and excluded from the integration of the cost function to remove singularity at the origin due to rational parameterization in (25). Although the proposed feedback control stabilizes the system for the entire $\mathbf{X}$, we design a local optimal linear quadratic regulator (LQR) controller for $\mathcal{N}$ to guarantee the optimality of control costs through the entire region.

By calculating the integrals in the cost function in terms of coefficient vectors of the polynomials and substituting non-negativity constraints with SOS constraints, (27) can be expressed as a SOS problem as below:

$$\begin{aligned}\min_{\substack{\boldsymbol{\mathcal{C}}_a,\boldsymbol{\mathcal{C}}_{c_j},\boldsymbol{\mathcal{C}}_{s_j}\\ j=1,\ldots,m}} \quad & \mathbf{d}_1^\top \boldsymbol{\mathcal{C}}_a + \beta\sum_{j=1}^m \mathbf{d}_2^\top \boldsymbol{\mathcal{C}}_{s_j}\\ \text{s.t.} \quad & (26) \in \Sigma[\mathbf{x}],\ a(\mathbf{x}) \in \Sigma[\mathbf{x}],\\ & (\mathbf{s}(\mathbf{x}) - \mathbf{c}(\mathbf{x})) \in \Sigma[\mathbf{x}],\\ & (\mathbf{s}(\mathbf{x}) + \mathbf{c}(\mathbf{x})) \in \Sigma[\mathbf{x}],\end{aligned} \tag{28}$$

where $\mathbf{s}(\mathbf{x})$ are the polynomials equal to $|\mathbf{c}(\mathbf{x})|$; $\boldsymbol{\mathcal{C}}_a$, $\boldsymbol{\mathcal{C}}_{c_j}$ and $\boldsymbol{\mathcal{C}}_{s_j}$ are the vectors containing coefficients of $a(\mathbf{x})$, $c_j(\mathbf{x})$, and $s_j(\mathbf{x})$ in terms of the monomial vector, $\boldsymbol{\Psi}(\mathbf{x}) = [1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^2, \ldots]^\top$ such that

$$a(\mathbf{x}) = \boldsymbol{\mathcal{C}}_a^\top \boldsymbol{\Psi}(\mathbf{x}), c_j(\mathbf{x}) = \boldsymbol{\mathcal{C}}_{c_j}^\top \boldsymbol{\Psi}(\mathbf{x}), s_j(\mathbf{x}) = \boldsymbol{\mathcal{C}}_{s_j}^\top \boldsymbol{\Psi}(\mathbf{x}),$$

for $j = 1, \ldots, m$, with $\boldsymbol{\Psi}(\mathbf{x})$ having the maximum degree of $a(\mathbf{x})$, $c_j(\mathbf{x})$, and $s_j(\mathbf{x})$; and the coefficients $\mathbf{d}_1$ and $\mathbf{d}_2$ are calculated as

$$\mathbf{d}_1 = \int_{\mathbf{X}_1} \frac{q(\mathbf{x})\boldsymbol{\Psi}(\mathbf{x})}{b(\mathbf{x})^\alpha} d\mathbf{x},\ \mathbf{d}_2 = \int_{\mathbf{X}_1} \frac{\boldsymbol{\Psi}(\mathbf{x})}{b(\mathbf{x})^\alpha} d\mathbf{x}. \tag{29}$$

B. Optimal Control with $\mathcal{L}_2$ norm of feedback control

Following the same parameterization in (25), $\mathcal{L}_2$ optimal control problem in (13) is restated as:

$$\begin{aligned}\min \quad & \int_{\mathbf{X}_1} \frac{q(\mathbf{x})a(\mathbf{x})}{b(\mathbf{x})^\alpha} + \frac{\mathbf{c}(\mathbf{x})^\top \mathbf{R}\mathbf{c}(\mathbf{x})}{a(\mathbf{x})b(\mathbf{x})^\alpha} dx\\ \text{s.t.} \quad & (26) \geq 0,\ a(\mathbf{x}) \geq 0,\end{aligned} \tag{30}$$

where we exclude a small neighborhood of the origin to avoid singularity at the origin. To convert (30) into a SOS problem, we first reformulate (30) as follows:

$$\begin{aligned}\min \quad & \int_{\mathbf{X}_1} \frac{q(\mathbf{x})a(\mathbf{x})}{b(\mathbf{x})^\alpha} + \frac{w(\mathbf{x})}{b(\mathbf{x})^\alpha} d\mathbf{x}\\ \text{s.t.} \quad & (26) \geq 0,\ a(\mathbf{x}) \geq 0,\\ & \mathbf{M}(\mathbf{x}) = \begin{bmatrix} w(\mathbf{x}) & \mathbf{c}(\mathbf{x})^\top \\ \mathbf{c}(\mathbf{x}) & a(\mathbf{x})\mathbf{R}^{-1}\end{bmatrix} \succcurlyeq 0,\end{aligned} \tag{31}$$

where the positive semidefinite constraint of the polynomial matrix $\mathbf{M}(\mathbf{x})$ is a result of applying the Schur complement lemma on the $\mathcal{L}_2$ control cost bounded by $w(\mathbf{x})$, i.e., $\frac{\mathbf{c}(\mathbf{x})^\top \mathbf{R}\mathbf{c}(\mathbf{x})}{a(\mathbf{x})} \leq w(\mathbf{x})$;. Now, to algebraically express the positive semidefinite matrix $\mathbf{M}(\mathbf{x})$, we first introduce the following lemma:

Lemma 8 (Positive semidefinite polynomial matrix [18]): A $p \times p$ matrix $\mathbf{H}(\mathbf{x})$ whose entries are polynomials is positive semidefinite with respect to the monomial vector $\mathbf{z}(\mathbf{x})$, *if and only if*, there exist $\mathbf{D} \succcurlyeq 0$ such that

$$\mathbf{H}(\mathbf{x}) = \left(\mathbf{z}(\mathbf{x}) \otimes \mathbf{I}_p\right)^\top \mathbf{D}\left(\mathbf{z}(\mathbf{x}) \otimes \mathbf{I}_p\right),$$

where $\otimes$ denotes a Kronecker product (tensor product) and $\mathbf{I}_p$ is an identity matrix with dimension p.

Following Lemma 8, let $\mathbf{z}(\mathbf{x})$ be a monomial vector with $\deg(\mathbf{z}(\mathbf{x})) = \text{floor}(\frac{\deg(\boldsymbol{\Psi}(\mathbf{x}))}{2}) + 1$, then $\mathbf{M}(\mathbf{x})$ in (30) is positive semidefinite when there exists $\mathbf{D} \succcurlyeq 0$ such that $\mathbf{M}(\mathbf{x}) = \mathbf{H}(\mathbf{x})$. Using this result and also the integrals of the cost functions in terms of $\mathbf{\Psi}(\mathbf{x})$ shown in (29), a SOS problem equivalent to (31) can be formulated as follows:

$$\begin{aligned} \min_{\substack{\boldsymbol{\mathcal{C}}_a, \boldsymbol{\mathcal{C}}_w, \boldsymbol{\mathcal{C}}_{c_j} \\ j=1,\dots,m}} \quad & \mathbf{d}_1^\top \boldsymbol{\mathcal{C}}_a + \mathbf{d}_2^\top \boldsymbol{\mathcal{C}}_w \\ \text{s.t.} \quad & (26) \in \Sigma[\mathbf{x}],\ a(\mathbf{x}) \in \Sigma[\mathbf{x}], \\ & w(\mathbf{x}) - \mathbf{H}_{11}(\mathbf{x}) = 0, \\ & \mathbf{c}(\mathbf{x}) - \mathbf{H}_{12}(\mathbf{x}) = 0, \\ & a(\mathbf{x})\mathbf{R}^{-1} - \mathbf{H}_{22}(\mathbf{x}) = 0, \\ & \mathbf{D} \succcurlyeq 0, \end{aligned} \tag{32}$$

where $\mathbf{H}_{ij}(\mathbf{x})$ denotes a polynomial in ijth entry of $\mathbf{H}(\mathbf{x})$; and $\boldsymbol{\mathcal{C}}_w$ is a vector of coefficient of $w(\mathbf{x})$ in terms of $\mathbf{\Psi}(\mathbf{x})$, i.e., $w(\mathbf{x}) = \boldsymbol{\mathcal{C}}_w^\top \mathbf{\Psi}(\mathbf{x})$.

V. Simulation Results

In this section, we present simulation results for the model-based optimal control involving examples of systems with polynomial vector field.

Example 1 Consider a system with following dynamics:

$$\dot{x_1} = x_2 - x_1^3 + x_1^2,\ \ \dot{x_2} = u.$$

The finite-dimensional approximation of the infinite dimensional optimization problem is obtained as follows. For the stabilization equations (refer to (25)–(26)), the parameters are chosen: $\deg(a(\mathbf{x})) = 0:2$, $\deg(c(\mathbf{x})) = 0:5$, $\deg(s(\mathbf{x})) = 0:6$, and $\alpha = 4$. Also, the parameters for $\mathcal{L}_1$ and $\mathcal{L}_2$ optimal control costs are chosen with reference to (27) as follows: $\beta = 0.1$, $\mathbf{R} = 1$, $b(\mathbf{x}) = 4.7x_1^2 + 4.47x_1x_2 + 2.1x_2^2$ and $q(\mathbf{x}) = 3.5x_1^2 + x_2^2$. By solving the SOS optimization problems for both $\mathcal{L}_1$ and $\mathcal{L}_2$ control costs described in Section IV, an optimal control solution is obtained, $u(\mathbf{x}) = \frac{c(\mathbf{x})}{a(\mathbf{x})}$. In Fig. 1 and 2, we show the plot for the closed loop trajectories with control designed using $\mathcal{L}_1$ and $\mathcal{L}_2$ cost on the control input. The domain for simulation is assumed to be $\mathbf{X} = [-4,4] \times [-4,4]$. We notice that all the trajectories are attracted to the origin. The small red box around the origin denotes the small neighborhood $\mathcal{N}$ (Remark 1). We design a local optimal linear quadratic regulator (LQR) controller based on the linearization of the system which is active inside $\mathcal{N}$.

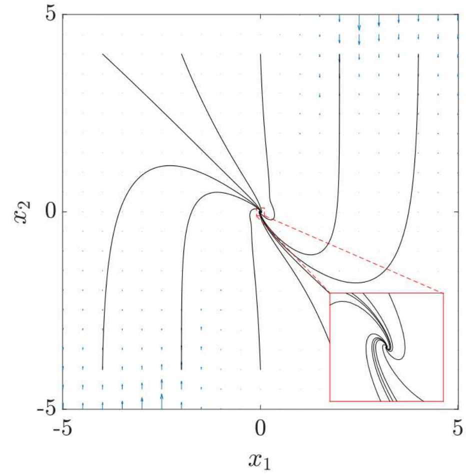

Fig. 1: Closed-loop trajectories of the system in Example 1 stabilised by the proposed method for $\mathcal{L}_1$ feedback control.

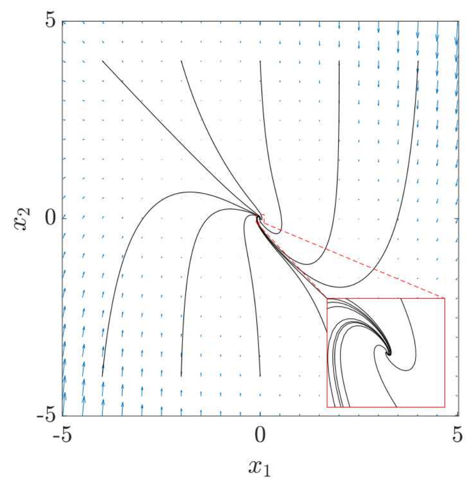

Fig. 2: Closed-loop trajectories of the system in Example 1 stabilised by the proposed method for $\mathcal{L}_2$ feedback control.

Example 2 Consider the dynamics of Van der Pol oscillator as follows:

$$\dot{x_1} = x_2,\ \dot{x_2} = (1 - x_1^2)x_2 - x_1 + u.$$

In this example, we will consider $\mathcal{L}_2$ feedback control. For the procedure of formulating finite-dimensional optimal control problems described in Section IV for the feedback control, we consider the parameters to (25)–(27) as follows: $\deg(a(\mathbf{x})) = 0:3$, $\deg(c(\mathbf{x})) = 0:6$, $\deg(s(\mathbf{x})) = 0:7$, $b(\mathbf{x}) = 5.8x_1^2 + 2.9x_1x_2 + 2.9x_2^2$, $\alpha = 4$, $q(\mathbf{x}) = 3.5x_1^2 + x_2^2$, and $\mathbf{R} = 1$. By solving the resulting SOS optimization problems in Section IV, we get the stabilized trajectories starting from arbitrary initial points in $\mathbf{X} = [-2,2] \times [-2,2]$, converging to the origin, as shown in Fig. 3. Similar to the previous example, the small red box in the figures represents the small neighborhood of the origin, $\mathcal{N} = [-0.1, 0.1] \times [-0.1, 0.1]$, where the control is switched to the local LQR controller once the trajectories are attracted to $\mathcal{L}$ towards the origin by the nonlinear optimal feedback controls synthesized from the proposed method.

Example 3: Consider the dynamics of Lorentz attractor:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 + u, \\ \dot{x}_3 &= x_1x_2 - \gamma x_3, \end{aligned}$$

where $\sigma = 10$, $\rho = 28$, and $\gamma = \frac{8}{3}$. To formulate finite-dimensional $\mathcal{L}_2$ optimal control problem described in Section IV, consider $\deg(a(\mathbf{x})) = 0:2$, $\deg(c(\mathbf{x})) = 0:9$, and $\deg(s(\mathbf{x})) = 0:10$, $b(\mathbf{x}) = 39.15x_1^2 + 60.82x_1x_2 + 23.7x_2^2 + 0.2x_3^2$, $\alpha = 4$, $q(\mathbf{x}) = 3.5x_1^2 + x_2^2 + x_3^2$, and $\mathbf{R} = 1$. The result of $\mathcal{L}_2$ control synthesized from our proposed method is shown in Fig. 5. The closed-loop trajectories are stabilized, starting from arbitrary initial conditions in $\mathbf{X} = [-4,4] \times [-4,4] \times [-4,4]$. The LQR control for the linearized system is applied to the region near the origin,

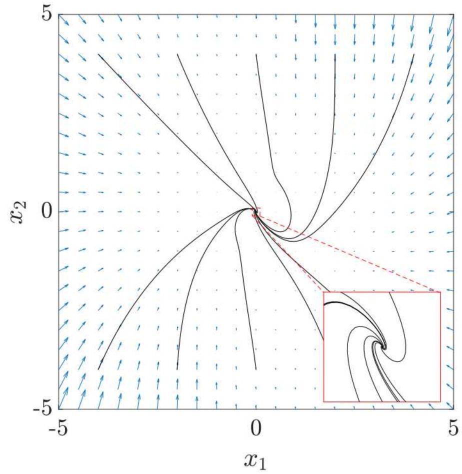

Fig. 3: Dynamics of Van der Pol oscillator stabilized by the proposed method for $\mathcal{L}_2$ feedback control case.

$\mathcal{N} = [-0.1, 0.1] \times [-0.1, 0.1] \times [-0.1, 0.1]$ as denoted by the red box in Fig. 5.

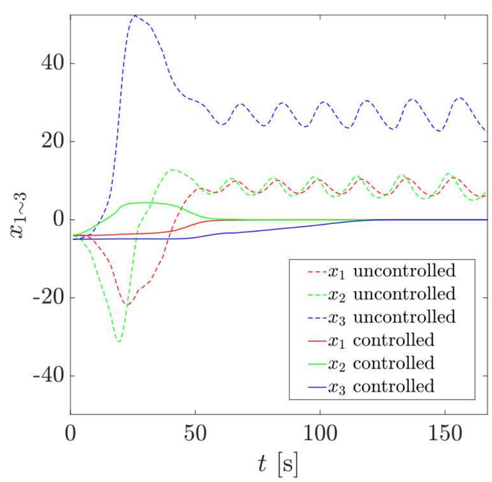

Fig. 4: Dynmics of Lorentz attractor stabilised by $\mathcal{L}_2$ feedback control. The red and blue plots represent open and closed loop control repectively.

VI. Conclusion

A systematic convex optimization-based framework is provided for optimal control of nonlinear system. The optimal control problem is formulated in the dual space of density function leading to infinite dimensional convex optimization problem for optimal control. The proposed approach use Sum of Square (SoS) optimization framework for the finite dimensional approximation of the optimization problem. Future research efforts will focus on extending the framework to data-driven setting, where the explicit knowledge of system dynamics is not assumed to be known.

References

[1] I. M. Ross, *A primer on Pontryagin's principle in optimal control*. Collegiate publishers, 2015.

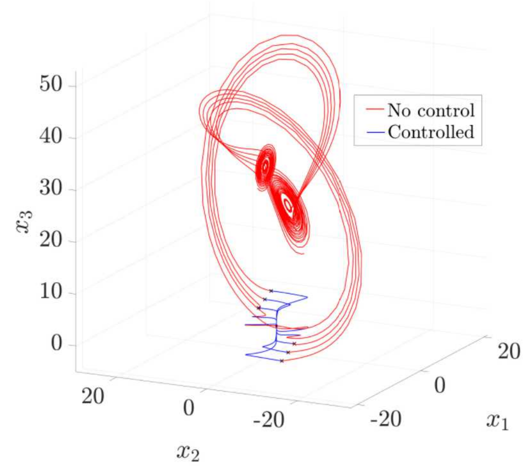

Fig. 5: Trajectories in states vs. time of Lorentz attractor simulated from open-loop as well as $\mathcal{L}_2$ feedback controlled dynamics, starting from some disturbed initial points converge to the origin while open-loop dynamics shows chaotic behavior.

[2] W. H. Fleming and R. W. Rishel, *Deterministic and stochastic optimal control*. Springer Science & Business Media, 2012, vol. 1.
[3] R. M. Murray, Z. Li, S. S. Sastry, and S. S. Sastry, *A mathematical introduction to robotic manipulation*. CRC press, 1994.
[4] S. Prajna, P. A. Parrilo, and A. Rantzer, "Nonlinear control synthesis by convex optimization," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 310–314, 2004.
[5] M. Vidyasagar, *Nonlinear systems analysis*. SIAM, 2002.
[6] A. Rantzer, "A dual to Lyapunov's stability theorem," *Systems & Control Letters*, vol. 42, no. 3, pp. 161–168, 2001.
[7] U. Vaidya and P. G. Mehta, "Lyapunov measure for almost everywhere stability," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 307–323, 2008.
[8] U. Vaidya, P. Mehta, and U. Shanbhag, "Nonlinear stabilization via control lyapunov meausre," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1314–1328, 2010.
[9] A. Raghunathan and U. Vaidya, "Optimal stabilization using lyapunov measure," in *Submitted to American Control Conference*, Seattle, WA, 2008.
[10] R. Rajaram, U. Vaidya, and M. Fardad, "Connection between almost everywhere stability and advection PDE," in *Submitted to IEEE Conference on Decision and Control*, New Orleans, LA, 2007.
[11] U. Topcu, A. Packard, P. Seiler, and G. Balas, "Help on sos [ask the experts]," *IEEE Control Systems Magazine*, vol. 30, no. 4, pp. 18–23, 2010.
[12] P. A. Parrilo, "Semidefinite programming relaxations for semialgebraic problems," *Mathematical Programming*, vol. 96, pp. 293–320, May 2003.
[13] P. A. Parrilo and B. Sturmfels, "Minimizing polynomial functions," 2001.
[14] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, May 2000.
[15] M. Laurent, *Sums of Squares, Moment Matrices and Optimization Over Polynomials*. New York, NY: Springer New York, 2009, pp. 157–270.
[16] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, `http://arxiv.org/abs/1310.4716`, 2013.
[17] P. Seiler, "Sosopt: A toolbox for polynomial optimization," 2013.
[18] C. W. Scherer and C. W. J. Hol, "Matrix sum-of-squares relaxations for robust semi-definite programs," *Mathematical Programming*, no. 6, pp. 189–211, June 2006. [Online]. Available: https://doi.org/10.1007/s10107-005-0684-2