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Non-Diffusive Volume Advection with A High Order Interface Reconstruction Method

J. W. Liu, J. Yao

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NON-DIFFUSIVE VOLUME ADVECTION WITH A HIGH ORDER INTERFACE RECONSTRUCTION METHOD

JERRY LIU[1, 2], JIN YAO[1]

[1]Lawrence Livermore National Laboratory
7000 East Avenue, Livermore, CA 94551, USA

[2]Duke University
120 Science Dr., Durham, NC 27708, USA

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Abstract.
We show that non-diffusive volume advection in two-dimensions is achieved with several benchmark problems using a newly developed high-order volume of fluids (VOF) interface reconstruction method.

1. A new VOF interface reconstruction method using circular/corner facets (linear facets are a degenerate case of arcs). We create a circular interface facet in each mixed zone by matching neighbor volume with a hybrid Newton's-bisection method and the local solution is final. In the general case, the new VOF interface reconstruction has 3rd order accuracy and can be easily made seamless. The new method addresses intrinsic issues with Young's method such as gaps between interface facets in the case of a curved interface, and inability to define curvature nor identify corners.

2. A non-diffusive volume advection scheme. In an ALE advection step, a well-defined interface can be carried over through a Lagrange step and used to compute volume distribution into a relaxed mesh. Then, an interface reconstruction step is performed to redefine the interface in the relaxed mesh. We must point out that the interface carried over is also a solution of interface reconstruction because all the volume fractions in the relaxed mesh are naturally matched. We provide an interface tracking method compatible with our reconstruction scheme, where it is granted to use the prior info as an initial guess to capture sub-mesh resolution features. As a result, we are able to treat multiple facets inside a single mixed cell and obtain highly accurate, non-diffusive solution for advection problems with rather coarse meshes. We show our solutions for two-dimensional incompressible flows with two materials with a) the X + O diagonal translation; b) the Zalesak rotational test; and c) the single vortex spiral test.

1 INTRODUCTION

Fixed-grid methods for simulating interfacial flows include interface tracking and interface capturing methods, which use explicit and implicit representations of the fluid interface respectively. Interface tracking methods have been used to achieve highly accurate representations of interface, including features finer than mesh resolution, but these methods can struggle with volume conservation. Furthermore, in the case of topology changes, additional explicit methods are often required. A key interface capturing method is volume of fluid (VOF), which maintains volume fraction info for each mesh cell. The VOF method enforces volume conservation and naturally handles topology changes but requires an interface reconstruction step.

Piecewise linear interface calculation (PLIC) methods are commonly used for interface reconstruction within VOF methods. An interface normal is obtained for each mixed cell based on an estimate of the gradient of the volume fraction function, and a piecewise linear interface is formed by matching volume fractions. In general, PLIC methods achieve only second-order accuracy given sufficient mesh resolution. Additionally, piecewise linear interfaces are unable to directly represent curvature or sharp cusps within a mesh cell, typically resulting in large discontinuities in the interface. With coarser meshes or flows with heavy deformation, PLIC methods often result in unphysical topology changes (“flotsam” and “jetsam”).

We implement a new high-order interface reconstruction method using a piecewise circular representation of the interface. Circles are chosen for simple analytical expressions of points and areas of intersection between interface facets and mesh cells. Instead of relying on an estimate of interface slope, we use an iterative method to find the unique circular arc that matches three neighboring volume fractions exactly. To represent cusps, we extend two neighboring facets to intersect at a corner within a single mesh cell and match volume fractions using an iterative optimization method.

Volume advection is handled using geometric intersection algorithms. Facets are carried over via a Lagrangian step and updated flux polygons are intersected with neighboring relaxed mesh cells to compute new volume fractions. When the interface has features finer than mesh resolution, we employ an interface tracking algorithm using interface info from the previous time step carried over to detect when cells require multiple disconnected facets. We can compute volume fractions accurately in these cases and we can detect topology changes caused by intersecting facets, so the interface tracking algorithm is compatible with interface reconstruction.

Our approach is entirely local and accurately represents curvature and cusps of the interface. We show gaps between neighboring facets are reduced and C0 continuity is achieved with a simple corrective scheme. Our proposed interface reconstruction method has 3^{rd} order spatial accuracy and perfectly reconstructs interfaces consisting of piecewise linear or circular components (including all 2D polygons) given sufficient mesh resolution. We demonstrate non-diffusive volume advection in several benchmark advection problems. Although this work focuses on two-material flows in 2D, our ideas can be generalized to multi-material or 3D cases.

2 HIGH-ORDER INTERFACE RECONSTRUCTION SCHEME (WITHOUT PRIOR INFO)

2.1 Assumptions on true interface and local mesh resolution

We assume the true interface is C0 and piecewise C2 continuous, possibly with cusps.

Typically, the mesh resolution is fine enough that the true interface lies only within a fraction of the cells. In addition, the intersection of the intersect with most mixed cells should consist of only a single connected component, and most mixed cells should have exactly two mixed neighbors. We call such mixed cells *regular*.

For interface reconstruction without prior info, we assume the input mesh is sufficiently fine. In C2 neighborhoods of the interface, this means that mixed cells must be regular (a sufficient condition is for the radius of curvature to be at least 3 times the local zone size), although we also discuss methods to address cases when this assumption does not hold. In neighborhoods around cusps, mixed cells are likely not regular no matter the mesh resolution. Instead, we assume that any cusps, if present, are spaced apart enough such that at least one regular mixed cell lies between any pair of cusps. This gives us sufficient info to determine when a cusp exists and to accurately reconstruct it.

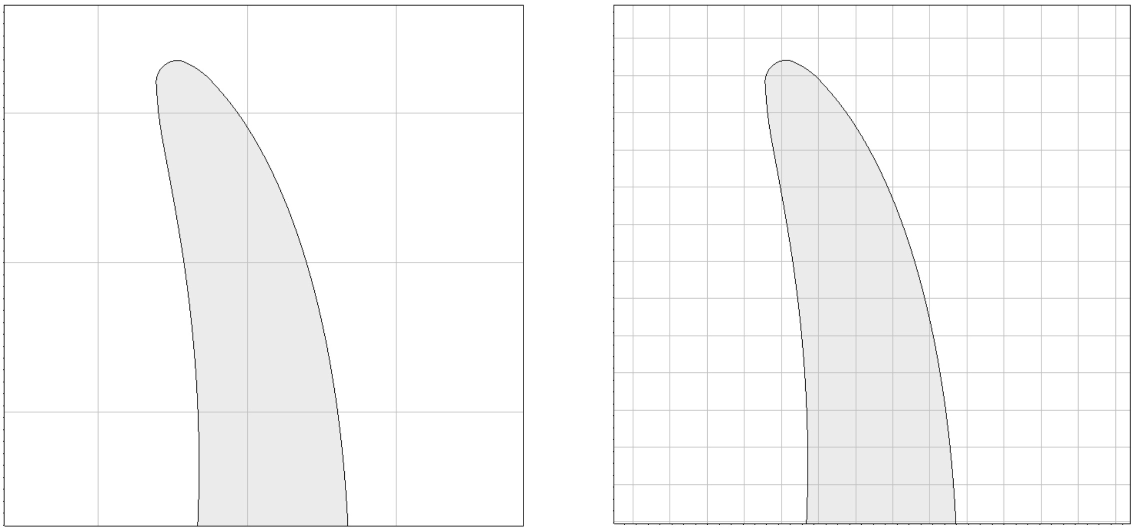

Figure 1: Left: an insufficient resolution, mixed cells are not regular. Right: a sufficient resolution.

We note that in typical cases these conditions are fulfilled for most mixed cells, and that violations only occur within small neighborhoods. We rely on our front tracking method, described later, in cases of extreme sub-mesh level interface features.

2.2 Orientation of interface facets

A *facet* is the solution of interface reconstruction within a single mixed cell. For regular mixed cells, assuming a single interface component, we note that we can determine the orientation of the desired facet only from the volume fractions of the immediate neighboring cells. The reconstructed interface must travel through the two edges of the regular cell shared with its two mixed neighbors. The other edges must correspond to neighbors that are either full or empty. We choose the orientation of the facet such that the interiors of both fluids are consistently oriented (moving along the reconstructed interface yields a walk around the shared boundary).

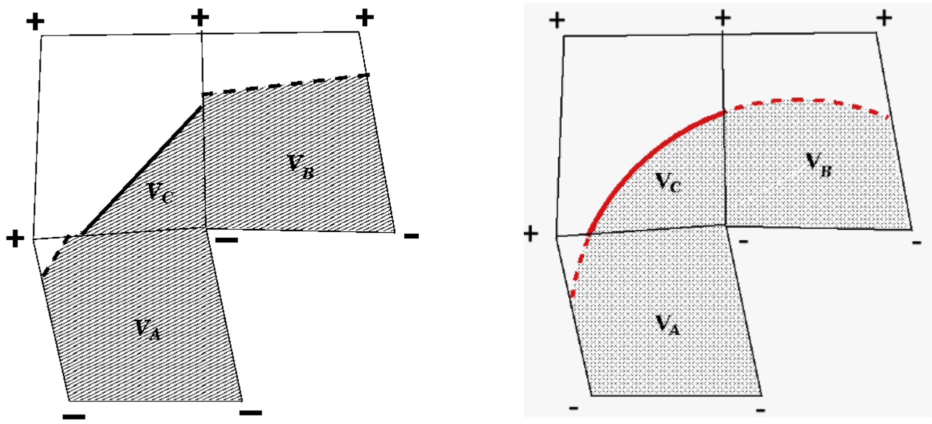

Figure 2: Regular mixed cell C and its neighbors A and B. Walk from B to C to A is counterclockwise from interior of dark fluid, and clockwise from interior of light fluid. V_A, V_B, and V_C are the corresponding partial volumes. Left: linear facets; right: circular facets.

2.2.1 Treatment of non-regular mixed cells by merging

We note that our assumptions on the mesh resolution may not be entirely satisfied. There are cases in which non-regular mixed cells appear in a small neighborhood, even when mesh resolution is generally sufficient, which still grant enough info to reconstruct a unique facet. We resolve the issue by merging ambiguous neighborhoods and treating them as single mixed cells. Because merged zones are now regular and because our method does not require any assumptions on mesh structure, we may run our interface reconstruction algorithm normally.

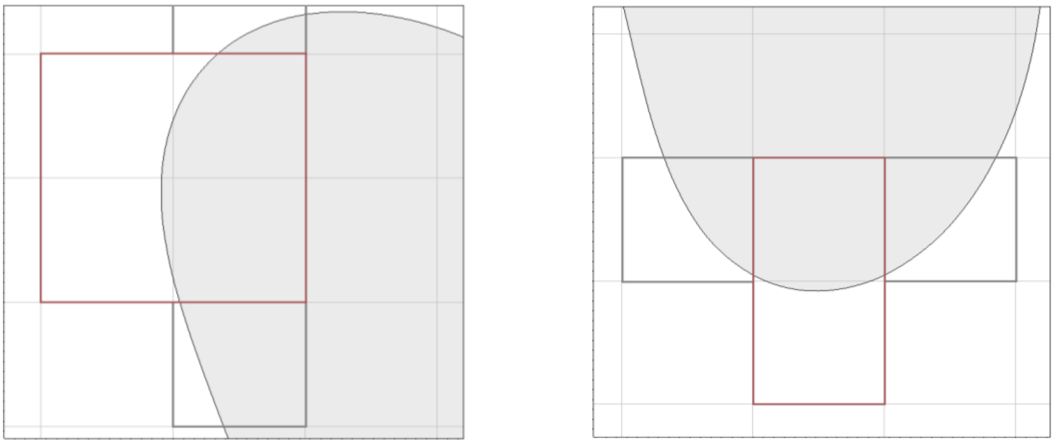

Figure 3: Ambiguous regions become regular mixed zones when merged (outlined in red).

Note that although merging makes local mesh resolution coarser, uniqueness of our facet fitting algorithm provides more consistent results than attempting to guess the appropriate neighbors. Additionally, assuming the true interface curvature does not change too drastically, a circular facet is still accurate enough to capture the curvature of ambiguous regions.

2.3 Uniqueness of reconstructed interface and detection of cusps

After orientations are determined for each regular mixed cell, three local neighbors z_A, z_B, z_C and their volume fractions v_A, v_B, v_C are identified and ordered with a walk that keeps fluid interiors consistent. Because a circle has three degrees of freedom in 2D (center coordinates and curvature), given three polygons, three volume fractions, and an orientation defining the direction of the interior, there exists in general a unique circle defined by these conditions. (Note that a line is a degenerate case of a circle.)

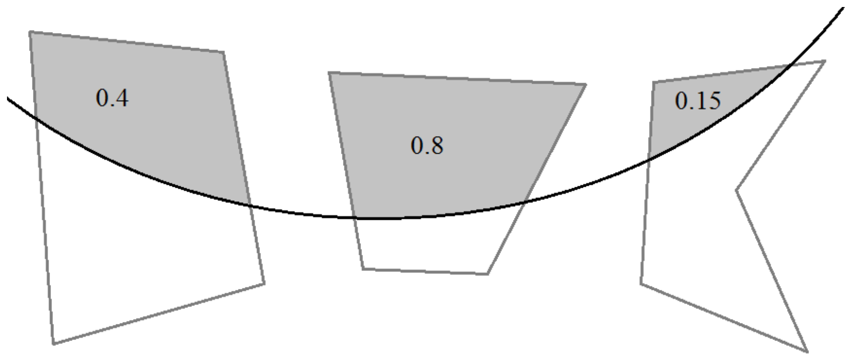

Figure 4: Three partial volume fractions uniquely determine a circle with three degrees of freedom.

In the presence of cusps, the unique circle defined above will differ significantly from the true interface. Indeed, if a cusp is present, the unique circular facets in adjacent mixed cells defined by their volume fractions must exhibit large changes in both predicted interface normals and curvatures. Additionally, given sufficient mesh resolution, we know the edges of a corner will have similar normals and curvatures to nearby regular non-cusp facets. Thus, we can determine whether a cusp exists by checking for large changes in interface normals and curvatures across adjacent facet endpoints. We define a corner facet as the intersection of the closest two neighboring non-cusp facets and confirm whether the corner facet matches the volume fractions accurately.

2.4 Facet fitting

Our interface reconstruction scheme attempts to fit facets from low order to high order: linear, corners with only linear edges, circular, and corners with curved edges. In this way we ensure our reconstructed interface matches the order of the true interface in all local regions.

Linear facets:
Rather than relying on an estimate of the interface normal, we find the line matching volume fractions v_A and v_B. Note that since a line has two degrees of freedom, two polygons and their volume fractions define a unique linear facet. A Newton's method is applied to obtain the solution. If v_C happens to also be matched by this linear solution, the interface must be linear in this region and a linear facet in z_C is obtained.

Corners with linear edges:
Each neighborhood of unpredicted mixed cells may signal the presence of a corner with linear edges. Corners are formed by extending the nearest linear facets until they intersect at a point. If a proposed corner facet matches the unpredicted volume fractions sufficiently, Newton's method is applied on the coordinates of the corner point to optimize the match on volume fractions.

By this stage of the algorithm, we have fully reconstructed all planar polygons.

Circular facets:
Circular facets are now fit to all remaining unpredicted mixed cells. The linear fit from the linear facet stage is used to set the initial guess of a circular facet fit. A hybrid Newton's-bisection method is used to find the unique curvature and coordinates of the circle's center. A near quadratic convergence is consistently observed.

Corners with curved edges:
The final stage checks for the presence of curved-edge corners. Adjacent facets that exhibit large change in interface normals or curvatures are either resulting from unrepresented cusps or from mesh coarseness. Curved-edge corners are formed by extending the nearest non-corner facets until they intersect at a point. Note that by our assumption on mesh resolution, there always exists at least one proper linear or circular facet between any two cusps. As with straight-edged corners, if a proposed corner facet matches the unpredicted volume fractions sufficiently, Newton's method is applied on the coordinates of the corner point to optimize the match on volume fractions.

2.5 C0 continuity enforcement

Because our proposed interface reconstruction achieves 3rd order spatial accuracy given a smooth interface, gaps between adjacent facets are small. Thus, a simple corrective scheme is sufficient to achieve C0 continuity while preserving accurate interface curvature. Given two neighboring facets, the rightmost endpoint of the left facet and the leftmost endpoint of the

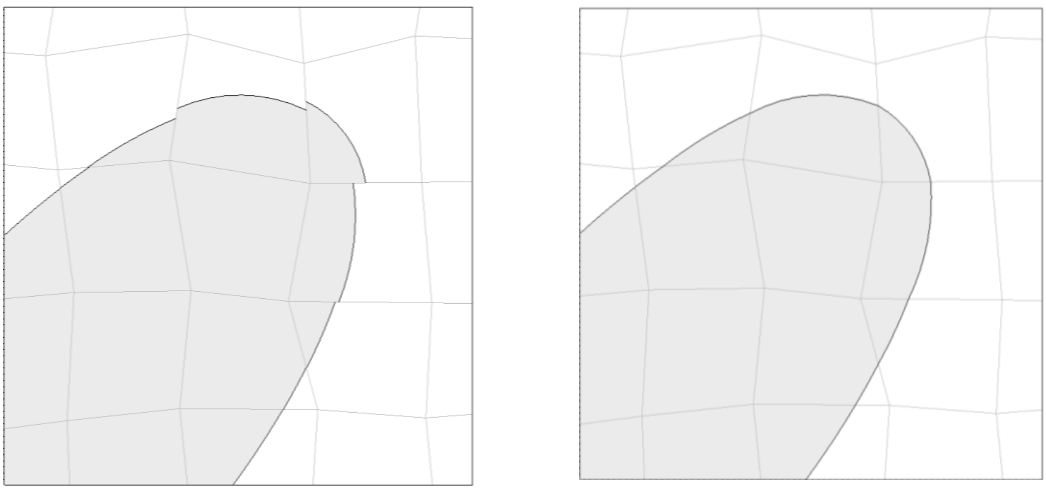

Figure 5: Left: reconstructed interface without C0 enforcement. Right: after C0 correction.

right facet are reset to the midpoint of the facet gap. Curvature of each mixed cell is again optimized to match volume fractions. Note that because the linear/circular facet fitting algorithms determine endpoint locations by checking neighboring cells, facet gaps are guaranteed to lie on the same edge. This operation provides a seamless interface which stays 3rd order accurate in space. Our entire interface reconstruction method is local and final.

3 EQUIVALENCE OF RECONSTRUCTION AND DIRECT TRACKING

3.1 Updating volume fractions

A linear/circular facet is characterized by three control points: the two endpoints and its midpoint. Each of these points is associated with a pair of logical parameters that map a quadrilateral zone to a logical square. In the Lagrange phase of an ALE advection, a circular facet is carried over with the logical parameters of the control points. The orientation is set by the order of these points. A seamless interface stays seamless after a Lagrange step. The volume fluxes distributed into the relaxed mesh are computed by finding geometric intersections. Updated interface facets are intersected with the polygonal regions obtained from intersecting the advected and relaxed meshes. Volume fractions for the relaxed mesh are totaled and employed for interface reconstruction at the next time step.

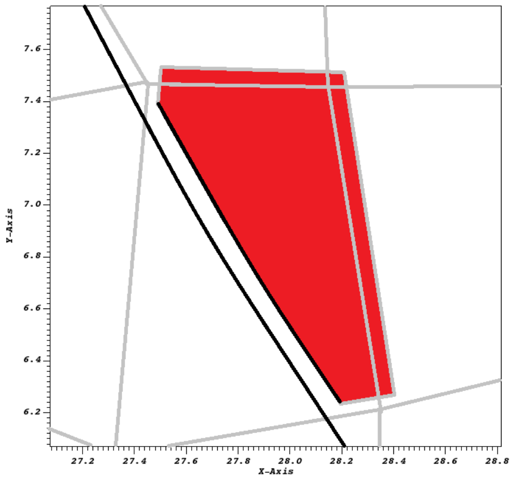

Figure 6: Updating volume fractions. Advected volume is in red. Volume fluxes into relaxed mesh is computed by taking intersection of advected region with relaxed mesh cells.

3.2 Interface tracking as a solution to reconstruction problem

We point out that a seamless interface directly carried through the Lagrange motion is by nature a solution of interface reconstruction on the relaxed mesh, because all the volume fractions in relaxed zones are naturally matched. In this sense, interface tracking, and its VOF (volume of fluids) reconstruction are *equivalent*. The only difference is the interface carried over may not be the converged solution of the interface reconstruction method of choice. Nevertheless, the information carried over can be taken as granted and used as an initial guess to accelerate the convergence of our reconstruction scheme.

This is particularly helpful with complex interface geometries or coarse meshes. In cases when multiple disconnected sections of the interface lie in a single mesh cell, attempting to represent the interface with a single facet induces topology changes where none should exist (creating “flotsam” and “jetsam”). We propose to combine a direct interface tracking with our reconstruction method to identify complex fluid structures.

3.3 Interface tracking and compatibility with VOF

Our interface tracking scheme uses the same Lagrange step formulation as in 3.1, using three control points. Instead of computing volume fluxes, explicit intersection points of updated facets with the relaxed mesh are stored. Using this info, we can detect when the updated interface intersects a mixed cell with multiple disconnected components.

Within a mixed cell, each component is remerged, preserving volume fractions of updated facets and thus ensuring local mass conservation. Interface normals and curvatures of adjacent updated facets are compared to determine when remerging should occur. This allows for accurate representation of sub-mesh resolution interfacial features. Proper remerging combats the formation of undulations that often occur with high-order interface tracking methods.

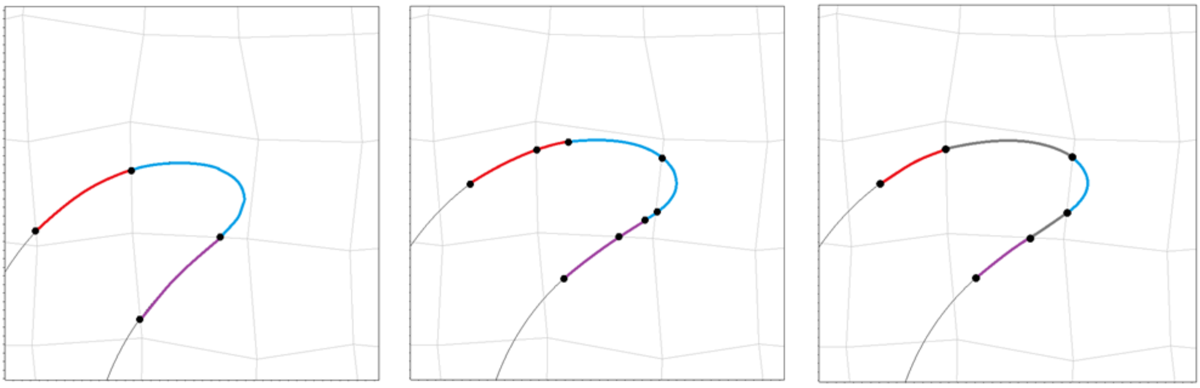

Figure 7: Interface tracking method. Left: interface facets from reconstruction. Middle: after Lagrangian step. Right: after remerging.

For full compatibility with our VOF method, we can compute volume fractions given the output of our tracking method, even with multiple disconnected facets in a cell. For each mixed cell, we simply form a walk along the interface facets and cell edges with consistent orientation to trace a polygon with linear and circular edges. The volumes of each polygon are then summed, and the resulting volume fractions can be used for the next round of interface reconstruction.

A consequence of our volume fraction computing method is that we can detect the presence of topology changes caused by intersecting facets. In this case, polygons formed by our walk are overlapping. We propose to use the interface reconstruction scheme to accurately represent the new interface when a topology change occurs.

4 NUMERICAL STATIC RECONSTRUCTION TESTS

In all our numerical tests exact volume fractions are computed with target geometries and used as input for the proposed interface reconstruction scheme.

4.1 Perfect reconstruction of planar polygons

Given sufficient resolution, our interface reconstruction scheme can reconstruct to arbitrary precision any 2D shapes consisting entirely of lines or circular arcs joined at sharp cusps. This includes all polygons and circles, as well as shapes with circular arc sides.

Furthermore, the proposed interface reconstruction method does not require mesh regularity as shown below. Because we cast all the components of our method as geometric problems with unique solutions, there is no constraint on mesh structure. We demonstrate with tests on perturbed Cartesian meshes.

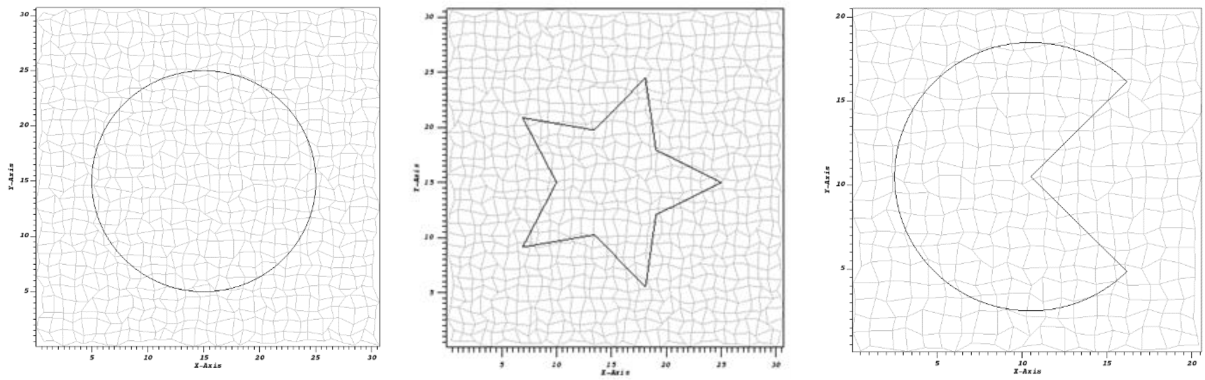

Figure 8: Interface reconstructions with our method on perturbed Cartesian meshes (with a tolerance 10-11). Curvatures are accurately represented and corners are sharp.

4.2 Reconstruction of random ellipses

We demonstrate the 3rd order spatial accuracy of our interface reconstruction method by reconstructing randomly oriented ellipses. Ellipses used have minor and major axis lengths $\frac{\sqrt{12}}{5}$ and $\frac{\sqrt{2}}{5}$ respectively and are rotated arbitrarily to avoid any biases. We show the average results over 100 trials. Our method shows 1st order convergence of curvature and 3rd order convergence of facet gaps, consistent with a 3rd order spatial accurate method. This is expected because we use circular facets as our geometric primitive, which have three degrees of freedom.

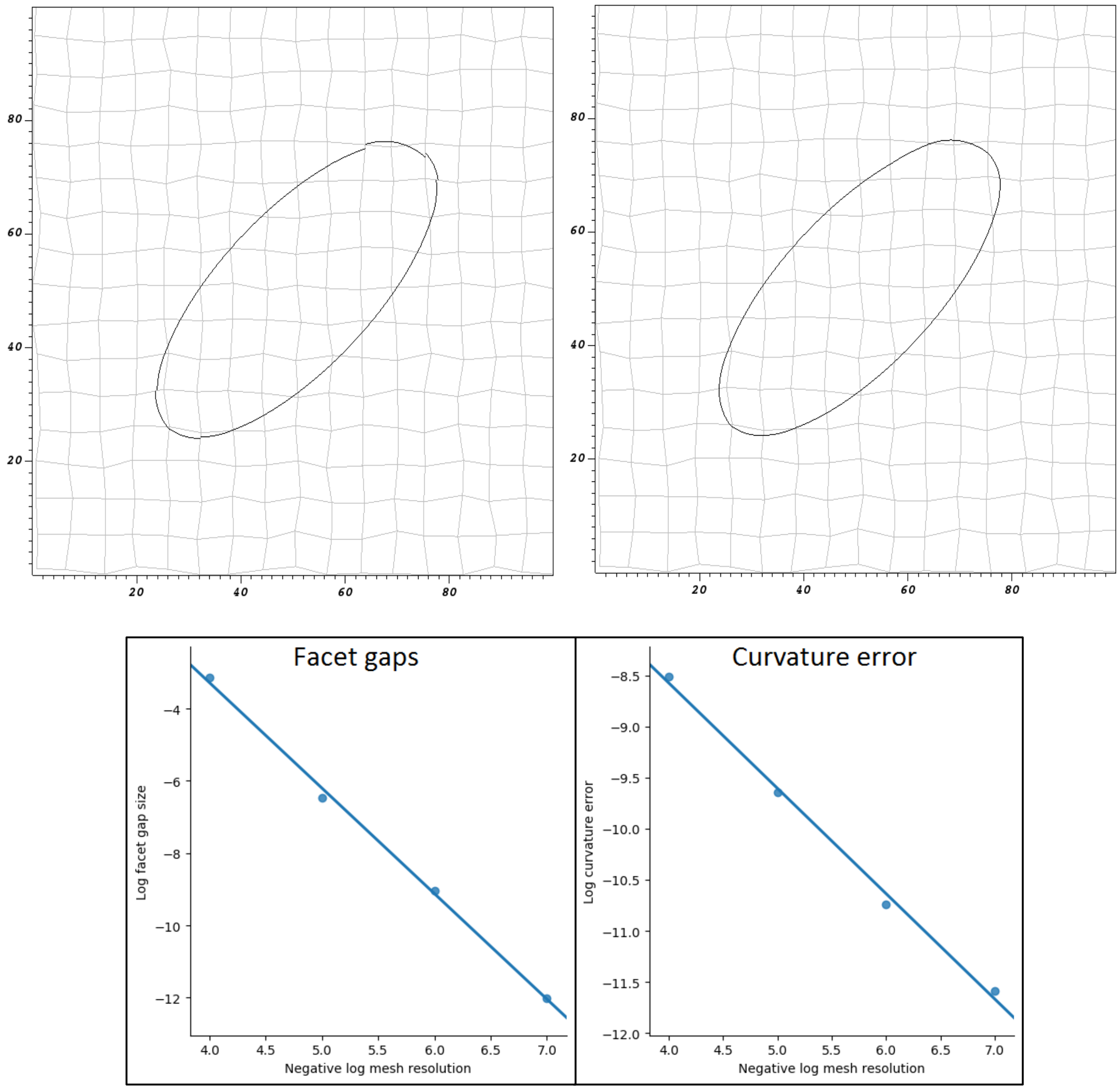

Figure 9: Top: interface reconstructions of randomly oriented ellipse. Without C0 continuous correction scheme (left) and with (right). Bottom: reconstruction method achieves 3rd order convergence of facet gaps (left) and 1st order convergence of curvature (right).

5 NUMERICAL VOLUME ADVECTION TESTS

We demonstrate non-diffusive volume advection with several benchmark problems. In our tests the meshes are set back to unit Cartesian ones after each Lagrange step.

5.1 x+o

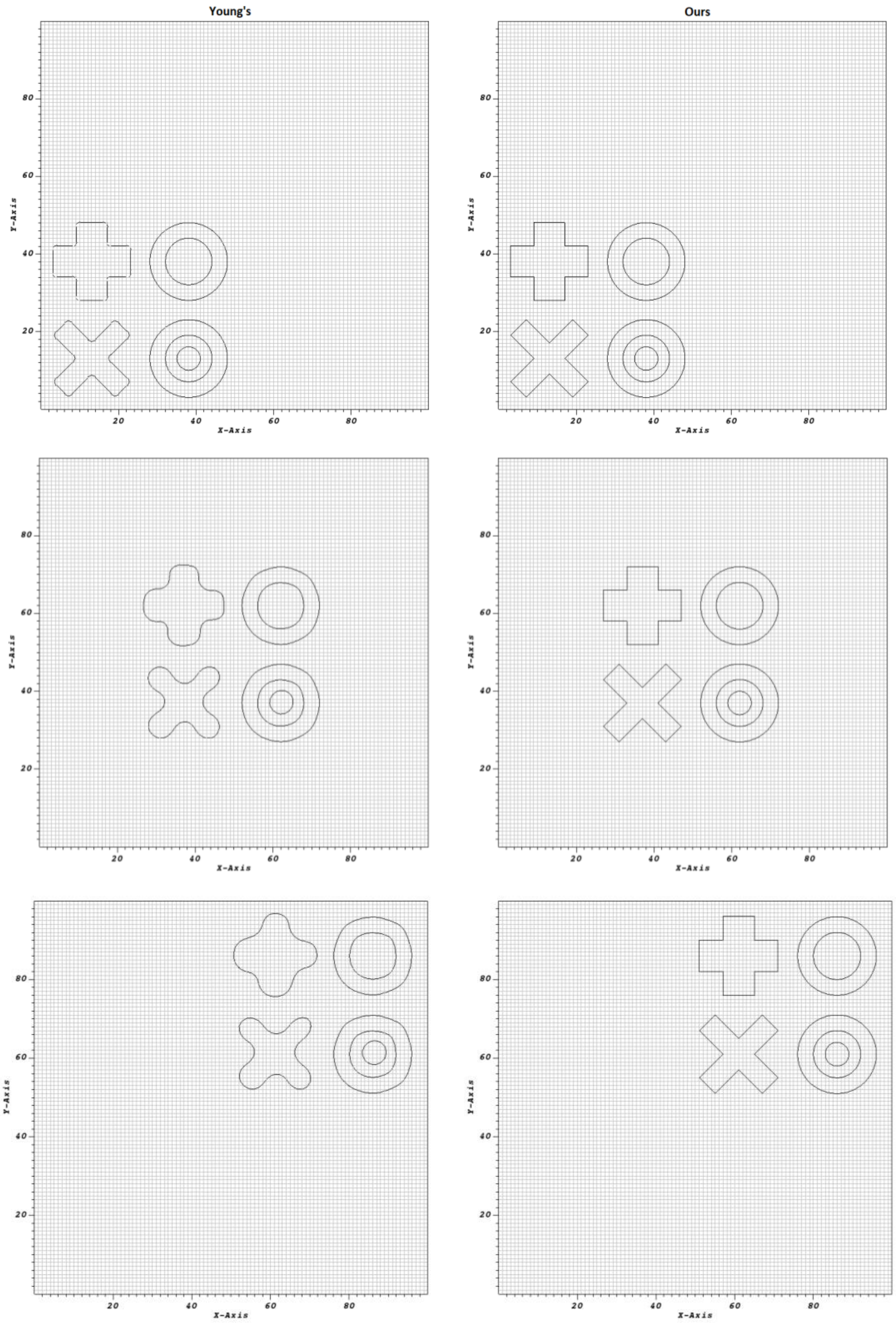

Figure 10: Diagonal translation of x+o shapes. Initial setup (top), middle (middle), end (bottom). Young's (left) method diffuses sharp cusps of x's and distorts curvature of o's, whereas ours (right) remains accurate.

5.2 Zalesak slotted disk rotation

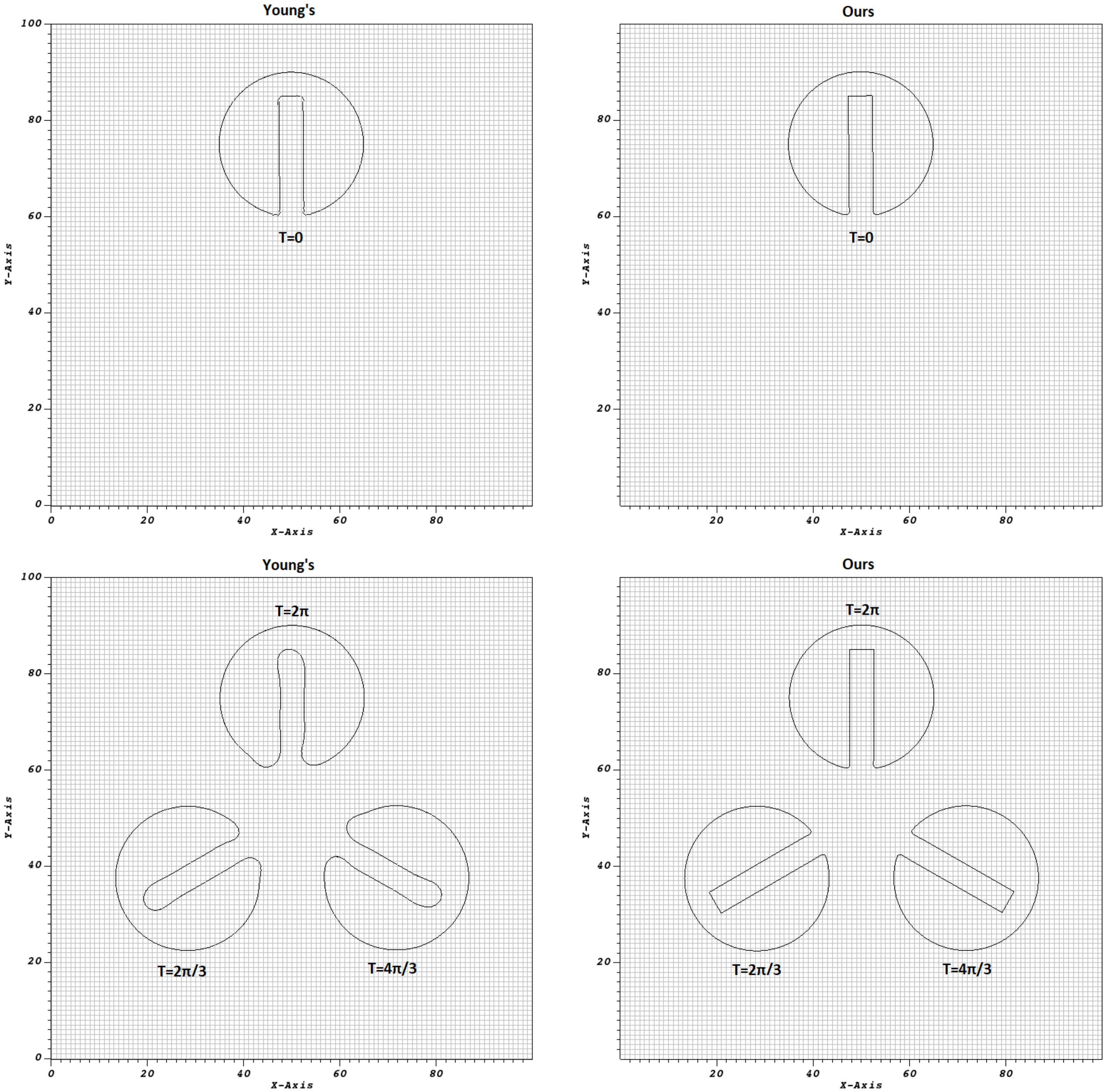

Figure 11: Rotation of slotted disk. Top: initial reconstructed interface. Bottom: reconstructed interfaces at rotation angles $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, and 2π. Young's (left) diffuses sharp cusps of slotted disk, whereas ours (right) remains accurate.

5.3 Rider-Kothe time-reversing vortex

A circle is placed within a time-reversed vorticity field defined via stream function field

$$\boldsymbol{\psi}(x, y, t) = \frac{1}{\pi} \sin^2(\pi x) \sin^2(\pi y) \cos\left(\frac{\pi t}{T}\right)$$

with parameter T. The maximum fluid deformation occurs at time t=T/2 and returns to the initial setup at time t=T.

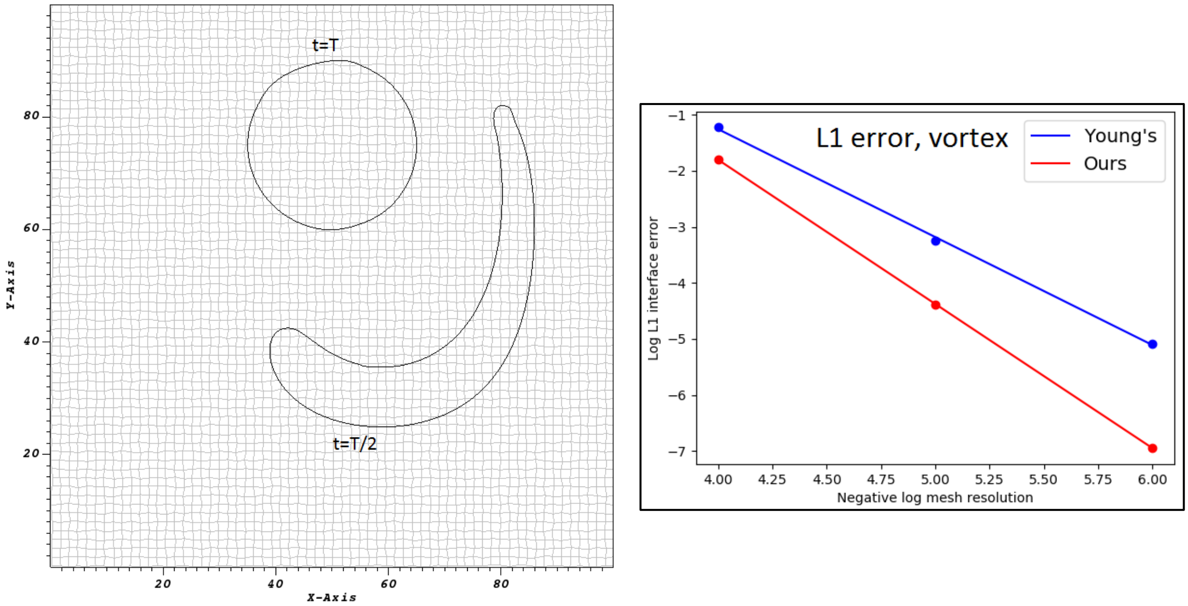

Figure 12: Left: reconstructed interfaces using our method, no prior info, T=2. Right: approximately 3rd order spatial accuracy is achieved, compared to only 2nd order accuracy of Young's.

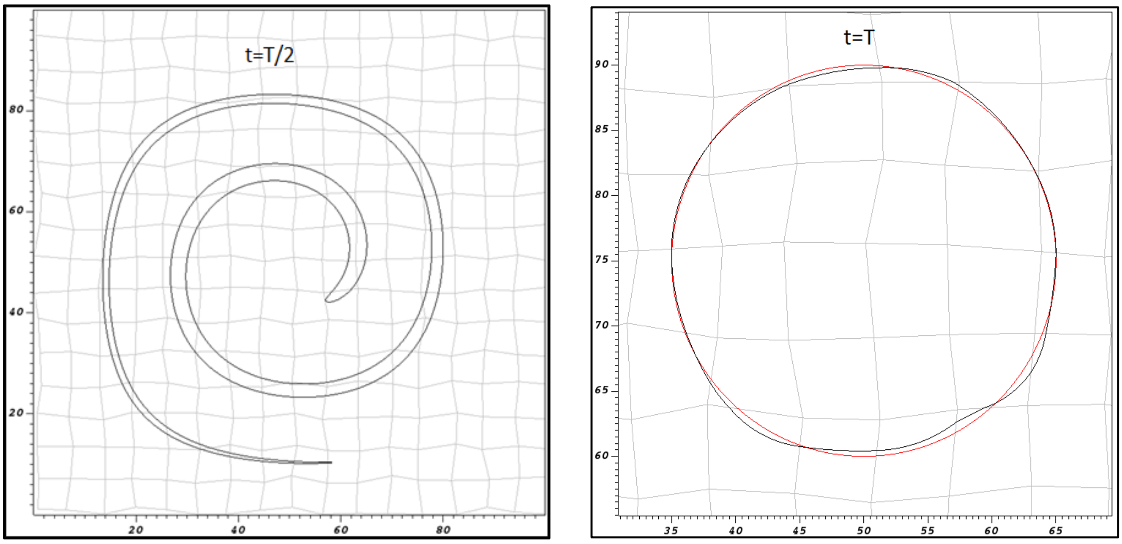

Figure 13: Reconstructed interfaces using our method with prior info, T=8. Stronger deformation and coarser mesh can be handled.

6 CONCLUSIONS

- We propose a high-order interface reconstruction scheme using a combination of linear, circular, and corner facets. Our approach accurately represents curvature and cusps while ensuring a seamless interface, addressing key shortcomings of PLIC methods. Given sufficient mesh resolution, typical mixed cells have exactly two mixed neighbors: we can determine the orientation of the interface and obtain a circular facet matching all three volume fractions using only this local information. A hybrid Newton's-bisection method is used to fit circular facets, and we observe near quadratic convergence consistently. Because circular facets have three degrees of freedom, our method ensures uniqueness of reconstructed interfaces. Facets are fit from low to high order to identify sections of the interface that are linear or where a cusp exists. We guarantee C0 continuity with a simple corrective scheme.
- We demonstrate the effectiveness and accuracy of the new interface reconstruction method on static tests. We show our method can reconstruct planar polygons and circles to arbitrary precision and has 3^{rd} order spatial accuracy on general smooth interfaces. We achieve non-diffusive volume advection on several benchmark problems.
- We point out that interface tracking and reconstruction are compatible. Tracking is achieved by updating a reconstructed seamless interface via a Lagrange step, and volume fractions for a tracked interface can be computed. We demonstrate the ability of the tracking method to capture sub-mesh resolution features of a complex interface on the Rider-Kothe vortex benchmark.

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