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Geometrically Symmetric Quadrature Rules for Singular Integrals in the Method-of-Moments Implementation of the Electric-Field Integral Equation

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Outline

- Introduction
- Triangle Quadrature Rules
- Logarithmic Singularities in the Test Integrand of the EFIE
- Numerical Experiments for (Near-)Singular and Far Interactions
- Summary

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- Introduction
  - The Method of Moments Implementation of the EFIE
  - Existing Approaches for (Near-)Singularities
  - This Work
- Triangle Quadrature Rules
- Logarithmic Singularities in the Test Integrand of the EFIE
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- Summary

The Method of Moments Implementation of the EFIE

- Surfaces are discretized using planar or curvilinear mesh elements
- 4D integrals are evaluated over source and test elements
- Green's function yields (near-)singularities in higher-order derivatives
  - In scalar and vector potential terms
  - Singularities: test and source elements share one or more edges or vertices
  - Near-Singularities: test and source elements are otherwise close

Existing Approaches for (Near-)Singularities

- Approaches for inner, source-element integral
  - Singularity subtraction
  - Singularity cancellation through variable transformation
  - Hybrid schemes that combine subtraction and cancellation
- Approaches for outer, test-element integral
  - Outer product of 1D quadrature rules
  - Series of variable transformations and integration reordering
  - Other approaches for MFIE and CFIE

This Work

- Development of geometrically symmetric quadrature rules
- Characterization of logarithmic singularities in the test integral

Outline

- Introduction
- Triangle Quadrature Rules
  - Overview
  - Triangles
  - Approach 1: Optimization for Moderate Number of Functions
  - Approach 2: Quadrilateral Subdomains
- Logarithmic Singularities in the Test Integrand of the EFIE
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Overview

- An n-point quadrature rule exactly integrates a sequence of n_f functions $\mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), \dots, f_{n_f}(\mathbf{x})\}$, such that

$$\int_A \mathbf{f}(\mathbf{x}) dA = \sum_{i=1}^{n} w_i \mathbf{f}(\mathbf{x}_i)$$

- In 1D, $n_f = 2n$ and, for polynomials, $\mathbf{f}(x) = \{1, \dots, x^{2n-1}\}$
- In 2D, $n_f \stackrel{?}{=} 3n$,
  - This is unproven
  - If rules are symmetric, the efficiency can be significantly lower

Challenges to Generate

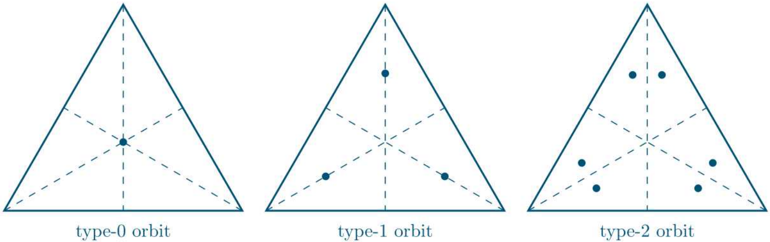
- Regardless of dimension and function sequence, equations for computing quadrature rules are stiff and highly dependent upon initial guess
- In multiple dimensions, for a given number of points, n_f is unknown

Triangles

- Quadrature rules for triangles useful for evaluating surface integrals
- Several authors have computed symmetric rules for polynomials
  - Rules do not converge monotonically or rapidly for singular integrands
  - For 1D, rules have been developed for singular functions
- Geometrically symmetric rules are desirable
  - Mapping is straightforward
  - Points are not more concentrated at a single vertex

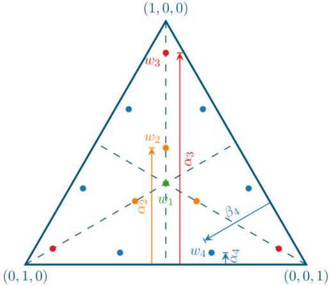
Symmetric Rules for Triangles

- Invariant to rotation and reflection about the medians for equilateral triangles
- Triangles can be isoparametrically transformed to other triangles
- Rules are constructed from a combination of orbits, such that $n = n_0 + 3n_1 + 6n_2$



Approach 1: Overview

- Goal is to efficiently integrate polynomials and singularities
- Compute points & weights through optimization – nonlinear least squares
- This approach uses polynomial rules as a baseline
  - Initial guesses near the polynomial rule
  - Same orbit counts for each n
- Replace higher polynomial degrees with singular functions
- Attempt to increase number of functions integrated



Approach 1: Computation

- Given a function sequence, we formulate problem as unconstrained optimization problem in barycentric coordinates:

$$\underset{\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w}}{\arg\min}\, F(\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w}),$$

where

$$F(\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w}) = \sum_{j=1}^{n_f} \left(\frac{\tilde{I}_{f_j} - I_{f_j}}{I_{f_j}}\right)^2,$$

$$\tilde{I}_{f_j} = \sum_{i=1}^{n} w_i' f_j(\alpha_i,\beta_i), \qquad I_{f_j} = \int_0^1 \int_0^{1-\beta} f_j(\alpha,\beta)\,d\alpha\, d\beta,$$

with the expectation that $F(\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w}) = 0$

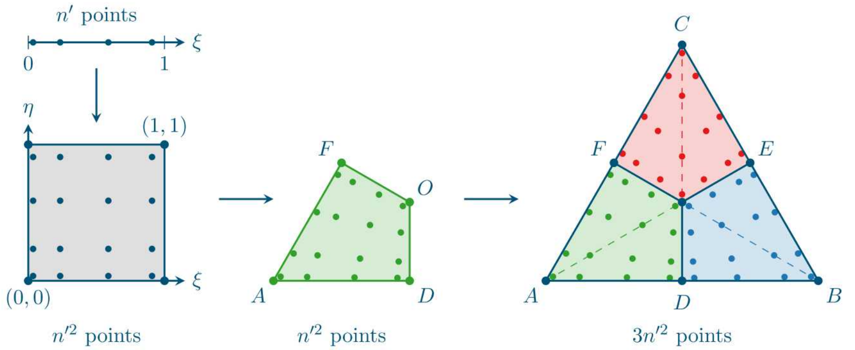
- We only consider interior points

Approach 1: Function Sequence

- Weigh number of singular functions against maximum polynomial degree
- Ability to integrate polynomials includes ability to integrate cross terms (e.g., x^3 includes x^2y)
- Ability to integrate singular functions does not extend to cross terms
- Three approaches to address this issue:
  - Use 2D characterization of singularity, if available
  - Use 1D characterization of singularity, assume cross terms are not essential
  - Include cross terms for 1D characterization and reduce polynomial degree
- Alternatively, one can use Approach 2

Approach 2: Overview

- In multiple dimensions, number of integrable functions not straightforward
- Computation is expensive and multiple solutions exist
- For large n_f, we employ n'-point 1D rules that integrate 1D function sequences, such that $n = 3n'^2$



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  - Coplanar Domains
  - Perpendicular Domains
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Singularities in Scalar Potential and Vector Potential

Singular integrals in EFIE when using MoM take the form

$$I_s = \int_{A_\mathcal{T}} \nabla \cdot \mathbf{\Lambda}_\mathcal{T}^j \int_{A_\mathcal{S}} \frac{e^{-jkR}}{R} \nabla \cdot \mathbf{\Lambda}_\mathcal{S}^i dA_\mathcal{S} dA_\mathcal{T},$$

$$I_v = \int_{A_\mathcal{T}} \mathbf{\Lambda}_\mathcal{T}^j \cdot \int_{A_\mathcal{S}} \frac{e^{-jkR}}{R} \mathbf{\Lambda}_\mathcal{S}^i dA_\mathcal{S} dA_\mathcal{T},$$

where I_s appears in scalar potential, I_v appears in vector potential

- $A_\mathcal{S}$ and $A_\mathcal{T}$ are source- and test-element surfaces
- $R(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T}) = \|\mathbf{x}_\mathcal{S} - \mathbf{x}_\mathcal{T}\|_2$
- $\mathbf{x}_\mathcal{S}$ and $\mathbf{x}_\mathcal{T}$ are source and test points
- $\mathbf{\Lambda}_\mathcal{T}^j(\mathbf{x}_\mathcal{T})$ is test basis function associated with edge j
- $\mathbf{\Lambda}_\mathcal{S}^i(\mathbf{x}_\mathcal{S})$ is source basis function associated with edge i

Scalar Potential

When $\mathbf{\Lambda}_\mathcal{T}^j$ and $\mathbf{\Lambda}_\mathcal{S}^i$ are linear (e.g., RWG), $\nabla \cdot \mathbf{\Lambda}_\mathcal{T}^j$ and $\nabla \cdot \mathbf{\Lambda}_\mathcal{S}^i$ are constants:

$$I_s = C_1 \int_{A_\mathcal{T}} \int_{A_\mathcal{S}} \frac{e^{-jkR}}{R} dA_\mathcal{S} dA_\mathcal{T}$$

Taylor-series expansion test integrand about R:

$$f(\mathbf{x}_\mathcal{T}) = \sum_{p=0}^{\infty} \frac{(-jk)^p}{p!} \int_{A_\mathcal{S}} R(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T})^{p-1} dA_\mathcal{S}$$

Odd powers of R yield unbounded derivatives along boundaries of $A_\mathcal{S}$
Even powers of R remain smooth and integrable

Vector Potential

When $\boldsymbol{\Lambda}_\mathcal{T}^j(\mathbf{x}_\mathcal{T}) = \mathbf{x}_\mathcal{T} - \mathbf{x}_j$ and $\boldsymbol{\Lambda}_\mathcal{S}^i(\mathbf{x}_\mathcal{S}) = \mathbf{x}_\mathcal{S} - \mathbf{x}_i$ (unnormalized RWG)

- $\mathbf{x}_j$ is vertex of test element opposite edge j
- $\mathbf{x}_i$ is vertex of source element opposite edge i

$$\boldsymbol{\Lambda}_\mathcal{T}^j \cdot \boldsymbol{\Lambda}_\mathcal{S}^i = (\mathbf{x}_\mathcal{T} - \mathbf{x}_j) \cdot (\mathbf{x}_\mathcal{S} - \mathbf{x}_i) = \left(\tilde{\mathbf{x}} + \frac{\mathbf{x}_\mathcal{T} - \mathbf{x}_\mathcal{S}}{2} - \mathbf{x}_j\right) \cdot \left(\tilde{\mathbf{x}} - \frac{\mathbf{x}_\mathcal{T} - \mathbf{x}_\mathcal{S}}{2} - \mathbf{x}_i\right)$$
$$= D_0 + D_1 R + D_2 R^2,$$

$\tilde{\mathbf{x}}(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T}) = (\mathbf{x}_\mathcal{S} + \mathbf{x}_\mathcal{T})/2$, $\qquad \phi(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T})$ is angle between $(\mathbf{x}_\mathcal{T} - \mathbf{x}_\mathcal{S})$ and $(\mathbf{x}_j - \mathbf{x}_i)$,

$D_0(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T}) = \|\tilde{\mathbf{x}}\|_2^2 - (\mathbf{x}_i + \mathbf{x}_j) \cdot \tilde{\mathbf{x}} + \mathbf{x}_i \cdot \mathbf{x}_j$, $\quad D_1(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T}) = \frac{\|\mathbf{x}_j - \mathbf{x}_i\|_2}{2} \cos\phi$, $\quad D_2 = -1/4$

$$I_v = \int_{A_\mathcal{T}} \int_{A_\mathcal{S}} D_0 \frac{e^{-jkR}}{R} dA_\mathcal{S} dA_\mathcal{T} + \int_{A_\mathcal{T}} \int_{A_\mathcal{S}} D_1 e^{-jkR} dA_\mathcal{S} dA_\mathcal{T} + D_2 \int_{A_\mathcal{T}} \int_{A_\mathcal{S}} e^{-jkR} R \, dA_\mathcal{S} dA_\mathcal{T}$$

Taylor series expansion leads to integer powers of R (odd powers yield singularities)

Singularities in Scalar Potential and Vector Potential

Scalar and vector potential contain singularities of the form

$$\int_{A_\mathcal{S}} R(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T})^q dA_\mathcal{S}, \quad \text{for } q = -1,\, 0,\, 1, \dots$$

Coplanar Domains: $q=-1$

For $q=-1$,

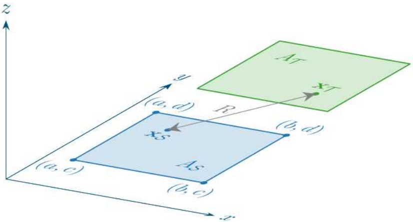
$$\int_{A_\mathcal{S}} R(\mathbf{x}_\mathcal{S},\mathbf{x}_\mathcal{T})^{-1} dA_\mathcal{S} = \int_c^d \int_a^b \frac{1}{\sqrt{(x_\mathcal{T}-x_\mathcal{S})^2+(y_\mathcal{T}-y_\mathcal{S})^2}} dx_\mathcal{S} dy_\mathcal{S}$$

$$= \sum_{i=1}^{4} \left\{ \alpha_i \ln\left[\beta_i + \sqrt{\alpha_i^2+\beta_i^2}\right] - \alpha_i \ln\left[\gamma_i + \sqrt{\alpha_i^2+\gamma_i^2}\right] \right\}$$

$\alpha = \{y_\mathcal{T}-c,\, y_\mathcal{T}-d,\, x_\mathcal{T}-a,\, x_\mathcal{T}-b\}$,

$\beta = \{x_\mathcal{T}-a,\, x_\mathcal{T}-b,\, y_\mathcal{T}-c,\, y_\mathcal{T}-d\}$,

$\gamma = \{x_\mathcal{T}-b,\, x_\mathcal{T}-a,\, y_\mathcal{T}-d,\, y_\mathcal{T}-c\}$



Coplanar Domains: Vertex Singularities ($q = -1$)

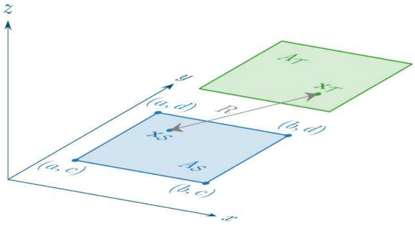
Approaching edges of $A_\mathcal{S}$, where $\beta_i = 0$ or $\gamma_i = 0$,

$$\lim_{\beta_i \to 0} \alpha_i \ln \left[\beta_i + \sqrt{\alpha_i^2 + \beta_i^2} \right] = \alpha_i \ln |\alpha_i|,$$

$$\lim_{\gamma_i \to 0} \alpha_i \ln \left[\gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2} \right] = \alpha_i \ln |\alpha_i|$$

At vertices of $A_\mathcal{S}$, where $\alpha_i = 0$, $\alpha_i \ln |\alpha_i|$ is singular (unbounded derivatives)

$$\alpha = \{y_\mathcal{T} - c, \, y_\mathcal{T} - d, \, x_\mathcal{T} - a, \, x_\mathcal{T} - b\},$$
$$\beta = \{x_\mathcal{T} - a, \, x_\mathcal{T} - b, \, y_\mathcal{T} - c, \, y_\mathcal{T} - d\},$$
$$\gamma = \{x_\mathcal{T} - b, \, x_\mathcal{T} - a, \, y_\mathcal{T} - d, \, y_\mathcal{T} - c\}$$



Coplanar Domains: Edge Singularities ($q = -1$)

On edges of $A_\mathcal{S}$ not at vertices, $\alpha_i = 0$, and β_i and γ_i have opposite signs

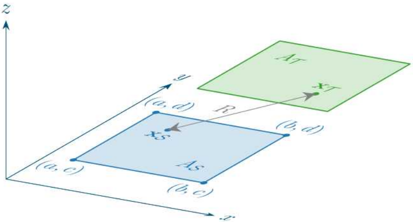
Taylor series expansions of logarithm argument:

$$\beta_i + \sqrt{\alpha_i^2 + \beta_i^2} = \beta_i + |\beta_i| + \frac{\alpha_i^2}{2|\beta_i|} + \mathcal{O}(\alpha_i^3), \qquad \text{when } \beta_i < 0, \approx \frac{\alpha_i^2}{2|\beta_i|}$$

$$\gamma_i + \sqrt{\alpha_i^2 + \gamma_i^2} = \gamma_i + |\gamma_i| + \frac{\alpha_i^2}{2|\gamma_i|} + \mathcal{O}(\alpha_i^3), \qquad \text{when } \gamma_i < 0, \approx \frac{\alpha_i^2}{2|\gamma_i|}$$

These yield terms with $\alpha_i \ln|\alpha_i|$ singularities

$$\alpha = \{y_\mathcal{T} - c,\, y_\mathcal{T} - d,\, x_\mathcal{T} - a,\, x_\mathcal{T} - b\},$$
$$\beta = \{x_\mathcal{T} - a,\, x_\mathcal{T} - b,\, y_\mathcal{T} - c,\, y_\mathcal{T} - d\},$$
$$\gamma = \{x_\mathcal{T} - b,\, x_\mathcal{T} - a,\, y_\mathcal{T} - d,\, y_\mathcal{T} - c\}$$



Coplanar Domains: Series Expansions and Higher Powers

Along an edge of $A_\mathcal{S}$, a series expansion of $q=-1$ integrand as $\alpha_i \to 0$ is

$$1,\ \alpha_i,\ \alpha_i \ln|\alpha_i|,\ \alpha_i^2,\ \alpha_i^3,\ \alpha_i^4,\ \alpha_i^5,\ldots$$

For $q=1$, $\int_{A_\mathcal{S}} R(\mathbf{x}_\mathcal{S},\mathbf{x}_\mathcal{T})dA_\mathcal{S}$, yield additional terms, including singular terms

$$\alpha_i^3 \ln\left[\beta_i+\sqrt{\alpha_i^2+\beta_i^2}\right], \qquad \alpha_i^3 \ln\left[\gamma_i+\sqrt{\alpha_i^2+\gamma_i^2}\right]$$

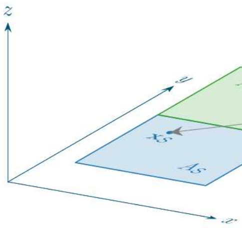
and series expansion

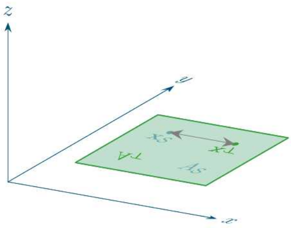
$$1,\ \alpha_i,\ \alpha_i^2,\ \alpha_i^3,\ \alpha_i^3 \ln|\alpha_i|,\ \alpha_i^4,\ \alpha_i^5,\ldots$$

Trend continues for odd powers of R, yielding $\alpha_i^{q+2}\ln|\alpha_i|$ from

$$\alpha_i^{q+2} \ln\left[\beta_i+\sqrt{\alpha_i^2+\beta_i^2}\right], \qquad \alpha_i^{q+2} \ln\left[\gamma_i+\sqrt{\alpha_i^2+\gamma_i^2}\right]$$

Coplanar Domains: Singular Examples





$A_\mathcal{T}$ and $A_\mathcal{S}$ share an edge:
Shared edge and vertices have singularities

$A_\mathcal{T} = A_\mathcal{S}$:
Entire boundary has singularities

Perpendicular Domains: $q = -1$

For $q = -1$,

$$\int_{A_\mathcal{S}} R(\mathbf{x}_\mathcal{S}, \mathbf{x}_\mathcal{T})^{-1} dA_\mathcal{S}$$

$$= \int_0^a \int_c^d \frac{1}{\sqrt{x_\mathcal{T}^2 + (y_\mathcal{T} - y_\mathcal{S})^2 + z_\mathcal{S}^2}} dy_\mathcal{S} dz_\mathcal{S}$$

$$= - x_\mathcal{T} \arctan \frac{a y_c}{x_\mathcal{T} R_c} + x_\mathcal{T} \arctan \frac{a y_d}{x_\mathcal{T} R_d}$$

$$- \frac{y_c}{2} \ln \left[x_\mathcal{T}^2 + y_c^2\right] + \frac{y_d}{2} \ln \left[x_\mathcal{T}^2 + y_d^2\right]$$

$$+ y_c \ln \left[a + R_c\right] - y_d \ln \left[a + R_d\right]$$

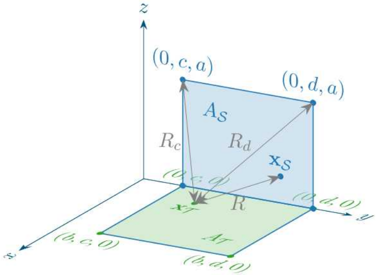
$$+ a \ln \left[y_c + R_c\right] - a \ln \left[y_d + R_d\right]$$

$$R_c^2 = x_\mathcal{T}^2 + (y_\mathcal{T} - c)^2 + a^2,$$

$$R_d^2 = x_\mathcal{T}^2 + (y_\mathcal{T} - d)^2 + a^2,$$

$$y_c = y_\mathcal{T} - c,$$

$$y_d = y_\mathcal{T} - d$$



Perpendicular Domains: Vertex Singularities ($q = -1$)

Approaching shared vertices from inside $A_\mathcal{T}$, $x_\mathcal{T} = 0$,

$$\lim_{x_\mathcal{T}\to 0} \frac{y_c}{2} \ln\left[x_\mathcal{T}^2 + y_c^2\right] = y_c \ln|y_c|,$$

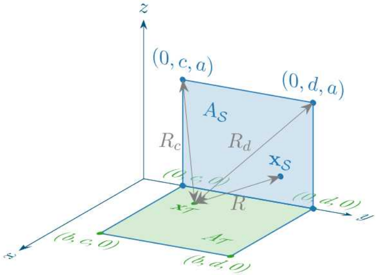
$$\lim_{x_\mathcal{T}\to 0} \frac{y_d}{2} \ln\left[x_\mathcal{T}^2 + y_d^2\right] = y_d \ln|y_d|$$

$$R_c^2 = x_\mathcal{T}^2 + (y_\mathcal{T} - c)^2 + a^2,$$
$$R_d^2 = x_\mathcal{T}^2 + (y_\mathcal{T} - d)^2 + a^2,$$
$$y_c = y_\mathcal{T} - c,$$
$$y_d = y_\mathcal{T} - d$$

At shared vertices, where $y_c = 0$ or $y_d = 0$, $y_c \ln|y_c|$ or $y_d \ln|y_d|$ is singular

Trend continues for other odd q powers, with $y_c^{q+2}\ln|y_c|$ and $y_d^{q+2}\ln|y_d|$

Arctangent terms are also singular at shared vertices but behave differently



One-Dimensional Characterization

- Series expansion about singularity location
- Expansion alternates between monomials and singularities
- From coplanar and perpendicular cases,

$$\mathbf{f}(x) = \{1,\, x,\, x\ln x,\, x^2,\, x^3,\, x^3\ln x,\, x^4,\, x^5,\, x^5\ln x, \dots\}$$

Two-Dimensional Characterization

- Expansions alternate between monomials and singularities
- From coplanar case,

$$1$$
$$x$$
$$x\ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$$
$$x\ln\big(y+\sqrt{x^2+y^2}\big)$$
$$x^2,\ xy$$
$$x^3,\ x^2y$$
$$x^3\ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$$
$$x^3\ln\big(y+\sqrt{x^2+y^2}\big)$$
$$x^4,\ x^3y,\ x^2y^2$$
$$x^5,\ x^4y,\ x^3y^2$$
$$x^5\ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$$
$$x^5\ln\big(y+\sqrt{x^2+y^2}\big)$$
$$x^6,\ x^5y,\ x^4y^2,\ x^3y^3$$
$$x^7,\ x^6y,\ x^5y^2,\ x^4y^3$$
$$x^7\ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$$
$$x^7\ln\big(y+\sqrt{x^2+y^2}\big)$$
$$x^8,\ x^7y,\ x^6y^2,\ x^5y^3,\ x^4y^4$$
$$x^9,\ x^8y,\ x^7y^2,\ x^6y^3,\ x^5y^4$$

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Overview

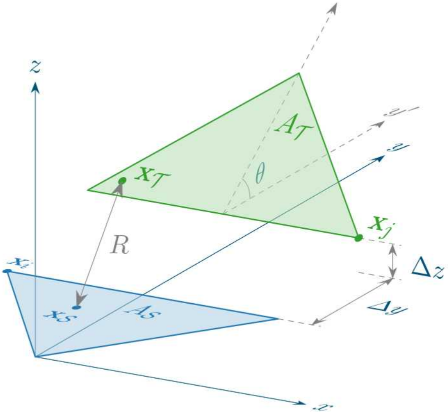
For convenience, we assume k is real and $\lambda = 1$ m

Scalar Potential

$$I_{s,c} = \int_{A_\mathcal{T}} \int_{A_\mathcal{S}} \frac{\cos(2\pi R)}{R} dA_\mathcal{S} dA_\mathcal{T}$$
$$I_{s,s} = \int_{A_\mathcal{T}} \int_{A_\mathcal{S}} \frac{\sin(2\pi R)}{R} dA_\mathcal{S} dA_\mathcal{T}$$

Vector Potential

$$I_{v,c} = \int_{A_\mathcal{T}} (\mathbf{x}_\mathcal{T} - \mathbf{x}_j) \cdot \int_{A_\mathcal{S}} \frac{\cos(2\pi R)}{R} (\mathbf{x}_\mathcal{S} - \mathbf{x}_i) dA_\mathcal{S} dA_\mathcal{T}$$
$$I_{v,s} = \int_{A_\mathcal{T}} (\mathbf{x}_\mathcal{T} - \mathbf{x}_j) \cdot \int_{A_\mathcal{S}} \frac{\sin(2\pi R)}{R} (\mathbf{x}_\mathcal{S} - \mathbf{x}_i) dA_\mathcal{S} dA_\mathcal{T}$$



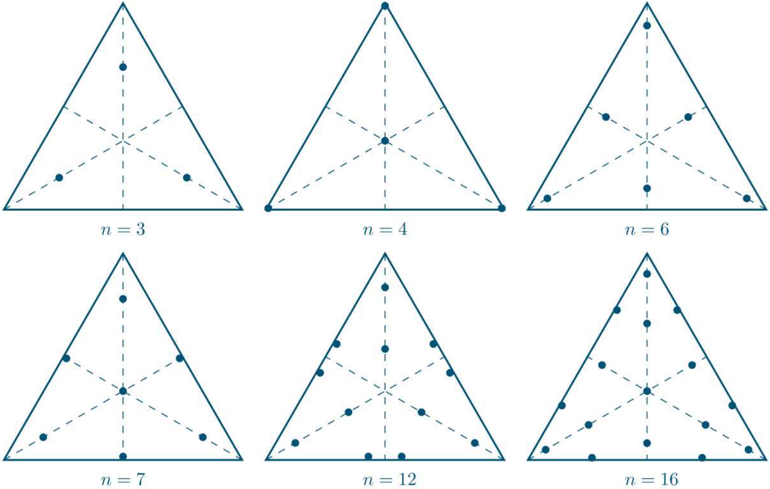
$A_\mathcal{S}$ has vertices $(0\,\text{m}, 0\,\text{m})$, $(1/20\,\text{m}, 1/20\,\text{m})$, and $(-1/20\,\text{m}, 1/20\,\text{m})$

$A_\mathcal{T}$ has same shape

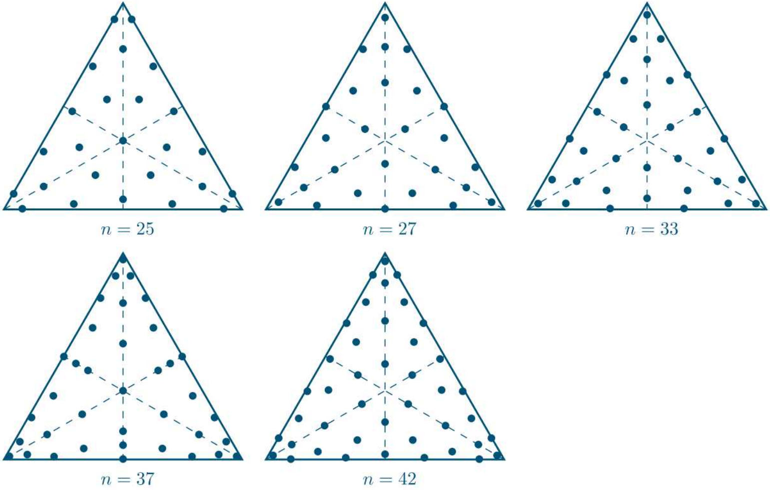
Approach 1 Function Sequences

1D Singularities	2D Singularities
1	1
x	x
$x \ln x$	$x \ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$
$x^2,\ xy$	$x \ln\big(y+\sqrt{x^2+y^2}\big)$
$x^3,\ x^2y$	$x^2,\ xy$
$x^3 \ln x$	$x^3,\ x^2y$
$x^4,\ x^3y,\ x^2y^2$	$x^3 \ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$
$x^5,\ x^4y,\ x^3y^2$	$x^3 \ln\big(y+\sqrt{x^2+y^2}\big)$
$x^5 \ln x$	$x^4,\ x^3y,\ x^2y^2$
$x^6,\ x^5y,\ x^4y^2,\ x^3y^3$	$x^5,\ x^4y,\ x^3y^2$
$x^7,\ x^6y,\ x^5y^2,\ x^4y^3$	$x^5 \ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$
$x^7 \ln x$	$x^5 \ln\big(y+\sqrt{x^2+y^2}\big)$
$x^8,\ x^7y,\ x^6y^2,\ x^5y^3,\ x^4y^4$	$x^6,\ x^5y,\ x^4y^2,\ x^3y^3$
$x^9,\ x^8y,\ x^7y^2,\ x^6y^3,\ x^5y^4$	$x^7,\ x^6y,\ x^5y^2,\ x^4y^3$
$x^9 \ln x$	$x^7 \ln\big(y-1+\sqrt{x^2+(y-1)^2}\big)$
$x^{10},\ x^9y,\ x^8y^2,\ x^7y^3,\ x^6y^4,\ x^5y^5$	$x^7 \ln\big(y+\sqrt{x^2+y^2}\big)$

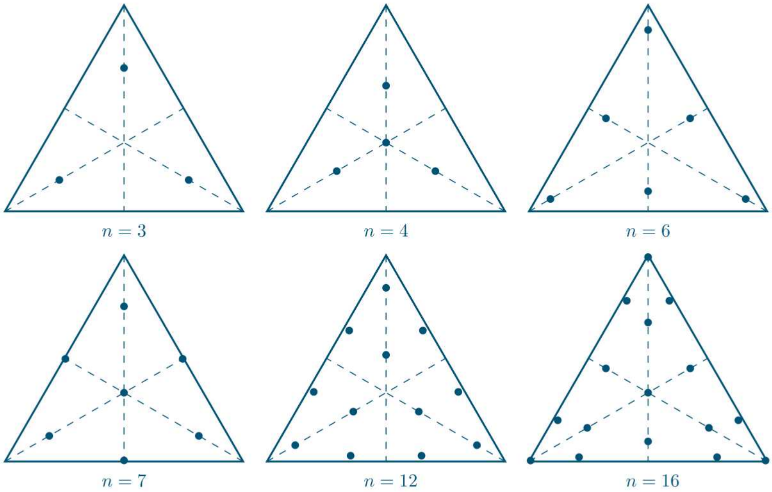
Approach 1, 1D Singularities



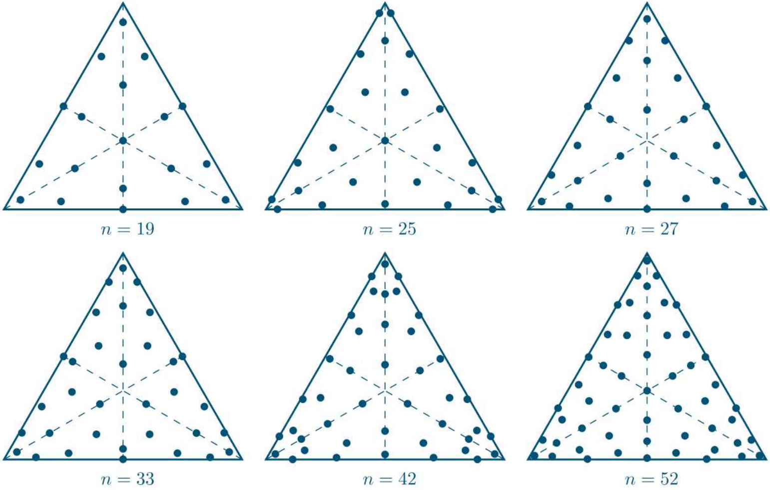
Approach 1, 1D Singularities (continued)



Approach 1, 2D Singularities



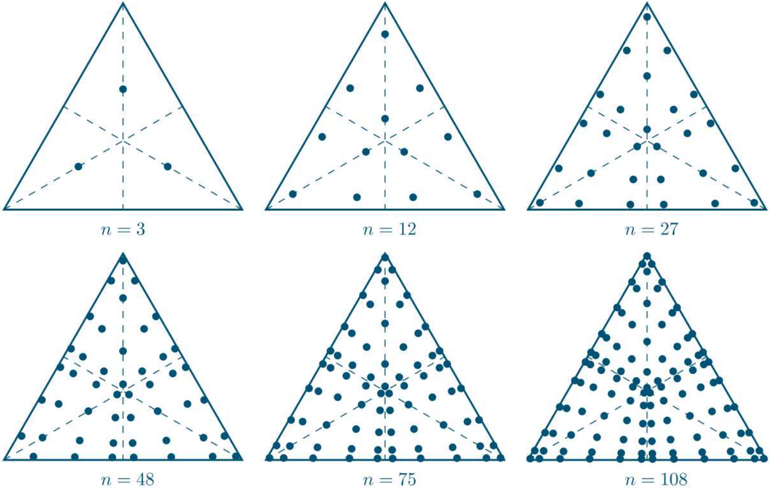
Approach 1, 2D Singularities (continued)



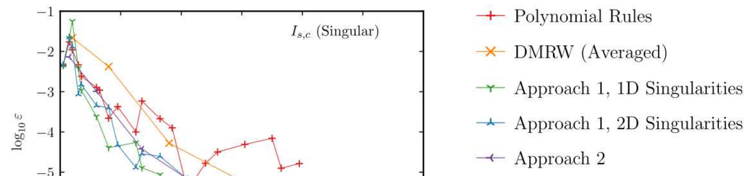
Approach 2 Function Sequence

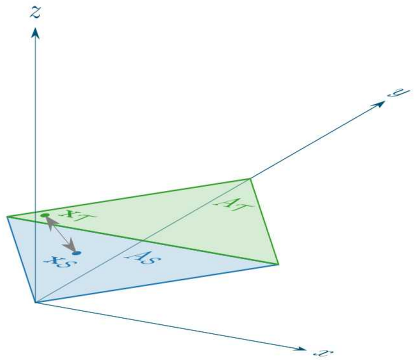
$$\mathbf{f}(x) = \{1,\, x,\, x\ln x,\, x^2,\, x^3,\, x^3\ln x,\, x^4,\, x^5,\, x^5\ln x, \ldots\}$$

Approach 2

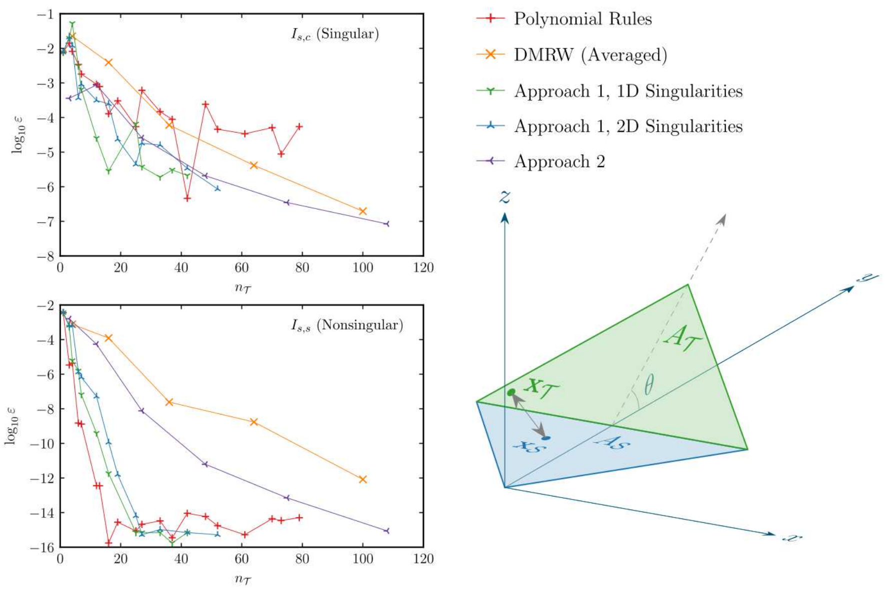


Case 1: Scalar potential, singular interaction, $\theta = 0°$, $\Delta y = 0$, and $\Delta z = 0$

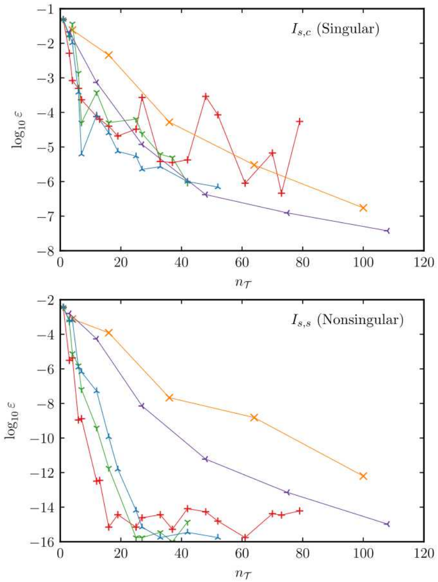


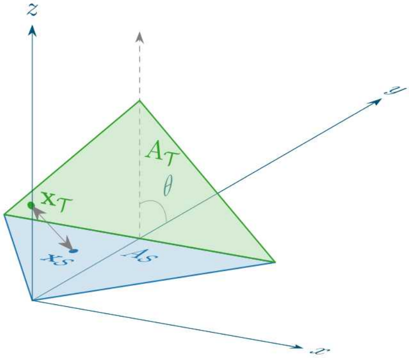


Case 2: Scalar potential, singular interaction, $\theta = 45°$, $\Delta y = 0$, and $\Delta z = 0$

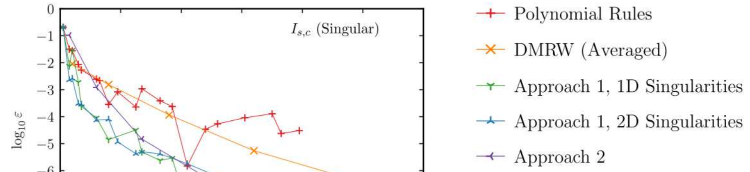


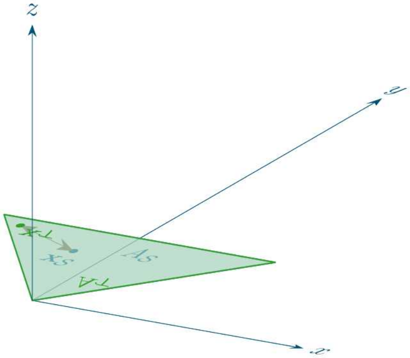
Case 3: Scalar potential, singular interaction, $\theta = 90°$, $\Delta y = 0$, and $\Delta z = 0$



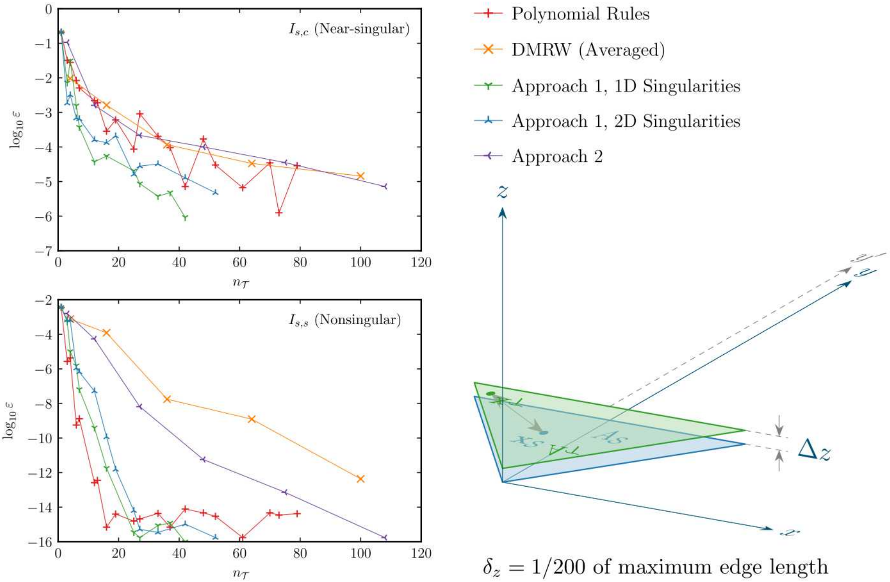


Case 4: Scalar potential, singular interaction, $\theta = 180°$, $\Delta y = 0$, and $\Delta z = 0$



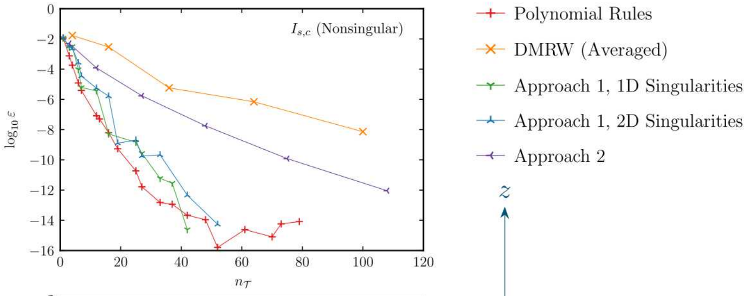


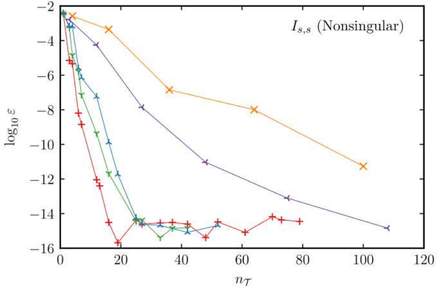
Case 5: Scalar potential, near-singular interaction, $\theta = 180°$, $\Delta y = 0$, and $\Delta z = \delta_z$

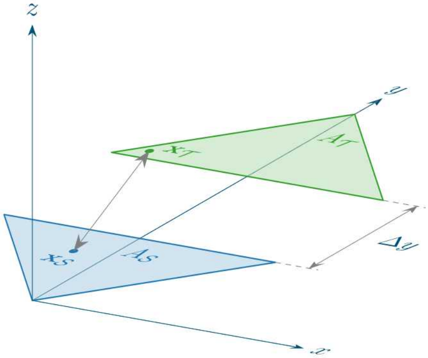


$\delta_z = 1/200$ of maximum edge length

Case 6: Scalar potential, far interaction, $\theta = 0^\circ$, $\Delta y = \delta_y$, and $\Delta z = 0$

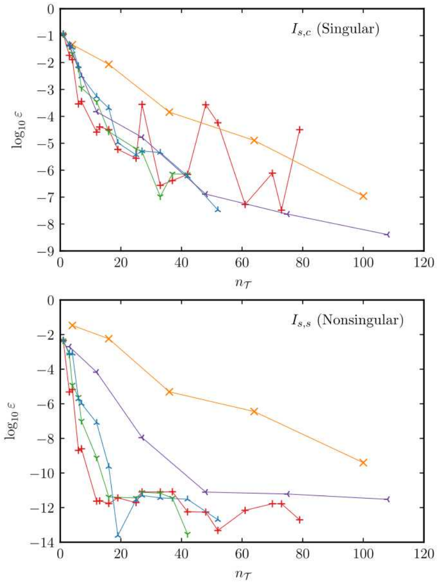


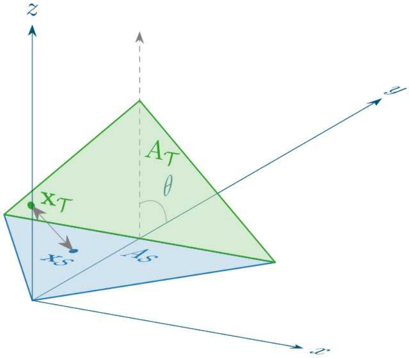




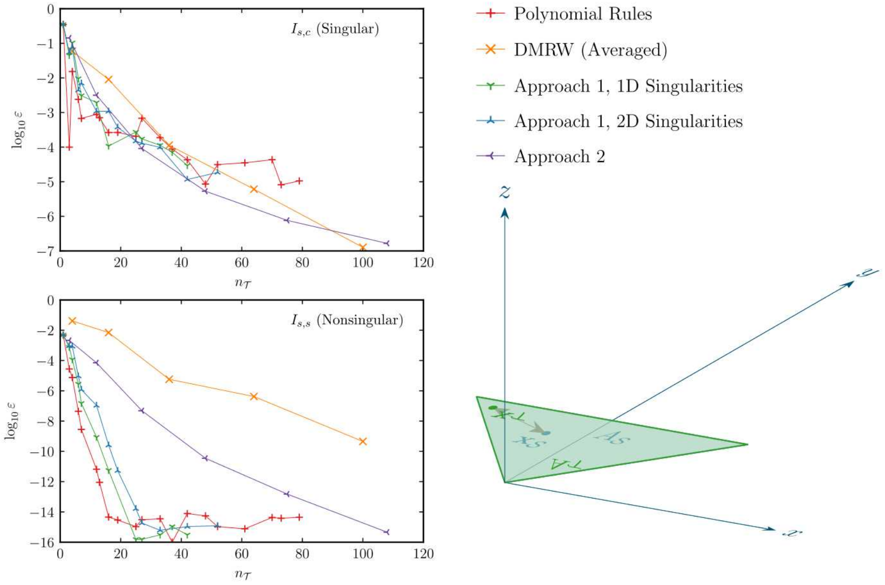
$\delta_y \approx 1.25 \times$ (maximum edge length)

Case 7: Vector potential, singular interaction, $\theta = 90°$, $\Delta y = 0$, and $\Delta z = 0$





Case 8: Vector potential, singular interaction, $\theta = 180°$, $\Delta y = 0$, and $\Delta z = 0$



Outline

- Introduction
- Triangle Quadrature Rules
- Logarithmic Singularities in the Test Integrand of the EFIE
- Numerical Experiments for (Near-)Singular and Far Interactions
- Summary
  - Concluding Remarks

Summary

- Introduced 2 symmetric quadrature approaches for arbitrary functions
- Motivated by need to integrate singular test integrands in EFIE
- Approach 1
  - Generally most efficient for singular integrands – outperformed polynomial rules by orders of magnitude
  - Similar efficiency to polynomial rules for nonsingular integrands
- Approach 2
  - More efficient than polynomial rules for singular integrands
  - Error decreases monotonically relative to number of integration points
  - Points are cheap to compute (from 1D rules)

Additional Information

- B. Freno, W. Johnson, B. Zinser, S. Campione
  Symmetric triangle quadrature rules for arbitrary functions
  Computers & Mathematics with Applications (2020) `arXiv:1909.01480`

- B. Freno, W. Johnson, B. Zinser, D. Wilton, F. Vipiana, S. Campione
  Symmetric numerical integration techniques for singular integrals in the method-of-moments implementation of the electric-field integral equation
  `arXiv:1911.02107`

Questions?

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