SAND2018-13793C

On Minimum Detectable Velocity

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ABSTRACT

In comparing system performance for ground moving target indicator (GMTI) radar systems, various metrics are used. It is highly desirable that the metric be simple and powerful. Ideally it is a single number, or a plot. It is often the case that a single number is not sufficient to describe the radar performance under all operational conditions. In spite of this, it is still common to attempt to use a simple metric, such as the minimum detectable velocity (MDV). This paper discusses the concept of minimum detectable velocity with the goal of showing what this metric attempts to communicate, and what may not be properly communicated by this metric without careful attention. Basic parameters that affect the minimum detectable velocity are presented.

Keywords: radar; minimum detectable velocity; ground moving target indicator; clutter

1 INTRODUCTION

One of the oft-cited measures of the performance of a GMTI system is the minimum detectable velocity, or MDV. The meaning behind the MDV quoted for a system can be vague. As the name implies, MDV is a based upon a radar detection background. However, often MDV is presented as a “rule-of-thumb” type relationship. The reason for the rule-of-thumb relationship is that it can be difficult to establish all of the parameters that go into the full detection problem. For example, it is often difficult to agree upon what radar cross-section (RCS) statistics to use for the desired target. At other times, only a quick calculation is desired to bound a problem. In any case, this memorandum will present both approaches.

2 Detection

At the center of the “minimum detectable velocity” (MDV) is the classic radar detection problem. It is well-known that the detection problem is a strong function of the signal-to-interference-plus-noise ratio (SINR). For moving target detection in the presence of clutter, additional spatial/temporal processing is employed to reduce the clutter interference. We developed the equation for signal-to-noise ratio after clutter attenuation in the appendix with corresponding assumptions, and we repeat it here:

$$SNR_{CA} = SNR_0 \cdot L_p \tag{1}$$

where SNR_0 is the standard signal-to-noise ratio of the matched filter in the absence of clutter, e.g., for exoclutter GMTI from [1], and L_p is the SINR loss term. The SINR loss is a consequence of clutter attenuation processing. Note for this paper, we will tend to fix the value of SNR_0 which means that the SINR loss is the key parameter in those circumstances. To do so, we will assume that the antenna area is fixed. As opposed to other articles on clutter

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attenuation, we want to emphasize that often the volume available to the radar system designer is fixed. For example, we will hold this antenna length, D_a, constant when we vary the number of subapertures available to the designer, and this fixes the antenna gain for this paper in that case.

Returning to the detection problem, given the parameters that go into equation (1) along with the target model (e.g., Swerling model), this sets the probability of detection versus the false-alarm rate. As we will discuss, minimum detectable velocity implies some assumed value of the probability of detection and some desired false-alarm rate, although this is rarely explicitly specified.

3 Minimum detectable velocity

In Ward's classic report on space-time-adaptive processing (STAP) [2], he states that the "minimum detectable velocity (MDV) is defined as the velocity closest to that of the mainlobe clutter at which *acceptable* SINR loss is achieved" where SINR is the signal-to-interference-plus-noise-ratio. The key phrase in Ward's definition is the value for the "acceptable SINR loss". As shown in the previous section, this SINR loss has important implications that we will examine the in the remainder of this paper.

3.1 MDV and "rule-of-thumb"

Ward suggests a couple example values of possible acceptable SINR loss of 12 dB, and 5 dB. He provides a reasonable justification by noting that the system designer may recoup the 12 dB processing loss by choosing to sacrifice half of the operating range when compared to the no processing loss case. Likewise, he notes that the 5 dB loss case corresponds to sacrificing 25% of the operating range relative to no processing loss. Other factors are chosen by other system designers (e.g., 10 dB in [3]); although the most common "rule-of-thumb" appears to be 5 dB.

To illustrate the implications of the rule-of-thumb we start with the 2-subaperture case where we assume a very large clutter-to-noise ratio. For this case, equation (31) from the appendix becomes:

$$L_p = \left| \sin\left[\frac{\pi}{\lambda} \left(\frac{D_a}{2} \right) \sin\theta_d \right] \right|^2 \tag{2}$$

A couple of things will be mentioned at this point. Equation (2) yields a null at the boresight. We can steer this null in signal processing if desired, but will ignore this for simplicity. We will also assume broadside imaging, and that the antenna aligns with the velocity vector for simplification.

A further simplification is that we will assume that the moving target has a velocity and location such that the *total Doppler* observed by the radar for the target is zero. Under the assumptions in this paper this says:

$$f_D = 0 = \frac{2}{\lambda} V_x \sin\theta_d - \frac{2}{\lambda} v_r \tag{3}$$

or:

$$\sin\theta_d \approx \frac{v_r}{V_x} \tag{4}$$

We typically prefer that the mover velocity be in the ground plane, therefore we rewrite equation (4) as:

$$\sin\theta_d \approx \frac{v_{gr}\cos\psi}{V_x} \tag{5}$$

Note that the sign in equation (4) is unimportant because of symmetry arguments and because the expression in equation (2) is in terms of power.

We plug equation (5) back into equation (2) to get:

$$L_p = \left[\sin\left(\frac{\pi D_a v_{gr}\cos\psi}{2\lambda V_x}\right)\right]^2 \tag{6}$$

Figure 1 shows a plot the SINR loss from equation (6) versus the ground velocity. For this plot we assume a center frequency of 15 GHz and a grazing angle of 20º. We have highlighted the 5 dB loss target velocity in this figure.

The equation illustrates two of the most important quantities in determining the SINR loss; hence MDV, are the antenna dimension (in wavelengths) and the platform velocity. It is obvious that higher platform velocities result in a higher MDV, and larger antenna sizes relative to the wavelength result in lower MDV, if all else is equal in equation (1).

As we have stated, the typical rule-of-thumb is that the minimum detectable velocity is the velocity of the mover which results in a 5 dB SINR loss (from [4-6] and other references)[‡]. In an ideal case, using the assumptions given above and in the appendix with no other outside error sources, equation (6) gives the SINR gain (loss). For the 2-subaperture case this becomes:

$$MDV \approx \frac{2\lambda V_x}{\pi D_a\cos\psi}\arcsin\left(10^{-5/20}\right) \approx 0.38\frac{\lambda V_x}{D_a\cos\psi} \tag{7}$$

This matches what we see in the dashed lines in Figure 1. It is interesting to note from the equation, that the 5 dB SINR loss point is at about 1/3rd of the beamwidth.

We will see that the general form of equation (7) is:

$$MDV \approx \kappa\frac{\lambda V_x}{D_a\cos\psi} \tag{8}$$

where κ is a scalar that depends upon the specified SINR loss and various other parameters which we will expand upon further in this paper.

Equation (8) is a function of the assumed SINR loss that meets some performance requirement. The following equation generalizes to other "acceptable rules-of-thumb" for the SINR loss for 2-subapertures:

$$MDV \approx \frac{2\lambda V_x}{\pi D_a\cos\psi}\arcsin\left(10^{-SINR_{loss}/20}\right) = \frac{2\lambda V_x}{\pi D_a\cos\psi}\arcsin\left(10^{10\log_{10}\left[L_p\right]/20}\right) \tag{9}$$

Figure 2 shows the values of κ as a function of the assume SINR loss from equation (9). Note that in the figure we assume the acceptable SINR loss is a positive value in deciBels.

[‡] As noted above that although 5 dB SINR loss is mentioned most often in literature, other less conservative values than 5 dB are also occasionally used in literature.

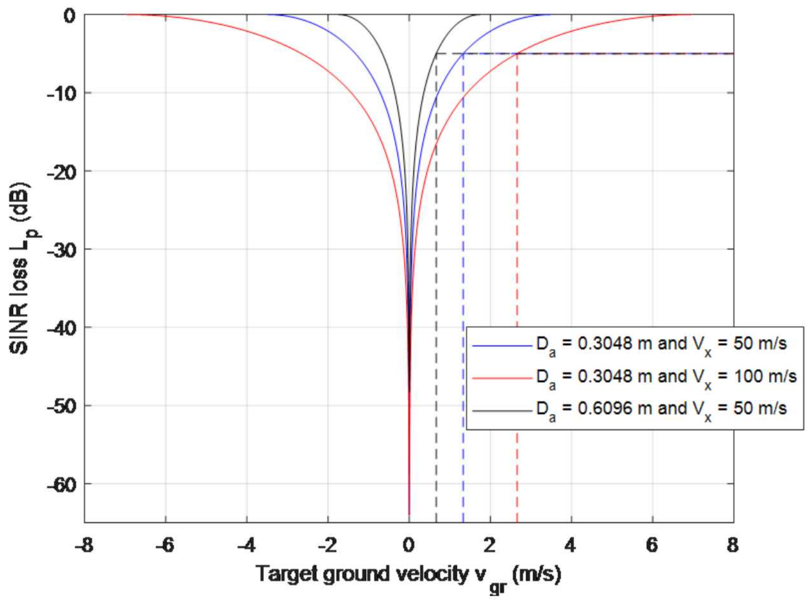

Figure 1: SINR loss example versus target ground velocity for 2-subaperture case

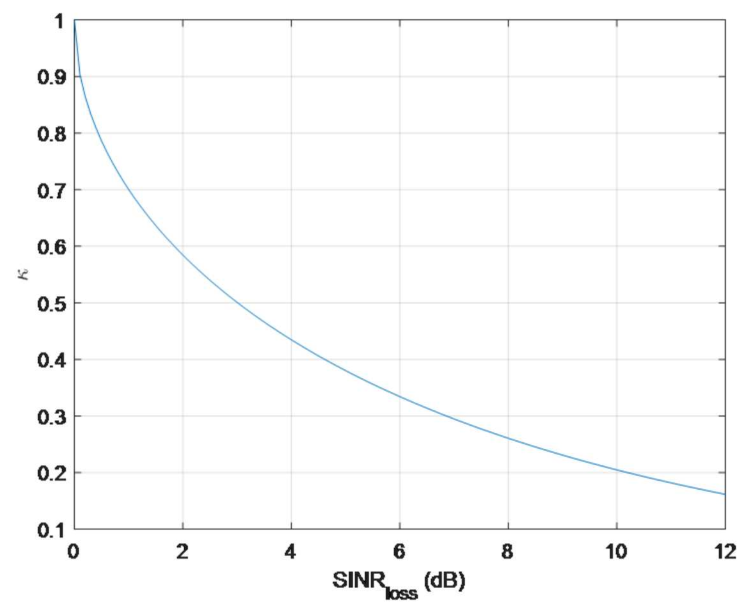

Figure 2: κ versus $SINR_{loss}$ for two sub-apertures

Although it is not part of this paper, we point out an interesting side note that the equation for MDV in exoclutter follows the same form with $\kappa \approx 1$ [7].

3.2 MDV and detection

The rule-of-thumb presented in the previous section has some underlying assumptions. To better understand those, we look at the MDV as a radar detection problem, as implied by the "detectable" part of the term MDV. We assume that the basics of radar detection are known to the reader (see [8] for example).

There are several additional important parameters that go into this detection. All of these parameters need to be specified before the calculation can be performed. We need to know the desired probability of detection (P_d), the desired probability of false alarm (P_{fa}), and the RCS statistics of the target, including its mean RCS, as well as the RCS model, such as Swerling type. At this point, it is obvious that by the use of a rule-of-thumb we can avoid the requirement of the knowledge of these parameters, which is the attraction of using a rule-of-thumb. The truth is that by using a rule-of-thumb we really made an implicit assumption about these parameters.

The key parameter to any basic detection problem is the signal-to-noise ratio (SNR). Endoclutter GMTI is no different. The only difference is that endoclutter GMTI adds a filter that attempts to filter out all targets with zero velocity. As with all filters, endoclutter attenuation has a finite roll-off. The result is that non-zero velocity targets are also attenuated, which in turn reduces the effective SNR input into the detection problem.

It should be noted that the "noise" in the detection problem is actually any undesired interference. In endoclutter GMTI, one source of additional noise is any clutter that is not perfectly cancelled. In this paper, we assume perfect cancellation of clutter, i.e., that the resulting interference after filtering is only due to thermal noise.

Now we relate this to an example. Assume that we want to know the minimum detectable velocity with a P_d=0.9 and a P_{fa}=10^{-6} for a 0 dBsm Swerling I target. Assume that the thermal noise power is equal to a -26 dBsm target. From standard detection curves, we need SNR_{CA}=21 dB to attain the desired P_d for the given P_{fa} . This would mean that we need to limit SINR loss to L_p=-5 dB, which by malice aforethought is the "rule-of-thumb" case. So for this case, given D_a, λ, V_x, and ψ, we can solve equation (7) for the MDV. In the more general case, where the L_p does not happen to match the rule of thumb, we can modify equation (7) as follows:

$$MDV \approx \frac{2\lambda\, V_x}{\pi D_a \cos\psi} \arcsin\left(10^{-\left\{10\log_{10}(\sigma_t/\sigma_n)-SNR_{Pd}\right\}/20}\right) \tag{10}$$

Obviously the exponential term in the equation is the available SINR loss in dB that permits us to meet the desired P_d and P_{fa} for the target and noise statistics. Note that the term in the braces in (10) must be negated because L_p by definitions above is less than or equal to one.

We now look at what changes in equation (10) versus the rule-of-thumb. The simplest thing to do is to keep everything the same in the previous example with the exception of changing the average target return. Assume that we wish to keep the same P_d and P_{fa}, however, we change the target average return to +3 dBsm. This means we can handle an SINR loss of 8 dB which is less conservative than 5 dB and therefore the MDV is reduced. Using equation (10) it becomes:

$$MDV \approx \frac{2\lambda\, V_x}{\pi D_a \cos\psi} \arcsin\left(10^{-8/20}\right) \approx 0.26 \frac{\lambda\, V_x}{D_a \cos\psi} \tag{11}$$

If, on the other hand, the target average return is -3 dBsm, we can only handle and SINR loss of 2 dB and the MDV performance is degraded:

$$MDV \approx \frac{2\lambda V_x}{\pi D_a \cos\psi} \arcsin\left(10^{-2/20}\right) \approx 0.58 \frac{\lambda V_x}{D_a \cos\psi} \tag{12}$$

Note that in this case, as suggested by Ward, the system designer may choose to trade-off some parameter, such as range in equation (13), to buy back the MDV performance.

The κ values used in these examples of can be found from equation (9), or Figure 1, both of which assume the 2-subaperture array case.

3.3 Other factors in MDV

There are many other factors that influence the MDV which we will not expound upon in much detail this paper. Some of these include the various assumptions made in this paper, such as broadside imaging, and ignoring practical temporal processing, etc. Other factors are given in Klemm [9], and other references.

One factor that we want to examine further here is the number of receiver elements. Recall that in this paper we operate under the assumption that the antenna dimension D_a is fixed. Figure 3 shows a plot the SINR loss versus the ground velocity for M=2, 3, and 4 receivers based on equation (31). For this plot we assume a center frequency of 15 GHz and a grazing angle of 20º.

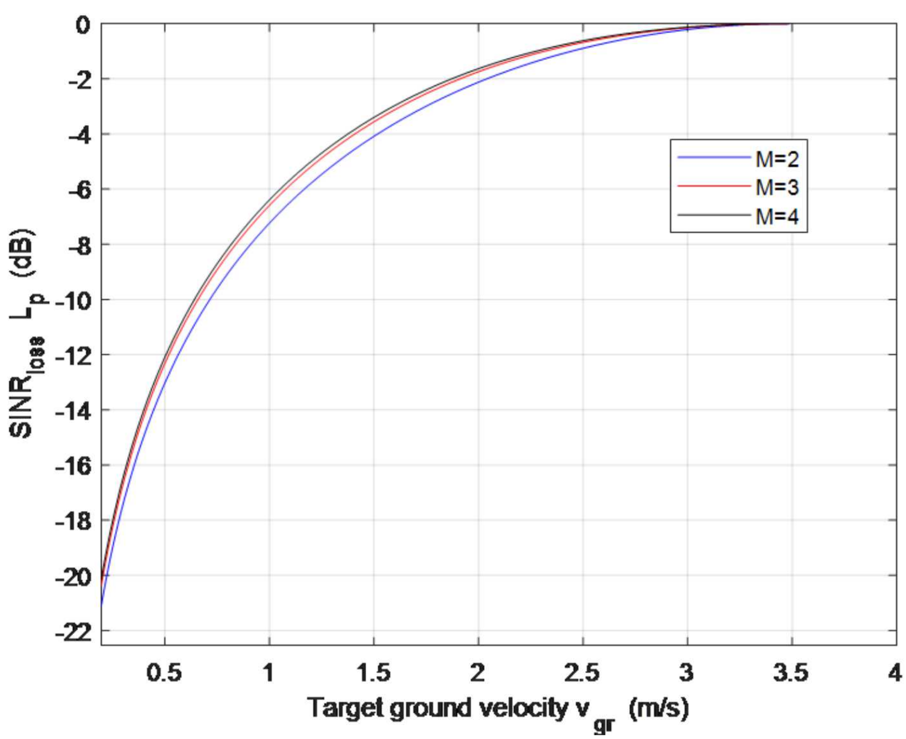

Figure 3: SINR loss as a function of the number of subapertures for a fixed antenna width

For the general $M > 2$ case, there is not a simple analytic expression for κ and MDV so it has to be determined numerically. For the previous example, we recall that for $M = 2$ and SINR loss of 5 dB, we $\kappa \approx 0.38$. For the example above at SINR loss of 5 dB, for $M = 3$ then $\kappa \approx 0.35$, and for $M = 4$ then $\kappa \approx 0.34$.

It is important to note that increasing the number of subapertures without increasing the antenna size has only minimal effect on the SINR loss, and no improvement in the antenna gain under the assumptions in this paper. Therefore, there is only a slight improvement in the MDV in this case[§].

4 SUMMARY & CONCLUSIONS

This paper emphasizes that the key radar measure of minimum detectable velocity comes from the standard radar detection problem. The SINR loss is a unique factor in the detection problem for endoclutter ground moving target indication. This SINR loss is a processing loss that occurs as a side-effect of clutter attenuation from multiple subapertures. Key variables that affect SINR loss were presented. We presented how the common "rule-of-thumb" values for this SINR loss come about, and show that these are just assumptions made to simplify the standard detection problem analysis and present a common language for evaluating system performances. These can come at the price of loss of fidelity of the performance model.

ACKNOWLEDGEMENTS

The authors acknowledge the important initial work done by our retired Sandia colleague, Tom Cordaro, during the mid-1990's through mid-2000's that laid a foundation for Sandia's work in this area.

Sandia National Laboratories is a multimission laboratory managed and operated by National Technology and Engineering Solutions of Sandia, LLC., a wholly owned subsidiary of Honeywell International, Inc., for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-NA-0003525.

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§ There are good reasons other than MDV for additional subapertures. These include improved geolocation for slow movers, and improved detection and geolocation in a contested environment. These topics are outside the scope of this paper.

SYMBOLS

Symbols used in this paper are:

a - is the target amplitude

A_A - is antenn area

D_a - the dimension of the antenna in azimuth

F_N - is noise figure

$\mathbf{I}$ - is the identity matrix

k - is the Boltzman constant

L_p - is the loss term due to clutter attenuation processing (developed in the appendix)

L_{rad} - is various radar loss terms

M - is the number of subapertures used in the processing

MDV - is the minimum detectable velocity

P_{avg} - is the average transmit power

R - is the range to the target

$\mathbf{R}_{\eta\eta}$ - is the covariance of the interference plus noise

$SINR(\theta)$ - is the SINR from the optimal filtering in interference plus noise versus target location

SNR_0 - is the matched filter signal-to-noise ratio if the only interference were thermal noise (absence of clutter)

SNR_{CA} - is the available signal-to-noise ratio including clutter attenuation processing

$SNR_{loss} = -10\log_{10}(L_p)$ in dB

SNR_{Pd} - is the SNR required to achieve the desired P_d for the given P_{fa} (in dB)

T - is the noise temperature

T_{cpi} - is the coherent processing interval time

v_r - is the radial velocity of the mover in the slant plane

$\mathbf{v}_c$ - is the steering vector corresponding to the clutter

$\mathbf{v}_d$ - is target steering vector

v_{gr} - is the radial velocity projected into the ground plane

V_x - is the velocity of the platform in the azimuth direction

ψ - is the grazing angle of the radar

κ - is a factor that scales the minimum detectable velocity

σ_n^2 - the variance of the thermal noise

$\mathbf{w}_{opt}$ - is the optimal filter

α - is a scalar the depends upon the selected detector (e.g., MVDR)

$\beta_c = \dfrac{\sigma_c^2}{\sigma_n^2}$ - is the clutter-to-noise-ratio (cnr)

λ - is the radar wavelength

η_{ap} - is the antenna efficiency

θ_c - is the direction of the clutter

θ_d - is the angle off of the boresight of the antenna to the target direction

σ_n - is the equivalent RCS of the thermal noise in meters-squared

σ_t - is the radar cross-section (RCS) of the target in meters-squared which we assume generally is $|a|^2$

$$\Omega_c = \frac{2\pi D_a \sin\theta_c}{\lambda M}$$

$$\Omega_d = \frac{2\pi D_a \sin\theta_d}{\lambda M}$$

APPENDIX

SNR for Detection

This section of the appendix gives the signal-to-noise ratio (SNR) equation for the endoclutter ground moving target indicator (GMTI) processing. We introduce a version of the radar equation modified from the reference [1] to account for clutter attenuation processing:

$$SNR_{CA} = SNR \cdot L_p = \left[\frac{P_{avg} T_{cpi} \left(\eta_{ap} A_A / \lambda \right)^2 \sigma_t}{\left(4\pi\right) R^4 \left(kTF_N\right)} L_{rad} \right] L_p \tag{13}$$

A few comments about the difference between this equation and that in the reference. We have written the equation to purposely emphasize the term, L_p, which plays an important role in this paper. This is an additional processing loss which occurs as a result of performing clutter attenuation.

For completeness, other differences with [1] include that we have made the "loss" terms into "gain" terms by moving them to the numerator, so they will be less than or equal to one. We have lumped all the other loss terms into L_{rad}. We have re-written the equation to use the radar wavelength, λ.

Also, we note that A_A is proportional to the along-track antenna dimension, D_a. In this paper, we tend to hold the antenna area constant. This results in the processing loss being the main variable parameter which affects the SNR_{CA} in the above equation for our analysis. In other words, we have fixed the antenna gain for this paper when we vary the other quantities, such as the number of subapertures.

Additionally, we imply in the equation that the clutter suppression is ideal down to the thermal noise floor, although one could include additional loss from imperfect clutter cancellation into the L_{rad} term.

SINR Loss

This section of the appendix develops the SINR loss equation used in this paper, and is related to the work in [10] and others. We will assume an ideal implementation with *M* equally spaced phase centers along a uniform linear array. We also assume uniform clutter statistics where the power return is σ_c^2, and thermal noise for each channel that is σ_n^2. We assume a target vector given by:

$$\mathbf{s} = a\mathbf{v}_d \tag{14}$$

It is well known that the optimal filter is:

$$\mathbf{w}_{opt}^H = \alpha \mathbf{R}_{\eta\eta}^{-1} \mathbf{v}_d^H \tag{15}$$

In the case of optimum filter, the signal-to-interference-plus-noise (SINR) loss, or processing loss is given as:

$$L_p\left(\theta\right) = \frac{SINR\left(\theta\right)}{SNR_0} \tag{16}$$

The matched filter SNR is always greater than or equal to the SINR, and is given by:

$$SNR_0 = M\left(\frac{|a|^2}{\sigma_n^2}\right) \tag{17}$$

We now turn our attention to the SINR which is given by:

$$SINR = \frac{|a|^2\left|\mathbf{w}_{opt}^H \mathbf{v}_d\right|^2}{\mathbf{w}_{opt}^H \mathbf{R}_{\eta\eta}\mathbf{w}_{opt}} = \frac{|a|^2\left|\mathbf{v}_{opt}^H \mathbf{R}_{\eta\eta}^{-1}\mathbf{v}_d\right|^2}{\mathbf{v}_d^H \mathbf{R}_{\eta\eta}^{-1}\mathbf{R}_{\eta\eta}\mathbf{R}_{\eta\eta}^{-1}\mathbf{v}_d} = |a|^2\left|\mathbf{v}_{opt}^H \mathbf{R}_{\eta\eta}^{-1}\mathbf{v}_d\right| \tag{18}$$

he interference-plus-noise covariance can be written as:

$$\mathbf{R}_{\eta\eta} = \sigma_c^2 \mathbf{v}_c \mathbf{v}_c^H + \sigma_n^2 \mathbf{I} \tag{19}$$

We acknowledge that the covariance model in equation (19) is simplified in many respects, but it will help to highlight key quantities of interest pertinent to this memorandum.

We can invert the covariance using the famous Sherman-Morrison formula to yield:

$$\mathbf{R}_{\eta\eta}^{-1} = \left(\frac{1}{\sigma_n^2}\right)\left[\mathbf{I} - \left(\frac{M\beta_c}{1+\beta_c M}\right)\left(\frac{\mathbf{v}_c\mathbf{v}_c^H}{M}\right)\right] \tag{20}$$

We plug this back into the above equation for SINR:

$$\begin{aligned}
|a|^2\left|\mathbf{v}_d^H \mathbf{R}_{\eta\eta}^{-1}\mathbf{v}_d\right| &= \left(\frac{|a|^2}{\sigma_n^2}\right)\left|\mathbf{v}_d^H\left[\mathbf{I} - \left(\frac{M\beta_c}{1+\beta_c M}\right)\left(\frac{\mathbf{v}_c\mathbf{v}_c^H}{M}\right)\right]\mathbf{v}_d\right| \\
&= \left(\frac{|a|^2}{\sigma_n^2}\right)\left|\left[M - \left(\frac{M\beta_c}{1+\beta_c M}\right)\left(\frac{\mathbf{v}_d^H\mathbf{v}_c\mathbf{v}_c^H\mathbf{v}_d}{M}\right)\right]\right| \\
&= \left(\frac{|a|^2}{\sigma_n^2}\right)\left|\left[M - \left(\frac{M\beta_c}{1+\beta_c M}\right)\left(\frac{\left|\mathbf{v}_d^H\mathbf{v}_c\right|^2}{M}\right)\right]\right| \\
&= M\left(\frac{|a|^2}{\sigma_n^2}\right)\left|\left[1 - \left(\frac{M\beta_c}{1+\beta_c M}\right)\left(\frac{\left|\mathbf{v}_d^H\mathbf{v}_c\right|^2}{M^2}\right)\right]\right|
\end{aligned} \tag{21}$$

We will now define a couple of terms:

$$\rho_{cnr} = \left(\frac{M\beta_c}{1+\beta_c M}\right) \tag{22}$$

and

$$\rho_{dc} = \frac{\mathbf{v}_d^H \mathbf{v}_c}{M} \tag{23}$$

We note that both ρ_{cnr} and ρ_{dc} have the properties of correlation coefficients, for example both have magnitudes between 0 and 1.

We re-write the SINR equation as:

$$SINR = M\left(\frac{|a|^2}{\sigma_n^2}\right)\left|1-\rho_{cnr}\left|\rho_{dc}\right|^2\right| \tag{24}$$

Note that $|a|^2/\sigma_n^2$ is accounted for in equation (13). Also note that the M in this equation says for the uniform linear array the individual subarrays are only D_a/M in width. Thus although A_A/M is the area for each subaperture in equation (1), we gain back the factor of M in processing. In the rest of this appendix we will normalize to the latter term in equation (24) and recognize that that this term after normalization is just the L_p in equation (1). L_p is bound between 0 and 1, with 1 being the matched filter (or standard beamformer) case.

Now we can write the SINR/processing loss that we will use in this memorandum concisely as:

$$L_p(\Omega_d, \Omega_c) = \left|1-\rho_{cnr}\left|\rho_{dc}\right|^2\right| \tag{25}$$

We defined Ω_d and Ω_c in the symbols section, but we will drop these terms in the main text for notational simplicity.

Assuming a uniform linear array (ULA) that results in a target steering vector of:

$$\mathbf{v}_d^T = \begin{bmatrix} 1 & e^{-j\Omega_d} & e^{-j2\Omega_d} \dots & e^{-j(M-1)\Omega_d} \end{bmatrix} \tag{26}$$

and the steering vector for the clutter is:

$$\mathbf{v}_c^T = \begin{bmatrix} 1 & e^{-j\Omega_c} & e^{-j2\Omega_c} \dots & e^{-j(M-1)\Omega_c} \end{bmatrix} \tag{27}$$

This leads to:

$$M\rho_{dc} = \mathbf{v}_d^H \mathbf{v}_c = 1 + e^{-j(\Omega_d-\Omega_c)} + e^{-j2(\Omega_d-\Omega_c)} + \dots + e^{-j(M-1)(\Omega_d-\Omega_c)} \qquad (28)::$$

or:

$$\rho_{dc} = e^{-j(M-1)(\Omega_d-\Omega_c)/2} \frac{\sin\left[M\left(\Omega_d-\Omega_c\right)/2\right]}{M\sin\left[\left(\Omega_d-\Omega_c\right)/2\right]} \qquad (29)::$$

The last term in called the Dirichlet sinc or the aliased sinc function, which is familiar in digital signal processing.

Plugging this back into the SINR loss gives:

$$L_p\left(\Omega_d,\Omega_c\right)=\left|1-\rho_{cnr}\left|\frac{\sin\left[M\left(\Omega_d-\Omega_c\right)/2\right]}{M\sin\left[\left(\Omega_d-\Omega_c\right)/2\right]}\right|^2\right| \tag{30}$$

To reduce the "clutter" a bit further in this equation, we assume that the clutter angle is broadside so that:

$$\begin{aligned} L_p\left(\Omega_d,\Omega_c=0\right)&=\left|1-\rho_{cnr}\left|\frac{\sin\left[M\,\Omega_d/2\right]}{M\sin\left[\Omega_d/2\right]}\right|^2\right| \\ &=\left|1-\rho_{cnr}\left|\frac{\sin\left[\pi MD_a\sin\theta_d/\left(\lambda M\right)\right]}{M\sin\left[\pi D_a\sin\theta_d/\left(\lambda M\right)\right]}\right|^2\right| \end{aligned} \tag{31}$$

Equation (31) will be the main equation that we use in the main body of this paper.