Title: SU(n) and Quantum SU(n) Symmetries in Physical Systems

Author(s): Makaruk, Hanna Ewa

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SU(n) and Quantum SU(n) Symmetries in Physical Systems

Hanna Makaruk,
P-2 Applied and Fundamental Physics Group,
Los Alamos National Laboratory,
Los Alamos, NM, US
hanna_m@lanl.gov

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Abstract

Presence of *SU(n)* or other Lie group symmetry in a physical system is its powerful, usually underutilized property. In many cases it allows for finding analytical solutions to nonlinear differential equations describing this system. Power of the method is presented on diversified examples from mathematical physics: Lie-group symmetries in finding solutions of generalized, multidimensional theory of gravity; analytical Dirac–equation solutions for description of conducting polymers; stability of qubit states in quantum computers; spatial defects in condensed matter; reconstruction of 3D object from its 2D tomographic image; significant improvement of numerical solutions stability for Euler equations. The next question after obtaining such Lie group symmetric solution is: does a generalized solution with appropriate quantum group symmetry exists for the given physical system, and if yes what is the physical meaning of the deformation parameter q introduced by such solution. In many cases it can be identified. Any *SU(n)* solution is by its nature singular, assuming a perfect symmetry of the physical system discussed. Such solution gives a powerful insight to theoretical physics, yet the assumption may be too demanding for experimental applications. Deformation parameter *q* from a quantum group symmetry allows for a continuum of solutions, more applicable to experiments.

UNAM Campo Quarto, Cuautitlan, Mexico,
August 2000


Yucatan, 2006


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Introduction

- Numerical methods are the standard approach to solving Nonlinear Differential Equations (NDE), and linear differential equations on manifolds with curvature.

- Analytical solutions remain useful:

  - verification of computer codes
  - enhancing numerical stability
  - analysis how a change of a parameter value or initial conditions modifies the solution (it is a non-intuitive if process is nonlinear.)

- Analytical solutions are in general difficult to find – new methods are needed.

- NDEs solutions are not unique. Correct selection of the relevant solution is crucial.

- Lie- group symmetries and quantum group symmetries are very effective tools in finding new analytical solutions for various NDEs.

- Applying a physical system's symmetry produces the NDE solution describing this system, not just any of the possible solutions.

Applying analytical solutions strengthens not substitutes numerical methods.

Overview

- Applying SU(n) and other Lie group symmetries of physical systems to solving differential equations
  - Analytical Dirac equation solutions for description of conducting polymers
  - Numerical stability of Euler equation

- Never impose a non-existing symmetry on a physical system
  - Reconstruction of 3D object from its 2D tomographic image
  - Multi layer neural net

- Quantum group type generalizations of the Lie group symmetry solutions
  - Stability of qubit states in quantum computers
  - Generalized gravitation theory
  - Classification and evolution of dislocations in continuous media

Genuine Lie and quantum group symmetries allow for finding new analytical solutions for nonlinear differential equations.


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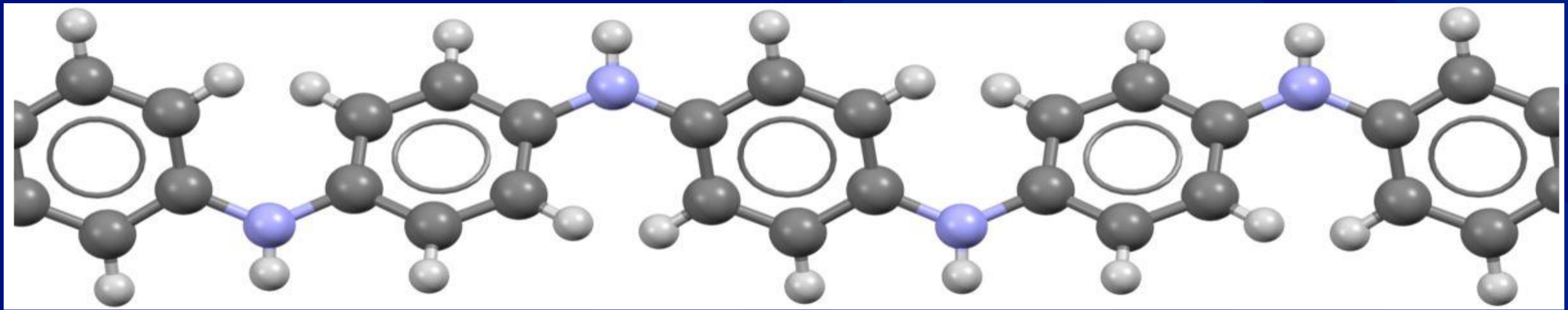
Applying SU(n) and other Lie Group Symmetries of Physical Systems to Solve Differential Equations

- Nonlinear Differential Equations (NDEs) and some linear differential equations on curved manifolds present a significant challenge in finding the analytical solutions.
- Most methods of finding a solution start from guessing the type of function (trigonometric, exponential, polynomial, logarithmic…) and finding the correct parameters.
- Not all solutions have physical meaning. (Example: Maxwell equations solution traveling back in time.)
- Physical solutions of NDE obey the symmetry of the physical system the equations describe.
- Symmetry of the physical system provides the initial “guess” of the form of the solution.
- Let us see how the SU(n) based method works on real examples…

Analytical Dirac Equation Solutions for Description of Conducting Polymers



Conducting polymer polyaniline: grey carbon, purple nitrogen, white hydrogen

- Polarons are charge carriers inside extremely long macromolecules of polymers like polyaniline, allowing electric conductivity.
- Usual polyaniline samples are prepared so that macromolecules underlie SU(2) symmetry - it gives 1D conductivity to a 3D sample of the material
- Creation/annihilation of polarons is a relativistic phenomenon –it requires Dirac not Schrödinger description.

- A. Proń, J. Laska, J.E Österholm. P. Smith, Polymer, **34**, (1993), 4235
- J. Laska, M. Trznadel, A. Proń, Materials Science Forum, **122**, (1993), 177
- J. Laska, PhD thesis, Warsaw University of Technology, (1994)
- H. Makaruk, *Dirac description of energy levels of polarons in polyaniline*, Mod. Phys. Lett. B, **9**, (1995), 543-551;
- H. Makaruk, *Multidimensional Quantum Description of Organic Conductors,* Acta Phys. Pol. B, **27**, (1996), 2747-2754

Construction of a Spinor Bundle over SU(2) Manifold

- Dirac equation on a curved SU(2) manifold becomes nonlinear.
- SU(2) and S^3 manifolds are equivalent => spinor structures over SU(2) and S^3 are isomorphic.
- Spinor structure construction:

$$
\begin{array}{ccc}
Spin(3) \cong SU(2) & \rightarrow & SU(2) \times SU(2) \cong Spin(4) \\
\downarrow & & \downarrow \\
SO(3) & \rightarrow & SO(4) \\
 & & \downarrow \\
 & & S^3
\end{array}
$$

where

- horizontal arrows mean actions of groups on spaces, vertical arrows are covering maps.
- Dirac operator acts on the vector bundle associated with the principal bundle representing spinor structure.

Spectrum of the Dirac operator on S^3 : $\pm(n+3/2)\ m_o$,
where n – integer, m_o – experimental scaling factor.
The energy spectrum experimentally measured for polarons in polyaniline has energy levels equally spaced, exactly as the above spectrum of the Dirac operator over SU(2).
In contrast, numerical Schrödinger-based approximations were not equally spaced and were missing one of the energy levels measured in experiments.

Numerical Stability of Euler Equation

- Euler equation for incompressible fluid is applied in a wide range of numerical computations including weather prediction.
- Galerkin method is a standard numerical approach to solving Euler equation. It is numerically unstable.
- Numerical instability limits achievable evolution time of the system, for example time of accurate weather prediction.

su(n) –based method of solving Euler equation

Structure constants of *su(N)*-> Structure constants of *sdiff(T^2)* when *N*--> ∞,

su(N) – Lie algebra; *N* odd; *sdiff(T^2)* – area preserving diffeomorphisms of 2D torus;

- Solutions of Euler equations associated with su(N) algebras converge to solutions of Euler equations of incompressible fluid on a 2D torus.

- SU(N) based approximation of Euler equation is significantly less computationally efficient than the Galerkin method, yet computational resources are becoming more and more affordable.

SU(N) based approximation of Euler equation is computationally stable for significantly (order of magnitude) longer system evolution time than the standard Galerkin method.

- Z. Peradzyński, R. Owczarek, H. Makaruk, *On group-theoretic finite-mode approximation of 2 dimensional ideal hydrodynamics I. The proof,* LA-UR 11-06945, 2011

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Never impose non-existing symmetry on a system

- Mathematics exists by itself: differential equations have solutions regardless of the physical validity of the assumptions.

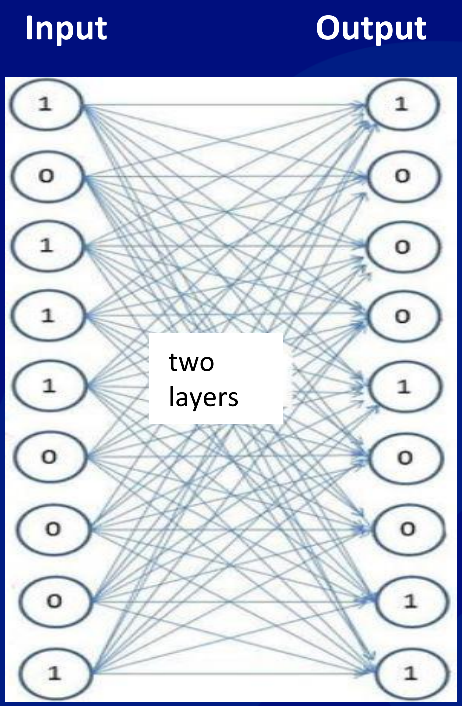
- Making up a symmetry to solve NDEs allows to find a solution, unfortunately not the correct one.

- No one knowingly picks a wrong solution, so why does it happen so frequently in science?
  - Implicit assumptions are not examined.
  - Intuition from linear equations is misleading for nonlinear ones.
  - Incorrect solutions may initially look quite reasonable.

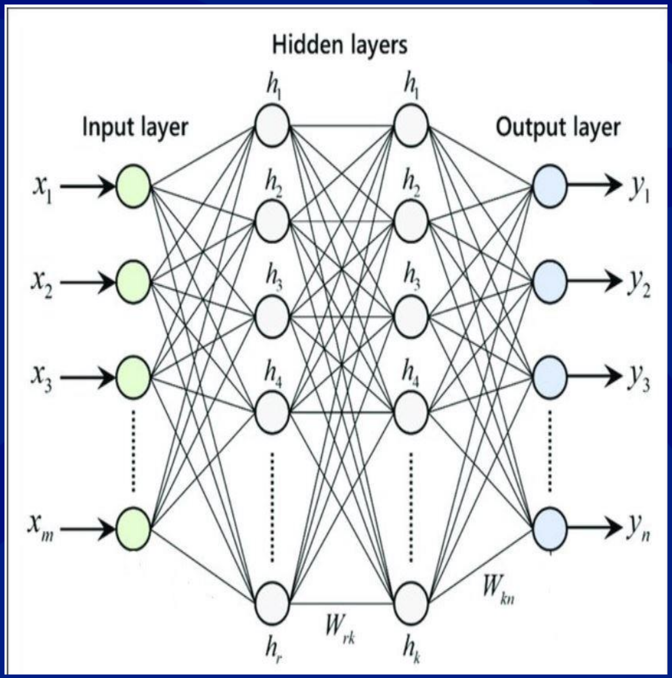
Imposing a non-existing symmetry can be dangerously misleading. It produces reasonably looking yet truly incorrect results.

Multi Layer, Sparsely Connected Neural Net

- In 90' early neural nets had two layers, maximizing number of connections from each noddle.
- Neural net expert Valeriu Beiu had a strong intuition that deep, multi layer neural net architecture with sparsely connected nodes was optimal, yet cost function calculations seemed to show the opposite.
- Problem: applying in calculation n-D circumscribing balls as appear limits of the real n-D cost function volumes was inversing the order of bigger and smaller volumes.
- Calculation of the real n-D volumes corrected the problem.
- Almost all modern AI architecture is multi layer with sparsely connected nodes.



Two layers neural network with densely connected nodes.



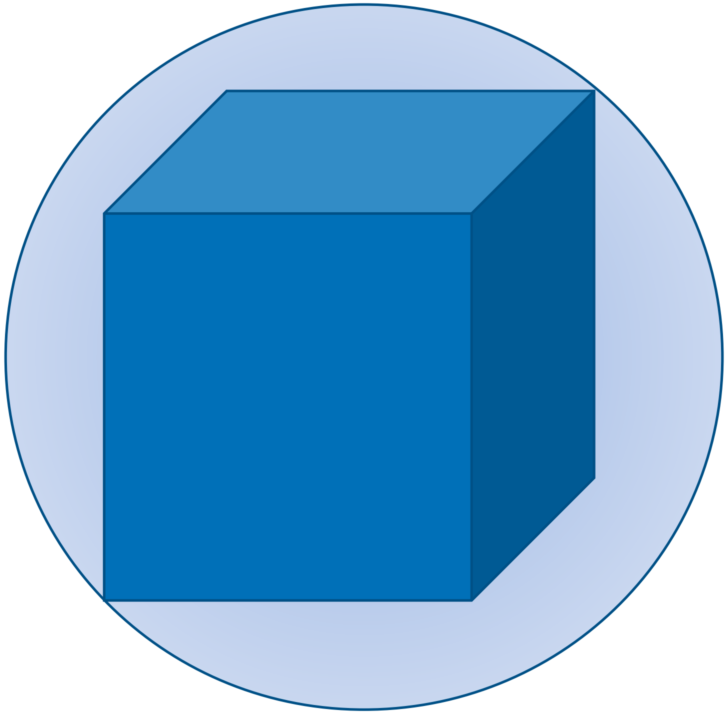
Multi layers neural network with sparsely connected nodes.

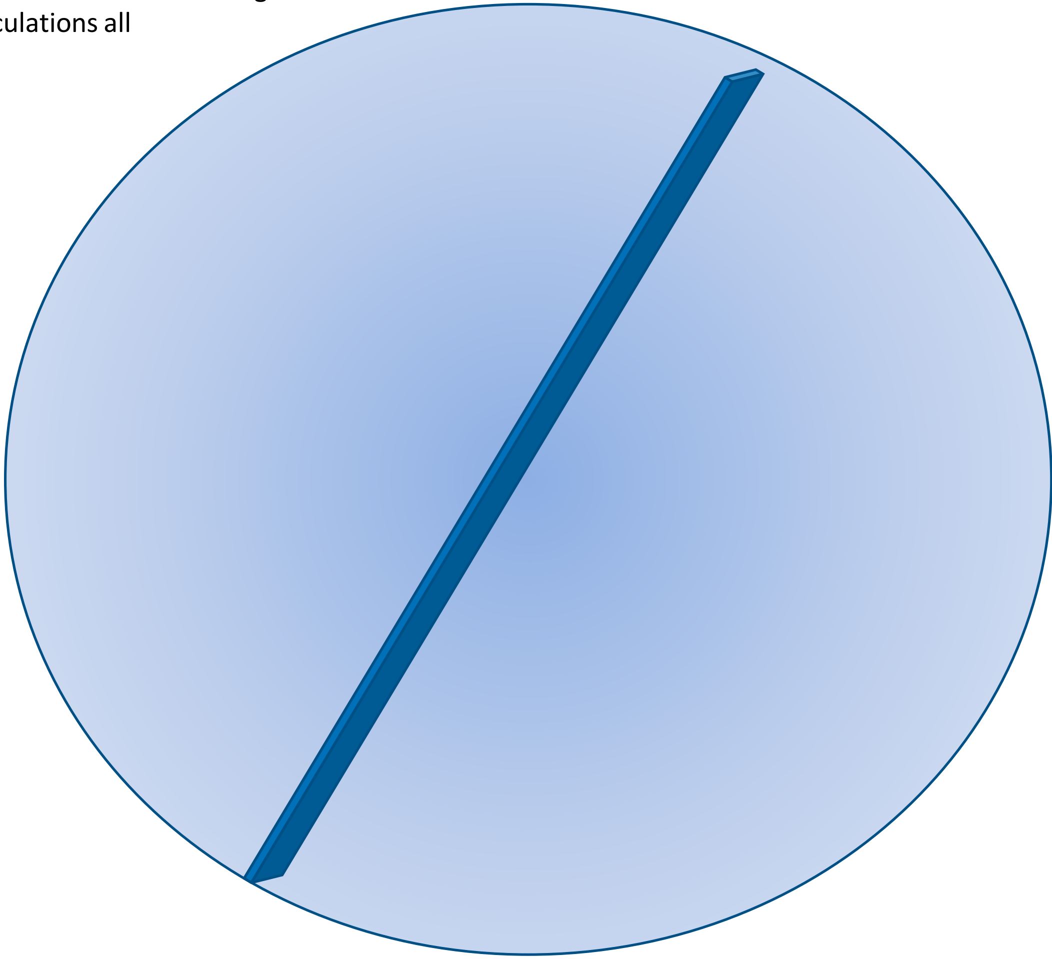
Advantage of multi layer small fan-in neural net over two layers infinite fan-in net becomes visible when n-dimensional volumes were calculated exactly, not upper-limited by n-D balls.

V. Beiu, H.E. Makaruk, *Deeper Sparsely Nets are Optimal*, Neural Processing Letters, 8, (1998);

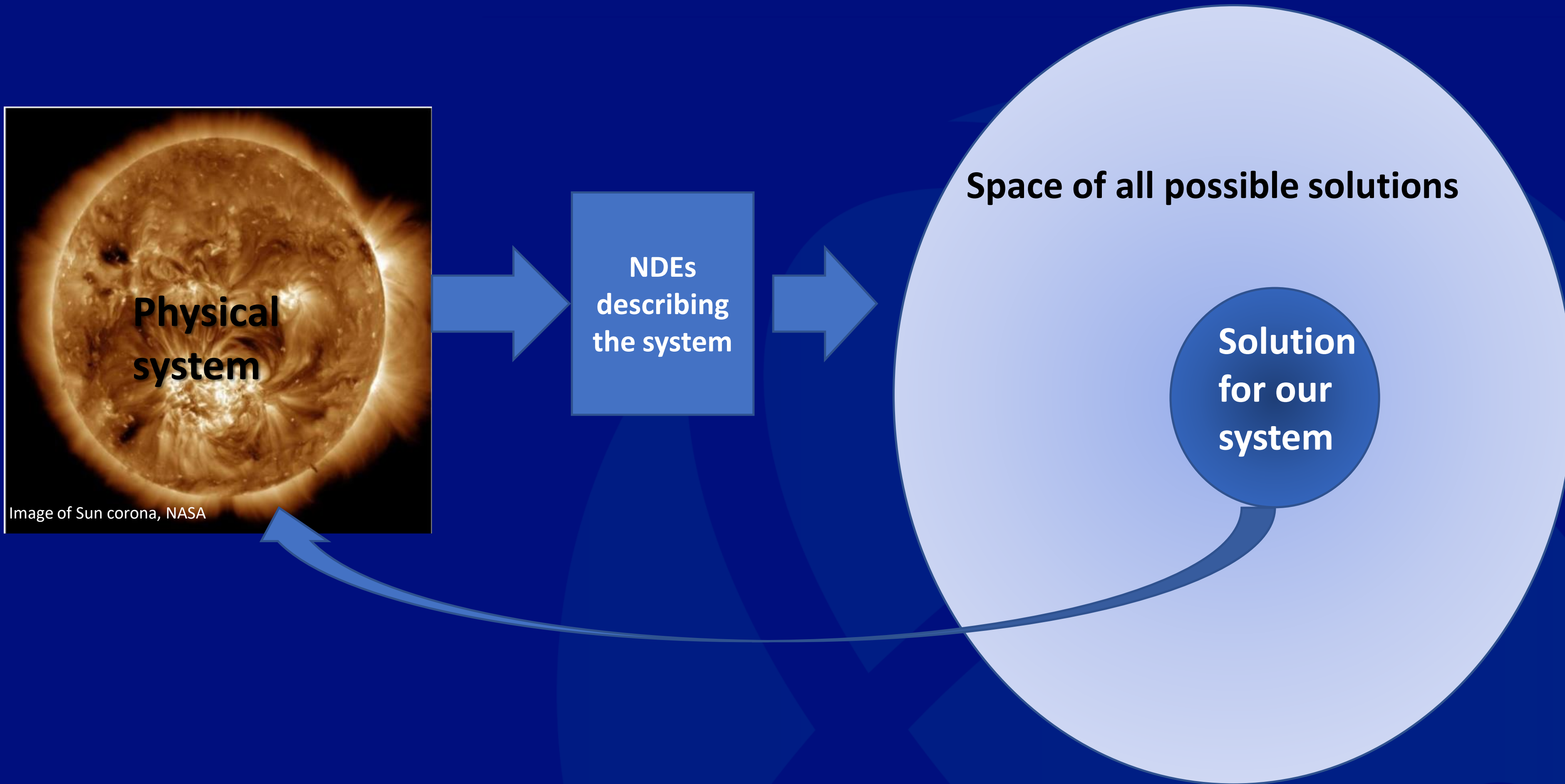
How was it possible to inverse order of bigger and smaller volumes by a simple and self-consistent approximation?

- Calculation of a N-D volumes bounded by analytically defined N-1 surfaces are demanding.
- N-D ball volume is known for any N, so it is easy to substitute in calculations all complicated N-D volumes by N-D balls circumscribing them.
- Easy does not mean correct.





The bigger of two volumes can be circumscribed in a smaller ball, when all of its dimensions are almost equal. The smaller volume, which has a very different size in each dimensions requires a bigger ball to circumscribe it in.
For high dimensional objects the problem illustrated here is getting exacerbated with each additional dimension.



We do not need to find any solution of the given set of nonlinear differential equations (NDEs), we need exclusively the solution related to the physical system in question.

Abel Inversion: Reconstruction of 3D Object from its 2D Tomographic Image

Direct Abel transform (a kind of Radon transform) is equivalent to making a physical radiograph, integrating along paths through the object. It maps densities (g/cm^3) to areal densities (g/cm^2).

$$F(y) = \int_y^\infty \frac{f(r)r}{\sqrt{r^2 - y^2}} dr$$

Inverse Abel transform reconstructs density (g/cm^3) from areal density (g/cm^2), <u>assuming the axial symmetry</u>:

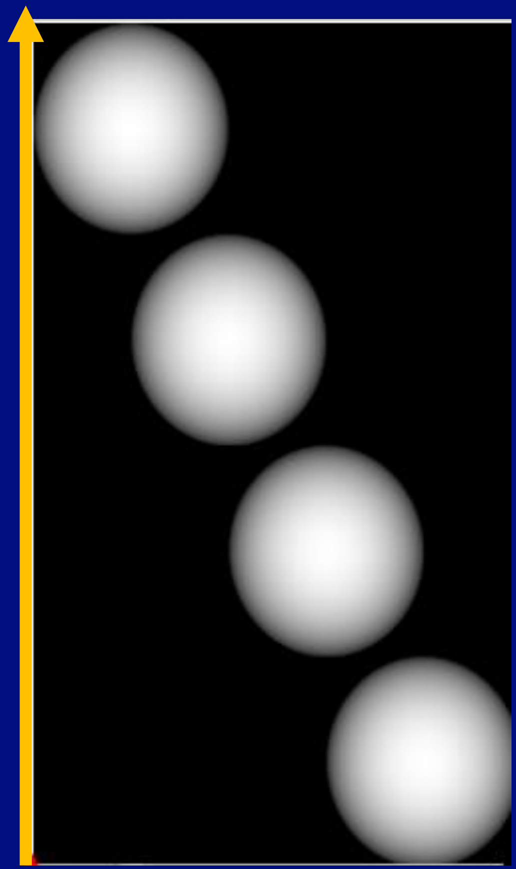
$$f(r) = -\frac{1}{\pi}\int_r^\infty \frac{dF(y)}{dy}\frac{dy}{\sqrt{y^2 - r^2}}$$

- Any deviation from the axial symmetry (axis shift, shape deviation) systematically alters reconstructed density values.
- Small alterations escape common sense checking of numerical results. Only drastic effects like negative density areas are noticeable.
- After loss of axial symmetry other additional information needs to be provided to make the solution unique.
- Additional analytical transformation of the results of failed inverse Abel called Generalization of Inverse Abel Transform allows for reconstruction of the original densities from a single radiogram despite loss of the axial symmetry.

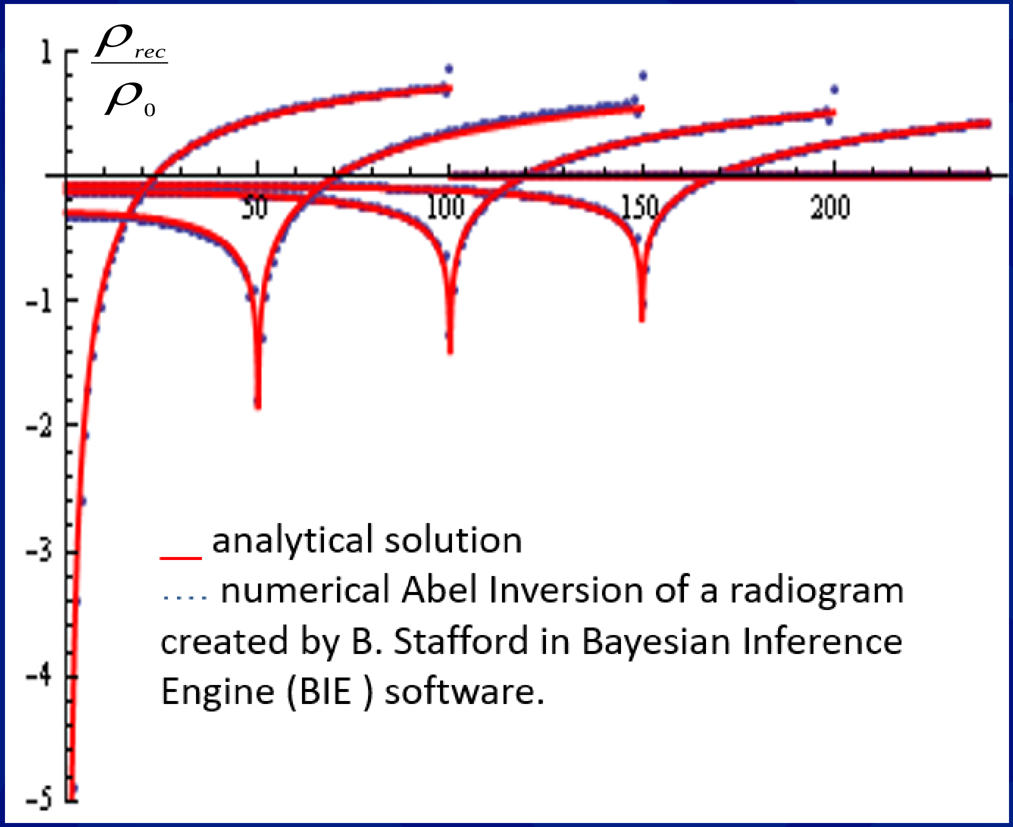
Abel Inversion reconstruction of a non axially symmetric object is mathematically doable, yet it produces nonphysical, sometimes even negative densities.

- Hanna Makaruk, Robert Owczarek, *Analytical Approach to Geometric Errors in Radiography: Inverse Abel Transform for a Homogeneous Ball*, LA-UR-11-07118, Hadronic Journal, vol.35, No5, October 2012, 509

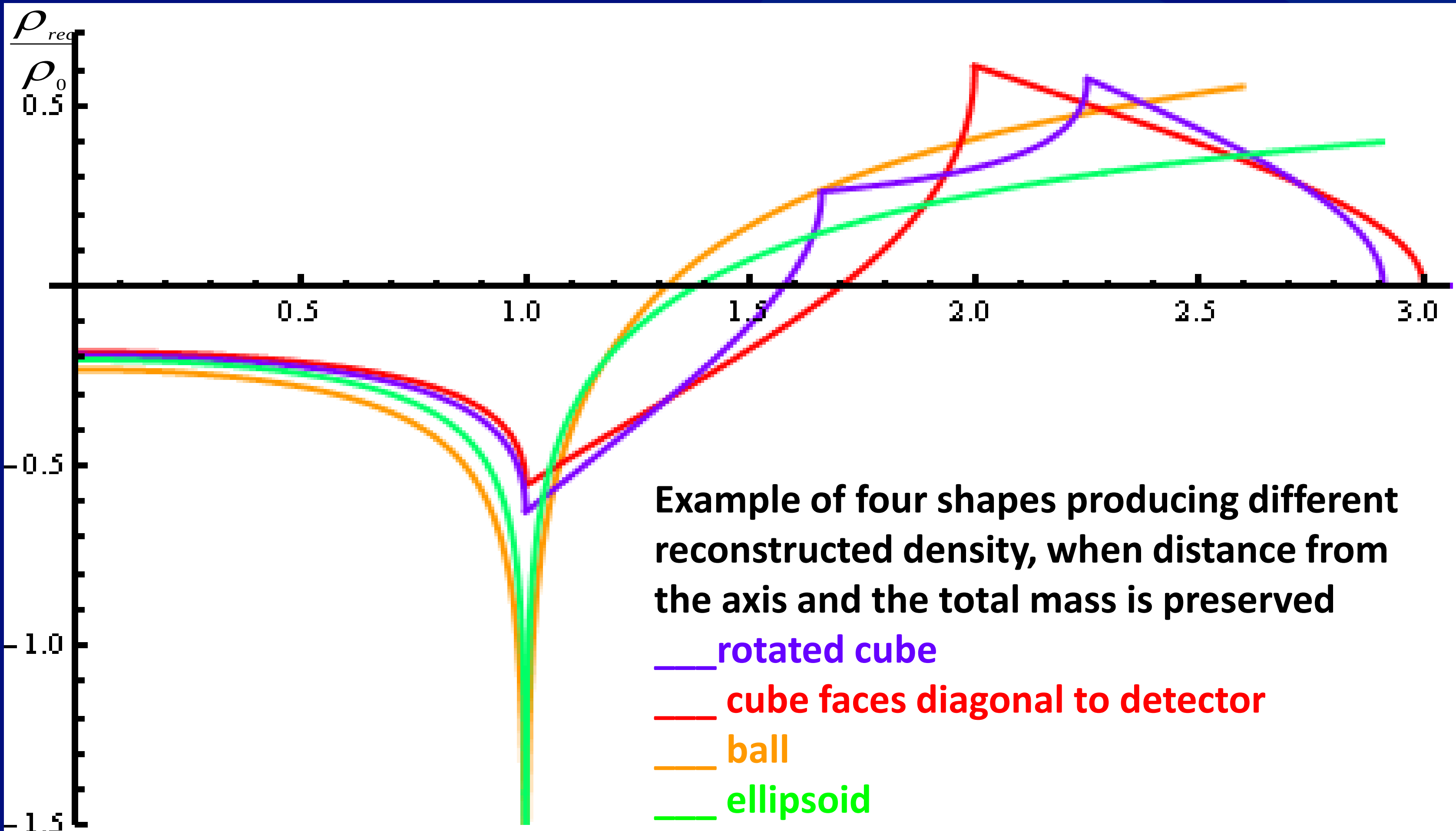
Abel Inverse Transform for off-axis objects creates unphysical negative densities



Steel balls shifted from inversion axis



Abel Inverse density reconstruction from a radiogram of 3D off – axis blobs of matter



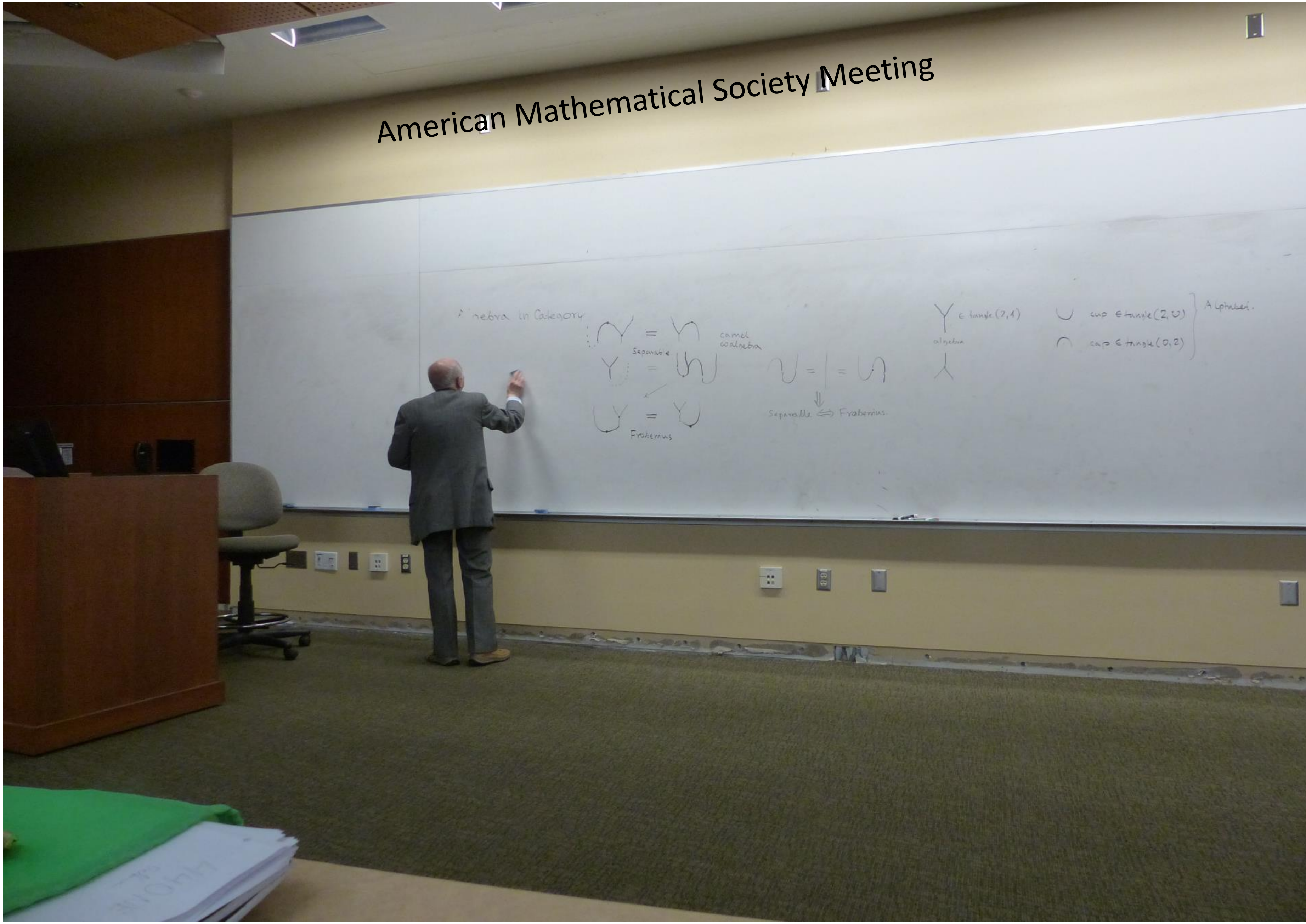
Example of four shapes producing different reconstructed density, when distance from the axis and the total mass is preserved

___rotated cube

___ cube faces diagonal to detector

___ ball

___ ellipsoid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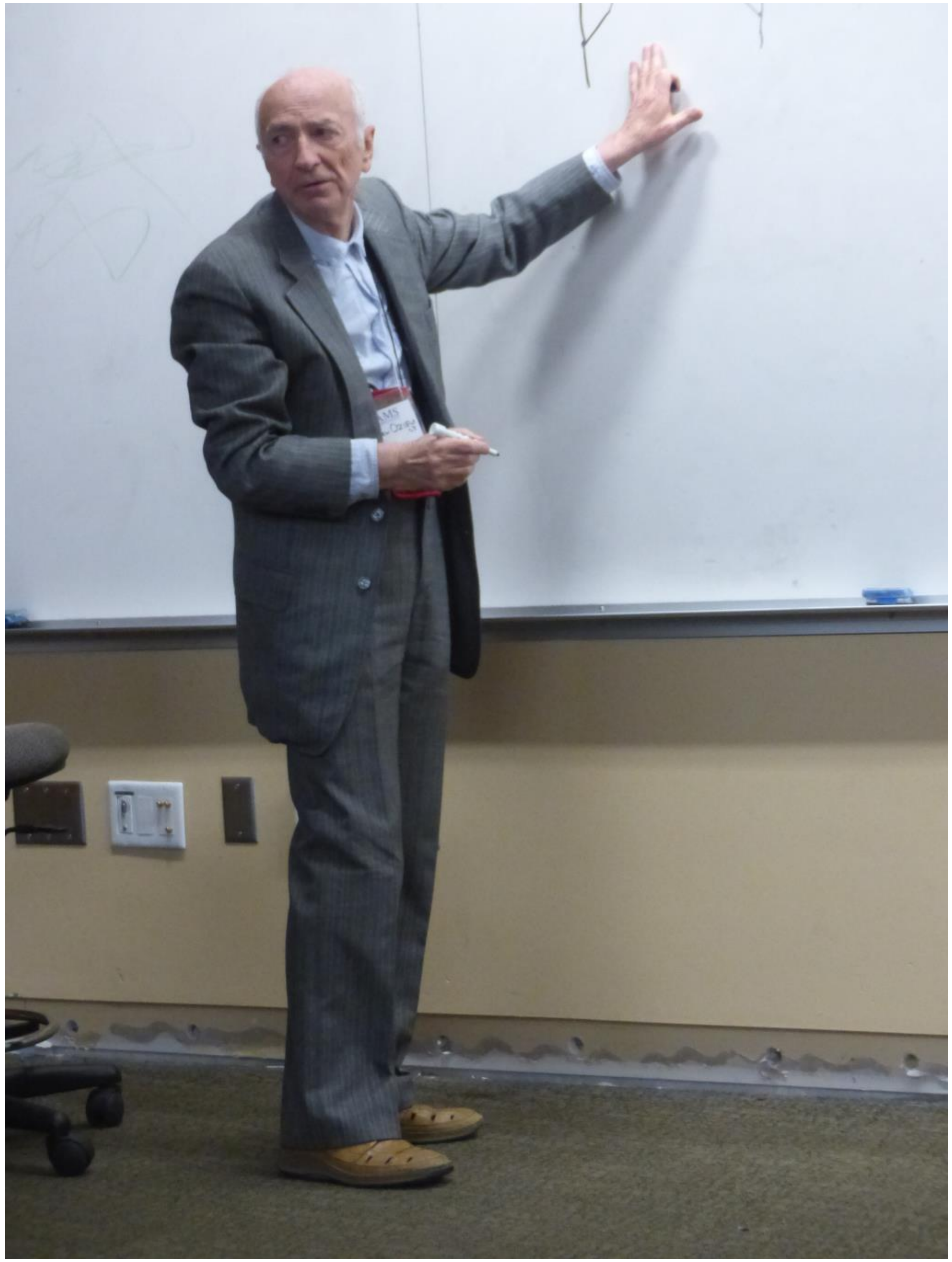
nebra in Category
Separable
camel
coalgebra
Frobenius
Separable ⟺ Frobenius
∈ tangle (2,1)
algebra
cup ∈ tangle (2, 0)
cap ∈ tangle (0, 2)
Alphabet.

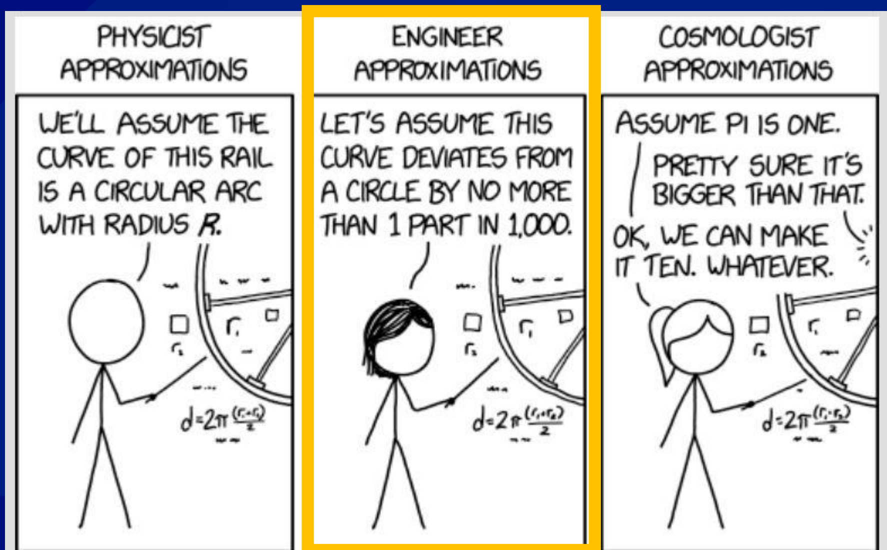


Mathematics is exact. Engineering introduces a finite precision. Quantum groups to the rescue!

- Lie – group symmetric NDE solutions are exact, singular. No neighborhood of approximate solutions exists around them.
- Real physics/engineering implementations are achievable with a finite precision only

  => Lie symmetric solutions should be impossible to achieve in macroscopic physical world.
- Multiple examples show that Lie-symmetric solutions properly describe experimental data, how is it possible?
- Lie-group symmetry is a very reach mathematical structure. Not all of the properties are necessary for the NDE solution.
- In many cases assumptions can be relaxed from Lie group to a quantum group, preserving existence of the solution.
- This generalization changes a singular Lie-solution into a 1D continuous family of solutions parametrized by quantum group deformation parameter.
- Deformation parameter allows for flexible experimental implementation of the solution.
- Deformation parameter connects between Lie groups allowing physical systems for evolution between two Lie-symmetrical solutions.
- Multi –parameter quantum groups are responsible for higher than 1D continuous families of solutions.
- Lie –symmetric solution gains by its quantum group generalization a neighborhood, it is no longer singular.
- This allows for macroscopic physical implementation.

Physical meaning of the quantum group deformation parameter is the key to application of the quantum group symmetric solutions.



Stability of qubit states in quantum computers

- In quantum computing minimizing appearance of quantum state errors is a fundamental issue.
- Zanardi and Rassetti predicted existence of error protected quantum states preserved during the evolution.
- This quantum state error-avoiding property is based on a dynamical Lie algebra - symmetry of the state.
- It is extremely challenging to physically build quantum states possessing exact Lie – group symmetry.

- Analysis of the quantum state evolution preservation shows that possessing a Haar measure by a system's symmetry is the only mathematical property necessary for construction of an error -avoiding quantum state.
- Locally compact Lie groups and locally compact quantum groups both possess Haar measure.
  => quantum state symmetry can be weaker than a Lie - group, a quantum group symmetry is sufficient.
- This observation generalizes a discrete solution to a 1-parameter family of solutions.
- In a physical system allowing a one parameter deformation means that a restricted deformation of the "perfect" quantum state becomes possible. This makes a fundamental difference in engineering.

Implementation of the continuous family of quantum group symmetric solutions is experimentally much easier than achieving a singular Lie-group symmetric state.

- P. Zanardi, M. Rassetti, *Noiseless quantum codes,* Phys. Rev. Lett., **79,** 3306-3309, (1997)
- P. Zanardi, M. Rassetti, *Error avoiding quantum codes,* Mod Phys. Lett. B, **25**, 1085-1093
- M. Durdevich, H. Makaruk, R. Owczarek, *Generalized noiseless quantum codes utilizing quantum enveloping algebras*, J. Phys. A: Math. Gen., 34, (2001), 1423-1437

Discussion about relativity theory with Professor Mickiewicz

Lie Algebra Solutions of Generalized Theory of Gravity

- Complete classification of the 3D and 4D Lie algebras is known.
- All these algebras have been checked, if any of them fulfills the Einstein equation in field theory formulation expressed by Weitzenböck invariants.
- All four 4D Lie-algebraic solutions that exist, have been identified with known cosmological solutions of the Einstein equation:
- Abel algebra solution is identified with the flat vacuum solution of the Einstein equations.
- Solutions A1 x Bianchi VIo and A1 x Bianchi VII also possess vanishing Riemann tensors – these are flat cosmological solutions too.
- The truly 4D Lie algebra known as $A_{4,6}$ in Patera –Winternitz classification is equivalent to a non-flat cosmological solution, known as the Petrov solution with parameter $k^2=4/3$
- More Lie algebraic solutions have been recently found for multidimensional formulation of Einstein equation.
- We have shown that when a quantum group generalization of any of the existing solutions is defined, such quantum group automatically fullfills the Einstein equation.

General relativity has quantum group solutions.

- J.J. Sławianowski, *Field of linear frames as a fundamental self interacting system*, Rep. Math. Phys., **22,** 323-371, 1985
- J.J. Sławianowski, *Space time as a micromorphic continuum*, Int. J. Theor. Phys., **29,** 1177-1184, 1990.
- H. Makaruk, *Lie Groups and Quantum Groups Applied as a Tool in Finding Solutions in Some Field Theories*, Rep. Math. Phys., 36, (1995), 347-353;
- H. Makaruk *Real Lie Algebras of Dimension d less or equal 4 which fulfill the Einstein Equation*, Rep. Math. Phys., 32, (1993), 375-383;

Classification and Evolution of Dislocations in Continuous Media

- Trzęsowski and Sławianowski provided classification of all possible 3D dislocations in continuous media using Bianchi classification of all 3D Lie algebras.
- This classification includes dislocations described by the simple 3D algebras, for which the quantum group generalizations exist.
- Construction used for Lie algebras can be repeated for quantum groups.
- Quantum group parameter ***q*** is in this theory a deformation parameter, which describes a continuous evolution of a given dislocation between Abel group symmetry and a dislocation with symmetry provided by a Lie group counterpart of a given quantum group.
- Formation of dislocations in various materials is a process crucial for many branches of engineering. It describes materials' aging, for example changing properties of concrete due to aging of buildings and other constructions.

Quantum group parameter *q* describes the process of dislocation formation responsible in engineering for materials aging.

- A. Trzęsowski, J.J. Sławianowski, *Global Invariance and Lie-algebraic description in the theory of dislocations,* Int. J. Theor. Phys., **29**, 1239-1249, 1990
- A. Trzęsowski, , *Gauge theory of dislocations*, Int. J. Theor. Phys., **26**, 1059-1068, 1987
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- H. Makaruk, *Description of Dislocations: Quantum Groups Methods*, J. Tech. Phys., 37, (1996), 95-100

Summary and Further Work

- Finding the correct analytical solution for sets of Nonlinear Differential Equations (NDEs) describing a physical system is still important, it strengthens the numerical approach. Examples:
  - Stability of Euler equation.
  - Correcting Abel Inversion software.
  - Optimizing architecture of neural network chips for AI applications.
  - Error protected states in quantum computers.

- Application of the SU(n) or other Lie symmetry is an effective method of finding analytical solutions of NDEs.

- Imposing a non-existing symmetry on the NDEs produces a misleading non-physical solution.

- Lie group symmetric solutions are singular.

- Lie –symmetric solution gains by its quantum group generalization a neighborhood, is no longer singular. As such they can be implemented in macroscopic physical systems.

- Hypothesis – Lie-symmetric applicable to physics posses neighborhoods, even if our constructions have not found these neighborhoods yet.

It is only a beginning, the symmetry method in solving nonlinear differential equations has much more to offer.


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