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Transforming a Simple Structure Model to Represent a Complex Dynamic System with Unknown Boundary Restraints

J. Woodall[1]; M. Hossain[2]; A. Maji[3], PE; and F. Moreu[4], PE

[1]Department of Civil, Construction, and Environmental Engineering, University of New Mexico, Albuquerque, New Mexico

[2]Department of Mechanical Engineering, University of New Mexico, Albuquerque, New Mexico

[3]Sandia National Laboratories, Albuquerque, New Mexico

[4]Assistant Professor, Department of Civil, Construction, and Environmental Engineering, University of New Mexico, MSC01 1070, 1 University of New Mexico, Albuquerque, NM 87131. Email: fmoreu@unm.edu

ABSTRACT

Imposing a boundary condition on a structure can significantly alter its dynamic properties. However, sometimes the specifics of the new boundary conditions are not known. When the effects of a boundary condition are uncertain or there is not enough information, engineers need to excite the complex structure to obtain these modified properties. In order to experimentally obtain the new properties, engineers need multiple experiments and many outputs for interpolation in order to sufficiently represent the entire structure. The researchers attached a stinger to a cantilever beam, acting as a new transverse restraint of unknown properties. This paper presents a conversion expression that predicts the dynamic behavior of any point in the system with the new boundary condition. This expression relies only on one impact hammer experiment with one output and the model of the stinger-free cantilever beam, referred to as the simple structure. Researchers

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estimated the Transfer Function (TRF) of the beam and compared it with an experimentally measured TRF to validate the method. The mean absolute error of the estimated TRF compared to the experimental TRF is 1.99 dB. This demonstrates the use of the proposed method for estimating unmeasured TRFs in a system with an uncertain boundary condition using a single input, single output (SISO) test and a model of the simple structure.

Keywords: Transfer Function; Acceleration; Vibration; Finite Element; Boundary Conditions.

INTRODUCTION

Researchers use models to design experiments prior to their implementation for safety, economy, or laboratory efficiency. This pre-test analysis helps to establish input design as well as sensor and input locations [1-5]. However, differences between generated models and real-world systems often cause misrepresentation of the system, which results in errors. Consequently, researchers must take uncertainties into account when using models to design an experiment [6]. Even though it is important to have models to design tests, sometimes there is not enough information available to inform these models.

Boundary conditions are an important part of models, but are often difficult to represent accurately, especially if its dynamic properties are unknown. Researchers use models to design experiments to test structures for vibration, shock, and impact analysis [7], but analytical models must properly represent the structure to be effective [8]. Examples of boundary conditions include fixed support on one end, lateral fixtures attached along the structure, and interaction with another structure [9]. These boundary conditions complicate the structure and the model must include their effects to represent it properly [10]. However, for dynamic analysis, theoretical models do not always consider the physical boundary conditions [11]. The model of a simple structure that does not represent the dynamic effects of any supports or attachments on it will result in errors if a researcher uses it inappropriately.

The properties of the restraints in structural models are not always available. Modeling elastic restraints such as stingers on a beam typically requires understanding their dynamic properties [12-13]. Specifically, researchers have developed methods to create a modal model for both the complex, coupled structure, and its independent components [14]. However, they assume sufficient knowledge concerning the nature of the boundary conditions, which is not always the case. While researchers have investigated methods to model uncertainties in these parameters [15-16], the quality of a model ultimately depends on the quality of the parameters assumed.

Researchers design models of complex structures for pre-test analysis with substructuring. Substructuring uses either an analytical model such as an FE model or a modal model that consists of measured TRFs. Each of these has inherent strengths and weaknesses [17]. A FE model requires a high degree of computational power and storage, in addition to being difficult to accurately represent the setup [18]. Researchers have investigated methods to reduce and simplify models to be more computationally accessible while maintaining a reasonable degree of accuracy, but the problem is still unsolved [19-21]. Likewise, using an experiment based modal model requires sensors to measure the outputs at every relevant location. The number of sensors can be limited, and adding sensors changes the dynamic properties, often requiring an analytical model to remove their effects [22-24]. In summary, substructuring is a useful tool, with its own limitations as well.

This paper provides and tests a novel method that combines the model of a simple structure and a SISO experiment with the complex structure to estimate the TRFs of output locations not found in the test. In this paper the researchers used a cantilever beam with a stinger attached as a partial transverse restraint to be the complex structure. The researchers modified a model of the simple cantilever structure with a SISO test to yield a conversion expression, and then applied it to other locations in the model of the simple structure. This provided the estimated TRFs for any output. The researchers calculated this estimate and experimentally measured the TRF at the tip of the beam to compare the two and validate the estimate. With this, researchers estimated the dynamic effects of an added uncertain boundary condition. This method requires less foreknowledge than building a Finite Element Model for the entire system and is easier to implement than directly measuring the TRFs for every possible output.

EXPERIMENT OBJECTIVE AND RESEARCH METHODOLOGY

The objective of this research is to experimentally estimate the dynamic properties of a complex structure with knowledge of the simple system, only one SISO experiment, and no information of the complex boundary condition. The following sections describe the complex structure selected for this experiment, the simple system model, and the research methodology.

Complex Structure

The research team tested a steel cantilever beam which clamped to a stiff frame. Sensors measuring the output acceleration attached at the tip and the 1/3rd mark respectively. A stinger, representing a partial transverse restraint, connected at the 1/3rd location as well. This component rigidly attached to a strong wall and constituted the uncertain boundary condition/component attached to the simple structure, turning it into a complex structure. This attachment changed the dynamic properties of the system significantly, rendering a model of the simple structure obsolete by itself. Figure 1 displays this setup.

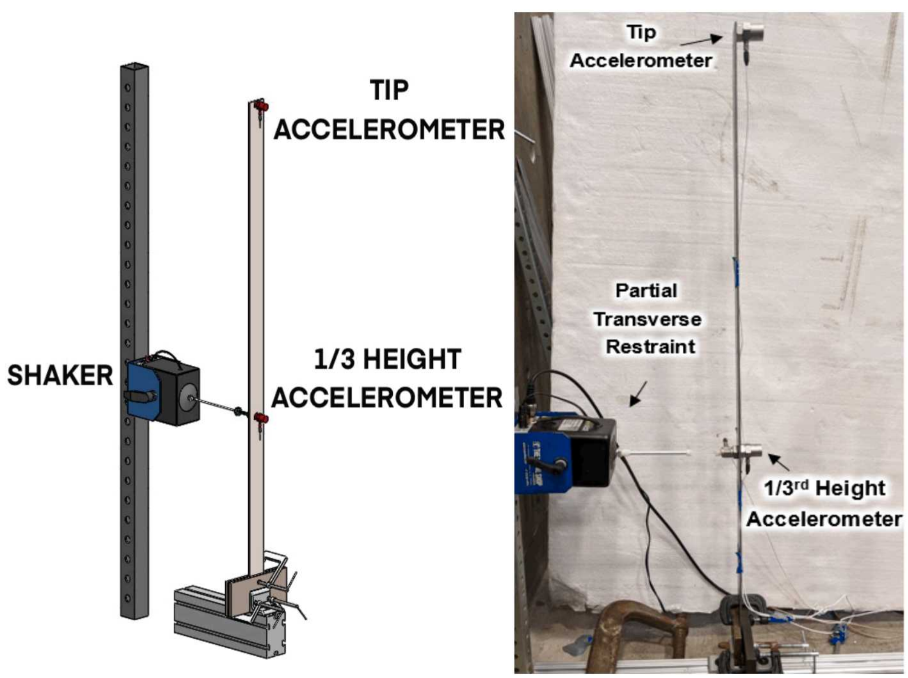


Fig. 1: Complex Structure Setup

Dynamic Model of Simple System

The researchers created a Finite Element (FE) model of the simple structure without a stinger attached using the same geometric and material properties as the experimental beam. An element with a hundred nodes represents the beam. The researchers calculated the TRFs of the FE model of the simple structure with the following expression:

$$H(\omega) = \frac{\overline{X(\omega) \cdot Y(\omega)^*}}{\overline{X(\omega) \cdot X(\omega)^*}} \tag{1}$$

where the H is a Transfer Function, ω is each frequency line, X is the input force, Y is the output response, * is the complex conjugate, and $^-$ is the average over multiple iterations.

The researchers conducted a test without the restraint attached as a pure cantilever beam to update the FE model of the simple structure. They struck the structure with an impact hammer at the 1/3rd location. Each test consisted of three independent strikes to the beam with a sampling rate of 8192 Hz. The recorded data channels consisted of the impact hammer and the accelerometer at the 1/3rd height (same location). It is worth mentioning that this experiment was conducted prior to attaching the stinger to the simple beam.

Table 1 shows the comparison of the simple cantilever structure between the FE model and the experiment. The two accelerometers were attached at the tip and 1/3rd height, respectively, since the simple model needs to represent a portion of the complex model. The errors of the first four modes of the structure are all under 2%, except for the fourth natural frequency which is 2.71%.

Table 1. Natural Frequency Comparison (Hz)

	Matlab Finite Element Model	Experiment	Difference	% Error
f_1	2.93	2.90	0.03	1.03
f_2	18.13	17.93	0.2	1.12
f_3	50.08	49.87	0.21	0.42
f_4	106.20	103.4	2.8	2.71

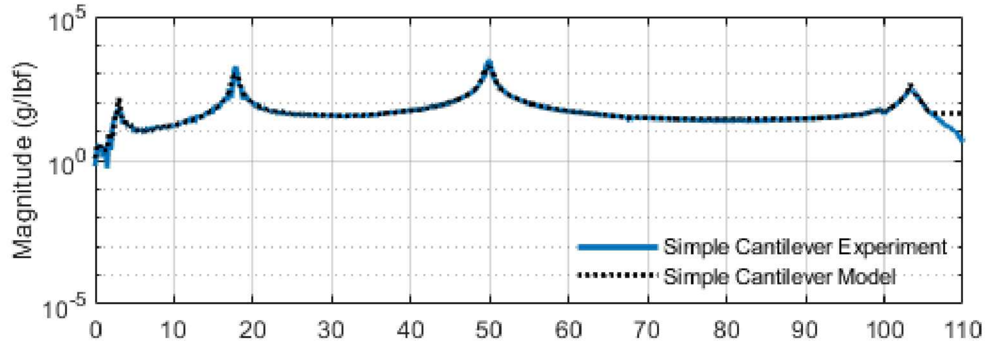
FE Model Update

The researchers updated the TRF from the simple model between 0 and 110 Hz applying Dynamic Time Warping to ensure peak and pole alignment between the model and the experiment for the first four modes. In order to evaluate the simple structure model, researchers estimated the error between the estimated and the experimentally measured TRFs of the simple structure. In this paper, researchers quantified the difference between two TRFS by using the decibel error [25];

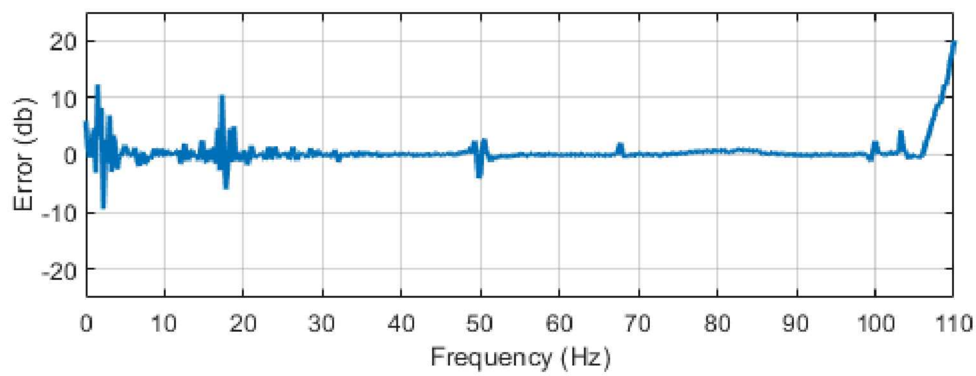
$$Error = 20\log_{10}\left(\frac{TRF\ Estimate}{TRF\ Standard}\right) \quad (2)$$

where Error is the error metric for a given frequency line, the TRF Estimate is the estimate for that line, and TRF Standard is that of the ground truth. Across a range of frequencies, the error is the average absolute decibel error of those frequencies.

Figure 2 compares the TRF from the FE model of the simple structure and the experiment with the simple structure at 1/3rd of the height, with respect to an input force at the same location. Figure 2(a) shows the TRF amplitudes of both the model and the measured experiment. Figure 2(b) shows their decibel error. The largest errors in the TRF of the model of the simple structure are in the regions near the peaks and poles, with a maximum error of 3.3 dB outside these regions. The average error between the TRFs of the dynamic model and the experiment is 0.85 dB between 0 and 110 Hz. The researchers prioritized maintaining the fidelity of the peaks in the model of the simple structure and chose this model for the derivation of the complex experiment estimation.



(a)



(b)

Fig. 2: Model Vs. Experimentally Measured TRFs at 1/3 Height (a) Magnitude (b) Decibel Error

The researchers used this model to estimate the dynamic properties of the complex structure by combining it with the results of one SISO experiment, as described in the next section.

Research Methodology

The objective of this research is to develop a method that combines the model of a simple structure with the results of a SISO complex structure test to estimate the TRFs of locations not measured in the test. This offers a tool to use in designing experiments which eliminates the need to create a full model with the complex structure and is more versatile than experimentally creating a modal model. Figure 3 summarizes the proposed methodology.

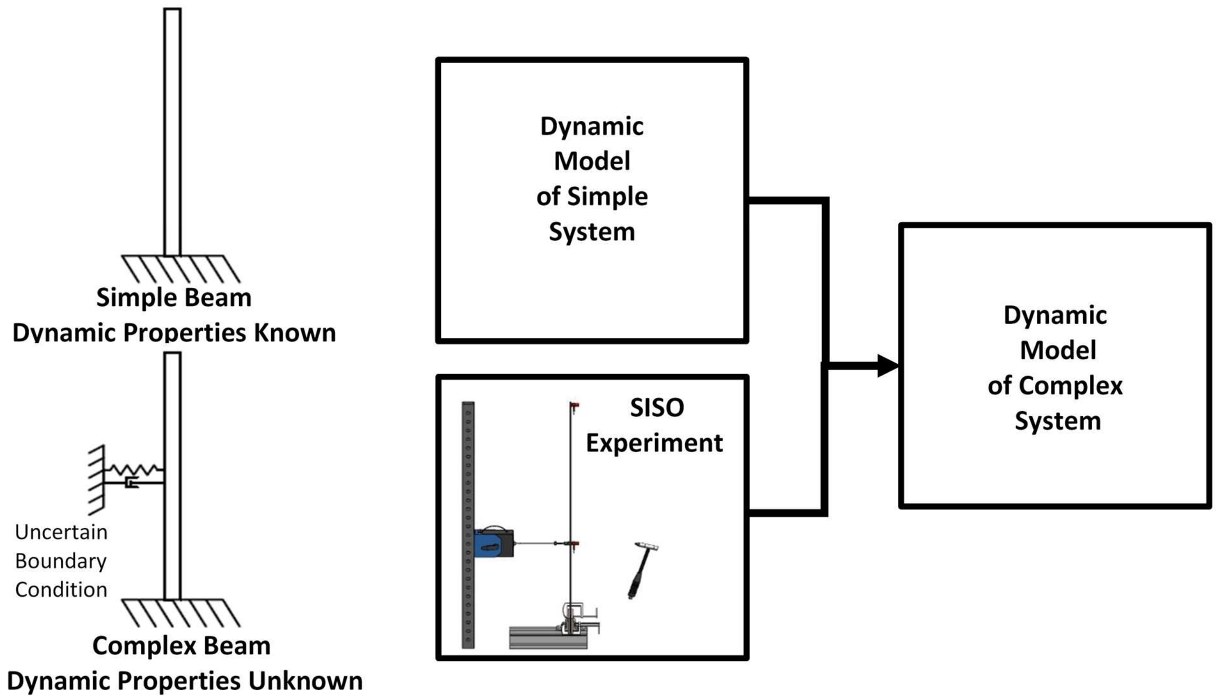


Fig. 3: Combining the model of a simple structure with a SISO Experiment to estimate complex structure TRFs

This method assumes that the system is fully linear; the restraint exerts only an axial force with no rotational or translational effect; and that the dynamic model of the model of the simple structure is an accurate representation of the system.

TRANSFORMATION DERIVATION

Boundary Condition

In the example this paper presents, the researchers hit the beam with the hammer at the same location that the restraint was connected to the structure, and collected the response on that same location with an accelerometer. The researchers assumed that if the stinger is connected to a given location, then there is access to that location. The possibility of limited experimental access is one of the motivations for developing this method as a first step to enable a new experimental technique. In a more complex example where the input is not at the same location, more terms are involved but the derivation process remains the same.

In the experiment the researchers used a load cell between the stinger and the beam to measure the force the inactive stinger applies. The methodology makes assumptions about the relationship between the force of the stinger and the input force, and the researchers used this data channel to validate those assumptions. The presence of the load cell is omitted from descriptions of the experiment in the rest of the paper since the details of this validation step are not critical and the method is developed for the case where there is no load cell in the tip of the stinger.

For this derivation, the researchers assumed linear superposition and considered the stinger to be a dependent input to the structure. With this assumption, a TRF of the simple structure with respect to the input force is identical to the TRF of the complex structure with respect to the combined input force from the hammer and reactionary force applied by the stinger. Assuming the use of a model representative of the simple structure, the expression for the output of the complex structure within the frequency domain is then:

$$A = H_{Mod} * (F_{Ham} + F_{LC}) \tag{3}$$

where A is the output acceleration, H_{Mod} is the TRF from the model of the simple cantilever structure for a single arbitrary location, F_{Ham} is the applied input force by the hammer, and F_{LC} is the reactionary force of the restraint.

H_{Mod} is distributed between the two forces which constitute the total input.

$$A = H_{Mod} * (F_{LC}) + H_{Mod} * (F_{Ham}) \tag{4}$$

This treats the applied input and the reactionary force from the restraint as two separate inputs with their own relationships to the output. The hammer is the only independent input and the reactionary force applied by the stinger consists of a function of the hammer's input and the system properties. H_{Sting} relates the output force of the restraint to the input force applied by the impact hammer, similar to a traditional TRF:

$$F_{LC} = H_{Sting} * F_{Ham} \tag{5}$$

where H_{Sting} is the experimental TRF of the reactionary force of the stinger to the input force of the impact hammer.

Substituting this term into equation (3);

$$A = H_{Mod} * F_{Ham} + H_{Mod} * H_{Sting} * F_{Ham} \tag{6}$$

Application

Let H_{Test} denote the TRF of the complex structure which relates the output acceleration to applied input force. Dividing both sides of Equation 6 by F_{Ham} leaves the definition of H_{Test} on the left. Reorganize the right side of the equation to obtain:

$$H_{Mod} = \frac{H_{Test}}{1 + H_{Sting}} \tag{7}$$

This estimates the simple structure TRF based only off on TRFs from the test with the complex structure. Moreover, the inverting the expression obtains the TRF of the complex structure with respect to the applied force based upon a model of the simple structure. Rearranging equation 7:

$$H_{Test} = H_{Mod} * \left(1 + H_{Sting}\right) \tag{8}$$

The conversion function developed is independent of the output with no related data channels or TRFs. This indicates that the factor is the same irrespective of the output. As a result, a straightforward test obtains the conversion function without considering the load cell measuring the force applied by the extra connection. This is because the conversion function is equivalent to the ratio between the two TRFs of interest for a given input and output, since this ratio is the same for every Degree of Freedom (DOF). For converting a model of a simple structure to represent a more complex structure then, the conversion expression is:

$$Conversion\ Expression = \left(1 + H_{Sting}\right) = \frac{H_{Test}}{H_{Mod}} \tag{9}$$

H_{Test} represents a single point. This conversion expression yields an estimate for the TRF of any other point, as according to the equation

$$H_X = H_{ModX} * \frac{H_{Test}}{H_{Mod}} \tag{10}$$

where H_X is the estimated TRF of a point not measured in the complex experiment and H_{ModX} is the TRF for that point from the simple model. This conversion expression relates the TRF from a model of the simple structure for any output to the TRF of the complex structure for the same location. With this, researchers can estimate any output of a complex structure using the model of the simple structure and only one SISO experiment. The following section describes the experiment which the researchers conducted to validate this method and examine its accuracy and limitations.

EXPERIMENT AND RESULTS

Researchers performed an experiment to obtain data for the estimation and validation of the estimate, then used this data to follow the Methodology and estimate a TRF for the complex structure. The researchers used the TRF calculated for the same location in the experiment to validate this estimate.

SISO Experiment

Figure 4 shows the experiment. The researchers conducted three tests. Each of the experiments consisted of one strike to the beam with a hammer at the 1/3rd location, allotting 15 seconds per strike to record the full vibration. Researchers used a sampling rate of 8192 Hz for all channels in the experiment. The data channels were: the impact hammer, the accelerometer at the 1/3rd height to estimate the tip TRF, and the accelerometer at the tip used to validate the estimation.

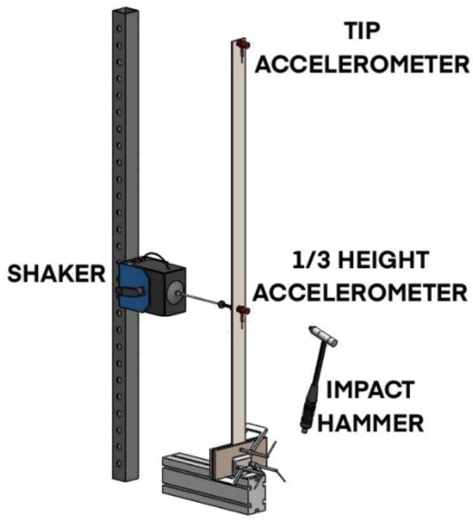


Fig. 4: Experiment with Complex Structure

Complex Structure Measured TRF

Researchers averaged the three impact events together to calculate the TRFs using the data from the tip accelerometer and used it as the TRF ground truth to validate the new method. The contribution of the new method is that in experimental implementation the research team does not need to install an accelerometer at every location where the TRF is of interest. They can instead estimate the TRFs of these locations with only one SISO test in combination with the model of the simple beam.

Complex Structure TRF Estimation

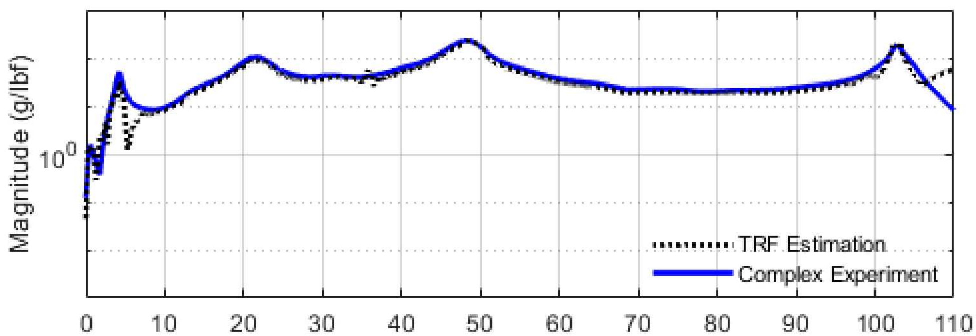
The researchers obtained the conversion expression by calculating the ratio of the TRF from the model of the simple structure at 1/3rd the height with the TRF from the complex structure using the data from the accelerometer for the same location. This constitutes the conversion factor applied to the model of the simple structure. By using this conversion factor with a model of a simple structure, researchers can estimate the TRF of the complex structure at any output location. For simplicity, researchers validated the TRF estimation at the tip of the complex structure where they added one accelerometer. It is worth mentioning that this TRF has been estimated without using the data from the accelerometer at the tip of the beam.

Results and Analysis

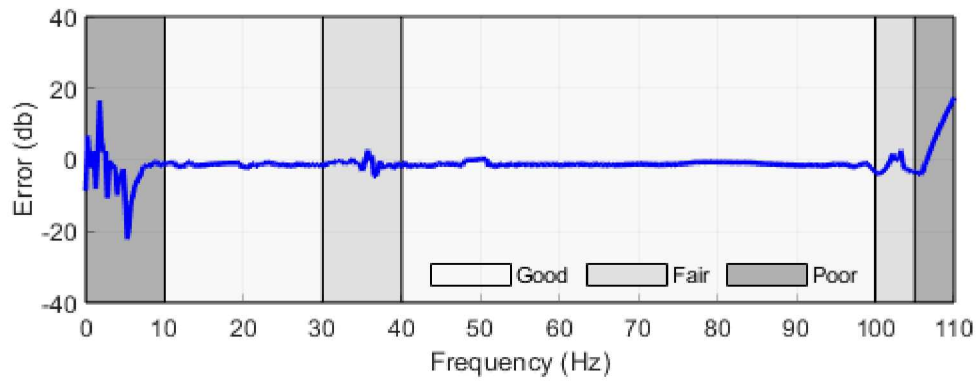
The researchers compared the TRF estimation of the complex structure with the experimentally calculated TRF from the tip accelerometer. This accelerometer was present only for the validation of the estimation and is not necessary for the method.

Researchers divided the TRF error into discrete frequency regions to better analyze the parts of the TRFs and assigned a rating to each region. The regions were all multiple of 5 Hz in and if two regions had the same rating they were combined. Each region was rated based on the mean and maximum errors simultaneously within certain dB error ranges, assigning the lower rating of the two if they disagreed. Good rating corresponds to mean and maximum errors below 2 dB and 4 dB, respectively. Fair rating corresponds to mean and maximum errors ranging between 2 to 4 dB and 4 to 8 dB, respectively. Poor rating corresponds to mean or maximum errors above 4 dB and 8 dB, respectively. The researchers analyzed the estimate TRF first by performance over these regions, and then by the performance at natural frequencies.

Figure 5 shows the results of comparing both the estimated and experimentally measured TRFs for the beam response at its tip with respect to an input force at 1/3rd height of the structure. Figure 5(a) plots the estimate and measured TRFs together, while Figure 5(b) shows the decibel error between them shading each region according to their rating.



(a)



(b)

Fig. 5: Estimated Vs. Experimentally Measured TRFs at the tip (a) Magnitude (b) Decibel Error

In general, the TRF estimation tends to slightly underestimate the experimental TRF of the the complex structure. Table 2 shows the mean and maximum errors for each region.

Table 2: Comparison of Estimated and Measured TRFs Frequency Ranges

Frequency (Hz)	0-10	10-30	30-40	40-100	100-105	105-110
Mean Error (dB)	5.08	1.36	1.45	1.27	2.29	7.43
Max Error (dB)	22.26	2.52	4.65	3.41	4.08	17.28
Interpretation	Poor	Good	Fair	Good	Fair	Poor

The dB error between the 10-30 Hz range and 40-100 Hz has a rating of good. This includes natural frequencies 2 and 3, and it is worth noticing that the error of natural frequencies 1 and 4 is also low but their vicinities have either fair or poor estimations. For the frequency regions 10-30

Hz and 40-100 Hz, it is also important to note that the mean error has a similar magnitude to the maximum error, indicating that the error is stable within these regions. The 0-10 Hz region rates poor for TRF estimation, with the largest error of 22.26 dB at 5.25 Hz. This error occurs immediately after the 1st mode located at 4.25 Hz and misrepresents the estimated TRF. The 30-40 Hz region rates fair, with a maximum error of 4.65 dB at 36.75 Hz. This is the location of a pole in the simple system TRF for the 1/3rd height point. In this location the discrepancy between the real system and the model propagates to the TRF estimate of the complex structure. the 105-110 Hz region rates poor with a maximum error of 17.28 dB at the last point of the TRF (110 Hz). Finally, for the entire range of 0-110 Hz, the TRF estimate mean and maximum errors are 1.99 dB and 22.26 dB, respectively.

Table 3 shows the natural frequencies of the estimated and measured TRFs respectively, describing the frequency, damping, and magnitudes for each mode.

Table 3: Comparison of Estimated and Measured TRF Natural Frequencies

	Mode 1			**Mode 2**			**Mode 3**			**Mode 4**		
	Exp	Est	Error	Exp	Est	Error	Exp	Est	Error	Exp	Est	Error
Frequency (Hz)	4.25	4.25	0%	21.5	21.5	0%	48.25	48.25	0%	102.75	102.5	0.24%
Damping (%)	6.79	12.98	91.96%	1.86	2.1	12.9%	0.87	1.03	18.39%	0.37	0.21	43.24%
Magnitude (g/lbf)	52.49	30.95	4.42 dB	105.2	85.41	1.81 dB	231.7	226.1	0.21 dB	186.8	190.8	0.18 dB

The natural frequencies from the estimated TRF match those of the measured TRF, with the exception of the 4th mode with a 0.25 Hz discrepancy. The damping of the estimated TRF for the 2nd and 3rd modes is good, at 12.9% and 18.39% error respectively. The damping of the 1st and 4th modes match poorly, at 91.96% error and 43.24% error respectively. For the first three modes the TRF estimates a higher damping, except for in the 4th mode which is lower. The damping in the

1st mode has error in the model of the simple structure as well, with the model underestimating the damping of the system. The error of the damping in the 4th mode could be due to high frequency non-linearities. The magnitudes of the modes of the estimated TRF generally match those of the measured TRF. The 2nd, 3rd, and 4th natural frequencies have a rating of Good, with errors of 1.81, 0.21, and 0.18 dB respectively. The 1st mode only has a rating of fair at 4.42 dB. The peaks are the most important part of a TRF and represent the highest priority when addressing errors.

The large errors in the TRF estimation in the first and forth mode coincide with the large errors in the model of the simple structure. This manifests most as the error within 0-10 Hz and 105-110 Hz which is shared by both systems. A shared error occurs at 36.75 Hz as well, where the simple structure TRF also has a pole.

One possible source of error can be the transverse and rotational forces the stinger applies to the beam. The derivation assumes these forces to be negligible, which is acceptable if the interaction between the force and the stinger is in fact small in comparison with the input forces applied to the structure. If these forces are not negligible, then their impedance could add rigidity to the system, resulting in errors in the modes.

These results constitute the first step to enable researchers to efficiently estimate the unmeasured TRF for any output of a complex structure using only the model of a simple structure and a SISO experiment. In the future, researchers plan to conduct experiments to better inform the model of the simple structure that will minimize the errors in this experimental technique. Additionally, researchers will conduct a parametric study that will quantify the effect of transverse and rotational forces of the stinger in the derivation in relation to the input force. The quantification of these errors will contribute to further develop the proposed method and increase its significance to enable higher reliability of the estimation for experimental settings.

CONCLUSIONS

Researchers measured the error between the estimated and the measured TRFs, and the method proves to be accurate for most of the natural frequencies within the frequency range the researchers examined. Based on these results, the researchers conclude that the method helps efficiently estimate the unmeasured outputs for a cantilever beam with a partial transverse restraint. While more accurate methods of determining TRFs for test-planning exist, this method presents advantages in certain settings. Typically, methods to perform a pretest analysis with a complex structure involve either a FE model representing the entire system, or the generation of a modal model with an experimental process. As compared to using a pure FE model, the proposed method does not require knowledge of the boundary condition's dynamic properties. Additionally, this method also has advantages over experimentally making a modal model, which necessitates the application of a sensor at every relevant output. This can be expensive, and by its nature imposes a change on the dynamic system by the addition of the sensor mass. However, the method presented only requires one measured output while still estimating the TRF of any other output, offering the versatility of a full model. As a result, the method presented is useful under certain circumstances. It is valid when a system behaves linearly, when there is sufficient information to create a model of the simple structure, when only a rough and quick estimate of the TRFs are needed beforehand, and when there are limited input locations to be used. This project estimates the unmeasured TRF of a complex structure with an uncertain boundary condition by transforming a model of a simple structure with data obtained from one SISO experiment with the complex structure. However, the same process can apply to more complex situations as well, with more boundary conditions or other input locations.

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