# Implementing a projection-based reformulation and decomposition algorithm for global optimization of a class of mixed integer bilevel linear programs

## Introduction

- Bilevel programs are mathematical programs with optimization problems in their constraint.
- Bilevel programs are used in cybersecurity modeling
- A general bilevel mixed integer linear program:

$$
\begin{aligned}
\textbf{(P0)} \quad & \min_{x^u, y^u, x^{l0}, y^{l0}} \; c_R^t x^u + c_Z^t y^u + d_R^t x^{l0} + d_Z^t y^{l0} \\
\text{s.t.} \quad & A_R x^u + A_Z y^u + B_R x^{l0} + B_Z y^{l0} \le r \\
& x^u \in \mathbb{R}_+^{m_R},\; y^u \in \mathbb{Z}_+^{m_Z},\; x^{l0} \in \mathbb{R}_+^{n_R},\; y^{l0} \in \mathbb{Z}_+^{n_Z} \\
& \left(x^{l0}, y^{l0}\right) \in \operatorname*{argmax}_{(x^l, y^l)} \{\; w_R^t x^l + w_Z^t y^l : \\
& \qquad P_R x^l + P_Z y^l \le s - Q_R x^u - Q_Z y^u \\
& \qquad x^l \in \mathbb{R}_+^{n_R},\; y^l \in \mathbb{Z}_+^{n_Z} \}
\end{aligned}
$$

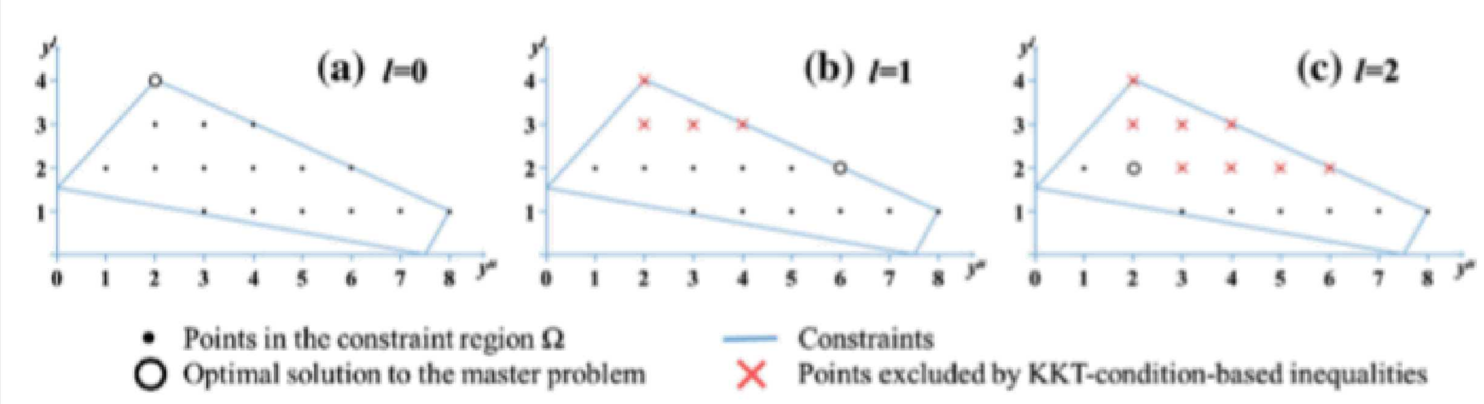
## Algorithm

**Algorithm.** Projection-based reformulation and decomposition through CCG method

**Step 1 (Initialization)**
  Set $LB = -\infty$, $UB = +\infty$, $\xi = 0$, $k = 0$, and $\underline{Y}_0^L \leftarrow \varnothing$.
**Step 2 (Lower Bounding)**
  Solve problem (P5).
  Denote the optimal solution as $\left(x_k^{u,*}, y_k^{u,*}, x_k^{l0,*}, y_k^{l0,*}\right)$.
  Set $LB$ to the optimal objective value $\Theta_k^*$.
**Step 3 (Termination)**
  **if** $UB - LB < \xi$, **then** Terminate and return optimal solution.
**Step 4 (Subproblem 1)**
  Solve problem (P6) at $\left(x_k^{u,*}, y_k^{u,*}\right)$.
  Denote the optimal solution as $\left(\hat{x}_k^l, \hat{y}_k^l\right)$ and optimal objective value as $\theta_k\left(x_k^{u,*}, y_k^{u,*}\right)$.
**Step 5 (Subproblem 2)**
  Solve problem (P7) at $\left(x_k^{u,*}, y_k^{u,*}\right)$ and $\theta_k\left(x_k^{u,*}, y_k^{u,*}\right)$
  **if** Feasible **then**
    Denote the optimal solution as $\left(x_k^{l,*}, y_k^{l,*}\right)$
    Set $UB = \min\left\{UB, c_R^t x_k^{u,*} + c_Z^t y_k^{u,*} + \Theta_{o,k}\left(x_k^{u,*}, y_k^{u,*}\right)\right\}$.
    Set $\widetilde{y}_k^l = y_k^{l,*}$.
  **else (Infeasible Problem)**
    Set $\widetilde{y}_k^l = \hat{y}_k^l$.
  **end**
**Step 6 (Tightening the Master Problem)**
  Create new variables $\left(x^{l,j}, \pi^j\right)$ and constraint (54) corresponding to $y^{l,j} = \widetilde{y}_k^l$.
  Set $\underline{Y}_{k+1}^L = \underline{Y}_{k+1}^L \cup \left\{\widetilde{y}_k^l\right\}$ and $k = k + 1$.
**Step 7 (Loop)**
  **if** $UB - LB < \xi$, **then**
    Terminate and return the optimal solution.
  **else**
    Go to step 2.
  **end**

- Master Problem is initially the high point problem (does not ensure optimality of the lower level problem) and adds constraints through column generation
- Subproblem 1 is the lower level problem given the optimal upper level solution from the previously solved Master Problem
- Subproblem 2 either selects the most optimistic lower level solution of solutions found in Subproblem 1 or indicates that the lower level solution found in Subproblem 1 is infeasible for the Master Problem

## A Classic Toy Example




$$
\begin{aligned}
& \min_{y^u, y^{l0}} \; -y^u - 10y^{l0} \\
\text{s.t.} \quad & y^u \in \mathbb{Z}_+,\; y^{l0} \in \mathbb{Z}_+ \\
& y^{l0} \in \arg\max_{y^l} \left\{ \begin{array}{l} -y^l : \\ -25y^u + 20y^l \le 30 \\ y^u + 2y^l \le 10 \\ 2y^u - y^l \le 15 \\ -2y^u - 10y^l \le -15 \\ y^l \in \mathbb{Z}_+ \end{array} \right\}
\end{aligned}
$$

## Implementation

$$
\begin{aligned}
\textbf{(P9)} \quad & \min_{x^{l,j}, t^j} \; e^t t^j && (78) \\
\text{s.t.} \quad & P_R x^{l,j} - t^j \le s - Q_R x^u - Q_Z y^u - P_Z y^{l,j} && (79) \\
& x^{l,j} \in \mathbb{R}_+^{n_R},\; t^j \in \mathbb{R}_+^{n_L} && (80)
\end{aligned}
$$

$$
\left[e^t t^j = 0\right] \Rightarrow \begin{bmatrix} w_R^t x^{l0} + w_Z^t y^{l0} \ge w_R^t x^{l,j} + w_Z^t y^{l,j} \\ P_R x^{l,j} \le s - Q_R x^u - Q_Z y^u - P_Z y^{l,j} \\ P_R^t \pi^j \ge w_R^t,\; x^{l,j} \perp (P_R^t \pi^j - w_R^t) \\ \pi^j \perp (s - Q_R x^u - Q_Z y^u - P_R x^{l,j} - P_Z y^{l,j}) \\ x^{l,j} \in \mathbb{R}_+^{n_R},\; \pi^j \in \mathbb{R}_+^{n_L} \end{bmatrix},\; \forall y^{l,j} \in \underline{Y}_k^L \subseteq Y^L \qquad (82)
$$

$$P_R \lambda^j \ge 0,\; x^{l,j} \perp P_R \lambda^j,\; \forall y^{l,j} \in \underline{Y}_k^L \subseteq Y^L \qquad (83)$$

$$e - \lambda^j \ge 0,\; t^j \perp \left(e - \lambda^j\right),\; \forall y^{l,j} \in \underline{Y}_k^L \subseteq Y^L \qquad (84)$$

$$\lambda^j \perp \left(s - Q_R x^u - Q_Z y^u - P_Z y^{l,j} - P_R x^{l,j} + t^j\right),\; \forall y^{l,j} \in \underline{Y}_k^L \subseteq Y^L \qquad (85)$$

- The algorithm is implemented in Pyomo making use of Pyomo's packages mpec (mathematical programs with equilibrium constraints) and GDP (generalized disjunctive programming)
- The logical implication constraint required an approximate implementation:

$e^t t^j \ge \epsilon \;\cap$ CONSTRAINT BLOCK

## Exact Reformulation

- Goal: formulate without epsilon approximate disjunction
- Reformulate the "in-the-projection-set" LP as a feasibility problem
- The LP is infeasible iff dual is unbounded or infeasible
- Use disjunction:

$$
\begin{pmatrix} P_R x^{l,j} \le s - Q_R x^u - Q_Z y^u - P_Z y^{l,j} \\ x^{l,j} \ge 0 \\ \textit{CONSTRAINT BLOCK} \end{pmatrix} \cap \begin{pmatrix} d \ge 0 \\ P_R^T d \ge 0 \\ d^T (s - Q_R x^u - Q_Z y^u - P_Z y^{l,j}) \le -1 \end{pmatrix}
$$

- Introduce new variables to represent quadratic terms and linearize with McCormick inequalities
- Requires bounded variables

## Future Work

- Test the algorithm against toy examples with known solutions on each iteration
- Optimize the algorithm
- Continue to test reformulations of the approximation
- Compare and benchmark with two other recent bilevel solvers: Fischetti and Ralphs
- Apply to DCOPT model for power supply cybersecurity

PYOMO

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Dajun Yue, Jiyao Gao, Bo Zeng, Fengqi You. A projection-based reformulation and decomposition algorithm for global optimization of a class of mixed integer bilevel linear programs. *Journal of Global Optimization*, 73: 27-57,2019.