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Cost of Classical Strong Simulation of the T-Gate Magic State

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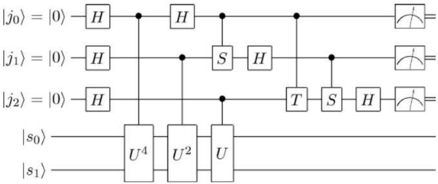
Tensoring Magic State Gives You Universality



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  - k-tensored T-gate magic states $|T\rangle^{\otimes k}$
  - then acted on by Clifford unitaries U_C
  - then partial traced over to obtain a marginal over any qudit

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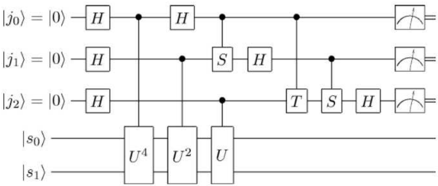

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Previous methods have relied on Monte Carlo numerics and stop converging at $k > 7$.
We show how to get to $k > 14$ using an algebraic method.

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It has been postulated that $\chi(T)$ grows slowest with increasing number of qubits

Computational Cost

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Remark: Trivial Tensor Bound

Let χ_k be the stabilizer rank of $|\Psi\rangle^k$. Since the tensor product of two stabilizer states is a stabilizer state, it follows that $\chi_t \leq \chi_k^{t/k}$.

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[Bravyi *et al*. PRL 116 (2016): 250501, & Quantum 3 (2019): 181]

Numerical Monte Carlo findings:

qubit k	1	2	3	4	5	6	7
χ_k	2	2	3	4	6	7	12
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Can we push this further?! Not with numerical techniques. Really want to know if can do better than trivial tensor bound for large t. This is why χ_6 is a big deal.

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Given a set of operators $R(\boldsymbol{x})$, indexed by $\boldsymbol{x} \in (\mathbb{Z}/p\mathbb{Z})^n$, that are Hilbert-Schmidt orthogonal, any operator can be represented as

$$\hat{A} = d^{-1} \sum_{\boldsymbol{x}} \mathrm{Tr}(\hat{R}(\boldsymbol{x})\hat{A})\hat{R}(\boldsymbol{x}) \equiv \sum_{\boldsymbol{x}} A(\boldsymbol{x})\hat{R}(\boldsymbol{x}).$$

Computation

$$P = \mathrm{Tr}\left[\langle 0|\hat{U}_C|T\rangle^{\otimes k}|0\rangle^{\otimes(n-k)}\right]$$

$$P \overset{=}{\text{WWM}} \sum_{\boldsymbol{x}'\in D}\left[\prod_{i=1}^{t}\rho_T(\boldsymbol{x}'_i)\prod_{j=t+1}^{n}\delta(x'_{q_j})\right].$$

for

$$D = \left\{\boldsymbol{x}' \middle| \left(\mathcal{M}^{-1}\left(\boldsymbol{x}' - \frac{\boldsymbol{\alpha}}{2}\right) - \frac{\boldsymbol{\alpha}}{2}\right)_{n+1} \mod 3^2 = 0\right\}$$

Basic computational primitive is quadratic Gauss sums. Each one costs $\mathcal{O}(n^3)$. We want to know how few we need.

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Hilbert space	WWM
Stab state	classical probability distribution
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WWM formalism is simpler for odd-prime-dimensional qudits and so here we focus on qutrits ($d = 3$)

Using WWM to Bound Stab Rank

[Kocia & Love, arXiv:1810.03622] Prior Algebraic Findings:

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Pushing Forward

Three Qutrit T-Gate Magic State:

$$\rho(\boldsymbol{\mathcal{M}}_{C_{1,3}^2 C_{2,3}}\boldsymbol{x}) = 3 \sum_{\substack{y_{q_1}, y_{q_2} \\ \in \mathbb{Z}/3\mathbb{Z}}} \exp\left[\frac{2\pi i}{9}\left(7y_{q_1}^3 + 8x_{q_1}^3\right)\right] \mathcal{A}_3(y_{q_1}, y_{q_2}, \boldsymbol{x}) \times \left[\delta(\neg(y_{q_1} - x_{q_1})) + \delta(y_{q_1} - x_{q_1})\delta(\Delta)\right],$$

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Extrapolating to higher t counts using the trivial tensor bound, this result shows that $3^{\frac{\log 8}{3 \log 3} t} \approx 3^{0.63t}$ quadratic Gauss sums can represent t magic states, for t a multiple of 3.

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Assuming a similar reduction occurs for qubits at $t = 12$ as for qutrits, the WWM method is able to algebraically search a space that consists of

$$> 8.3 \times 10^{\mathbf{13000}}$$

possible stabilizer states!

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