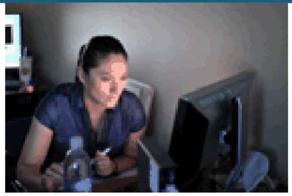
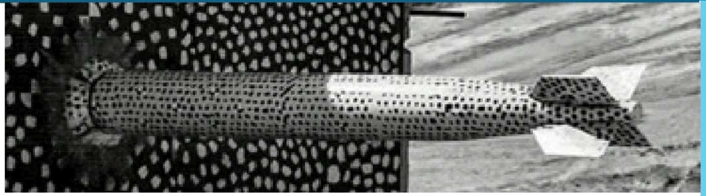
SAND2020-10188C

Regression Based Approach for Robust Finite Element Analysis on Arbitrary Grids (REBAR)





Paul Kuberry, Nathaniel Trask, Jake Koester, Pavel Bochev

Sandia National Laboratories

ASME® 2020 IMECE®
International Mechanical Engineering Congress & Exposition®
VIRTUAL CONFERENCE
Nov 16–19, 2020

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia LLC, a wholly owned subsidiary of Honeywell International Inc. for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA0003525.

Overview

Goal

Enable accurate and stable simulations on poor-quality meshes.

Our Approach

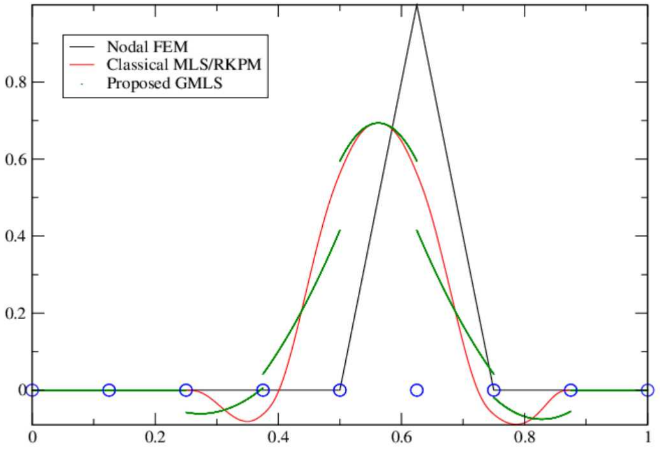
The construction of standard FEM basis functions is tie … Therefo … n
quality degrades due to deformation or use of fast mes … y suffers

We break the connection between mesh-quality and so … ping finit
spaces with meshfree tools, so that approximation pov … sh qualit

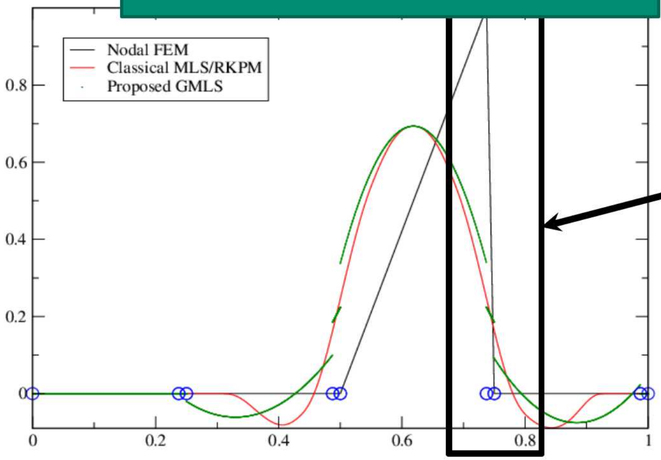
FEM shape functions have **infinite gradient** for sliver elements

Gradient of our approach **robust** to slivers

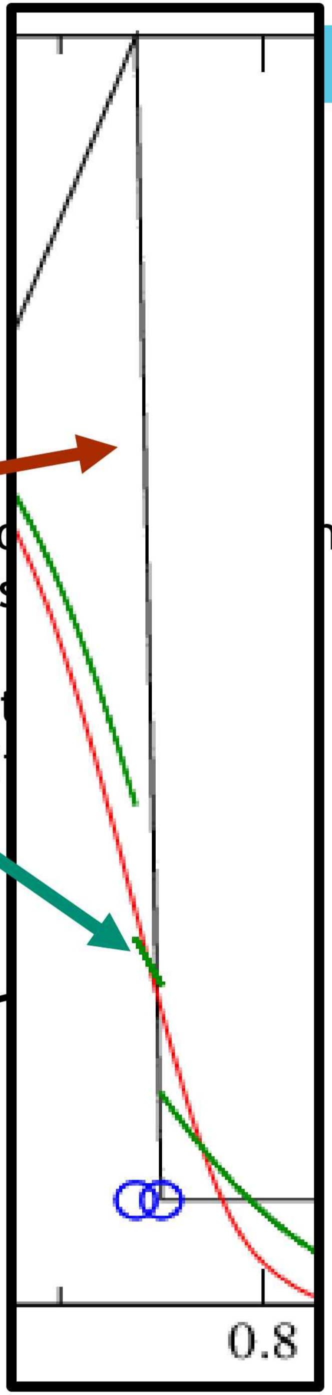
1D Example



"Good" mesh



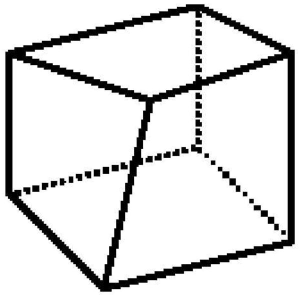
"Bad" mesh



Overview

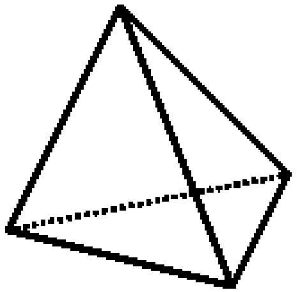
How is it done today? Limits?

Standard FEM basis functions interpolate degrees of freedom tied to underlying mesh entities. Below are the two most common 3D elements used for solid mechanics.



Hexahedron

Often requires **human-intervention to complete the meshing process***



DOI:10.1109/GMAP.2002.1027501

Tetrahedron

Open challenge to automatically create high-quality meshes for **complex geometries**

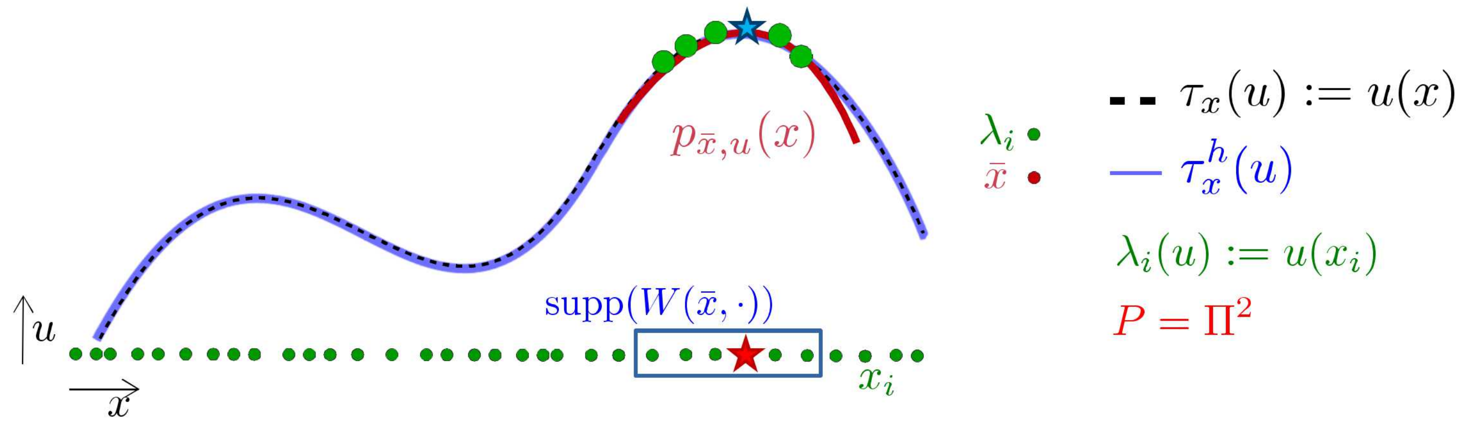
Our approach is **applicable to general polyhedra**, allowing impact to both workflows while **endowing next-generation automatic tet-meshers with improved robustness at less cost.**

*Hardwick, M., Clay, R., Boggs, P., Walsh, E., Larzelere, E., Altshuler, A.: DART system analysis. SAND2005-4647.

Generalized Moving Least Squares

$$p_{\bar{\mathbf{x}},u} = \arg\min_{p \in P} \sum_{i \in I_{\bar{\mathbf{x}}}} \left(\lambda_i(u) - \lambda_i(p)\right)^2 W(\bar{\mathbf{x}}, \mathbf{x}_i)$$

$$\tau^h_{\bar{\mathbf{x}}}(u) := \tau_{\bar{\mathbf{x}}}(p_{\bar{\mathbf{x}},u}),$$



- A rigorous mathematical framework for approximating linear functionals from scattered degrees of freedom
- Provides necessary theoretical infrastructure to **back approach with rigorous proofs**
- **Compadre Toolkit** in Trilinos provides **GPU-accelerated** software to support easy implementation

* https://www.github.com/SNLComputation/compadre

Example of Symmetric Interior Penalty Discontinuous-Galerkin (SIPG) framework applied to reaction-diffusion

$$A(w,v) = \sum_{E \in \mathscr{T}_h} \int_E K \nabla w \cdot \nabla v + \int_\Omega \alpha w v + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{\sigma_e}{|e|^{\beta_0}} \int_e [w][v]$$

$$- \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{K \nabla w \cdot \mathbf{n}_e\}[v] - \sum_{e \in \Gamma_h \cup \Gamma_D} \int_e \{K \nabla v \cdot \mathbf{n}_e\}[w],$$

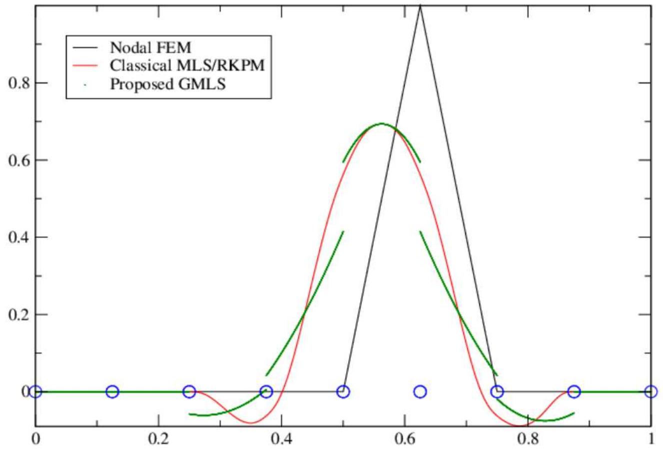
$$L(v) = \int_\Omega f v + \sum_{e \in \Gamma_N} \int_e v u_N - \sum_{e \in \Gamma_D} \int_e (K \nabla v \cdot \mathbf{n}_e) u_D + \sum_{e \in \Gamma_D} \frac{\sigma_e}{|e|^{\beta_0}} \int_e v u_D$$

Standard reaction-diffusion terms

Traditional SIPG stabilization terms

GMLS basis may provide better penalty parameters for DG than traditional finite elements

- GMLS basis amounts to **projection** of traditional meshfree elements **onto piecewise polynomial space**
- Require **stabilization** to handle variational crime from discontinuity **at cell interfaces**
- Mature DG technology allows **rigorous out-of-the-box treatment** of many problems



Variational form relation to Generalized Moving Least Squares (GMLS)

$a(u,v) = \sum_{k=1}^{N_e} a_k(u,v)$ and $f(v) = \sum_{k=1}^{N_e} f_k(v)$, where $a_k(\cdot,\cdot)$ and $f_k(\cdot)$ are restrictions of $a(\cdot,\cdot)$ and $f(\cdot)$ to element $\mathcal{K}_k$

$$(a_k(\boldsymbol{\phi},\boldsymbol{\phi}))_{st} = \int_{\mathcal{K}_k} \nabla\phi_s \cdot \nabla\phi_t dx$$

$$(f_k(\boldsymbol{\phi}))_s = \int_{\mathcal{K}_k} f\, \phi_s dx\,.$$

Let b_k denote the barycenter of element T_k.

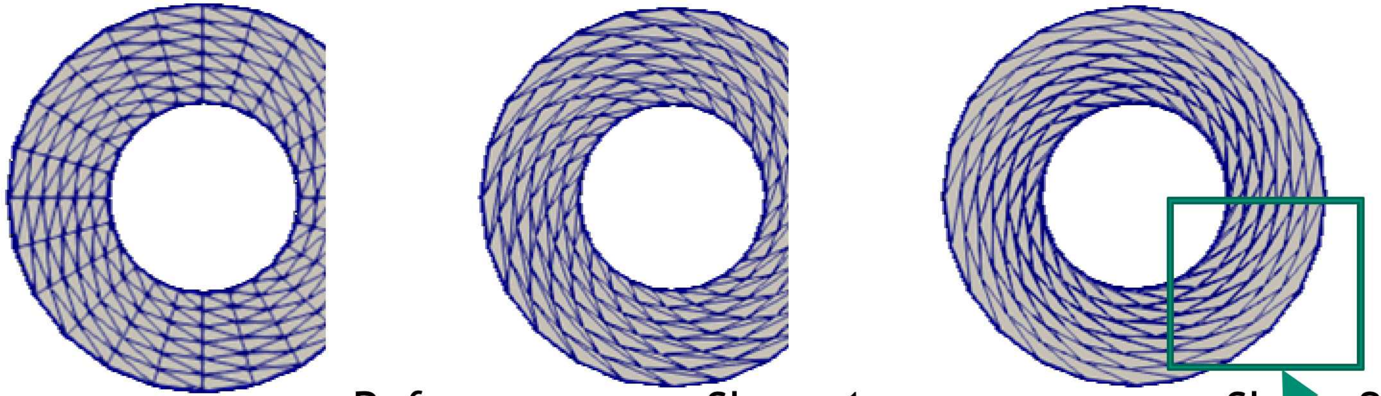
$$\widetilde{a}_k(u_j^k, u_i^k) := \boldsymbol{c}(\mathbf{e}_i^k; \boldsymbol{b}_k) \cdot a_k(\boldsymbol{\phi},\boldsymbol{\phi}) \cdot \boldsymbol{c}(\mathbf{e}_j^k; \boldsymbol{b}_k)$$

$$\widetilde{a}([u],[v]) := \sum_{\mathcal{K}_k \in \Omega^h} \widetilde{a}_k(u^k, v^k) \quad \text{and} \quad \widetilde{f}([v]) := \sum_{\mathcal{K}_k \in \Omega^h} \widetilde{f}_k(v^k)\,, \quad [u]_i := \sum_{\mathcal{K}_k \in \Omega^h} \chi_k u_i^k\,,$$

* P. Bochev, N. Trask, P. Kuberry, and M. Perego, **Mesh-hardened finite element analysis through a Generalized Moving Least-Squares approximation of variational problems**, *International Conference on Large-Scale Scientific Computing*, pp. 67-75. Springer, Cham, 2020. doi: 10.1007/978-3-030-41032-2_7.

Demonstrate feasibility of approach on poor quality meshes

rees of shear

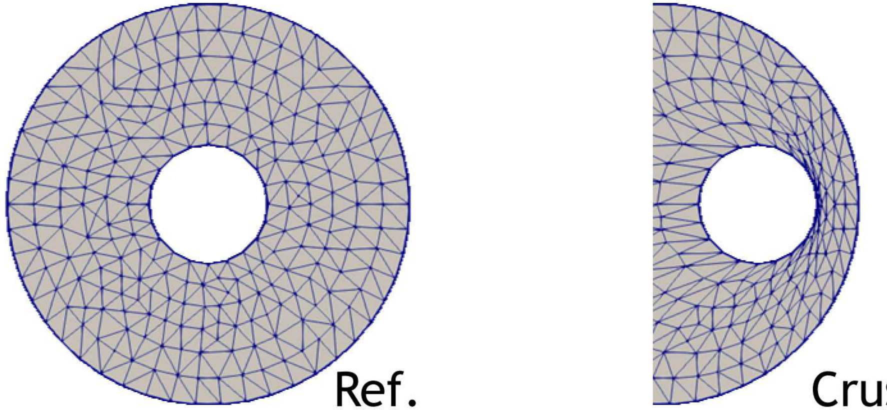


Ref. Shear 1 Shear 2

Consider 3 meshes representative of common use cases:

- Shear Many obtuse angle elements occurring in shear deformation
- Crushed Sliver elements occurring during penetration/contact
- Distorted box Degenerate elements caused by poor meshing

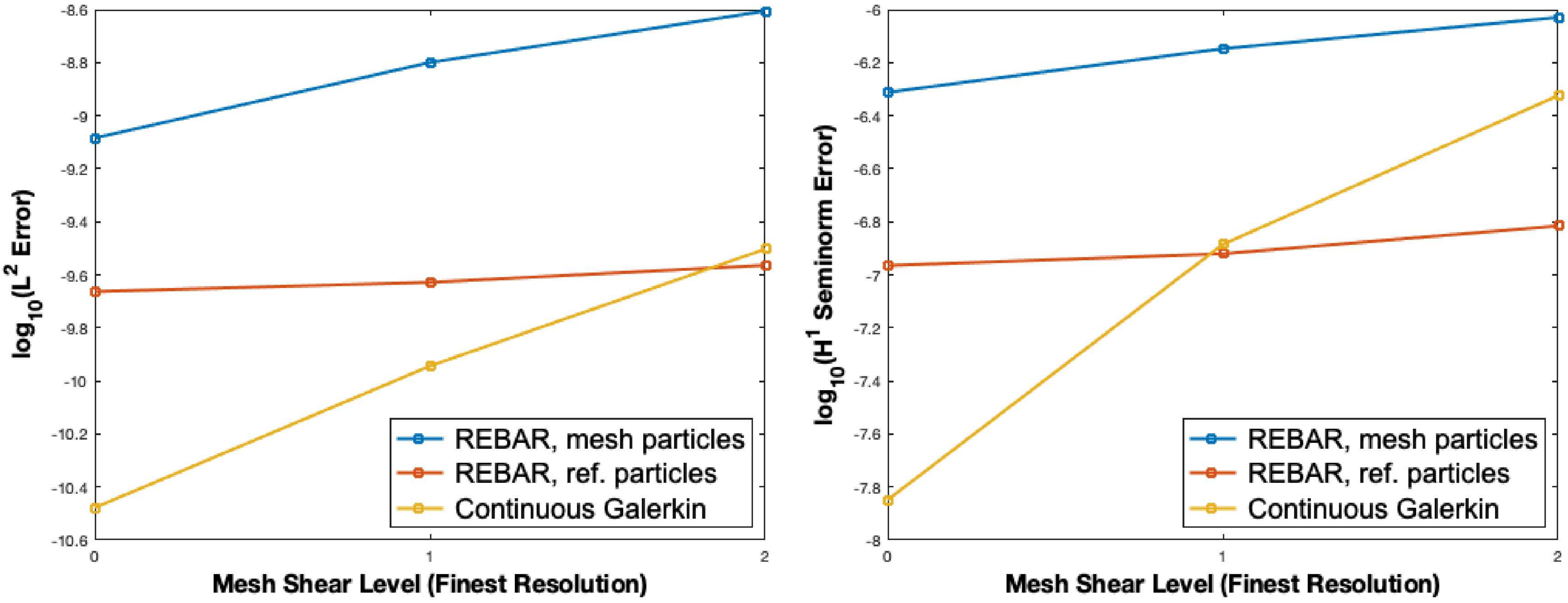
Annulus reference & crushed



Ref. Crushed

Pegu

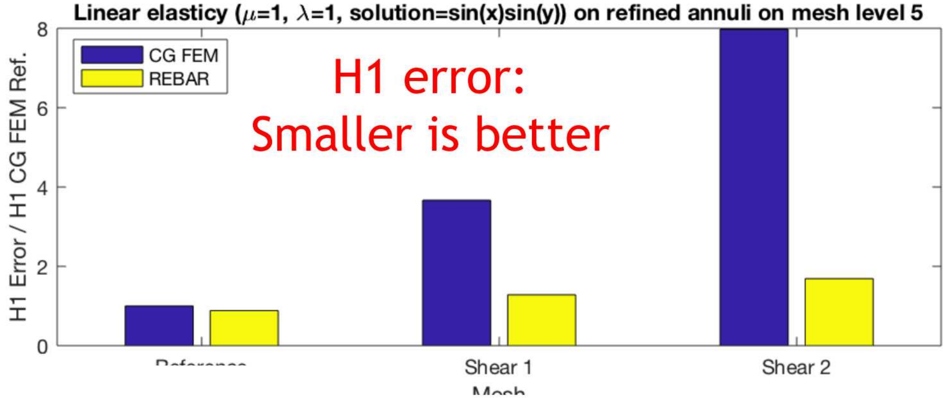
Demonstration of feasibility of approach on poor quality meshes (reaction-diffusion)

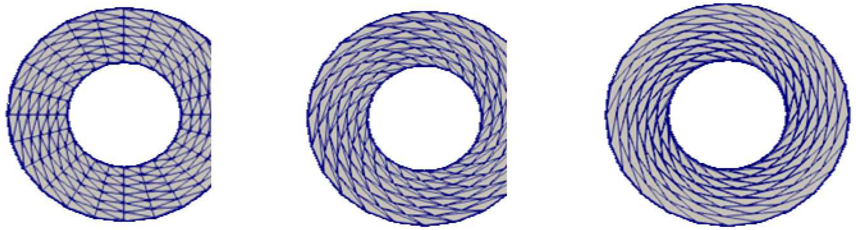


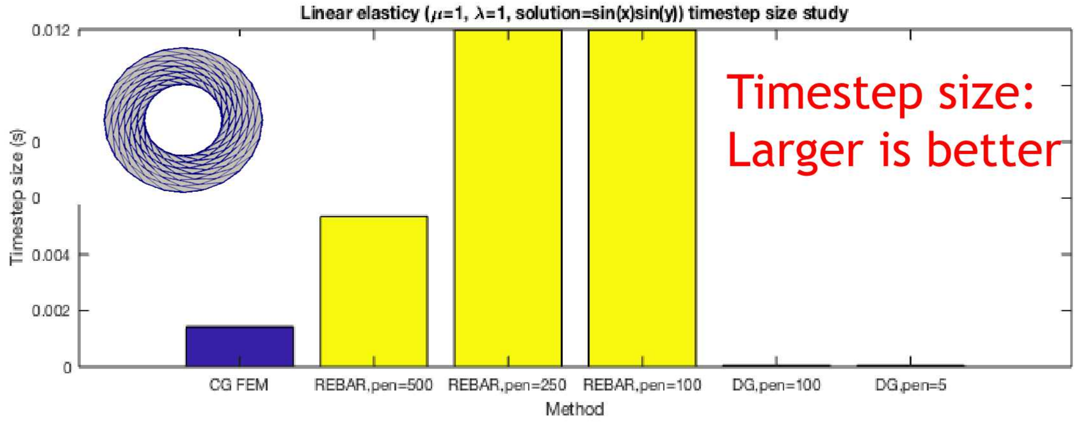
For L^2 and H^1 errors, our approach beats traditional CG-FEM using the reference particle set and has a smaller slope than CG-FEM with either particle set configuration

Demonstration of feasibility of approach on poor quality meshes (linear-elasticity)

More accurate solution while allowing larger timestep on low-quality meshes







L2 error on bad mesh: 5.5x lower than CG

H1 error on bad mesh: 5.73x lower than CG

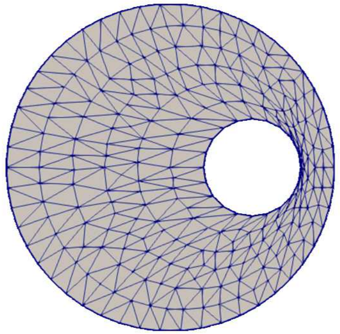
Stable timestep: 3.74-8.38x larger than CG

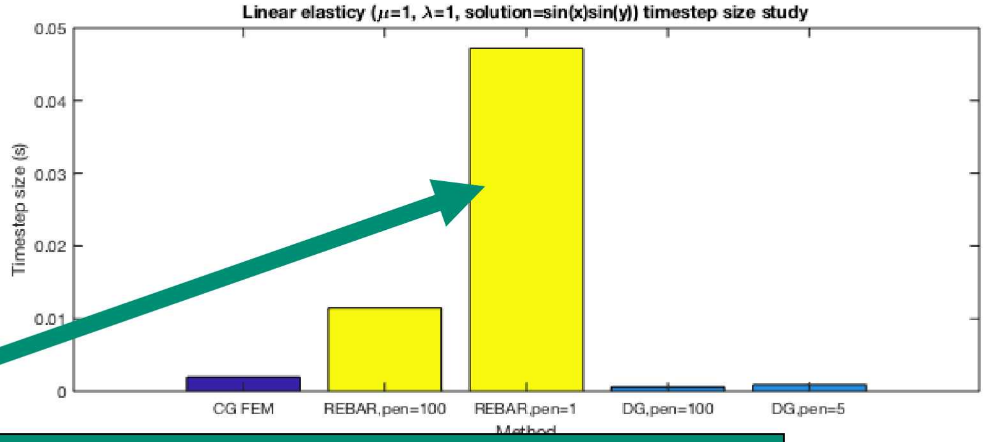
<u>Punchline</u>

~5 times better answer at ~3-8 times cheaper cost!

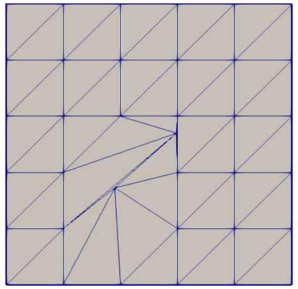
Demonstration of feasibility of approach on poor quality meshes

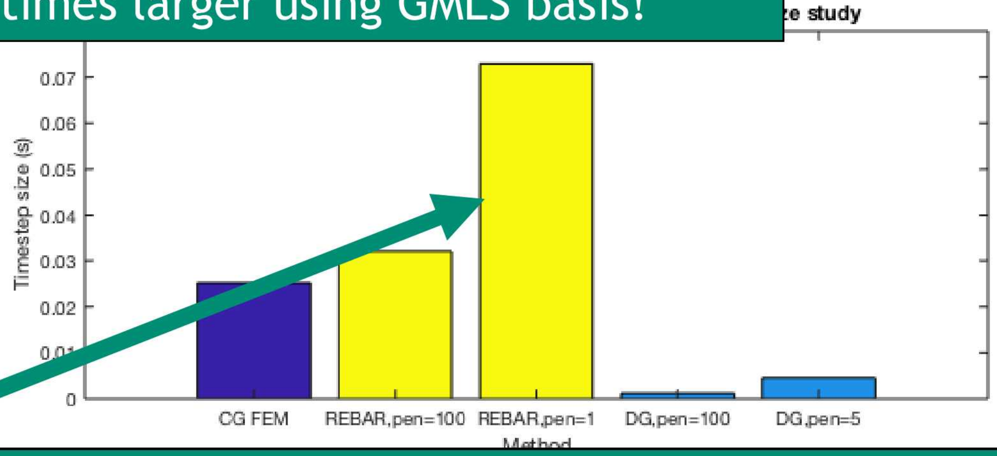
- **Improved timestep restriction resulting from discretization on poor-quality meshes [Cont.]**





Stable timestep is 5.86 times larger using GMLS basis!





GMLS basis timestep beats CG FEM with penalty of 100. With a penalty of 1, it does even better!
Big gains from investigating sharp estimates for the penalty.

Pressure stability for saddle-point problems (Stokes flow)

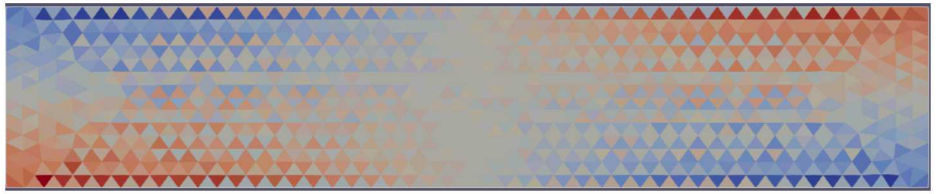
DG stabilization endows method with **stability in important engineering limits**, such as near-incompressibility (instrumental for handling material like rubber, or plastic deformation leading to ductile failure).

Divergence-free sinusoidal solution on Timoshenko mesh sequence

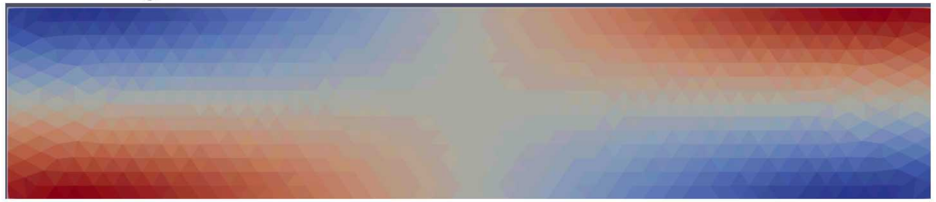
Velocity order=1, Pressure order=1			
Vel. L2 Error	**Rate**	**Vel. H1 Error**	**Rate**
1.12E-03	-	4.45E-02	-
2.02E-04	**2.47**	2.04E-02	**1.12**
4.10E-05	**2.30**	9.69E-03	**1.07**
9.33E-06	**2.13**	4.71E-03	**1.04**
2.10E-06	**2.15**	2.30E-03	**1.03**

Optimal convergence rates illustrate stability, even for polynomial pairs that are unstable for CG FEM.

Inf-sup unstable formulation:



vs. using GMLS basis in stabilized DG framework*:



Computed pressure field on Timoshenko mesh with divergence-free manufactured solution.

Smooth pressure field demonstrates no checker-boarding, which is characteristic of pressure locking.

* Burman, E., Hansbo, P. (2007). A unified stabilized method for Stokes' and Darcy's equations, *Journal of Computational and Applied Mathematics, 198(1)*, 35-51, https://doi.org/10.1016/j.cam.2005.11.022

Conclusions

- Developed variational PDE formulation using meshless techniques (GMLS), generating shape functions that are polynomial and discontinuous between cells
  - Mesh is used for integration (quadrature points and weights)
  - Solution lies on particle sets (**independent of mesh** vertices / midpoints / etc..)
  - Piecewise polynomial shape functions ensure **exact integration**

- Demonstrated on strongly-elliptic problem (reaction-diffusion) and linear elasticity that the approach:
  - **Optimal rates of convergence**
  - Performs well on poor quality meshes relative to traditional CG-FEM

- On saddle-point problem (Stokes flow), demonstrated that:
  - Minor, well-understood modification within **SIPG enables stability** for traditionally inf-sup unstable low-order velocity/pressure pairs
  - Inf-sup stability is maintained for velocity/pressure pairs of polynomial degree traditionally stable in CG-FEM (e.g., P2/P1)