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A consistent and stable quadrature scheme for Galerkin meshfree methods for applications in nonlinear solid mechanics

Joe Bishop

Engineering Sciences Center
Sandia National Laboratories
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Outline

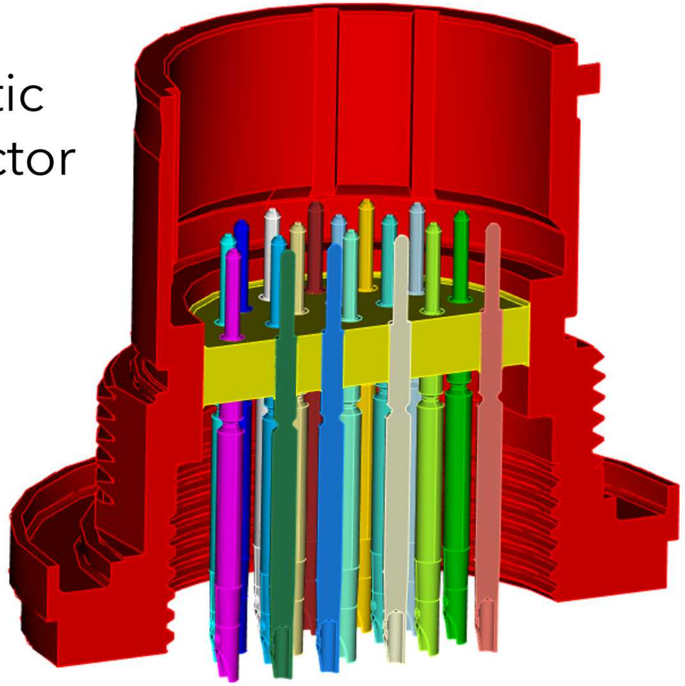
1. Motivation
2. Value proposition for meshfree
3. Modeling challenges for meshfree
4. New approach for meshfree quadrature
5. Verification example, elasticity
6. Summary

Motivation: rapid design-to-analysis

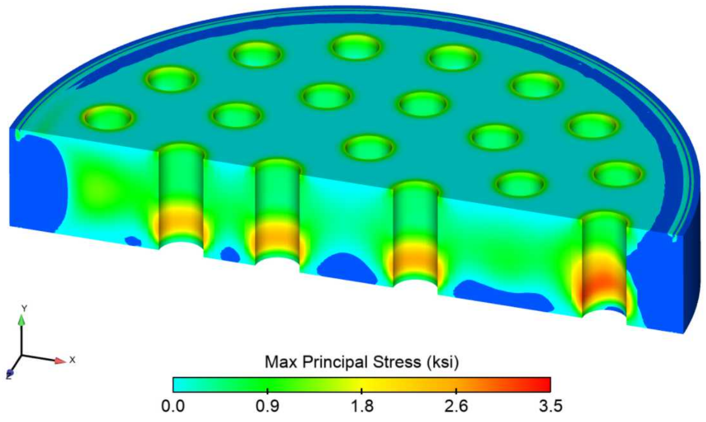
Possible discretization techniques

- FEA, tet/hex/polyhedral meshing
- FEA, isogeometric
- FEA, fictitious domain
- meshfree

hermetic connector



stress field



Value proposition for meshfree

- ease of discretization
  - use of point cloud and surface representation
- robust large deformation
  - shape functions defined on reference configuration
- local adaptivity
  - hp adaptivity – easy to change approximation order

Modeling challenges for meshfree

- surface representation
  - explicit vs. implicit representation
  - adaptivity on surface
- nonconvex domains
  - weight/shape functions around re-entrant corners
- assemblies of parts, material interfaces
- consistent quadrature, stabilization
  - stress points vs. nodal integration
  - integration consistency (to pass patch test)

Meshfree approach for this talk

- Use two meshfree clouds: one for d.o.f., one for quadrature
- Use sufficient number of quadrature points to avoid use of artificial stabilization.
- Use Reproducing Kernel (RK) shape function construction.
- For quadrature, define points and weights using shape functions
- For quadrature consistency, need to smooth shape function derivatives by projecting onto RK-d.o.f. cloud.

Governing equations (total-Lagrangian formulation)

strong form

$$\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{X}} : \boldsymbol{I} = \rho_0 \, \ddot{\mathbf{u}}$$

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \quad \text{on} \quad \Gamma_0^u \quad \text{and} \quad \boldsymbol{P} \cdot \boldsymbol{N} = \boldsymbol{t}_0 \quad \text{on} \quad \Gamma_0^t$$

weak form

find the trial functions $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega_0)$ such that

$$\int_{\Gamma_0^t} \boldsymbol{t}_0 \cdot \boldsymbol{v} \, dS - \int_{\Omega_0} \boldsymbol{P} : (\partial \boldsymbol{v} / \partial \boldsymbol{X}) \, d\boldsymbol{X} = \int_{\Omega_0} \rho_0 \, \ddot{\mathbf{u}} \cdot \boldsymbol{v} \, d\boldsymbol{X}$$

for all test functions $\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega_0)$

Moving Least Squares (Reproducing Kernel)

The MLS shape functions $\phi_I(\mathbf{X})$ are defined as a spatial modulation of the nodal weight functions.

$$\phi_I(\mathbf{X}) = c_I(\mathbf{X}) w_I(\mathbf{X})$$

where the modulation function $c_I(\mathbf{X})$ is found through a least square minimization process resulting in

$$c_I(\mathbf{X}) = \mathbf{g}^{\mathrm{T}}(\mathbf{X})\mathbf{A}^{-1}(\mathbf{X})\mathbf{g}(\mathbf{X}_I)$$

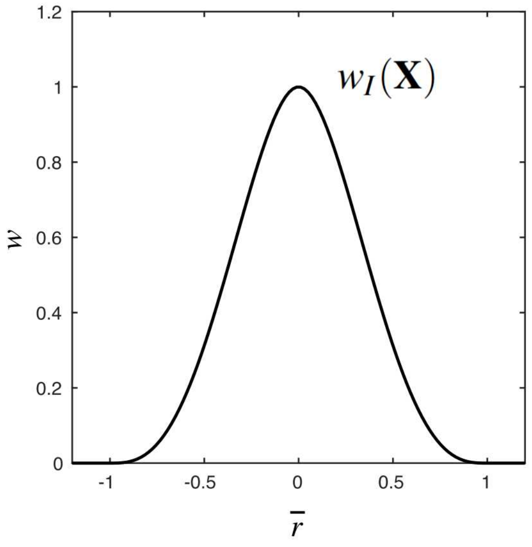
where

$$\mathbf{A}(\mathbf{X}) = \sum_{I \in \mathcal{N}} w_I(\mathbf{X})\mathbf{g}(\mathbf{X}_I)\mathbf{g}^{\mathrm{T}}(\mathbf{X}_I)$$ (sum over neighbors)

$$\mathbf{g}^{\mathrm{T}}(\mathbf{X}) = \{\, 1 \;\, X_1 \;\, X_2 \,\}$$ (linear reproducibility)

Note: shape function construction is algebraic.

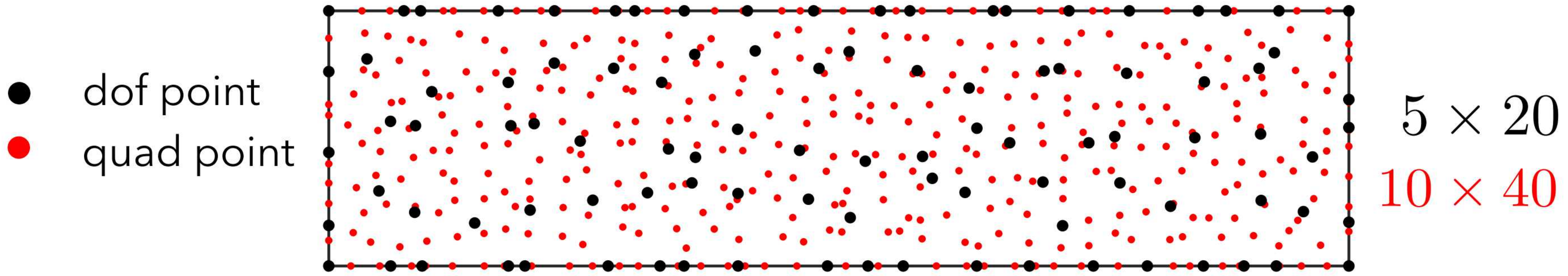
nodal weight function



circular or rectangular support

Meshfree approach

Use two meshfree clouds: one for d.o.f., one for quadrature.

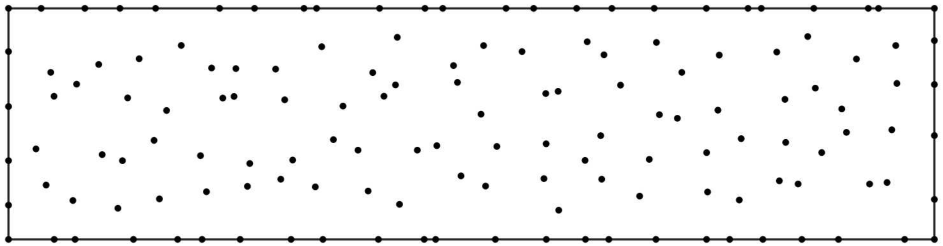


quad-to-dof point ratio = 4

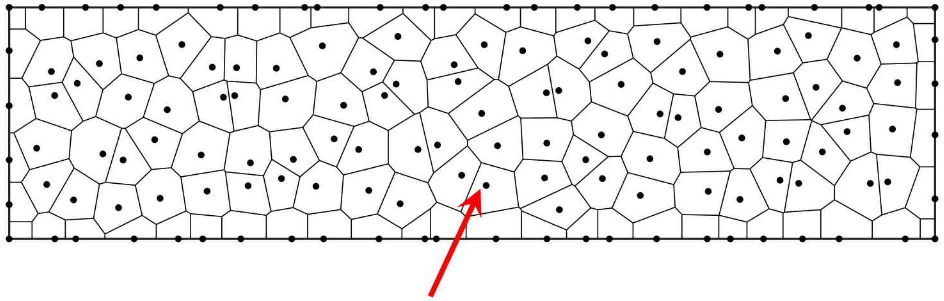
- What ratio of quad points to dof points is needed for stability?
- How to define quadrature weights?

How to define quadrature points?

quadrature points



Voronoi structure



Voronoi cell with area A_K

Quadrature weight defined as $\mathrm{w}_K \doteq A_K$

- Traditional approach is to construct a Voronoi diagram and assign the quadrature weight as the area of the Voronoi cell.
- This is challenging for geometrically complex domains, especially those that are non-convex.

Instead, can define using reproducing conditions: $\sum_I \phi_I(\mathbf{x}) = 1 \quad \sum_I \mathbf{x}_I \, \phi_I(\mathbf{x}) = \mathbf{x}$

Quadrature

Start with reproducing conditions $\sum_K \mathbf{x}_K \, \phi_K(\mathbf{x}) = \mathbf{x}$ (sum over K quadrature points)

Integrate both sides

(1) $$\int_\Omega \sum_K \mathbf{x}_K \, \phi_K(\mathbf{x}) = \int_\Omega \mathbf{x}$$

(2) $$\sum_K \mathbf{x}_K \int_\Omega \phi_K(\mathbf{x}) = \int_\Omega \mathbf{x}$$

Define quadrature weight as $\mathrm{w}_K = \int_\Omega \phi_K(\mathbf{x})$ then $\sum_K \mathrm{w}_K \, \mathbf{x}_K = \int_\Omega \mathbf{x}$

Quadrature points are just $\mathbf{x}_K$

This gives a linear-exact quadrature scheme.

Quadrature

Also, note that $$\sum_K \mathrm{w}_K = \sum_K \int_\Omega \phi_K(\mathbf{x}) = \int_\Omega \sum_K \phi_K(\mathbf{x}) = \int_\Omega 1 = V$$

Now have a second-order integration scheme that can integrate linear functions exactly.

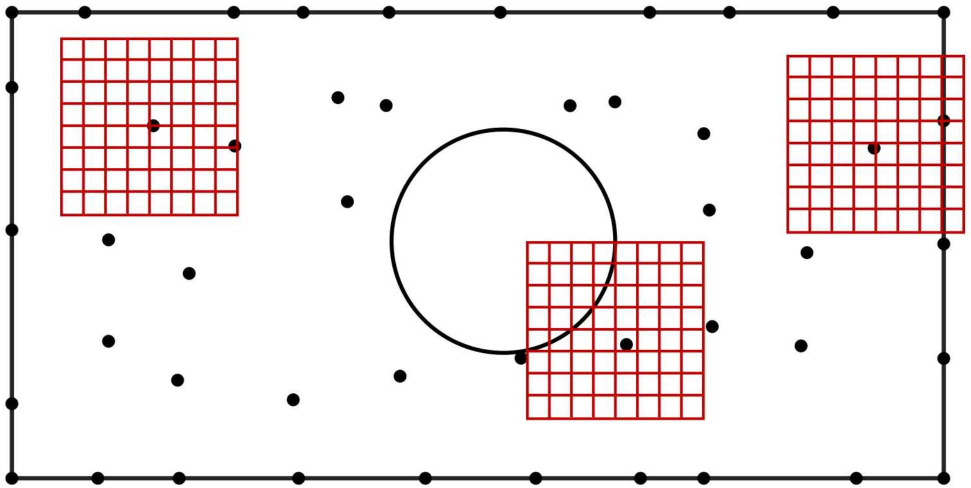
$$\sum_K \mathrm{w}_K = V \quad \text{and} \quad \sum_K \mathrm{w}_K \, \mathbf{x}_K = \int_\Omega \mathbf{x}$$

Can extend to higher-order integration using higher-order reproducing conditions.

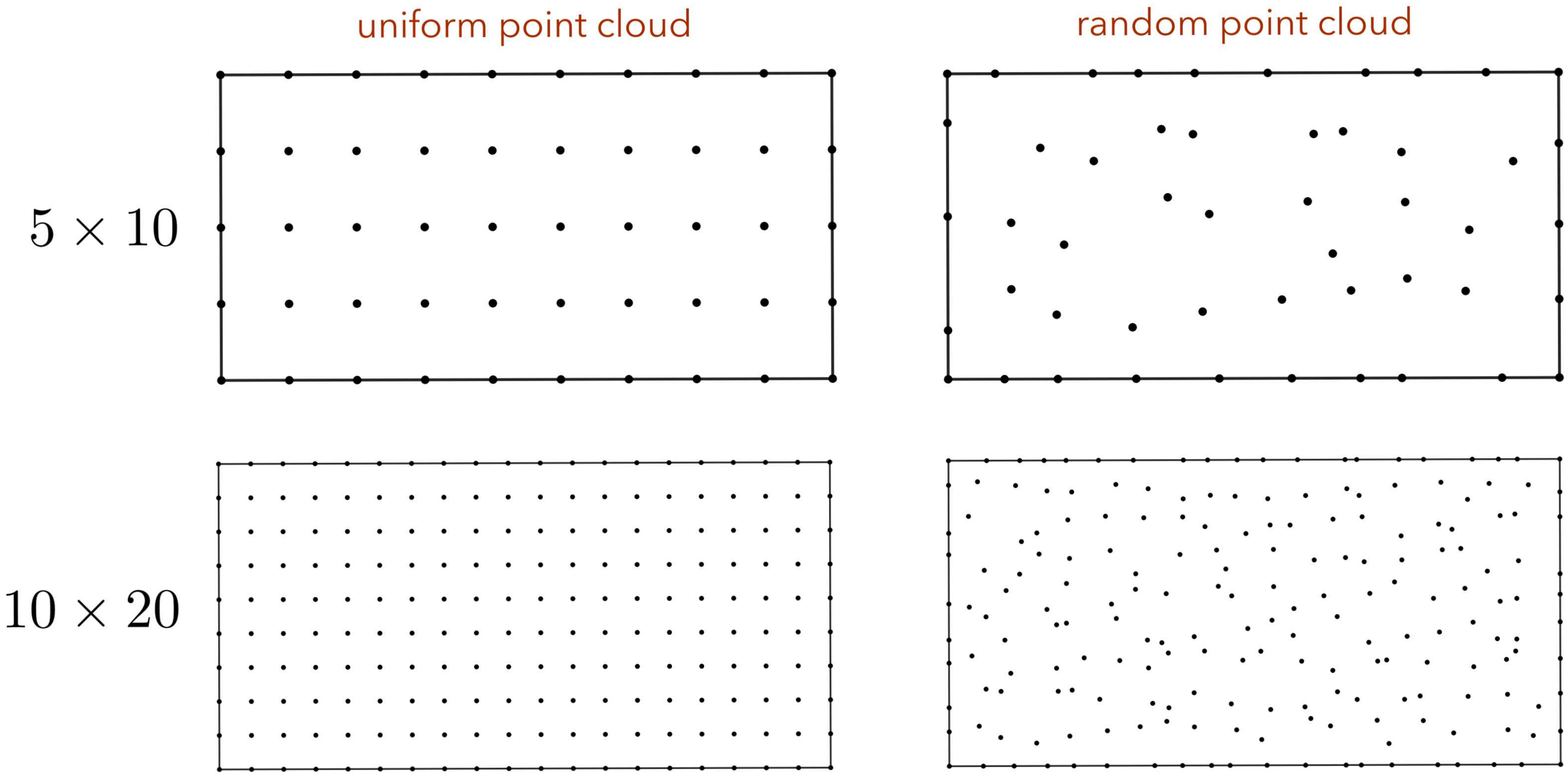
Quadrature

How to calculate? $\int_{\Omega} \phi_K(\mathbf{x})$

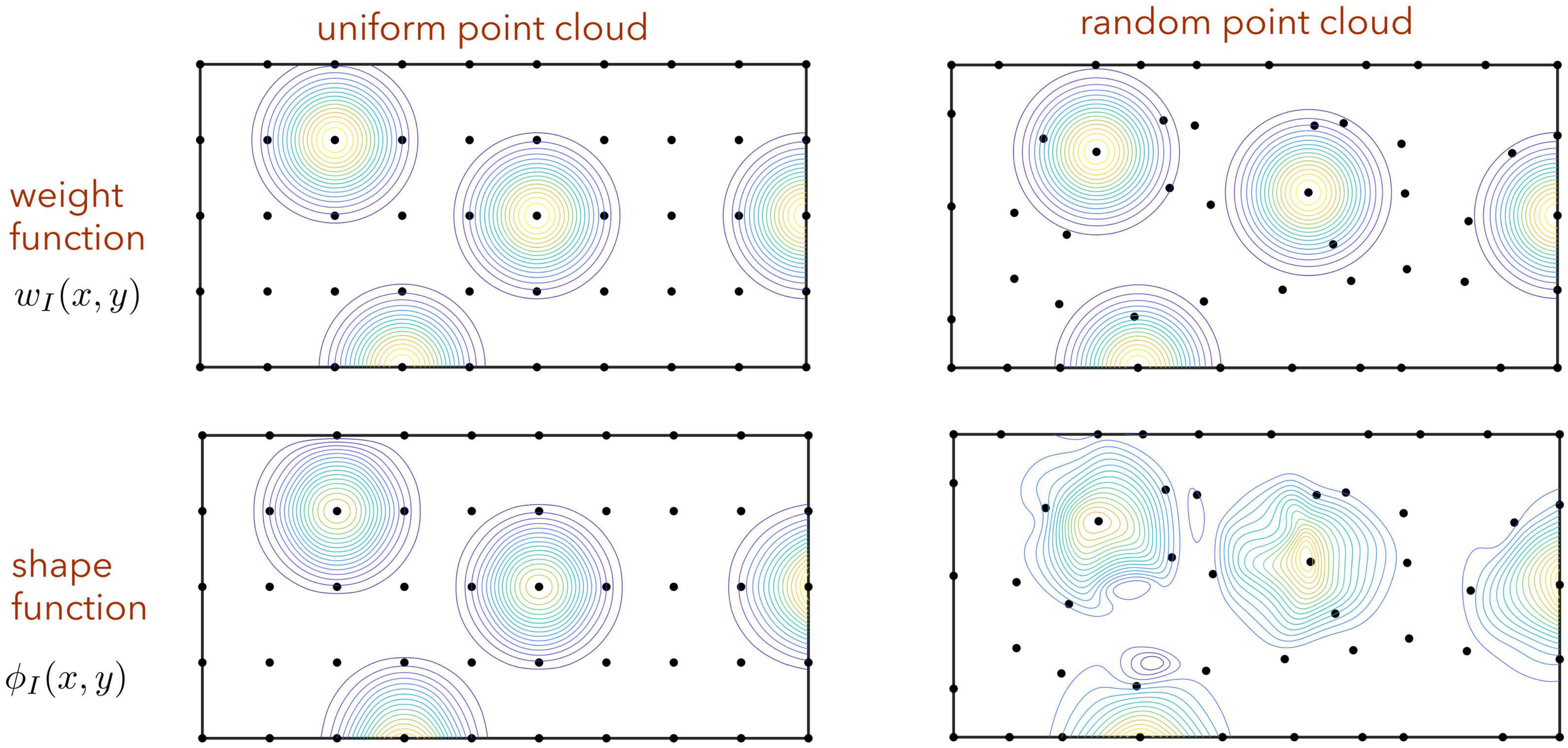
- Use a local virtual back-ground grid around each quadrature point.
- Calculate ϕ_K at each grid vertex.
- Integrate using trapezoidal rule for first-order RK, Simpson's rule for second-order RK.
- Use discontinuous techniques for cut cells.
- Only constructed locally (no global data structure needed)



Quadrature example



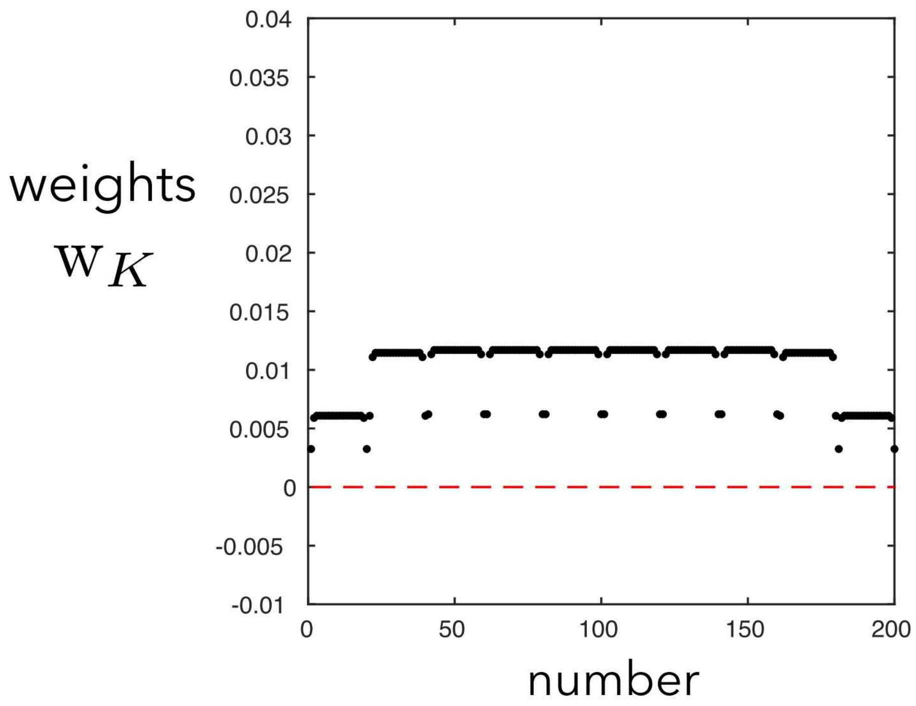
Quadrature example



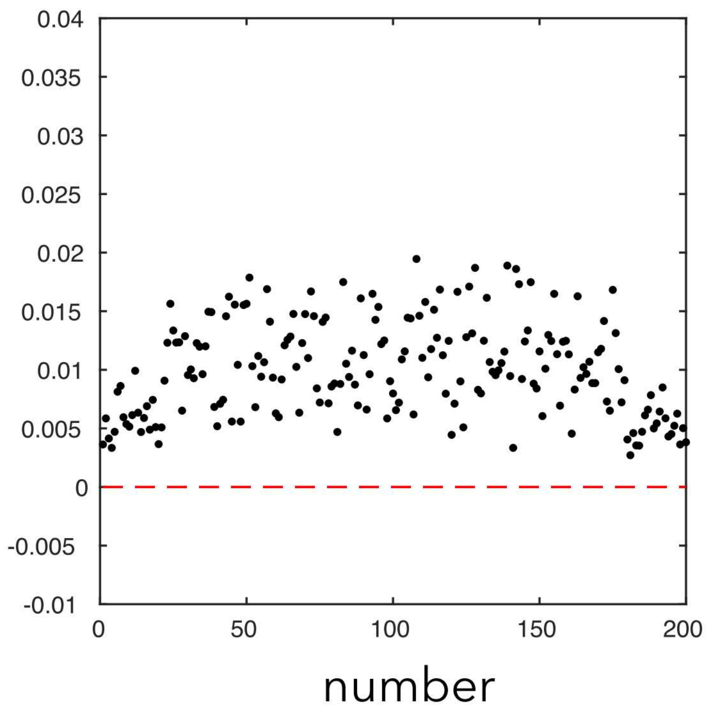
Quadrature weights

$$\mathrm{w}_K = \int_\Omega \phi_K(\mathbf{x})$$

uniform cloud



random cloud

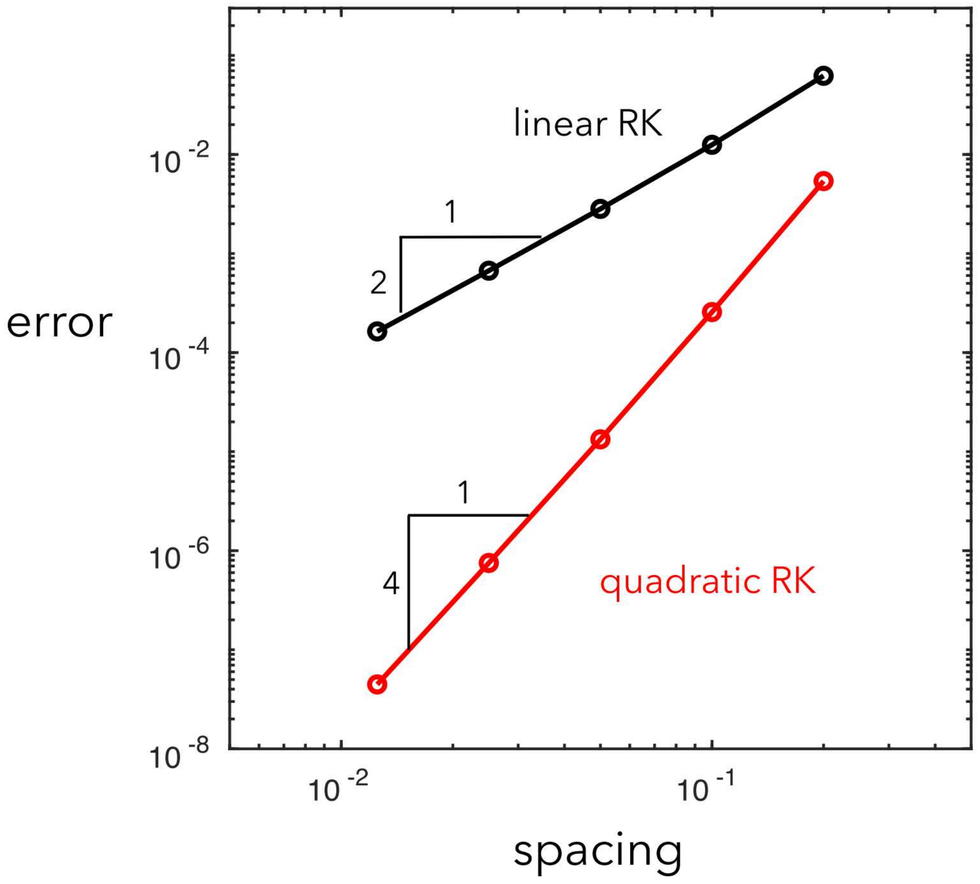


Note: For higher-order RK, will start to see some weights going negative which can cause stability issues.

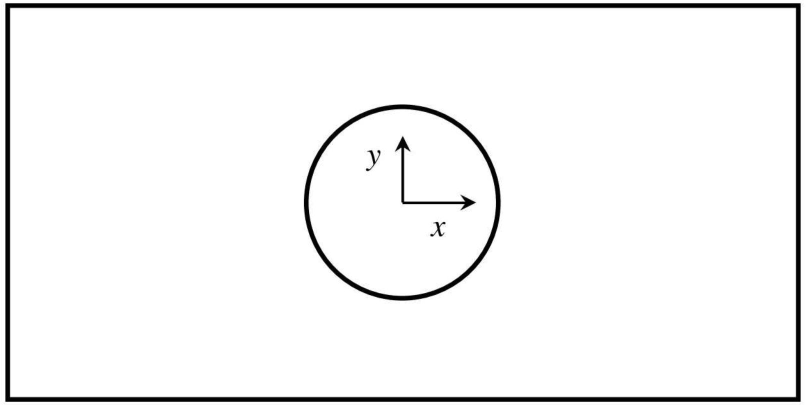
Integration convergence

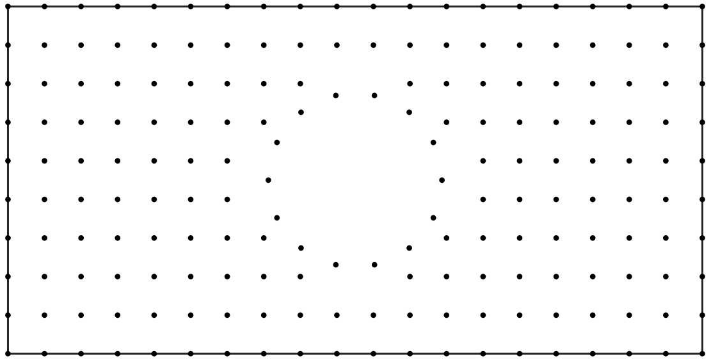
$$f(x, y) = \sin(\pi x/2)\sin(\pi y)$$

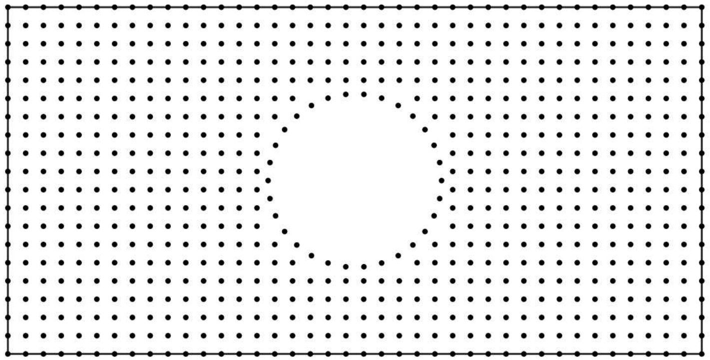
$$\text{error} = \left| \int f - \sum_i w_i f_i \right|$$



Example: Integration Convergence



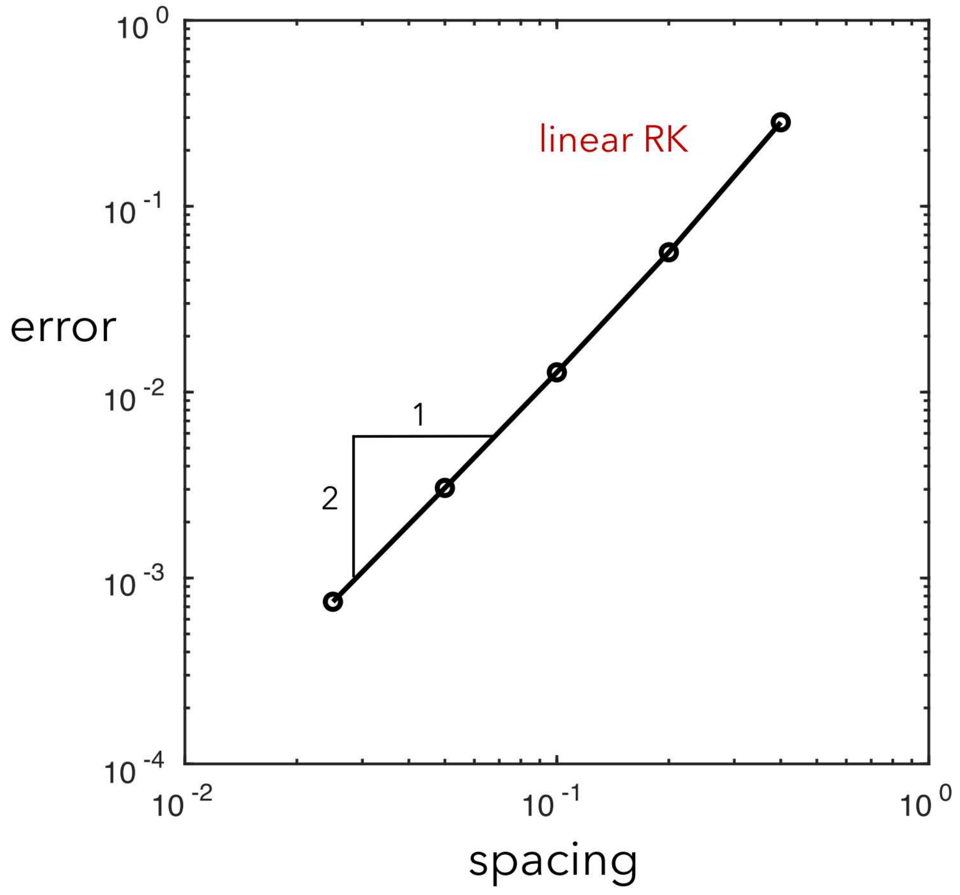




$$f(x,y) = \left[1 - \left(\frac{2x}{L_x}\right)^2\right]\left[1 - \left(\frac{2y}{L_y}\right)^2\right]$$

Integration convergence

$$\text{error} = \left| \int f - \sum_i w_i f_i \right|$$



Integration consistency

- Need to ensure integration consistency of shape function derivatives and quadrature scheme. (Krongauz & Belytschko, 1997; … Chen et al, 2001)
- Project the shape function derivatives to a consistent space to ensure integration consistency condition.
- Only performed once in a pre-processing step.
- Maintain the reproducing properties of the derivatives.

for interior node (by Gauss' thrm) $\int \nabla\phi_I = 0$ but, in general $\sum_K \mathrm{w}_K \nabla\phi_{IK} \neq 0$

To correct this, for each shape function I, project gradient onto quadrature shape function K.

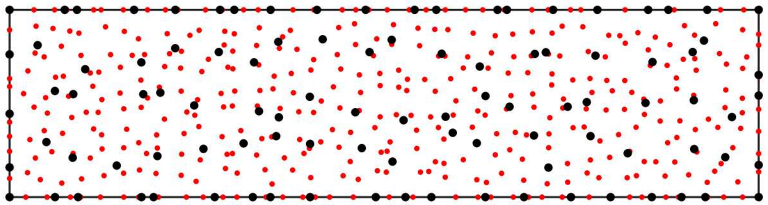
Define $$\overline{\nabla\phi}_{IK} \doteq \frac{1}{w_K}\int_\Omega \phi_K \, \nabla\phi_I$$

Verification example: cantilever beam with shear load

BVP

meshfree discretization

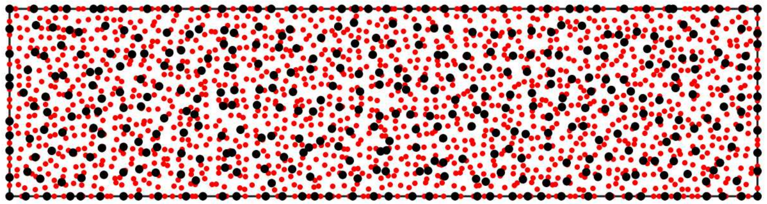
5×20



quad-to-dof point ratio = 4

● rk dof
● quad point

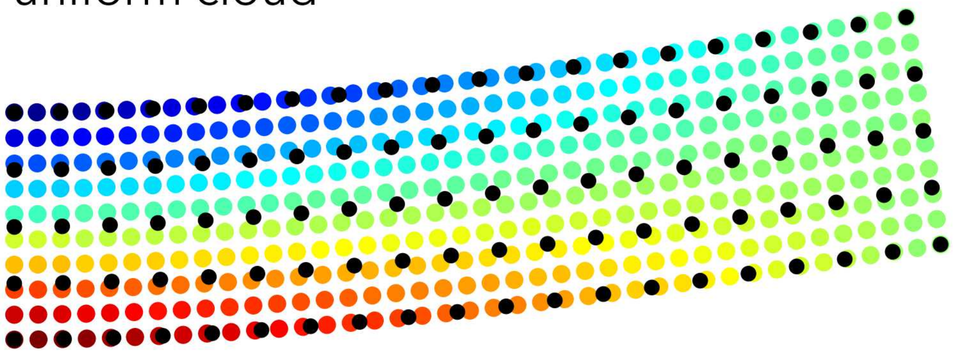
10×40



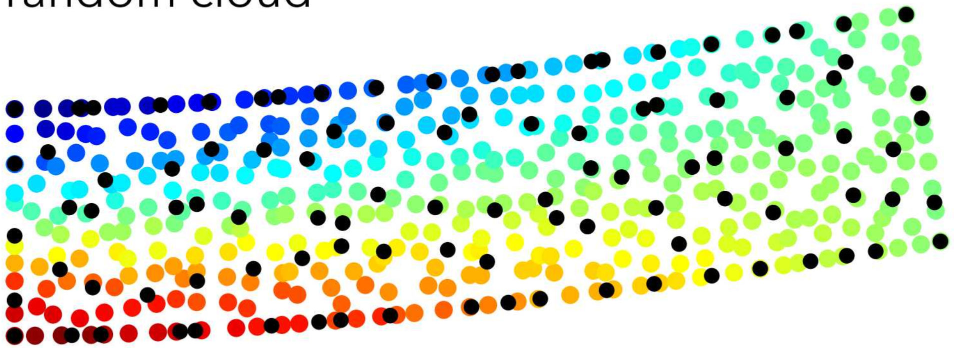
Beam with shear load

von Mises stress field

uniform cloud

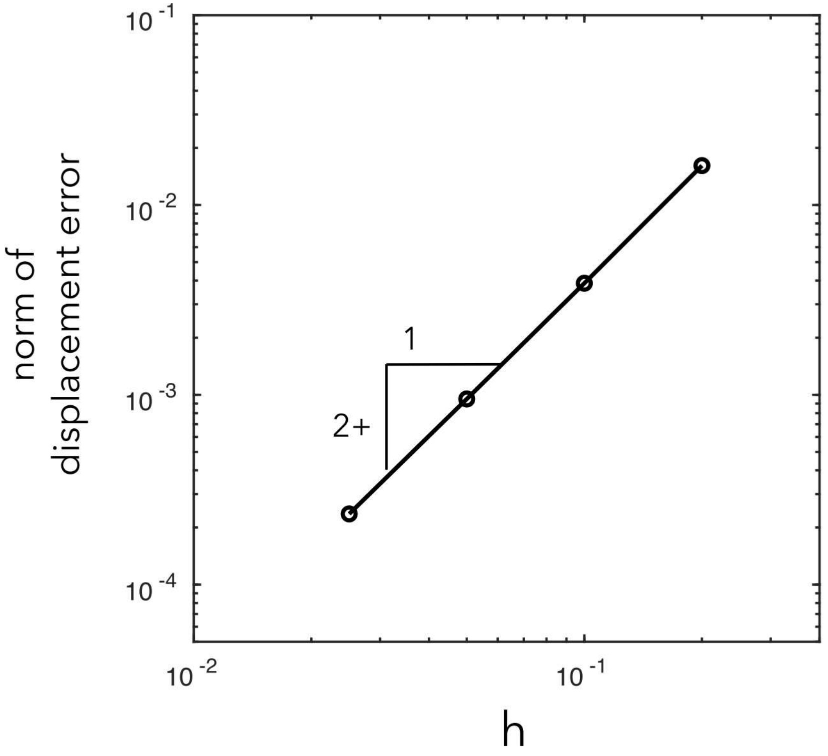


random cloud

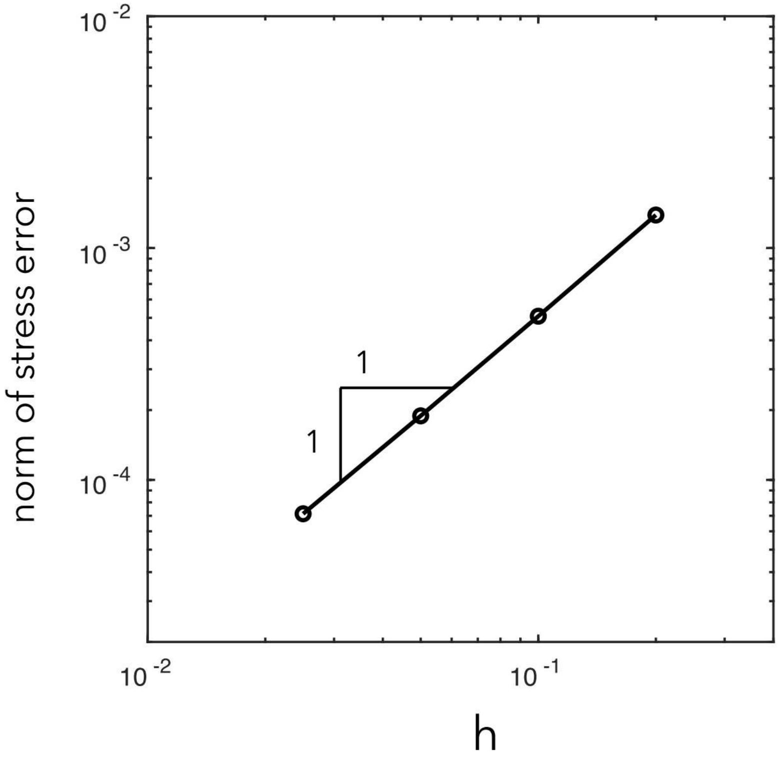


Beam with shear load: convergence

displacement error



stress error



Optimal rates of convergence

Summary

1. New approach to quadrature for meshfree methods based on reproducing conditions.
2. For 2D elasticity, need to project/correct shape function derivatives to be consistent with quadrature scheme.
3. Observed optimal convergence rates for 2D elasticity.
4. Exploring use of a local PDE on embedded domain to define weight functions on non-convex domain.