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On Highly Robust Efficient Solutions to Uncertain Multiobjective Linear Programs

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Outline

Notation

For all $\mathbf{y}', \mathbf{y}'' \in \mathbb{R}^p$, we write

$$\mathbf{y}' \leqq \mathbf{y}'' \text{ if } y'_k \leq y''_k \text{ for all } k = 1, \ldots, p;$$
$$\mathbf{y}' \leq \mathbf{y}'' \text{ if } y'_k \leq y''_k \text{ for all } k = 1, \ldots, p \text{ and } \mathbf{y}' \neq \mathbf{y}'';$$
$$\mathbf{y}' < \mathbf{y}'' \text{ if } y'_k < y''_k \text{ for all } k = 1, \ldots, p.$$

Motivation

- Conflicting goals may be present during the decision-making process, which suggests more than one objective function is needed
- Problems of concern may not only have conflicting objectives, but may also incorporate some level of uncertainty due to:
  - inaccurate data
  - imperfect modeling
  - lack of knowledge
  - volatility of the global environment

Robust Multiobjective Optimization

Multiobjective LP (MOLP):

$$\min_{\mathbf{x}} \quad \mathbf{C}\mathbf{x} = \begin{bmatrix} \mathbf{c}_1\mathbf{x} & \cdots & \mathbf{c}_p\mathbf{x} \end{bmatrix}^T$$
$$\text{s.t.} \quad \mathbf{x} \in X,$$

$\mathbf{c}_k \in \mathbb{R}^{1\times n}, k = 1, \ldots, p,$
$X = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \leqq \mathbf{b}, \mathbf{x} \geqq \mathbf{0}\},$
$\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$.

Definition

$\hat{\mathbf{x}} \in X$ is said to be an *efficient solution* to MOLP if there is no other $\mathbf{x} \in X$ s.t. $\mathbf{C}\mathbf{x} \leq \mathbf{C}\hat{\mathbf{x}}$. The *efficient set* is denoted $\mathrm{E}(X, \mathbf{C})$.

If uncertainty exists:

- robust multiobjective optimization
- variety of robustness concepts due to *multiple* objectives
- goal to solve uncertain MOLPs (UMOLPs) for *robust efficient* solutions

Uncertain Multiobjective Linear Programs (UMOLPs)

A UMOLP is

$$\{\mathrm{P}(\mathbf{u})\}_{\mathbf{u}\in U}.$$

In particular,

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} & \mathbf{C}(\mathbf{u})\mathbf{x} \\ \text{s.t.} & \mathbf{x} \in X \end{array} \right\}_{\mathbf{u}\in U}.$$

Terminology:

- $\mathrm{P}(\mathbf{u})$ is an *instance* of UMOLP
- $U \subseteq \mathbb{R}^q$ is the *uncertainty set* or *set of scenarios*
- $\mathbf{u}$ is a *realization* or *scenario*
- $X \subseteq \mathbb{R}^n$ is the polyhedral *feasible set*
- $\mathrm{E}(X, \mathbf{C}(\mathbf{u}))$ is the *efficient set* of $\mathrm{P}(\mathbf{u})$

Objective-wise Uncertainty

Definition

A UMOLP is of *objective-wise uncertainty* if the uncertainties of the cost vectors are independent of each other, i.e., if $U = U_1 \times \cdots \times U_p$, where $U_k \subseteq \mathbb{R}^{q_k}, k = 1, \ldots, p$, such that

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix} \mathbf{c}_1(\mathbf{u}_1) & \cdots & \mathbf{c}_p(\mathbf{u}_p) \end{bmatrix}^T$$

with $\mathbf{u} = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_p \end{bmatrix}^T \in U$ and $\mathbf{u}_k \in U_k, k = 1, \ldots, p$.

We only consider the UMOLP of objective-wise uncertainty with $U = U_1 \times \cdots \times U_p$ such that $U_k \subseteq \mathbb{R}^n, k = 1, \ldots, p$, and

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix} \mathbf{c}_1(\mathbf{u}_1) \\ \vdots \\ \mathbf{c}_p(\mathbf{u}_p) \end{bmatrix} = \begin{bmatrix} c_{11}u_{11} & \cdots & c_{1n}u_{1n} \\ \vdots & & \vdots \\ c_{p1}u_{p1} & \cdots & c_{pn}u_{pn} \end{bmatrix}.$$

Solution Concept: Highly Robust Efficiency

Definition

A solution $\mathbf{x}^* \in X$ to UMOLP is *highly robust efficient (HRE)* if for every $\mathbf{u} \in U$, there does not exist $\mathbf{x} \in X$ such that $\mathbf{C}(\mathbf{u})\mathbf{x} \leq \mathbf{C}(\mathbf{u})\mathbf{x}^*$. The *HRE set* of UMOLP is denoted by $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$.

Remark

The point $\mathbf{x}^* \in X$ *is an HRE solution to* UMOLP *if and only if* $\mathbf{x}^* \in \bigcap_{\mathbf{u} \in U} \mathrm{E}(X, \mathbf{C}(\mathbf{u}))$, *i.e.,* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} \mathrm{E}(X, \mathbf{C}(\mathbf{u}))$.

Research Objectives

What's missing in the literature on HRE solutions:

1. properties and characterizations of the HRE set
2. bound sets on the HRE set
3. a robust counterpart

Polyhedral Uncertainty Set Reduction

Existing (due to Ide and Schöbel (2015)):

- (Theorem 46) If the uncertainty set is a bounded polyhedron and the objective functions are affine w.r.t. $\mathbf{u} \in U$, then a solution is HRE w.r.t. U if and only if it is HRE w.r.t. the finite set of extreme points of U
- True for nonlinear uncertain multiobjective programs too
- Theorem does not hold if the objective-wise assumption is relaxed (cf. Example 48)

Consequently, we assume U is a finite set of scenarios given by

$$U := \{\mathbf{u}^1, \mathbf{u}^2, \ldots, \mathbf{u}^s\} \subseteq \mathbb{R}^q.$$

Properties of the HRE Set

Properties that immediately extend from the deterministic to uncertain setting:

Let the HRE set be nonempty. Then

1. the HRE set is
   - closed
   - not necessarily convex
   - either the entire set X or on the boundary of X
2. there exists an HRE extreme point
3. if a point in the relative interior of a face of X is HRE, then so is the entire face

A property that is *not* the same:

4. The HRE set is *not* necessarily connected

Example: Disconnected HRE Set

UMOLP:

$$\left\{ \begin{array}{lll} \min\limits_{\mathbf{x}} & \begin{bmatrix} 3u_{11} & -9u_{12} \\ -u_{21} & 9u_{22} \end{bmatrix} \mathbf{x} \\ \text{s.t.} & \mathbf{x} \in X_1 \end{array} \right\}_{\mathbf{u}_1 \in U_1 = \{(1,1)\}, \mathbf{u}_2 \in U_2 = \{(1,1),(2,-1/9)\}},$$

where $X_1 := \{\mathbf{x} \in \mathbb{R}^2 : -x_1 + 2x_2 \leq 6, x_1 + x_2 \leq 6, x_1, x_2 \geq 0\}$

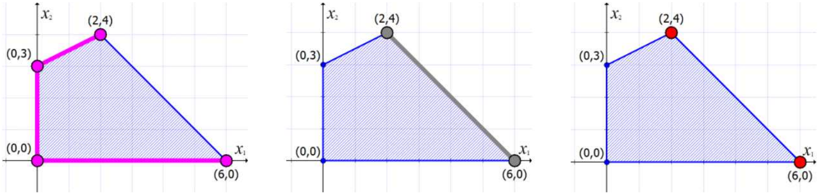


Figure: The efficient sets with $\mathbf{u} = (1,1,1,1)$ (purple) and $\mathbf{u} = (1,1,2,-1/9)$ (grey), as well as the HRE set (red)

Cones: Definitions

Let $K \subseteq \mathbb{R}^n$ be a cone.

Definition

K is *acute* if $\text{cl}(K) \subseteq H \cup \{\mathbf{0}\}$, where H is an open half-space whose generating hyperplane passes through the origin.

Definition

i. The *(positive) polar* of K is the cone
$$K^+ := \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} \geq 0 \text{ for all } \mathbf{x} \in K\}.$$

ii. The *strict (positive) polar* of K is the cone
$$K^{s+} := \{\mathbf{y} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} > 0 \text{ for all } \mathbf{x} \in K \setminus \{\mathbf{0}\}\}.$$

Cones: Definitions Ctd.

For an instance $\mathrm{P}(\mathbf{u})$, an *improving direction* is a vector $\mathbf{d} \in \mathbb{R}^n$ such that $\mathbf{C}(\mathbf{u})\mathbf{d} \leq \mathbf{0}$.

Definition

1. The *(closed) cone of improving directions* of $\mathrm{P}(\mathbf{u})$ for scenario $\mathbf{u} \in U$ is $K_{(\leqq)\leq}(\mathbf{C}(\mathbf{u})) := \{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}(\mathbf{u})\mathbf{d} \;(\leqq) \leq \mathbf{0}\}$.
2. The *(closed) cone of improving directions* of UMOLP for uncertainty set U is $K_{(\leqq)\leq}(\mathbf{C}(\mathbf{u}), U) := \bigcup_{\mathbf{u} \in U} K_{(\leqq)\leq}(\mathbf{C}(\mathbf{u}))$.

Definition

The *normal cone* to X at $\bar{\mathbf{x}} \in X$ is a convex cone

$$N_X(\bar{\mathbf{x}}) := \{\mathbf{p} \in \mathbb{R}^n : \mathbf{p}^T(\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \text{ for all } \mathbf{x} \in X\}.$$

Polar Cone Results

Proposition

The equality $K^{+}_{\leqq}(\mathbf{C}(\mathbf{u})) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{C}(\mathbf{u})^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *holds.*

Proposition

The equality $K^{+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} K^{+}_{\leqq}(\mathbf{C}(\mathbf{u}))$ *holds.*

Proposition

The equality $K^{+}_{\leqq}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{C}}^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$ *holds for some suitable matrix* $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$.

Strict Polar Cone Results

Proposition

If $K_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute, then

$$K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u})) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\mathbf{C}(\mathbf{u})^T \boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}.$$

Proposition

The equality $K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u}), U) = \bigcap_{\mathbf{u} \in U} K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u}))$ holds.

Proposition

If $K_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ is acute, then

$$K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u}), U) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = -\widetilde{\mathbf{C}}^T \boldsymbol{\lambda}, \boldsymbol{\lambda} > \mathbf{0}\}$$

for some suitable matrix $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$.

Acuteness Recognition: $K_{\leqq}(\mathbf{C}(\mathbf{u}))$

$K_{\leqq}(\mathbf{C}(\mathbf{u}))$ has two representations for each $\mathbf{u} \in U$

1. Inequality: $\{\mathbf{d} \in \mathbb{R}^n : \mathbf{C}(\mathbf{u})\mathbf{d} \leqq \mathbf{0}\}$
2. Generator: $\{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} = \mathbf{G}(\mathbf{u})^T \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0}\}$, where the columns of $\mathbf{G}(\mathbf{u})^T \in \mathbb{R}^{n \times r}$ are a finite set of generators of $K_{\leqq}(\mathbf{C}(\mathbf{u}))$

Note: 1 is given and 2 may be computed in SageMath

Theorem

For some $\mathbf{u} \in U$, let $K_{\leqq}(\mathbf{C}(\mathbf{u})) \neq \{\mathbf{0}\}$ be given in generator form. Then $K_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute if and only if $-\mathbf{G}(\mathbf{u})\mathbf{d} < \mathbf{0}$ is consistent.

Theorem

If $\dim(K^{+}_{\leqq}(\mathbf{C}(\mathbf{u}))) = n$, then $K_{\leqq}(\mathbf{C}(\mathbf{u}))$ is acute.

Acuteness Recognition Example

$$K_{\leqq}(\mathbf{C}(\mathbf{u}^1)) : \quad \left\{ \mathbf{d} \in \mathbb{R}^2 : \mathbf{d} = \begin{bmatrix} -3 & -9 \\ -1 & -1 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0} \right\}$$

$$K_{\leqq}(\mathbf{C}(\mathbf{u}^2)) : \quad \left\{ \mathbf{d} \in \mathbb{R}^2 : \mathbf{d} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \boldsymbol{\lambda}, \boldsymbol{\lambda} \geqq \mathbf{0} \right\}$$

System to check acuteness:

$$\begin{aligned} 3d_1 + d_2 &< 0 \\ 9d_1 + d_2 &< 0 \\ -3d_1 - d_2 &< 0 \\ d_1 - 2d_2 &< 0 \end{aligned}$$

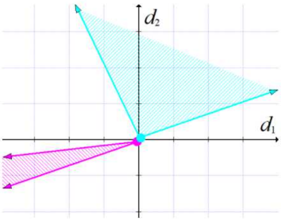


Figure: $K_{\leqq}(\mathbf{C}(\mathbf{u}^1))$ (purple) and $K_{\leqq}(\mathbf{C}(\mathbf{u}^2))$ (teal)

Characterization: Cone of Improving Directions

Theorem

$\mathbf{x}^* \in X$ is HRE if and only if $(K_{\leq}(\mathbf{C}(\mathbf{u}), U) + \{\mathbf{x}^*\}) \cap X = \emptyset$.

Sketch of Proof.

Intuition: The condition indicates that there is no feasible direction that is also improving. □

Example

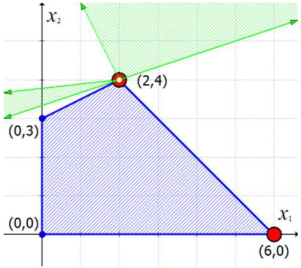


Figure: $K_{\leq}(\mathbf{C}(\mathbf{u}), U)$ (green) and HRE set (red)

Characterization: Normal Cone

Theorem

1. *Let* $K_{\leqq}(\mathbf{C}(\mathbf{u}))$ *be acute for all* $\mathbf{u} \in U$, *and* $\mathbf{x}^* \in X$. *Then* $\mathbf{x}^*$ *is HRE if and only if* $N_X(\mathbf{x}^*) \cap K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u})) \neq \emptyset$ *for all* $\mathbf{u} \in U$.
2. *Let* $K_{\leqq}(\mathbf{C}(\mathbf{u}))$ *be acute for all* $\mathbf{u} \in U$, *and* $\mathbf{x}^* \in X$. *If* $N_X(\mathbf{x}^*) \cap K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u}), U) \neq \emptyset$, *then* $\mathbf{x}^*$ *is HRE.*

Example

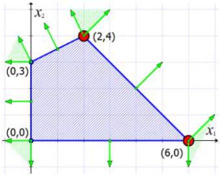


Figure: HRE set (red) and normal cones (green)

$K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u}), U) = \emptyset$

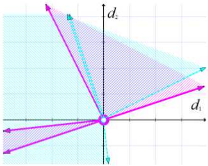


Figure: $K_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ (purple)

$N_X(\mathbf{x}^*) \cap K_{\leqq}^{s+}(\mathbf{C}(\mathbf{u}), U) = \emptyset$

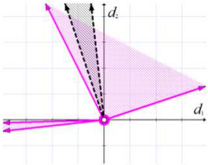


Figure: Strict polar cone (black)

Lower Bound Set

Theorem

If $K_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *is acute, then* $\mathrm{E}(X, \widetilde{\mathbf{C}}) \subseteq \mathrm{E}(X, \mathbf{C}(\mathbf{u}), U)$ *for some suitable matrix* $\widetilde{\mathbf{C}}^T \in \mathbb{R}^{n \times \tilde{p}}$.

As a direct consequence...

Corollary

If $K_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *is acute and* X *is bounded, then the HRE set is nonempty.*

How to find $\widetilde{\mathbf{C}}$?

- SageMath

Robust Counterpart

Definition

The *robust counterpart* of a UMOLP is an MOLP $\min_{\mathbf{x}\in X} \overline{\mathbf{C}}\mathbf{x}$ whose feasible and efficient solutions are feasible and HRE solutions to the UMOLP.

Theorem

If $K_{\leqq}(\mathbf{C}(\mathbf{u}), U)$ *is a polyhedral convex (finite) and acute cone, then* $\mathrm{E}(X, \mathbf{C}(\mathbf{u}), U) = \mathrm{E}(X, \overline{\mathbf{C}})$ *for some suitable matrix* $\overline{\mathbf{C}} \in \mathbb{R}^{\bar{p}\times n}$.

Sketch of Proof.

$K_{\leq}(\overline{\mathbf{C}}) = \{\mathbf{d} \in \mathbb{R}^n : \overline{\mathbf{C}}\mathbf{d} \leq \mathbf{0}\} = K_{\leq}(\mathbf{C}(\mathbf{u}), U)$. □

How to find $\overline{\mathbf{C}}$?

- Bemporad et al. (2001)

Robust Counterpart Example

Consider the UMOLP:

$$\left\{ \begin{array}{ll} \min\limits_{\mathbf{x}} \begin{bmatrix} u_{11} & -3u_{12} \\ u_{21} & u_{22} \end{bmatrix} \mathbf{x} & \\ \text{s.t.} & \mathbf{x} \in X_1 \end{array} \right\}_{\mathbf{u}_1 \in U_1, \mathbf{u}_2 \in U_2},$$

$$U_1 = \{(1,1)\}, U_2 = \{(1,-1),(1,1)\}$$

$$\mathbf{u}^1 = (1,1,1,-1) \qquad \mathbf{u}^2 = (1,1,1,1)$$

$$\mathbf{C}(\mathbf{u}^1) = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix} \quad \mathbf{C}(\mathbf{u}^2) = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$$

$$\boxed{\overline{\mathbf{C}} = \begin{bmatrix} 1 & -3 \\ 1 & -1 \end{bmatrix}}$$

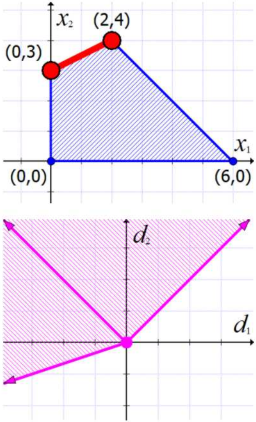


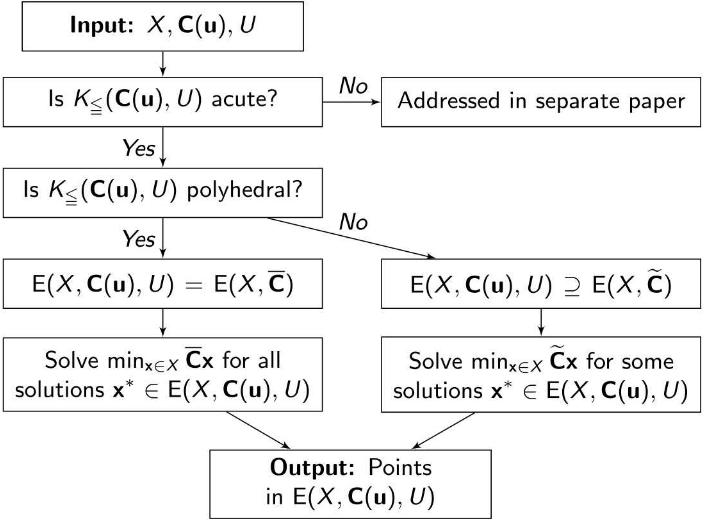
Figure: HRE set (red) and closed cone of improving directions of UMOLP (purple)

Contributions

Theoretical:

- Properties of the HRE set
- Characterizations of the HRE set
- Bound sets on the HRE set
- A robust counterpart for a special class of UMOLPs

Contributions: Computing HRE Solutions



Avenues for Future Research

1. Implement the proposed scheme for computing HRE solutions
2. Obtain additional lower bound sets
    - relax the acuteness assumption
3. Pursue further means to compute HRE solutions

The End

Thank you!

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